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Axioms and Models of Linear Logic

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Abstract. Girard's recent system of linear logic is presented in a way that avoids the two-level structure of formulae and sequents, and that minimises the number of primitive function symbols. A deduction theorem is proved concerning the classical implication as embedded in linear logic. The Hilbert-style axiomatisation is proved to be equivalent to the sequent formalism. The axiomatisation leads to a complete class of algebraic models. Various models are exhibited. On the meta-level we use Dijkstra's method of explicit equational proofs.

0. Introduction

0.0. Girard's recent system of linear logic [Gir87], may have an impact on computing science in various ways. On the one hand, it is a refinement of his earlier system F [Gir86], so it is related to type theory and functional programming [GiL87]. On the other hand, in [Gir86b], linear logic is presented as the logic of concurrent computation.

Girard developed linear logic in an investigation of the structure of natural deduction proofs in relation to certain classes of models: the qualitative domains. The usual logical connectors like "$\Rightarrow$" were broken up into more elementary linear connectives. The resulting logic consists of three parts: an intensional fragment with linear versions of disjunction, negation and entailment, a lattice theory fragment with least upper bounds and greatest lower bounds with respect to an ordering induced by linear entailment, and a modal fragment in which a modal operator is used to recover the power of ordinary logic.

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Our purpose in this paper is to present linear logic in a way that avoids the two level structure of formulae and sequents as used by Girard, and that minimises the number of primitive function symbols. From our axioms it is a small step to a complete class of models: the reduced topological Girard monoids, a generalisation of Girard's phase structures and of point set topology. We give some examples of these monoids and make a start on the structure theory. In particular, we give a new presentation of Girard's phase structures and we construct models based on ordered commutative groups and topological spaces.

Some of our results were obtained earlier by Avron [Avr87]. Our work arose independently. We give an axiomatic presentation of linear logic with fairly complete proofs in the meta-theory, without investigating some important alternatives, whereas, in [Avr87], many closely related theories of linear logic are compared, with proofs often left to the reader.

The Hilbert-style axiomatisation and the deduction theorem are presented in Section 1. In Section 2, we prove the equivalence of this axiomatisation to Girard's sequent calculus and we compare the two formalisms. In Section 3, the axiomatisation leads to a sound and complete class of models of linear logic. We investigate various subclasses of models in relation to additional logical postulates. Section 4 is an appendix, which contains a sketch of the lattice theoretical aspects of linear logic.

0.1. Implication, Relevance and Linear Entailment

In this section we want to provide some intuition for the meanings of the operators in linear logic.

In a classical proof, say of A, an intermediate result B may be used zero or more times. If it is not used at all, as in the implication

$$A \rightarrow (B \rightarrow A) \tag{0}$$

the antecedent B is regarded as irrelevant but harmless. In the field of relevance logic, cf. [Dun86], this fact is called the positive paradox and it is the reason for introducing a concept of relevant entailment $B \rightharpoonup A$, which requires that B be used at least once. Relevance logic admits the possibility that B is used more often. So it has the contraction axiom ([Dun86] p. 125)

$$(B \rightharpoonup (B \rightharpoonup A)) \rightharpoonup (B \rightharpoonup A) \tag{1}$$

Linear logic is characterised by its even stricter administration of the antecedents of an entailment. In fact, the linear entailment $B \multimap A$ requires that B be used precisely once in the proof of A. Therefore, it does not even satisfy formula (1).

To make a bold jump to computing hardware, one can say that the antecedent B in $B \multimap A$ represents an input wire of small capacity which can serve precisely one output wire A. This idea of linear logic as a formalism of hardware properties has been extended considerably [Gir86b]. Independently of whether this leads to useful hardware specifications, the idea is effective in providing intuition for the meaning of the operators of linear logic. For example, the linear negation "$\sim$", which satisfies $\sim\sim A = A$, interchanges input and output. The linear disjunction "$+$" (Girard's operator "$⅋$") stands for communicating cooperation. The intuitive understanding of the three operators "$\multimap$", "$\sim$", "$+$" must be supplemented by the postulate

$$B \multimap A = \sim B + A \tag{2}$$

In our development, we use "$\multimap$" as a defined operator.

