# Many simple cardinal invariants 

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#### Abstract

For $g<f$ in $\omega^{\omega}$ we define $\mathbf{c}(f, g)$ be the least number of uniform trees with $g$-splitting needed to cover a uniform tree with $f$-splitting. We show that we can simultaneously force $\aleph_{1}$ many different values for different functions $(f, g)$. In the language of [Blass]: There may be $\aleph_{1}$ many distinct uniform $\boldsymbol{\Pi}_{1}^{0}$ characteristics.


## 0. Introduction

[Blass] defined a classification of certain cardinal invariants of the continuum, based on the Borel hierarchy. For example, to every $\Pi_{1}^{0}$ formula $\varphi(x, y)=\forall n R(x \upharpoonright n, y\lceil n)$ ( $R$ recursive) the cardinal

$$
\kappa_{\varphi}:=\min \left\{\mathcal{B} \subseteq{ }^{\omega} \omega: \forall x \in{ }^{\omega} \omega \exists y \in \mathcal{B}: \varphi(x, y)\right\}
$$

is the "uniform $\Pi_{1}^{0}$ characteristic" associated to $\varphi$.
Blass proved structure theorems on simple cardinal invariants, e.g., that there is a smallest $\boldsymbol{\Pi}_{1}^{0}$ characteristic (namely, $\operatorname{Cov}(\mathcal{M})$, the smallest number of first category sets needed to cover the reals), and also that the $\Pi_{2}^{0}$-characteristics can behave quite chaotically. He asked whether the known uniform $\Pi_{1}^{0}$ characteristics $(\mathbf{c}, \mathbf{d}, \mathbf{r}, \operatorname{Cov}(\mathcal{M}))$ are the only ones or (since that is very unlikely) whether there could be a reasonable classification of the uniform $\Pi_{1}^{0}$ characteristics - say, a small list that contains all these invariants.
In this paper we give a strong negative answer to this question: For two $\boldsymbol{\Pi}_{1}^{0}$ formulas $\varphi_{1}, \varphi_{2}$ we say that $\varphi_{1}$ and $\varphi_{2}$ define "potentially nonequal characteristics" if $\kappa_{\varphi_{1}} \neq \kappa_{\varphi_{2}}$ is consistent. We say that $\varphi_{1}$ and $\varphi_{2}$ define "actually different characteristics", if $\kappa_{\varphi_{1}} \neq \varphi_{2}$.

We will find a family of $\Pi_{1}^{0}$-formulas indexed by a real parameter $(f, g)$, and we will show not only that there is a perfect set of parameters which defines pairwise potentially nonequal $\boldsymbol{\Pi}_{1}^{0}$-characteristics, but we produce a single universe in which (at least) $\aleph_{1}$ many cardinals appear as $\boldsymbol{\Pi}_{1}^{0}$-characteristics. (In fact it

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is also possible to produce a universe where there is a perfect set of parameters defining pairwise actually different $\boldsymbol{\Pi}_{1}^{\mathbf{0}}$-characteristics. See [Shelah 448a]).
If we want more than countably many cardinals, we obviously have to use the boldface pointclass. But the proof also produces many lightface uniform $\Pi_{1}^{0}$ characteristics.
For more information on cardinal invariants, see [Blass], [van Douwen], [Vaughan].

From another point of view, this paper is part of the program of finding consistency techniques for a large continuum, i.e., we want $2^{\aleph_{0}}>\aleph_{2}$ and have many values for cardinal invariants. We use a countable support product of forcing notions with an axiom A structure.

We will use invariants that were implicitly introduced in [Shelah $326, \S 2$ ], where it was proved that $\mathbf{c}(f, g)$ and $\mathbf{c}\left(f^{\prime}, g^{\prime}\right)$ (see below) may be distinct.
0.1 Definition: If $f \in{ }^{\omega} \omega$, we say that $\bar{B}=\left\langle B_{k}: k \in \omega\right\rangle$ is an $f$-slalom if for all $k,\left|B_{k}\right|=f(k)$. We write $h \in \bar{B}$ for $h \in \prod_{n} B_{n}$, i.e., $\forall n h(n) \in B_{n}$. (See figure 1) This is a $\Pi_{1}^{0}$-formula in the variables $h$ and $\bar{B}$.
Some authors call the set $\{h: h \in \bar{B}\}$ a "belt", or "uniform tree".
For example, $\prod_{n} f(n)$ is an $f$-slalom, because we identify the number $f(n)$ with the set of predecessors, $\{0, \ldots, f(n)-1\}$.

Figure 1: A slalom
0.2 Definition: Assume $f, g \in{ }^{\omega} \omega$. Assume that $\mathcal{B}$ is a family of $g$-slaloms, and $\bar{A}=\left\langle A_{k}: k \in \omega\right\rangle$ is an $f$-slalom.
We say that $\mathcal{B}$ covers $\bar{A}$ iff:
(*) for all $s \in \bar{A}$ there is $\bar{B} \in \mathcal{B}$ such that $s \in \bar{B}$
0.3 Definition: Assume $f, g \in{ }^{\omega} \omega$. Then we define the cardinal invariant $\mathbf{c}(f, g)$ to be the minimal number of $g$-slaloms needed to cover an $f$-slalom.
(Clearly this makes sense only if $\forall k f(k), g(k)>0$, so we will assume that from now on.)
This is a uniform $\Pi_{1}^{0}$-characteristic. (Strictly speaking, we are not working in ${ }^{\omega} \omega$, but rather in ${ }^{\omega}\left([\omega]^{<\omega}\right)$, but a trivial coding translates $\mathbf{c}(f, g)$ into a "uniform $\Pi_{1}^{0}$ characteristic" as defined above.)
Some relations between these cardinal invariants are provable in ZFC: For example, if $g<g^{\prime}<f^{\prime}<f$, then $\mathbf{c}\left(f^{\prime}, g^{\prime}\right) \leq \mathbf{c}(f, g)$. Also, $\mathbf{c}\left(f^{2}, g^{2}\right) \leq \mathbf{c}(f, g)$.
We will show that if $(f, g)$ is sufficiently different from $\left(f^{\prime}, g^{\prime}\right)$, then the values of $\mathbf{c}(f, g)$ and $\mathbf{c}\left(f^{\prime}, g^{\prime}\right)$ are quite independent, and moreover: if $\left\langle\left(f_{i}, g_{i}\right): i<\omega_{1}\right\rangle$ are pairwise sufficiently different, then almost any assignment of the form $\mathbf{c}\left(f_{i}, g_{i}\right)=\kappa_{i}$ will be consistent.

Similar results are possible for the "dual" version of $\mathbf{c}(f, g): \mathbf{c}^{d}(f, g):=$ the smallest family of $g$-slaloms $\bar{B}$ such that for every $h$ bounded by $f$ there are infinitely many $k$ with $h(k) \in B_{k}$, and for the "tree" version (a $g$-tree is a tree where every node in level $k$ has $g(k)$ many successors). See [Shelah 448a].

We thank Tomek Bartoszynski for pointing out the following known results about the cardinal characteristics $\mathbf{c}(f, g)$ :
For example, lemma 1.11 follows from Theorem 3.17 in [Comfort-Negrepontis]: Taking $\kappa=\alpha=\omega, \beta=n$, and letting $\mathcal{S} \subseteq n^{\omega}$ be a family of $\omega$-large oscillation, then no family of $n$-1-slaloms of size $<2^{\aleph_{0}}$ can cover $\mathcal{S}$. Indeed, whenever $F$ is a function on $\mathcal{S}$ such that for each $s \in \mathcal{S}, F(s)$ is a $n$-1-slalom covering $s$, then $F$ has to be finite-to-one and in fact at most $n-1$-to-one.

Also, since $\mathbf{c}(f, f-1)$ is the size of the smallest family of functions below $f$ which does not admit an "infinitely equal" function, i.e.,

$$
\mathbf{c}(f, f-1)=\min \left\{|G|: G \subseteq \prod_{n} f(n) \& \forall h \in \prod_{n} f(n) \exists g \in G \forall^{\infty} n f(n) \neq g(n)\right\}
$$

by [Miller] we have that the minimal value of $\mathbf{c}(f, f-1)$ is the smallest size of a set of reals which does not have strong measure zero.

Also, note that if $r$ is a random real over $V$ in $\prod_{n} f(n)$, and if $\sum_{n=1}^{\infty} 1 / f(n)=\infty$, then $\prod_{n}(1-1 / f(n))=0$, so $r$ cannot be covered by any $f-1$-slalom from $V$.
Conversely, if $\sum_{n=1}^{\infty} 1 / f(n)<\infty$, then for any function $h \in \prod_{n} f(n) \cap V$ there is a condition forcing that $h$ is covered by the $f-1$-slalom $(\{0, \ldots, f(k)-1\}-\{r(k)\}: k \in \omega)$.
Thus, if we add $\kappa$ many random reals with the measure algebra, a easy density argument shows that in the resulting model we have

$$
\mathbf{c}(f, f-1)= \begin{cases}\kappa=2^{\aleph_{0}} & \text { if } \sum_{n=1}^{\infty} 1 / f(n)=\infty \\ \aleph_{1} & \text { otherwise (use any } \aleph_{1} \text { many of the random reals) }\end{cases}
$$

That already shows that we can have at least two distinct values of $\mathbf{c}(f, g)$ and $\mathbf{c}\left(f^{\prime}, g^{\prime}\right)$.

