A Note on Series Parallel Irreducibility

by

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ABSTRACT

A new criterion for series-parallel irreducibility is given which makes no reference to underlying semigroups but involves only series-parallel connection operations.

A *semi-automaton* or *transition system* is a triple $\langle X, Q, M\rangle$ where X, Q are finite sets (of input symbols and internal states respectively), and $M\colon Q \times X \to Q$ is the transition function. (In the usual abuse of notation we write M for $\langle X, Q, M\rangle$.) In this note we shall characterize the semi-automata which are irreducible with respect to series-parallel decomposition. This augments the definition of Krohn and Rhodes [1] (see also Arbib's formulation in [2]), which in an essential way required the specification of output maps and thus held only for full automata, i.e., machines of the form $\langle S, Q, O, M, N\rangle$, where O is the outut set and N: $Q \to O$, the output function. Moreover, their definition of irreducibility for machines made direct reference to semigroups while the definition we shall give makes reference only to series-parallel connection operations. Except for changes in notation the presentation follows that of [2] (Chapters 3 and 5).

Let $S(M)$ denote the *semigroup* of M, i.e.,

$$S(M) = \{\tilde{M}(\ ,x)\colon Q \to Q | x \in X^*\},$$

where $\tilde{M}$ is M extended to X^*. Given a semigroup S, let M_S denote the semigroup transition system, i.e., M_S: $S^1 \times S \to S^1$ with $M_S(1, s) = s$ and $M_S(s, s') = ss'$ for all $s, s' \in S$. Note that $S(M_S) = S$.[1]

In the following we consider as usual only connected machines with specified starting state.

Given transition functions M_i: $Q_i \times X_i \to Q_i$, $i = 1, 2$, we say that M_2 *divides* M_1 (written $M_2|M_1$) if there exist $Q_1' \subseteq Q_1$ and maps g: $X_2 \to X_1^*$, h: $Q_1' \to Q_2$ (onto) such that

(1) Q_1' is closed under $g(X_2)^*$ and
(2) for all $q_1 \in Q_1'$, $s \in X_2$, $h(\tilde{M}_1(q_1, g(s)) = M_2(h(q_1), s)$.

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[1] S^1 is the smallest monoid containing S.

Given $\langle X, Q, M\rangle$ and a positive integer n define $\Pi^n M = \langle X, Q^n, \Pi^n M\rangle$ by $\Pi^n M(q_1,\cdots, q_n, s) = (M(q_1, s),\cdots, M(q_n, s))$ for all $(q_1,\cdots, q_n, s) \in Q^n \times X$. $\Pi^n M$ represents n copies of machine M (possibly in different states) which are run in parallel and are fed the same input symbol.

Definition. M_2 *π-divides* M_1 $(M_2|_\pi M_1)$ if there is a positive integer n such that $M_2|\Pi^n M_1$. We remark that division, and π-division are transitive relations.

M_2 *mutually π-divides* M_1 $(M_2 \equiv_\pi M_1)$ if $M_2|_\pi M_1$ and $M_1|_\pi M_2$. We require the following statements.

(1) $M_2|M_1$ implies $S(M_2)|S(M_1)$.[2]
(2) $S(M_2)|S(M_1)$ implies $M_2|M_{S(M_1)}$.
(3) $M_{S(M)}|_\pi M$.
(4) $S(\Pi^n M) = S(M)$.

Proofs may be found in Chapter 1 of [4]. Suffice it to say that (1) and (2) are well-known; (3) is a slight extension of Fact 2.14b, Chapter 5 of [3]. For (4) we note that

$$\widetilde{\Pi^n} M(q_1,\cdots, q_n, x) = (\tilde{M}(q_1, x),\cdots, \tilde{M}(q_n, x)),$$

and examining the Myhill equivalences relations, we have

$$\begin{aligned} x \equiv_{\Pi^n M} y &\Leftrightarrow \text{for all } (q_1, q_2,\cdots, q_n) \in Q^n,\ \widetilde{\Pi^n} M(q_1,\cdots, q_n, x) = \\ &\qquad \widetilde{\Pi^n} M(q_1,\cdots, q_n, y) \\ &\Leftrightarrow \text{for all } q \in Q,\ \tilde{M}(q, x) = \tilde{M}(q, y) \\ &\Leftrightarrow x \equiv_M y. \end{aligned}$$

Hence $S(\Pi^n M) = X^*| \equiv_{\Pi^n M} = X^*| \equiv_M = S(M)$.