Finally, there is an operator of modal weakening "?", pronounced as "why not", which represents storage. The validity of $A \multimap ?A$ means that every value can be stored. Here, the intuition can be guided by the definition of the irrelevant implication "$\rightarrow$" in

$$B \rightarrow A = ?(\sim B) + A. \tag{3}$$

0.2. Sequents or Axioms

In this section we discuss why we have chosen to use fewer operators than Girard and a one-level structure instead of Girard's sequent calculus.

Apart from the ingredients mentioned above, Girard's presentation of linear logic also has operators "$\otimes$" and "!". The duality "$\sim$" is a defined function in the sense that every atom a has its own dual atom $\sim a$, with $\sim\sim a = a$, and that duality of formulae is defined recursively by

$$\begin{aligned} &\sim(A+B) = \sim A \otimes \sim B \ \wedge \ \sim(A \otimes B) = \sim A + \sim B \\ &\wedge \ \sim ?A = !(\sim A) \ \wedge \ \sim !A = ?(\sim A). \end{aligned} \tag{4}$$

Above the level of the formulae with the operators "$+$", "$\otimes$", "?" and "!", Girard's system has a top level of sequents (lists of formulae) at which the derivability predicate "$\vdash$" is defined by some axioms and a number of derivation rules.

Instead of Girard's sequent calculus, we give a Hilbert-style axiomatisation in which we do not need the level of sequents, the negated atoms or the operators "$\otimes$" and "!". We have fewer rules and more axioms. The price we pay, is that our system does not allow cut-elimination, which is the central theorem of Girard's approach. In our view the advantages of our system are elegance and simplicity, and a direct connection with the model theory.

A point of secondary importance is that having abandoned the sequents of the object language we are free to introduce sequents at the meta-level for a convenient formulation of a deduction theorem. This theorem relates the implication given by (3) to a classical concept of derivability, as opposed to Girard's formalisms which stress the relationship of the entailment given by (2) to linear derivability.

0.3. Choices of Design and Notation

As indicated in formula (4), the duality operator "$\sim$" interchanges the operators "$+$" and "$\otimes$" and also "?" and "!". It follows that our choice to use "$+$" and "?" as the primitive operators is completely arbitrary.

Here	$\sim$	$+$	$\otimes$	$\sqcap$	$\sqcup$	$\rightarrow$	$\multimap$	!	?	$\perp$	$\top$	0	~ 0
Girard	$^{\perp}$	⅋	$\otimes$	&	$\oplus$	$\rightarrow$	$\multimap$	!	?	0	$\top$	$\perp$	1

Fig. 1

Linear logic has nine operators and four constants. For various reasons we have chosen to deviate slightly from Girard's notations, see Fig. 1.

We follow [Avr87] in using "$\sim$" and "$+$" for linear negation and disjunction, instead of Girard's "$^{\perp}$" and "⅋". In fact, in the model theory, operator "$+$" becomes the main operator of a commutative monoid. We prefer to use an additive notation in such monoids. Thus, we use "0" to denote the neutral element for "$+$". As Fig. 1 shows, this choice is incompatible with Girard's notation. Girard uses "0" for the smallest element with respect to the preorder induced by linear entailment, where "$\top$" is the biggest element. We prefer to use "$\perp$" for the smallest element. In our system the element $1 = \sim 0$ has no independent rôle.

The preorder mentioned above induces a lattice structure in the sense that formulae A and B have a greatest lower bound and a least upper bound. Here, we prefer to use the conventional notations $A \sqcap B$ and $A \sqcup B$, instead of $A \& B$ and $A \oplus B$ as used in [Gir87]. We present the lattice theory aspects in an appendix (Section 4) and leave the extra proofs to the reader.

Consequently, in the presentation in section 1 we only need the primitive function symbols "$\sim$", "$+$", "?", and the constant "0".

0.4. Explicit Proofs on the Meta-Level

We argue almost exclusively *about* the formal system, and not so much *inside* the system. For arguing about the system we use the ordinary logical connectives "$\equiv$", "$\Rightarrow$", "$\Leftarrow$", "$\wedge$", "$\neg$", ... The meta-level is separated consistently from the object level by the predicate symbols "$\models$" for validity and "$\vdash$" for derivability.