Contents of the paper: In section 1 we prove results in ZFC of the form

$$
\text { "If }(f, g) \text { is in relation } \ldots \text { to }\left(f^{\prime}, g^{\prime}\right) \text {, then } \mathbf{c}(f, g) \leq \mathbf{c}\left(f^{\prime}, g^{\prime}\right) \text { " }
$$

In section 2 we define a forcing notion $Q_{f, g}$ that increases $\mathbf{c}(f, g)$. (I.e., in $V^{Q_{f, g}}$, the $g$-slaloms from $V$ do not cover $\prod_{n} f(n)$.) Informally speaking, elements of $Q_{f, g}$ are perfect trees in which the size of the splitting is bounded by $f$, sometimes $=1$, but often (i.e., on every branch), much bigger than $g$.

In section 3 we show that, assuming $\left\{\left(f_{\xi}, g_{\xi}\right): \xi<\omega_{1}\right\}$ are sufficiently "independent", a countable support product $\prod_{\xi<\omega_{1}} Q_{\xi}^{\kappa \xi}$ of such forcing notions will force $\forall \xi \mathbf{c}\left(f_{\xi}, g_{\xi}\right)=\kappa_{\xi}$.
We use the symbol $\odot$ to denote the end of a proof, and we write $\Theta$ when we leave a proof to the reader.

## 1. Results in ZFC

1.1 Notation: Operations and relations on functions are understood to be pointwise, e.g., $f / g, g^{\varepsilon}, g<f$, etc. $\lfloor x\rfloor$ is the greatest integer $\leq x . \lim f$ is $\lim _{k \rightarrow \infty} f(k)$.
We write $f \leq^{*} g$ for $\exists n \forall k \geq n f(k) \leq g(k)$.
First we state some obvious facts:

### 1.2 Fact:

(1) $f \leq g$ iff $\mathbf{c}(f, g)=1$.
(2) $f \leq^{*} g$ iff $\mathbf{c}(f, g)$ finite.
(3) If $A:=\{k: g(k)<f(k)\}$ is infinite then $\mathbf{c}(f \upharpoonright A, g \upharpoonright A)=\mathbf{c}(f, g)$.
(4) If $\pi$ is a permutation of $\omega$, then $\mathbf{c}(f \circ \pi, g \circ \pi)=\mathbf{c}(f, g)$.1.2
(Strictly speaking, we define $\mathbf{c}(f, g)$ only for functions $f, g$ defined on all of $\omega$, so (3) should be formally rephrased as $\mathbf{c}(f \circ h, g \circ h)=\mathbf{c}(f, g)$, where $h$ is a 1-1 enumeration of $A)$
1.3 Convention: We will concentrate on the case where $\mathbf{c}(f, g)$ is infinite, so we will wlog assume that $g<f$. By (4), we may also wlog assume that $g$ is nondecreasing.

In these cases we will have that $\mathbf{c}(f, g)$ is infinite, and moreover an easy diagonal argument shows the following fact:

### 1.4 Fact:

$\mathbf{c}(f, g)$ is uncountable.1.4

Furthermore, we have the following properties:

### 1.5 Fact:

(1) (Monononicity) If $f \leq^{*} f^{\prime}, g \geq^{*} g^{\prime}$, then $\mathbf{c}(f, g) \leq \mathbf{c}\left(f^{\prime}, g^{\prime}\right)$.
(2) (Multiplicativity) $\mathbf{c}\left(f \cdot f^{\prime}, g \cdot g^{\prime}\right) \leq \mathbf{c}(f, g) \cdot \mathbf{c}\left(f^{\prime}, g^{\prime}\right)$.
(3) (Transitivity) $\mathbf{c}(f, h) \leq \mathbf{c}(f, g) \cdot \mathbf{c}(g, h)$.
(4) (Invariance) $\mathbf{c}(f, g)=\mathbf{c}\left(f^{-}, g^{-}\right.$) (where $f^{-}$is the function defined by $f^{-}(n)=f(n+1)$.
(5) (Monotonicity II) If $A \subseteq \omega$ is infinite, then $\mathbf{c}(f \upharpoonright A, g \upharpoonright A) \leq \mathbf{c}(f, g)$.1.5
1.6 Remark: (2) implies in particular $\mathbf{c}\left(f^{n}, g^{n}\right) \leq \mathbf{c}(f, g)$. See 3.4 for an example of $\mathbf{c}\left(f^{2}, g^{2}\right)<\mathbf{c}(f, g)$.

The following inequalities need a little more work.

### 1.7 Lemma:

(1) $\mathbf{c}(f \cdot\lfloor f / g\rfloor, f)=\mathbf{c}(f, g)$.
(2) $\mathbf{c}(f \cdot\lfloor f / g\rfloor, g)=\mathbf{c}(f, g)$.
(3) $\mathbf{c}\left(f \cdot\lfloor f / g\rfloor^{m}, g\right)=\mathbf{c}(f, g)$ for all $m \in \omega$.

Proof: (2) follows from (1) using transitivity, and (3) follows from (2) by induction, so we only have to prove (1).

Proof of (1): By monotonicity we only have to show $\leq$. So let $(N, \in)$ be a reasonably closed model of a large fragment of ZFC (say, $(N, \in)<\left(H\left(\chi^{+}\right), \in\right)$, where $\left.\chi=2^{\mathbf{c}}\right)$ of size $\mathbf{c}(f, g)$ such $\prod_{n} f(n)$ is covered by the set of all $g$-slaloms from $N$.

Define $h$ by $h(k):=f(k) \cdot\lfloor f(k) / g(k)\rfloor$. We can find a family $\left\langle B_{k}^{i}: i<f(k), k \in \omega\right\rangle$ in $N$ such that for all $k$, $\{0, \ldots, h(k)-1\}=\bigcup_{i<f(k)} B_{k}^{i}$, where $\left|B_{k}^{i}\right| \leq f(k) / g(k)$. We have to show that the set of $f$-slaloms from $N$ covers $\prod_{k} h(k)$.
So let $x$ be a function satisfying $\forall k x(k) \in \bigcup_{i<f(k)} B_{k}^{i}$. We can define a function $y \in \prod_{n} f(n)$ such that for all $k, x(k) \in B_{k}^{y(k)}$. So there is some $g$-slalom $\bar{C} \in N$ such that for all $k, y(k) \in C_{k}$.
Define $\bar{A}=\left\langle A_{k}: k \in \omega\right\rangle$ by $A_{k}:=\bigcup_{i \in C_{k}} B_{k}^{i}$. Then $\left|A_{k}\right| \leq\left|C_{k}\right| \cdot\left|B_{k}^{i}\right| \leq g(k) \cdot f(k) / g(k)=f(k)$, so $\bar{A}$ is an $f$-slalom in $N$, and for all $k, x(k) \in A_{k}$.
1.8 Lemma: Assume $f>g>0$. Assume that $\left\langle w_{i}: i \in \omega\right\rangle$ is a partition of $\omega$ into finite sets, and for each $i$ there are $\bar{H}^{i}=\left\langle H_{l}^{i}: l \in w_{i}\right\rangle$ satisfying (a)-(c). Then $\mathbf{c}\left(f^{\prime}, g^{\prime}\right) \leq \mathbf{c}(f, g)$.
(a) $\operatorname{dom} H_{l}^{i}=f^{\prime}(i)=\left\{0, \ldots, f^{\prime}(i)-1\right\}$
(b) $\operatorname{rng} H_{l}^{i} \subseteq f(l)=\{0, \ldots, f(l)-1\}$
(c) Whenever $\left\langle u_{l}: l \in w_{i}\right\rangle$ satisfies

$$
\begin{aligned}
& u_{l} \subseteq f(l) \\
& \left|u_{l}\right| \leq g(l)
\end{aligned}
$$

then $\left\{n<f^{\prime}(i): \forall l \in w_{i} H_{l}^{i}(n) \in u_{l}\right\}$ has cardinality $\leq g^{\prime}(i)$
Proof: To any $g$-slalom $\bar{B}=\left\langle B_{l}: l \in \omega\right\rangle$ we can associate a $g^{\prime}$-slalom $\bar{B}^{*}=\left\langle B_{i}^{*}: i \in \omega\right\rangle$ by letting

$$
B_{i}^{*}:=\left\{n<f^{\prime}(i): \forall l \in w_{i} H_{l}^{i}(n) \in w_{l}\right\}
$$

Conversely, to any function $x \in \prod_{i} f^{\prime}(i)$ we can define a function $x^{*}$ in $\prod_{n} f(n)$ by

$$
\text { if } l \in w_{i} \text {, then } x^{*}(l)=H_{l}^{i}(x(i))
$$

It is easy to check that if $x^{*}$ is in $\bar{B}$ then $x$ is in $\bar{B}^{*}$. The result follows.
1.9 Corollary: Assume $0=n_{0}<n_{1}<\cdots$, and let

$$
\begin{aligned}
f^{\prime}(i) & :=f\left(n_{i}\right) \cdot f\left(n_{i}+1\right) \cdots f\left(n_{i+1}-1\right) \\
g^{\prime}(i) & :=g\left(n_{i}\right) \cdot g\left(n_{i}+1\right) \cdots g\left(n_{i+1}-1\right)
\end{aligned}
$$

Then $\mathbf{c}\left(f^{\prime}, g^{\prime}\right) \leq \mathbf{c}(f, g)$.
Proof: Identify the set of numbers less than $f\left(n_{i}\right) \cdot f\left(n_{i}+1\right) \cdots f\left(f_{i+1}-1\right)$ with the cartesian product $\prod_{n_{i} \leq k<n_{i+1}} f(k)$, and let

$$
H_{l}^{i}: \prod_{n_{i} \leq k<n_{i+1}} f(k) \rightarrow f(l)
$$

be the projection onto the $l$-coordinate. We leave the verification of $1.8(\mathrm{c})$ to the reader.1.9
1.10 Lemma: If $g$ is constant, $f(k) \geq 2^{k}$, then $\mathbf{c}(f, g)=\mathbf{c}$.