PROPOSITION 1. *$S(M_2)|S(M_1)$ if and only if $M_2|_\pi M_1$.*

Proof. Assume that $S(M_2)|S(M_1)$. Then from (2), $M_2|M_{S(M_1)}$. Also from (3) $M_{S(M_1)}|_\pi M_1$ so by transitivity $M_2|_\pi M_1$.

Conversely, assume that $M_2|_\pi M_1$. Then for some n, $M_2|\Pi^n M_1$ so by (1) $S(M_2)|S(\Pi^n M_1)$. Recognizing that $S(\Pi^n M_1) = S(M_1)$ from (4) completes the proof.

We see that Proposition 1 allows re-interpretation of semigroup division in terms of π-division. This is not true for ordinary division; to make the converse of (1) hold, output maps have to be added to the semigroups as in Theorem 7.3.10 of [2]. The best that we can get from (1) and (2) is

(5) $S(M_2)|S(M_1)$ if and only if $M_2|M_{S(M_1)}$.

An interesting consequence of Proposition 1 is

COROLLARY 2. *$M_1 \equiv_\pi M_2$ if and only if $S(M_1) \cong S(M_2)$.*

Proof. Apply Proposition 1 twice.

The standard definitions of irreducibility are:

(a) A semigroup S is *irreducible* if whenever $S|S_2 \times_Z S_1$ then $S|S_2$ or $S|S_1$. (Here $S_2 \times_Z S_1$ is a semidirect product of S_1 by S_2 with connecting map Z.)

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[2]For semigroups S_i, $i = 1, 2$, $S_1|S_2$ if S_1 is a homorphic image of sub-semigroup of S_2.

(b) A machine M irreducible if whenever $M|M_2 \times_Z M_1$ then $M|M_2$ or $M|M_1$. (Here $M_2 \times_Z M_1$ is the series-parallel cascade of M_1 followed by M_2 with connecting map Z.)

(c) A machine M is *s-irreducible* if whenever $M|M_2 \times_Z M_1$ then $M|M_{S(M_2)}$ or $M|M_{S(M_1)}$.

We add the definition:

(d) A machine M is *π-irreducible* if whenever $M|M_2 \times_Z M_1$ then $M|_\pi M_2$ or $M|_\pi M_1$.

Theorems 8.3.6 and 8.3.7 ([2], p. 4) state that M is s-irreducible if and only if $S(M)$ is irreducible. On the other hand, while M is irreducible implies $S(M)$ is irreducible, the converse does not hold.[3] Using on Proposition 1 we can now show that the equivalence does hold for π-irreducibility.

THEOREM 3. *M is π-irreducible if and only if M is s-irreducible.*

Proof. M is π-irreducible $\Leftrightarrow$ if $M|M_2 \times_Z M_1$ then $M|_\pi M_2$ or $M|_\pi M_1 \Leftrightarrow$ if $M|M_2 \times_Z M_1$ then $S(M)|S(M_2)$ or $S(M)|S(M_1)$ (from Proposition 1) $\Leftrightarrow$ if $M|M_2 \times_Z M_1$ then $M|M_{S(M_2)}$ or $M|M_{S(M_1)}$ (from [5]) $\Leftrightarrow$ M is *s-irreducible.*

In conclusion, we have seen that the irreducibles are strictly included in the s-irreducibles which are co-extensive with the π-irreducibles. What this says is that although a machine M which is s-irreducible but not irreducible has a series-parallel decomposition into machines M_1, M_2 such that *neither* M_1 *nor* M_2 can simulate M, still it must be that by taking a suitable number of copies of either M_1 or M_2 we *can* simulate M, i.e., $M|_\pi M_1$ of $M|_\pi M_2$. Finally we note that Theorem 3 enables us to relate the s-irreducible machines given by the Krohn-Rhodes theory (the simple group and unit actions) entirely to machine decomposition operations without reference to semigroup concepts.

Added in proof: A related paper was presented at the Eleventh Annual Symposium on Switching and Automata Theory, Santa Monica, California.

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[3]Actually, these are proved for full machines but can easily be shown to be true for semi-automata.