Proof: Let $\forall k g(k)=n, f(k)=2^{k}$. Assume that $\prod_{l}{ }_{l} 2$ can be covered by $<\mathbf{c}$ many $g$-slaloms.
For any $\eta \in{ }^{\omega} 2$, the sequence $\bar{\eta}:=\langle\eta \upharpoonright l: l \in \omega\rangle$ is in $\prod_{l}{ }_{l}$ 2. But any $g$-slalom can contain only $n$ many such $\bar{\eta}$, i.e. for any $g$-slalom $\bar{B}=\left\langle B_{l}: l \in \omega\right\rangle$ we have

$$
\left|\left\{\eta \in{ }^{\omega} 2: \forall l \eta \upharpoonright l \in B_{l}\right\}\right| \leq m
$$

Since there are continuum many $\eta$ we need continuum many $g$-slaloms to cover $\prod_{l} f(l)$ (or equivalently, $\left.\prod_{l}{ }^{l} 2\right)$.
1.11 Lemma: If $f$ and $g$ are constant with $f>g$, then $\mathbf{c}(f, g)=\mathbf{c}$.

Proof: Using monotonicity wlog we assume that $f(k)=n+1, g(k)=n$ for all $k$. We will use 1.8. Let $\omega=\bigcup_{i \in \omega} w_{i}$ be a partition of $\omega$ where $\left|w_{i}\right|=n^{2^{i}}$.
Let $f^{\prime}(i)=2^{i}, g^{\prime}(i)=n$, and let $\left\langle H_{l}^{i}: l \in w_{i}\right\rangle$ enumerate all functions from $2^{i}$ to $n$.
We plan to show $\mathbf{c}(f, g) \geq \mathbf{c}\left(f^{\prime}, g^{\prime}\right)$ (so $\mathbf{c}(f, g)=\mathbf{c}$ by 1.10 ). We want to apply 1.8 , so fix a sequence $\left\langle u_{l}: l \in w_{i}\right\rangle$, where $u_{l} \subseteq f(l)$ and $\left|u_{l}\right| \leq g(l)$.
To show that the hypotheses of 1.8 are satisfied, fix $i_{0}$ and let

$$
A:=\left\{x<f^{\prime}\left(i_{0}\right): \forall l \in w_{i_{0}} H_{l}^{i_{0}}(x) \in u_{l}\right\}
$$

and assume $A$ has cardinality $>g^{\prime}\left(i_{0}\right)=n$. So let $x_{0}, \ldots, x_{n}$ be distinct elements of $A$. Let $H: f^{\prime}\left(i_{0}\right) \rightarrow n+1$ be a function satisfying

$$
\forall j \leq n H\left(x_{j}\right)=j
$$

$H$ is one of the functions $\left\{H_{l}^{i_{0}}: l \in w_{i_{0}}\right\}$, say $H=H_{l_{0}}^{i_{0}}$. Let $j_{0} \notin u_{l_{0}}$, then also

$$
x_{j_{0}} \notin\left\{x<f^{\prime}\left(i_{0}\right): H_{l_{0}}^{i_{0}}(x) \in u_{l_{0}}\right\} \supseteq A,
$$

contradicting $x_{j_{0}} \in A$.
1.12 Corollary: If $f>g$, and $\liminf _{k \rightarrow \infty} g(k)<\infty$, then $\mathbf{c}(f, g)=\mathbf{c}$.

Proof: This follows from 1.11, using monotonicity and monotonicity II.
We can now extend 1.7 as follows:
1.13 Theorem: If for some $\varepsilon>0, g^{1+\varepsilon} \leq f$, then for all $n, \mathbf{c}\left(f^{n}, g\right)=\mathbf{c}(f, g)$.

Proof: First we consider a special case: Assume that $g^{2} \leq f$. Then we get

$$
\mathbf{c}(f, g) \leq \mathbf{c}\left(f^{2}, g\right) \leq \mathbf{c}\left(f^{2}, f\right) \cdot \mathbf{c}(f, g) \leq \mathbf{c}\left(f^{2}, g^{2}\right) \cdot \mathbf{c}(f, g)=\mathbf{c}(f, g)
$$

Now we use this result on $(f, g)$, then on $\left(f^{2}, g\right)$, etc, to get

$$
\mathbf{c}(f, g)=\mathbf{c}\left(f^{2}, g\right)=\mathbf{c}\left(f^{4}, g\right)=\mathbf{c}\left(f^{8}, g\right)=\cdots
$$

and use monotonicity to get the general result under the assumption $g^{2} \leq f$.
Now we consider the general case $g^{1+\varepsilon} \leq f$ :
If $g$ does not diverge to infinity, we have already (by 1.12) $\mathbf{c}(f, g)=\mathbf{c}$. Otherwise we can find some $\delta>0$ such that for almost all $k$,

$$
\frac{f(k)}{g(k)} \geq g(k)^{\delta}+1
$$

so

$$
\left\lfloor\frac{f(k)}{g(k)}\right\rfloor \geq g(k)^{\delta}
$$

Now choose $m$ such that $m \cdot \delta>1$. Then $\lfloor f(k) / g(k)\rfloor^{m} \geq g$. By 1.7, $\mathbf{c}\left(f \cdot\lfloor f / g\rfloor^{m}, g\right)=\mathbf{c}(f, g)$ and so by monotonicity also $\mathbf{c}(f \cdot g, g)=\mathbf{c}(f, g)$. Since $g^{2} \leq f \cdot g$, we can apply the result from the special case above to get $\mathbf{c}(f, g)=\mathbf{c}\left(f^{n} \cdot g^{n}, g\right)$ so in particular, $\mathbf{c}\left(f^{n}, g\right)=\mathbf{c}(f, g)$.
If $f$ is not much bigger than $g$, the assumption in 1.7 and 1.13 may be false. For these cases, we can prove the following:

### 1.14 Lemma:

(1) $\mathbf{c}(2 f-g, f)=\mathbf{c}(f, g)$.
(2) $\mathbf{c}(2 f-g, g)=\mathbf{c}(f, g)$.
(3) $\mathbf{c}(f+m(f-g), g)=\mathbf{c}(f, g)$ for all $m \in \omega$.

Proof: The proof is similar to the proof of 1.7. Again we only have to show (1). Let ( $N, \in$ ) be a reasonably closed model of a large fragment of ZFC (say, $(N, \in) \prec\left(H\left(\chi^{+}\right), \in\right)$, where $\left.\chi=2^{\mathbf{c}}\right)$ of size $\mathbf{c}(f, g)$ such $\prod_{n} f(n)$ is covered by the set of all $g$-slaloms from $N$.
Define $h$ by $h(k):=f(k)+f(k)-g(k)$. We can find a family $\left\langle B_{k}^{i}: i<f(k), k \in \omega\right\rangle$ in $N$ such that for all $k,\{0, \ldots, h(k)-1\}=\bigcup_{i<f(k)} B_{k}^{i}$, where $\left|B_{k}^{i}\right|=2$ for $l<f(k)-g(k)$, and $\left|B_{k}^{i}\right|=1$ otherwise. We have to show that the set of $f$-slaloms from $N$ covers $\prod_{k} h(k)$.

So let $x$ be a function satisfying $\forall k x(k) \in \bigcup_{i<f(k)} B_{k}^{i}$. We can define a function $y \in \prod_{n} f(n)$ such that for all $k, x(k) \in B_{k}^{y(k)}$. So there is some $g$-slalom $\bar{C} \in N$ such that for all $k, y(k) \in C_{k}$.

Define $\bar{A}=\left\langle A_{k}: k \in \omega\right\rangle$ by $A_{k}:=\bigcup_{i \in C_{k}} B_{k}^{i}$. Thus $A_{k}$ is the union of $g(k)$ many sets, of which at most $f(k)-g(k)$ are pairs, and the others singletons. Thus $\left|A_{k}\right| \leq g(k)+(f(k)-g(k))=f(k)$, so $\bar{A}$ is an $f$-slalom in $N$, and for all $k, x(k) \in A_{k}$.

Similar to the proof of 1.13 we now get:

### 1.15 Lemma:

(1) If $2 g \leq f$, then for all $n, \mathbf{c}(n f, g)=\mathbf{c}(f, g)$.
(2) If for some $\varepsilon>0,(1+\varepsilon) g \leq f$, then for all $n, \mathbf{c}(n f, g)=\mathbf{c}(f, g)$.1.15

## 2. The forcing notion $Q_{f, g}$

2.1 Definition: We fix sequences $\left\langle n_{k}^{-}: k \in \omega\right\rangle$ and $\left\langle n_{k}^{+}: k \in \omega\right\rangle$ that increase very quickly and satisfy $n_{0}^{-} \ll n_{0}^{+} \ll n_{1}^{-} \ll n_{1}^{+} \ll \cdots$. In particular, we demand
(1) For all $k \prod_{j<k} n_{j}^{-} \leq n_{k}^{-}$
(2) $\lim _{k \rightarrow \infty} \frac{\log n_{k}^{+}}{\log n_{k}^{-}}=0$.
(3) $n_{k}^{-} \cdot n_{k}^{+}<n_{k+1}^{-}$.

We will only consider functions $f, g$ satisfying $n_{k}^{-} \leq g(k)<f(k) \leq n_{k}^{+}$. This is partly justified by 1.9 , and it also helps to keep the formulation of the main theorem relatively simple.
2.2 Definition: Let $X \neq \emptyset$ be finite, $c, d \in \omega$. A $(c, d)$-complete norm on $\mathbf{P}(X)$ is a map

$$
\|\|: \mathbf{P}(\mathbf{X})-\{\emptyset\} \rightarrow \omega
$$

mapping any nonempty $a \subseteq X$ to a number $\|a\|$ such that whenever $a=a_{1} \cup \cdots \cup a_{c} \subseteq X$, then for some $i_{1}, \ldots, i_{d} \in\{1, \ldots, c\},\left\|a_{i_{1}} \cup \cdots \cup a_{i_{d}}\right\| \geq\|a\|-1$. ( $|a|$ is the cardinality of the set $a$ )

A natural $(c, d)$-complete norm is given by $\|a\|:=\log _{c / d}|a| . c$-complete means $(c, 1)$-complete.
2.3 Definition: We call $(f, g, h)$ progressive, if $f, g, h$ are functions in ${ }^{\omega} \omega$, satisfying
(1) For all $k, n_{k}^{-} \leq g(k)<f(k) \leq n_{k}^{+}$
(2) For all $k, n_{k}^{-} \leq h(k)$
(3) $\lim _{k} \log \frac{f(k)}{g(k)} / \log h(k)=\infty$.

We call $(f, g)$ progressive, if there is a function $h$ such that $(f, g, h)$ is progressive(or equivalently, if $\left(f, g, n^{-}\right)$ is progressive, where $n^{-}$is the function defined by $\left.n^{-}(k)=n_{k}^{-}\right)$.
2.4 Remark: For example, if $f$ and $g$ satisfy (1), then $(f, g, g)$ is progressive iff $\log f / \log g \rightarrow \infty$. $\bigodot$2.4

In 2.6 we will define a forcing notion $Q_{f, g, h}$ for any progressive $(f, g, h)$. First we recall the following notation:
2.5 Notation: ${ }^{<\omega} \omega=\bigcup_{n}{ }^{n} 2$ is the set of finite sequences of natural numbers. For $s \in{ }^{<\omega} \omega,|s|$ is the length of $s$.

A tree $p$ is a nonempty subset of $<\omega_{\omega}$ with the properties

$$
\begin{aligned}
& \forall \eta \in p \forall k<|\eta|: \eta \upharpoonright k \in p \\
& \forall \eta \in p: \operatorname{succ}_{p}(\eta) \neq \emptyset, \text { where }
\end{aligned}
$$

$$
\operatorname{succ}_{p}(\eta):=\{\nu \in p: \eta \subset \nu,|\eta|+1=|\nu|\} .
$$

A branch $b$ of $p$ is a maximal linearly $\subseteq$-ordered subset of $p$. Every branch $b$ defines a function $\bar{b}: \omega \rightarrow \omega$ by $\bar{b}=\bigcup b$. We usually identify $b$ and $\bar{b}$, so we write $b \upharpoonright k($ instead of $(\bigcup b) \upharpoonright k)$ for the $k$ th element of $b$.
The set of all branches of $p$ is written as $[p]$.
For $\eta \in p$, we let

$$
p^{[\eta]}:=\{\nu \in p: \nu \subseteq \eta \text { or } \eta \subseteq \nu\}
$$

We let

$$
\begin{aligned}
\operatorname{split}(p) & :=\left\{\eta \in p:\left|\operatorname{succ}_{p}(\eta)\right|>1\right\} \quad(\text { the splitting nodes of } p) \\
\operatorname{split}_{n}(p) & :=\{\eta \in \operatorname{split}(p):|\{\nu \subset \eta: \nu \in \operatorname{split}(p)\}|=n\} \quad \text { (the } n \text {-th splitting level) }
\end{aligned}
$$

and we define the stem of $p$ to be the unique element of $\operatorname{split}_{0}(p)$.
2.6 Definition: Assume $f, g, h$ are as in 2.3. Then we define for all $k$, and for all sets $x$

$$
\|x\|_{k}:=\left\lfloor\frac{\log (|x| / g(k))}{\log h(k)}\right\rfloor
$$

and we define the forcing notion $Q_{f, g}$ (or more accurately, $Q_{f, g, h}$ ) to be the set of all $p$ satisfying
(1) $p$ is a perfect tree.
(2) $\forall \eta \in p \forall i \in \operatorname{dom}(\eta) \eta(i)<f(i)$.
(3) $\forall \eta \in \operatorname{split}_{n}(p)\left\|\operatorname{succ}_{p}(\nu)\right\|_{|\nu|} \geq n$.

We let $p \leq q$ (" $q$ extends $p$ ") iff $q \subseteq p$.
2.7 Remark: If we define

$$
p \sqsubseteq_{k} q \text { iff } p \leq q \text { and } \operatorname{split}_{k}(p) \subseteq q
$$

then $Q_{f, g, h}$ satisfies axiom A, and is in fact strongly ${ }^{\omega} \omega$-bounding, i.e., for name of an ordinal, $\underset{\sim}{\alpha}$, for any $p$ and for any $n$ there is a finite set $A$ and a condition $q \sqsupseteq_{n} p, q \Vdash \underset{\sim}{\alpha} \in A$. However, it will be more convenient to use the relation $\leq_{n}$ that is based on levels rather than splitting levels.
2.8 Definition: For $p, q \in Q, n \in \omega$ we define

$$
p \leq_{n} q \text { iff } p \leq q \text { and } p \cap{ }^{\leq n} \omega \subseteq q
$$

2.9 Notation: We will usually write $\|\eta\|_{p}$ instead of $\left\|\operatorname{succ}_{p}(\eta)\right\|_{|\eta|}$.
2.10 Remark: This forcing is similar to the forcing in [Shelah 326], but note the following important difference: Whereas in [Shelah 326] all nodes above the stem have to be splitting points, we allow many nodes to have only one successor, as long as there "many" nodes with high norm.

### 2.11 Remark:

(1) The norm $\|\cdot\|_{k}$ is $h(k)$-complete (hence also $n_{k}^{-}$-complete).
(2) If $c / d \leq h(k)$, then the norm is $(c, d)$-complete.
(3) If $\|a\|_{k}>0$, then $|a|>g(k)$.
(4) $\|f(k)\|_{k} \rightarrow \infty$ (so $Q_{f, g, h}$ is nonempty).2.11

We will see in the next section that this forcing (and any countable support product of such forcings) is proper and ${ }^{\omega} \omega$-bounding. For the moment, we only show why this forcing is useful in connection with $\mathbf{c}(f, g)$ :
2.12 Fact: Any generic filter $G \subseteq Q_{f, g}$ defines a "generic branch"

$$
r:=\bigcup_{p \in G} \operatorname{stem}(p)
$$

that avoids all $g$-slaloms from $V$.
Proof: Let $\bar{B}=\left\langle B_{k}: k \in \omega\right\rangle$ be a $g$-slalom in $V$, and let $p \in Q_{f, g}$ be a condition. Let $\eta \in p$ be a node satisfying $\|\eta\|_{p}>0$. Let $k:=|\eta|$. Then $\left|\operatorname{succ}_{p}(\eta)\right|>g(k)$ by $2.11(3)$, so there is $i \notin B_{k}, \eta \frown i \in p$. So $p^{\left[\eta^{-i]}\right.} \Vdash r(k)=i \notin B_{k}$.

## 3. The construction

In this section we will prove the following theorem:
3.1 Theorem (CH): Assume that $\left(f_{\xi}, g_{\xi}: \xi<\omega_{1}\right)$ is a sequence of progressive functions, witnessed by functions $h_{\xi}$ (see 2.3).
Let $\left(\kappa_{\xi}: \xi<\omega_{1}\right)$ be a sequence of cardinals satisfying $\kappa_{\xi}^{\omega}=\kappa_{\xi}$ such that whenever $\kappa_{\xi}<\kappa_{\zeta}$, then

$$
\lim _{k \rightarrow \infty} \min \left(\frac{f_{\zeta}(k)}{g_{\xi}(k)}, \frac{f_{\xi}(k)}{g_{\xi}(k)} / h_{\zeta}(k)\right)=0
$$

(or informally: either $f_{\zeta} \ll g_{\xi}$, or $f_{\xi} / g_{\xi} \ll h_{\zeta}$, or a combination of these two condition holds)
Then there is a proper forcing notion $P$ not collapsing cardinals nor changing cofinalities such that

$$
\Vdash_{P} \forall \xi: \mathbf{c}\left(f_{\xi}, g_{\xi}\right)=\kappa_{\xi}
$$

For the proof we use a countable support product of the forcing notions $Q_{f_{\xi}, g_{\xi}, h_{\xi}}$ described in the previous section.
3.2 Remark: The theorem is of course also true (with the same proof) if we have countably or finitely many functions to deal with.

If we are only interested in 2 cardinal invariants $\mathbf{c}\left(f^{\prime}, g^{\prime}\right), \mathbf{c}(f, g)$, then we can phrase the theorem without the auxiliary functions $h$ as follows: If $(f, g)$ and $\left(f^{\prime}, g^{\prime}\right)$ are progressive, and satisfy

$$
\min \left(\frac{f^{\prime}}{g}, \frac{\log (f / g)}{\log \left(f^{\prime} / g^{\prime}\right)}\right) \rightarrow 0
$$

then $\mathbf{c}(f, g)<\mathbf{c}\left(f^{\prime}, g^{\prime}\right)$ is consistent.
In particular, this shows that our result is quite sharp: For example, if for some function $d$ we have $\lim d=\infty$, $f^{\prime}=f^{d}, g^{\prime}=g^{d}$ (and $(f, g),\left(f^{\prime}, g^{\prime}\right)$ are progressive with the same $\left.n_{k}^{-}, n_{k}^{+}\right)$, then $\mathbf{c}(f, g)<\mathbf{c}\left(f^{\prime}, g^{\prime}\right)$ is consistent. On the other hand, $\mathbf{c}\left(f^{n}, g^{n}\right) \leq \mathbf{c}(f, g)$ for every fixed $n$.
Proof: Choose $h^{\prime}$ such that $\log h^{\prime} \approx 2 \log (f / g)$ whenever $\frac{f^{\prime}}{g} \geq \frac{\log (f / g)}{\log \left(f^{\prime} / g^{\prime}\right)} .\left(f^{\prime}, g^{\prime}, h^{\prime}\right)$ is progressive, and the assumptions of the theorem are satisfied. (Recall that $(f, g)$ is progressive, hence $\log f / g \gg \log n^{-}$, so $h^{\prime}$ will satisfy $\left.h^{\prime}(k) \geq n_{k}^{-}\right)$.
A similar simplified formulation of 3.1 is possible when we deal with only countably many functions.
3.3 Example: There is a family $\left\langle\left(f_{\xi}, g_{\xi}, g_{\xi}: \xi<\mathbf{c}\right\rangle\right.$ of continuum many progressive functions such that for any $\zeta \neq \xi$, $\min \left(\frac{f_{\xi}}{g_{\zeta}}, \frac{f_{\zeta}}{g_{\xi}}\right) \rightarrow 0$. [In particular, under CH we may choose any family $\left(\kappa_{\xi}: \xi<\omega_{1}\right)$ of cardinals satisfying $\kappa_{\xi}^{\omega}=\kappa_{\xi}$ and get an extension where $\mathbf{c}\left(f_{\xi}, g_{\xi}\right)=\kappa_{\xi}$.]
Proof: Let $\ell_{k}:=\left\lfloor\frac{1}{2} \sqrt{\log \frac{\log n_{k}^{+}}{\log n_{k}^{-}}}\right\rfloor$. (Here, "log" can be the logarithm to any (fixed) base, say 2.) Then $\lim _{k \rightarrow \infty} \ell_{k}=\infty$, and by invariance (1.5(4)) we may assume $\ell_{k} \geq 1$ for all $k$.

Let $T \subseteq 2^{<\omega}$ be a perfect tree such that for all $k$ we have $\left|T \cap 2^{k}\right|=\ell_{k}$, say, $T \cap 2^{k}=\left\{s_{1}(k), \ldots, s_{\ell_{k}}(k)\right\}$.
For any $x \in[T]$ (i.e., $x \in 2^{\omega}, \forall k x \upharpoonright k \in T$ ) we now define functions $f_{x}, g_{x}, h_{x}$ by:
If $x \upharpoonright k=s_{i}(k)$, then

$$
\begin{aligned}
f_{x}(k) & =\left(n_{k}^{-}\right)^{\ell_{k}^{2 i}} \\
h_{x}(k)=g_{x}(k) & =\left(n_{k}^{-}\right)^{\ell_{k}^{2 i-1}}
\end{aligned}
$$

We leave the verification that $\left(f_{x}, g_{x}, h_{x}\right)$ is indeed progressive to the reader. [Recall 2.4, and also note that $\log \log f_{x}(k) \leq 2 \ell_{k} \log \ell_{k}+\log \log n_{k}^{-}<\log \log n_{k}^{+}$. Finally, note that if $x \neq y$, then for almost all $k$ we have $\left.\min \left(\frac{f_{x}(k)}{g_{y}(k)}, \frac{f_{y}(k)}{h_{x}(k)}\right) \ll \frac{1}{n_{k}^{-}}.\right]$
3.4 Example: It is consistent to have $\mathbf{c}\left(f^{2}, g^{2}\right)<\mathbf{c}(f, g)$ (for certain $f, g$ ).

Proof: Let $\ell_{k}:=\left\lfloor\frac{1}{6} \log \frac{n_{k}^{+}}{n_{k}^{-}}\right\rfloor$. Assume $\ell_{k}>0$ for all $k$. Then, letting

$$
\begin{aligned}
& f(k):=\left(n_{k}^{-}\right)^{3 \ell_{k}} \\
& g(k):=\left(n_{k}^{-}\right)^{2 \ell_{k}} \\
& h(k):=n_{k}^{-}
\end{aligned}
$$

We have that $(f, g, h)$ and $\left(f^{2}, g^{2}, h\right)$ are progressive, and $\lim \frac{f}{g^{2}}=0$, so we can apply the theorem.3.4

### 3.5 Definition:

Let $\kappa$ be a disjoint union $\kappa=\bigcup_{\xi<\omega_{1}} A_{\xi}$, where $\left|A_{\xi}\right|=\kappa_{\xi}$.
For $\alpha<\kappa$, let $Q_{\alpha}$ be the forcing $Q_{f_{\xi}, g_{\xi}, h_{\xi}}$, if $\alpha \in A_{\xi}$, and let $P=\prod_{\alpha<\kappa} Q_{\alpha}$ be the countable support product of the forcing notions $Q_{\alpha}$, i.e., elements of $P$ are countable functions $p$ with $\operatorname{dom}(p) \subseteq \kappa$, and $\forall \alpha \in \operatorname{dom}(p) p(\alpha) \in Q_{\alpha}$.
For $A \subseteq \kappa$, we write $P \upharpoonright A:=\{p \upharpoonright A: p \in P\}$. Clearly $P \upharpoonright A<P$ for any $A$. In particular, $Q_{\alpha} \ll P$.
We write $\underset{\sim}{r}{ }_{\alpha}$ for the $Q_{\alpha}$-name (or $P$-name) for the generic branch introduced by a generic filter on $Q_{\alpha}$.
We say that $q$ strictly extends $p$, if $q \geq p$ and $\operatorname{dom}(q)=\operatorname{dom}(p)$.
3.6 Facts: Assume CH. Then
(1) each $Q_{\alpha}$ is proper and ${ }^{\omega} \omega$-bounding.
(2) $P$ is proper and ${ }^{\omega} \omega$-bounding.
(3) $P$ satisfies the $\aleph_{2}$-cc.
(4) Neither cardinals nor cofinalities are changed by forcing with $P$.

Proof of (1), (2): See below (3.23, 3.24)
Proof of (3): A straightforward $\Delta$-system argument, using CH .
(4) follows from (2) and (3).
3.6

We plan to show that $\Vdash_{P} \mathbf{c}_{\xi}=\kappa_{\xi}$ for all $\xi<\omega_{1}$.
3.7 Definition: If $p \in P, k \in \omega$, we let the level $k$ of $p$ be

$$
\begin{aligned}
\operatorname{Level}_{k}(p):=\{\bar{\eta}: \operatorname{dom}(\bar{\eta})=\operatorname{dom}(p) & \\
& \forall \alpha \in \operatorname{dom}(\bar{\eta}):|\bar{\eta}(\alpha)|=k, \bar{\eta}(\alpha) \in p(\alpha)\}
\end{aligned}
$$

We define the set of active ordinals at level $k$ as

$$
\operatorname{active}_{k}(p):=\{\alpha \in \operatorname{dom}(p):|\operatorname{stem}(p(\alpha))| \leq k\}
$$

3.8 Remark: Sometimes we identify the set $\operatorname{Level}_{k}(p)$ with the set

$$
\begin{aligned}
& \left\{\bar{\eta}: \operatorname{dom}(\bar{\eta})=\operatorname{active}_{k}(p), \forall \alpha \in \operatorname{dom}(\bar{\eta}):|\bar{\eta}(\alpha)|=k\right\} \\
& \quad=\left\{\bar{\eta} \operatorname{active}_{k}(p): \bar{\eta} \in \operatorname{Level}_{k}(p)\right\}
\end{aligned}
$$

3.9 Definition: We say that the $k$ th level is a splitting level of $p$ (or " $k$ is a splitting level of $p$ ") iff

$$
\exists \alpha \in \operatorname{dom}(p) \exists \eta \in \operatorname{split}(p(\alpha)):|\eta|=k
$$

3.10 Definition: If $\bar{\eta} \in \operatorname{Level}_{k}(p), \bar{\eta}^{\prime} \in \operatorname{Level}_{k^{\prime}}(p), k<k^{\prime}$, then we say that $\bar{\eta}^{\prime}$ extends $\bar{\eta}$ iff for all $\alpha \in \operatorname{dom}(\bar{\eta}), \bar{\eta}^{\prime}(\alpha)$ extends (i.e., $\left.\supseteq\right) \bar{\eta}(\alpha)$.
3.11 Definition: For $p, q \in P, k \in \omega$, we let

$$
p \leq_{k} q \text { iff } p \leq q \text { and } \forall \alpha \in \operatorname{dom}(p): p(\alpha) \leq_{k} q(\alpha) \text { and } \operatorname{active}_{k}(p)=\operatorname{active}_{k}(q)
$$

That is, we allow $\operatorname{dom}(q)$ to be bigger than $\operatorname{dom}(p)$, but for all new $\alpha \in \operatorname{dom}(q)-\operatorname{dom}(p)$ we require that $|\operatorname{stem}(q(\alpha))|>k$.
3.12 Definition: Let $A \subseteq P$. A set $D \subseteq P$ is
dense in $A$, if $\forall p \in A \exists q \in D: p \leq q$
strictly dense in $A$, if $\forall p \in A \exists q \in D: p \leq q$ and $\operatorname{dom}(p)=\operatorname{dom}(q)$
open in $A$, if $\forall p \in D \forall q \in A:(p \leq q$ implies $q \in D)$
almost open in $A$, if $\forall p \in D \forall q \in A:(p \leq q$ and $\operatorname{dom}(p)=\operatorname{dom}(q)$ implies $q \in D)$
These definitions can also be relativized to conditions above a given condition $p_{0}$. If we omit $A$ we mean $A=$ $P$.
3.13 Definition: If $\bar{\eta} \in \operatorname{Level}_{k}(p)$, we let $q=p^{[\bar{\eta}]}$ be the condition defined by $\operatorname{dom}(q)=\operatorname{dom}(p)$, and

$$
\forall \alpha \in \operatorname{dom}(q) q(\alpha)=p(\alpha)^{[\bar{\eta}(\alpha)]}
$$

3.14 Definition: If $p \Vdash \underset{\sim}{x} \in V$, and $\bar{\eta} \in \operatorname{Level}_{k}(p)$, we say that $\bar{\eta}$ decides $\underset{\sim}{x}$ (or more accurately, $p^{[\bar{\eta}]}$ decides $\underset{\sim}{x})$ if for some $y \in V, p^{[\bar{\eta}]} \Vdash \underset{\sim}{x}=\check{y}$.
First we simplify the form of our conditions such that all levels are finite.
3.15 Fact: The set of all conditions $p$ satisfying

I $\forall k\left|\operatorname{active}_{k}(p)\right|<\omega$, and moreover:
II For any splitting level $k$ there is exactly one pair $(\eta, \alpha)$ such that $\left|\operatorname{succ}_{p(\alpha)}(\eta)\right|>1$.
is dense in $P$.
3.16 Fact: If $p$ is in the dense set given by (I) and (II), then the size of level $k$ is $\leq n_{k-1}^{-} \cdot n_{k-1}^{+}<n_{k}^{-}$.

Proof: By induction.
$\bigodot_{3.16}$
From now on we will only work in the dense set of conditions satisfying (I) and (II).
3.17 Notation: For $p$ satisfying (I)-(II), we let $k_{l}=k_{l}(p)$ be the $l$ th splitting level. Let $\eta_{l}=\eta_{l}(p)$ and $\alpha_{l}=\alpha_{l}(p)$ be such that $\left|\eta_{l}(p)\right|=k_{l}(p), \eta_{l}(p) \in \operatorname{split}\left(p\left(\alpha_{l}\right)\right)$. We let $\zeta_{l}=\zeta_{l}(p)$ be such that $\alpha_{l} \in A_{\zeta_{l}}$.
We write $\|p\|_{k_{l}}$ for $\left\|\eta_{l}\right\|_{p\left(\alpha_{l}\right)}$, i.e., for $\left\|\operatorname{succ}_{p\left(\alpha_{l}\right)}\left(\eta_{l}\right)\right\|_{\zeta_{l}, k_{l}}$. (See figure 2)
3.18 Definition: If $p$ is a condition, $l \in \omega, \alpha^{*}:=\alpha_{l}(p), \eta^{*}:=\eta_{l}(p), \nu^{*} \in \operatorname{succ}_{p\left(\alpha^{*}\right)}\left(\eta^{*}\right)$, we can define a stronger condition $q$ by letting $q(\alpha)=p(\alpha)$ for all $\alpha \neq \alpha^{*}$, and

$$
q\left(\alpha^{*}\right):=\left\{\eta \in p\left(\alpha^{*}\right): \text { If } \eta^{*} \subset \eta, \text { then } \nu^{*} \subseteq \eta\right\}
$$

In this case, we say that $q$ was obtained from $p$ by "pruning the splitting node $\eta^{*}$."
To simplify the notation in the fusion arguments below, we will use the following game:

Figure 2: A condition satisfying (I) and (II)
3.19 Definition: For any condition $p \in P, G(P, p)$ is the following two person game with perfect information:

There are two players, the spendthrift and the accountant. A play in $G(P, p)$ last $\omega$ many moves (starting with move number 1) The accountant moves first. We let $p_{0}:=p, i_{0}:=0$.

In the $n$-th move, the accountant plays a pair $\left(\eta^{n}, \alpha^{n}\right)$ with $\eta^{n} \in p_{n-1}\left(\alpha^{n}\right),\left|\eta^{n}\right|=i_{n-1}$, and a number $b_{n}$. Player spendthrift responds by playing a condition $p_{n}$ and a finite sequence $\nu^{n}$ (letting $i_{n}:=\left|\nu^{n}\right|+1$ ) satisfying the following: (See Figure 3)
(1) $p_{n} \geq_{i_{n-1}} p_{n-1}$.
(2) $\nu^{n} \in p_{n}\left(\alpha^{n}\right)$
(3) $\left\|\nu^{n}\right\|_{p_{n}\left(\alpha^{n}\right)}>b_{n}$.
(4) $\nu^{n} \supset \eta^{n}$.
(5) For all $\alpha \in \operatorname{dom}\left(p_{n}\right)-\operatorname{dom}\left(p_{n-1}\right)$, $\left|\operatorname{stem}\left(p_{n}(\alpha)\right)\right|>\left|\nu^{n}\right|$.
(6) $\left|\operatorname{Level}_{\left|\nu^{n}\right|}\left(p_{n}\right)\right|=\left|\operatorname{Level}_{\left|\eta^{n}\right|}\left(p_{n}\right)\right|=\left|\operatorname{Level}_{\left|\eta^{n}\right|}\left(p_{n-1}\right)\right|$
(Remember that all conditions $p_{n}$ have to be in the dense set given by (I) and (II)) Player accountant wins iff after $\omega$ many moves there is a condition $q$ such that for all $n, p_{n} \leq q$, or equivalently, if the function $q$ with domain $\bigcup_{n} \operatorname{dom}\left(p_{n}\right)$, defined by

$$
q(\alpha)=\bigcup_{\alpha \in \operatorname{dom}\left(p_{n}\right)} p_{n}(\alpha)
$$

is a condition. Note that we have $\eta_{l}(q)=\nu^{l}, \alpha_{l}(q)=\alpha^{l}$, since the only splitting points are the ones chosen by spendthrift.
3.20 Fact: Player accountant has a winning strategy in $G(P, p)$.

Figure 3: stage $n$
Proof: We leave the proof to the reader, after pointing out that a finitary bookkeeping will ensure that the limit of the conditions $p_{n}$ is in fact a condition.

In particular, this shows that spendthrift has no winning strategy. Below we will define various strategies for the spendthrift, and use only the fact that there is a play in which the accountant wins.

The game gives us the following lemma:
3.21 Lemma: Assume that $p$ is a condition satisfying (I)-(II). For each $l$ let $\emptyset \neq F_{\eta_{l}} \subseteq \operatorname{succ}_{p\left(\alpha_{l}\right)}\left(\eta_{l}\right)$ be a set of norm $\left\|F_{\eta_{l}}\right\|_{k_{l}} \geq\left\|\operatorname{succ}_{p\left(\alpha_{l}\right)}\left(\eta_{l}\right)\right\| / 2$.
Then there is a condition $q \geq p, \operatorname{dom}(q)=\operatorname{dom}(p)$ such that for all $l$ :

$$
(*) \quad \text { If } \eta_{l}(p) \in q\left(\alpha_{l}(p)\right), \text { then } \operatorname{succ}_{q\left(\alpha_{l}(p)\right)}\left(\eta_{l}(p)\right) \subseteq F_{\eta_{l}}
$$

Proof: The condition $q$ can be constructed by playing the game. In the $n$-th move, spendthrift first finds a $\eta^{n} \supset \nu^{n}$ satisfying $\eta^{n}(i) \in F_{\eta_{i}}$ whenever this is applicable, and $\left\|\operatorname{succ}_{p_{n-1}}\left(\eta^{n}\right)\right\|>2 b_{n}$. Then spendthrift obtains $p_{n}$ by pruning (see 3.18) all splitting nodes of $p_{n-1}$ whose height is between $\left|\eta^{n}\right|$ and $\left|\nu^{n}\right|$ and further thinning out the successors of $\eta^{n}$ to satisfy $\operatorname{succ}_{p_{n}}\left(\eta^{n}\right)=F_{\eta^{n}}$ 。(Note that $F_{\eta^{n}} \subseteq \operatorname{succ}_{p_{n-1}}\left(\eta^{n}\right)=\operatorname{succ}_{p_{0}}\left(\eta^{n}\right)$.) In the resulting condition $q$ the only splitting nodes will be the nodes $\eta^{n}$, so $(*)$ will be satisfied. $\odot_{3.21}$ (Note that in general $\eta_{l}(q) \neq \eta_{l}(p)$, and indeed $k_{l}(q) \neq k_{l}(p)$, since many splitting levels of $p$ are not splitting levels in $q$ anymore.)
3.22 Lemma: Assume $\underset{\sim}{\tau}$ is a $P$-name of a function from $\omega$ to $\omega$, or even from $\omega$ into ordinals. Then the set of conditions satisfying (I)-(III) is dense and almost open.

III Whenever $k$ is a splitting level, then every $\bar{\eta}$ in level $k+1$ decides $\underset{\sim}{\tau} \upharpoonright k$.
Proof of (III): We will use the game from 3.19. We will define a strategy for the spendthrift ensuring that the condition $q$ the accountant produces at the end will satisfy (III).

In the $n$-th move, spendthrift finds a condition $r_{n} \geq_{i_{n-1}} p_{n-1}$ such that for every $\bar{\eta} \in \operatorname{Level}_{i_{n-1}}\left(r_{n}\right)$ the condition $\left(p_{n}\right)^{[\bar{\eta}]}$ decides $\underset{\sim}{\tau} \upharpoonright i_{n-1}+10$. Then spendthrift finds $\eta^{n} \in r_{n}\left(\alpha^{n}\right)$ satisfying the rules and obtains $p_{n}$ with $\eta^{n} \in p_{n}\left(\alpha^{n}\right)$ from $r_{n}$ by pruning all splitting levels between $i_{n-1}$ and $\left|\eta_{n}\right|$. $\square$ Since all levels of $q$ are finite, it is thus possible to find a finite sequence $\bar{B}=\left\langle B_{k}: k \in \omega\right\rangle$ in the ground model that will cover $\underset{\sim}{\tau}$. (I.e. $q \Vdash \underset{\sim}{\tau}(k) \in B_{k}$ ). The rest of this section will be devoted to finding "small" such sets $B_{k}$.
3.23 Corollary: $P$ is ${ }^{\omega} \omega$-bounding and does not collapse $\omega_{1}$.
3.24 Remark: Although it does not literally follow from 3.22, the reader will have no difficulty in showing that $P$ is actually $\alpha$-proper for any $\alpha<\omega_{1} . \bigodot$ Indeed, using the partial orders $\sqsubseteq_{n}$ from 2.7 , it is possible to carry out straightforward fusion arguments, without using the game 3.19 at all. However, the orderings $\leq_{n}$ are more easy to handle, since in induction steps we only have to take care of a single $\eta^{n}$, instead of a front.
3.25 Fact: $\vdash_{P} \forall \tau \in{ }^{\omega} \omega \exists B \subseteq \kappa$, $B$ countable, $B \in V$, and $\tau \in V[G \upharpoonright B]$.

Proof: Let $p$ be any condition and let $\underset{\sim}{\tau}$ be a name for a real. There is a stronger condition $q$ satisfying (I), (II) and (III). Let $B:=\operatorname{dom}(q)$. Clearly $q \Vdash \underset{\sim}{\tau} \in V[G \upharpoonright B]$.
3.26 Corollary: If $\lambda=|A|^{\omega}$, then $\Vdash_{P \upharpoonright A} 2^{\aleph_{0}} \leq \lambda$.

Proof: For each countable subset $B \subseteq A, \Vdash_{P \upharpoonright B} C H$. Since every real in $V[G]$ is in some such $V[G \upharpoonright B]$, the result follows.
3.27 Fact and Notation: If $p$ satisfies (II), then
(1) If $\bar{\eta}\left(\alpha_{l}\right)=\eta_{l}$, and $\nu \in \operatorname{succ}_{p\left(\alpha_{l}\right)}\left(\eta_{l}\right)$, then the requirement

$$
\bar{\eta}^{+\nu}\left(\alpha_{l}\right)=\nu
$$

uniquely defines an extension $\bar{\eta}^{+\nu}$ of $\bar{\eta}$ in $\operatorname{Level}_{k_{l}+1}(p)$.
(2) If $\bar{\eta}\left(\alpha_{l}\right) \neq \eta_{l}, \bar{\eta}$ has a unique extension $\bar{\eta}^{+} \in \operatorname{Level}_{k_{l}+1}(p)$. To simplify the notation in 3.33 below, we also define for this case, for any $\nu \in \operatorname{succ}_{p\left(\alpha_{l}\right)}\left(\eta_{l}\right), \bar{\eta}^{+\nu}:=\bar{\eta}^{+}$.
3.28 Fact: The set of conditions satisfying (IV) is strictly dense (but not almost open) in the set of conditions satisfying (I)-(II).

IV For all $l$ :

$$
\left|\operatorname{Level}_{k_{l}}(p)\right|<\min \left(\frac{\|p\|_{k_{l}}}{2}, n_{k_{l}}^{-}\right)
$$

For the proof, note that $\left|\operatorname{Level}_{k_{l}}(p)\right|=\left|\operatorname{Level}_{k_{l-1}+1}(p)\right|$.
3.29 Lemma: Assume $\underset{\sim}{\tau}$ is a $P$-name of a function $\in{ }^{\omega} \omega$, and $\Vdash_{P} \forall k \underset{\sim}{\tau}(k)<n_{k}^{+}$. Then the set of conditions satisfying (V) is strictly dense and almost open in the set give by (I), (II), (III). where

V Whenever $k$ is a splitting level, then every $\bar{\eta}$ in level $k$ decides $\underset{\sim}{\tau} \upharpoonright k$.
Proof: Fix $p$ satisfying (I), (II), (III), (IV).
Let $k_{l}:=k_{l}(p)$, etc. Let $m_{l}:=\left|\operatorname{Level}_{k_{l}}\right|$.
Proof: We will use 3.21. For each $l \in \omega, F_{\eta_{l}} \subseteq \operatorname{succ}_{p\left(\alpha_{l}\right)}\left(\eta_{l}\right)$ will be defined as follows: Let $m_{l}:=\left|\operatorname{Level} l_{k_{l}}(p)\right|$, and let $\bar{\eta}^{0}, \ldots, \bar{\eta}^{m-1}$ enumerate $\operatorname{Level}_{k}(p)$. Find a sequence

$$
\operatorname{succ}_{p\left(\alpha_{l}\right)}\left(\eta_{l}\right)=F^{0} \supseteq F^{1} \supseteq \cdots \supseteq F^{m} \quad \forall i\left\|F^{i+1}\right\|_{k} \geq\left\|F^{i}\right\|_{k}-1
$$

such that for all $i$ there exists $x^{i}$ such that for all $\nu \in F^{i+1}$ we have $p^{\left[\left(\bar{\eta}^{i}\right)^{+\nu}\right]} \Vdash \tau \backslash k=x$. It is possible to find such $F^{i+1}$ since $\|\cdot\|_{k}$ is $n_{k}^{-}$-complete, and there are only $n_{0}^{+} \cdot n_{1}^{+} \cdots n_{k-1}^{+}<n_{k}^{-}$many possible values of $\underset{\sim}{\tau} \upharpoonright k$. Finally, let $F_{\eta_{l}}:=F^{m}$. Applying 3.21 will yield the desired result.
3.30 Remark: Note that (V) in particular implies

Va Whenever $k$ is not a splitting level, then every $\bar{\eta}$ in level $k \operatorname{decides} \underset{\sim}{\tau}(k)$.
3.31 Proof that $\Vdash_{P} \mathbf{c}\left(f_{\xi}, g_{\xi}\right) \geq \kappa_{\xi}:$ (This proof is essentially the same as 2.12.)

Recall that $\underset{\sim}{r}$ is the generic real added by the forcing $Q_{\alpha}$. Working in $V[G]$, let $\mathcal{B}$ be a family of less than $\kappa_{\xi}$ many $g_{\xi}$-slaloms. We will show that they cannot cover $\prod f_{\xi}$, by finding an $\alpha$ such that $\underset{\sim}{r}$ is forced not to be covered.
There exists a set $A \in V$ of size $<\kappa_{\xi}$ such that $\mathcal{B} \subseteq V[G \upharpoonright A]$. Since $|A|<\kappa_{\xi}$ there is $\alpha \in A_{\xi}-A$.
Assume that $\bar{B}$ is a $g_{\xi}$-slalom in $V[G \upharpoonright A]$ covering $r_{\alpha}$. So in $V$ there are a $P \upharpoonright A$-name $\underset{\sim}{\bar{B}}$ and a condition $p$ such that

$$
\Vdash_{P\lceil A} \underset{\sim}{\bar{B}} \text { is a } g \text {-slalom }
$$

and

$$
p \Vdash_{P} \underset{\sim}{\bar{B}} \text { covers } r_{\alpha}
$$

We can find a node $\eta$ in $p(\alpha)$ with $\operatorname{succ}_{p(\alpha)}(\eta)$ having more than $g(|\eta|)$ elements. Increase $p \upharpoonright A$ to decide $\underset{\sim}{B}{ }_{|\eta|}$, then increase $p(\alpha)$ to make $r_{\alpha}$ avoid this set.
3.32 Fact: Fix $\xi^{*}$. Then the set of conditions $p$ satisfying

VI For all $l$ : If $\kappa_{\xi^{*}}<\kappa_{\zeta_{l}(p)}$, then

$$
\min \left(\frac{f_{\zeta_{l}(p)}\left(k_{l}\right)}{g_{\xi^{*}}\left(k_{l}\right)}, \frac{f_{\xi^{*}}\left(k_{l}\right)}{g_{\xi^{*}}\left(k_{l}\right)} / h_{\zeta_{l}(p)}\left(k_{l}\right)\right)<\frac{1}{\left|\operatorname{Level}_{k_{l}}(p)\right|}
$$

is dense almost open.
Proof: Write $F_{\zeta}$ for the function $\min \left(\frac{f_{\zeta}}{g_{\xi^{*}}}, \frac{f_{\xi^{*}}}{g_{\xi^{*}}} / h_{\zeta}\right)$. Recall that if $\kappa_{\zeta}<\kappa_{\xi^{*}}$, then $F_{\zeta}$ tends to 0 .

Fix a condition $p$, We will use the game $G(P, p)$. spendthrift will use the following strategy: Whenever $\alpha_{n} \in A_{\zeta}$ and $\kappa_{\zeta}<\kappa_{\xi^{*}}$, then spendthrift first find $m_{0}$ such that for all $m \geq m_{0}$ we have $F_{\zeta}(m)<$ $1 /\left|\operatorname{Level}_{h_{n-1}}\left(p_{n-1}\right)\right|$. Now find $\nu^{n} \supseteq \eta^{n}$ of length $>m_{0}$ with a large enough norm, and play any condition $p_{n}$ obeying the rules of the game. In particular, we must have $\left|\operatorname{Level}_{\left|\nu^{n}\right|}\left(p^{n}\right)\right|=\left|\operatorname{Level}_{\left|\eta^{n}\right|}\left(p^{n}\right)\right|$.
Clearly the condition resulting from the game satisfies the requirements.
3.33 Proof that $\Vdash_{P} \mathbf{c}\left(f_{\xi}, g_{\xi}\right) \leq \kappa_{\xi}$ : Fix $\xi$. We will write $f$ for $f_{\xi}$, etc.

Let

$$
A:=\bigcup\left\{A_{\zeta}: \kappa_{\zeta} \leq \kappa_{\xi}\right\}
$$

We will show that the $g$-slaloms from $V^{P \upharpoonright A}$ already cover $\prod f$. This is sufficient, because $\Vdash_{P}\left(2^{\aleph_{0}}\right)^{V^{P \upharpoonright A}} \leq$ $|A|=\kappa_{\xi}$.
Let $p_{0}$ be an arbitrary condition. Let $\underset{\sim}{\tau}$ be a name of a function $<f$. Find a condition $p \geq p_{0}$ satisfying (I)-(VI).

For each $l$ we now define sets $F_{\eta_{l}} \subseteq \operatorname{succ}_{p\left(\alpha_{l}\right)}\left(\eta_{l}\right)$ as follows:
(1) If $\alpha_{l} \in A$, then $F_{\eta_{l}}=\operatorname{succ}_{p\left(\alpha_{l}\right)}\left(\eta_{l}\right)$.
(2) If $f_{\zeta_{l}}\left(k_{l}\right) \leq g_{\xi}\left(k_{l}\right) /\left|\operatorname{Level}_{k_{l}}(p)\right|$, then again $F_{\eta_{l}}=\operatorname{succ}_{p\left(\alpha_{l}\right)}\left(\eta_{l}\right)$.
(3) Otherwise, we thin out the set $\operatorname{succ}_{p\left(\alpha_{l}\right)}\left(\eta_{l}\right)$ such that each $\bar{\eta}$ in $\operatorname{Level}_{k_{l}}(p)$ decides $\underset{\sim}{\tau}\left(k_{l}\right)$ up to at most $g\left(k_{l}\right) /\left|\operatorname{Level}_{k_{l}}(p)\right|$ many values.

Here is a more detailed description of case (3): Let $k=k_{l}, \zeta=\zeta_{l}$.
Note that if neither (1) nor (2) holds, then letting $c:=f_{\xi}(k), d:=g_{\xi}(k) /\left|\operatorname{Level}_{k}(p)\right|$, we have $c / d \leq h_{\zeta}(k)$.
Using $(c, d)$-completeness of the norm $\|\cdot\|_{\zeta, k}$ we define a sequence

$$
\operatorname{succ}_{p\left(\alpha_{l}\right)}\left(\eta_{l}\right)=L(0) \supseteq L(1) \supseteq \cdots \supseteq L\left(\left|\operatorname{Level}_{k}(p)\right|\right)
$$

as follows. Let $\bar{\eta}_{0}, \ldots, \bar{\eta}_{\left|\operatorname{Level}_{k}(p)\right|-1}$ be an enumeration of $\operatorname{Level}_{k}(p)$.
Given $L(i)$, we know that for each $\nu \in L(i)$ the sequence $\bar{\eta}_{i}^{+\nu}$, (i.e., the condition $p^{\left[\bar{\eta}_{i}^{+\nu}\right]}$ ) decides $\underset{\sim}{\tau}(k)$. (See 3.27.) since there only $\leq c$ many possible values for $\underset{\sim}{\tau}(k)$, we can use $(c, d)$-completeness to find a set $L(i+1) \subseteq L(i)$ and a set $C(i)$ such that
(a) $\|L(i+1)\| \geq\|L(i)\|-1$
(b) $|C(i)| \leq d$.
(c) For every $\nu \in L(i+1), p^{\left[\bar{\eta}_{i}^{+\nu}\right]} \Vdash \underset{\sim}{\tau}(k) \in C(i)$.

Now let $F_{\eta_{l}}$ be $L\left(\left|\operatorname{Level}_{k}(p)\right|\right)$, and let

$$
B_{k}:=\bigcup_{i} C(i)
$$

So $\left|B_{k}\right| \leq\left|\operatorname{Level}_{k}(p)\right| \cdot d \leq g(k)$.
Clearly $\left\|F_{\eta_{l}}\right\|_{\zeta_{l}, k_{l}} \geq\|p\|_{k_{l}}-\left|\operatorname{Level}_{k_{l}}(p)\right|>\frac{1}{2}\|p\|_{k_{l}}$.

This completes the definition of the sets $F_{\eta_{l}}$.
Let $q \geq p$ be the condition defined from $p$ using the $F_{\eta_{l}}$ (see 3.21 ). We will find a $P\lceil A$-name for a $g$-slalom $\underset{\sim}{\mathcal{B}}=\langle\underset{\sim}{\underset{\sim}{B}} k: k \in \omega\rangle$ such that

$$
q \Vdash \underset{\sim}{\underset{\sim}{B}} \text { covers } \underset{\sim}{\tau}
$$

If $k$ is not a splitting level, then every $\bar{\eta}$ in level $k$ decides $\underset{\sim}{\tau}(k)$ by (Va). So in this case we can let

$$
B_{k}:=\left\{i: \exists \bar{\eta} \in \operatorname{Level}_{k}(p), p^{[\bar{\eta}]} \Vdash \underset{\sim}{\tau}(k)=i\right\}
$$

This set is of size $\leq\left|\operatorname{Level}_{k}(p)\right|<g(k)$, and clearly $q \Vdash \underset{\sim}{\tau}(k) \in B_{k}$.
If $k$ is a splitting level, $k=k_{l}$, then there are three cases.
Case 1: $\alpha_{l} \in A$ : We define $\underset{\sim}{\underset{\sim}{B}}$ to be a $P \upharpoonright A$-name satisfying the following:

$$
\Vdash_{P \upharpoonright A} \underset{\sim}{\underset{\sim}{B}} k=\left\{i: \exists \bar{\eta} \in \operatorname{Level}_{k+1}(p), V \models p^{[\bar{\eta}]} \Vdash \underset{\sim}{\tau}(k)=i, \bar{\eta}\left(\alpha_{l}\right) \subseteq r_{\alpha_{l}}\right\}
$$

Thus, we only admit those $\bar{\eta}$ which agree with the generic real added by the forcing $Q_{\alpha_{l}}$. Clearly $\Vdash_{P \upharpoonright A}$ $\left|B_{k}\right| \leq \operatorname{Level}_{k}(p)<g(k)$, and $p \vdash_{P} \underset{\sim}{\tau}(k) \in B_{k}$.
Case 2: $f_{\zeta_{l}}(k) \leq g_{\xi}(k) /\left|\operatorname{Level}_{k}(p)\right|$.
So we have $\left|\operatorname{Level}_{k+1}(p)\right| \leq f_{\zeta_{l}}(k) \cdot\left|\operatorname{Level}_{k}(p)\right| \leq g(k)$, so we can let

$$
B_{k}:=\left\{i: \exists \bar{\eta} \in \operatorname{Level}_{k+1}(p), p^{[\bar{\eta}]} \Vdash \underset{\sim}{\tau}(k)=i\right\}
$$

This set is of size $\leq\left|\operatorname{Level}_{k+1}(p)\right| \leq g(k)$, and again $p \Vdash \underset{\sim}{\tau}(k) \in B_{k}$.
Case 3: Otherwise. We have already defined $B_{k_{l}}$ in $(\oplus)$. By condition (c) above, $q \Vdash \underset{\sim}{\tau}(k) \in B_{k}$.

So indeed $q \Vdash$ " $\underset{\sim}{\mathcal{B}}=\langle\underset{\sim}{\underset{\sim}{B}} k: k \in \omega\rangle$ is a $g$-slalom covering $\underset{\sim}{\tau} "$

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