On the meta-level we prefer to give explicit proofs in the style of [DiS90]. We use braces "{" and "}" to enclose comment within formal proofs. So, in order to prove that a predicates X and Y are equivalent, we may write

$$
\begin{aligned}
& X \\
\equiv\ & \{\text{arguments why } X \equiv Z \text{ holds}\}\ Z \\
\equiv\ & \{\text{arguments why } Z \equiv Y \text{ holds}\}\ Y
\end{aligned}
$$

If we only want to prove that X follows from Y, some of the symbols "$\equiv$" may be replaced by "$\Leftarrow$". If we want to prove X itself, we may prove that X follows from *true*. A similar format is used with other transitive relational operators. Notice that the formulae X, Y, Z may contain symbols like "$\equiv$" or "$\Leftarrow$". Therefore, we use the convention that a relational operator followed by comment represents a valid relation, and that it has the lowest possible priority.

Explicitness of proofs is aimed at for the several reasons. As the errors in programming practice are often of a trivial nature, the proofs in computer science should not neglect the trivialities. Another reason for explicit proofs is that in case of a change in the design one can more easily locate the places where changes in the proofs are needed.

In each section, the formulae are numbered consecutively. When refering to formulae of other sections, we write $i(j)$ to denote formula (j) of section i.

1. Axioms for Linear Logic

1.0. In this section we present a formal system equivalent to Girard's system PN2 [Gir87] without the lattice constants "$\top$" and "$\bot$" and the lattice operators "$\sqcap$" and "$\sqcup$".

The system consists of formulae with a validity predicate. The constant 0 is a formula. If A and B are formulae then $A+B$ and $\sim A$ and $?A$ are also formulae. There may be other ways to create formulae as well. The validity of a formula A is denoted by $\models A$; validity is characterised by ten postulates to be given in Sections 1.1 and 1.2 below. We write H to denote the set of formulae. So, H contains a constant 0. It has a binary operation "+" and two unary operations "$\sim$" and "?". There is a boolean function "$\models$" defined on H.

In Section 1.1 we present the postulates concerning validity in its relation to "+", "0" and "$\sim$". In Section 1.2 we present the remaining postulates, which concern the operator of modal weakening "?". In Sections 1.1 and 1.2, we also prepare the ground for the comparison in Section 2 with Girard's presentation and for the model theory of Section 3.

Section 1.3 contains a proof of the Leibniz property, which says that the validity of a formula remains unchanged if a subformula is replaced by a linearly equivalent one. The remainder of Section 1 is devoted to an investigation of the implication "$\multimap$" of 0(3). In order to show that "$\multimap$" is a classical implication, we prove a deduction theorem in which "$\multimap$" is compared with a classical concept of deduction for linear formulae.

1.1. The Intensional Kernel

As indicated by Girard, linear logic is built around a subsystem of extreme elegance. Following Avron, cf. [Avr87], this subsystem is called here the intensional kernel. It consists of "0", "+", "$\sim$" and "$\models$". In our axiomatisation, it has five postulates. We use free variables A, B, $C, \ldots$ to represent formulae. Universal quantification over these variables is understood implicitly.

Postulates. We start by postulating two derivation rules

$$\models A \quad \equiv \quad \models 0+A \tag{0}$$

$$\models A+B \ \wedge\ \models \sim B+C \ \Rightarrow\ \models A+C \quad \text{(the cut-rule)} \tag{1}$$

and three axiom schemes

$$\models \sim A + A \tag{2}$$

$$\models \sim(A + B) + (B + A) \tag{3}$$

$$\models \sim((A + B) + C) + (A + (B + C)) \tag{4}$$

□

In the remainder of this subsection we introduce the preorder "$\sqsubseteq$" that is induced by linear entailment, and a corresponding equivalence relation "$\approx$" on the set H of formulae. We show that the set $H/\approx$ of the equivalence classes forms a commutative monoid with a partial order. The results form an important part of the completeness result in Section 3.

On the set of formulae we define relation "$\sqsubseteq$" of valid entailment by

$$A \sqsubseteq B \;\equiv\; \models \sim A + B \tag{5}$$

Notice that $A \sqsubseteq B$ is equivalent to $\models A \multimap B$, with "$\multimap$" as defined in formula 0(2).

The axioms (2), (3) and (4) are now equivalent to

$$A \sqsubseteq A \tag{6}$$

$$A + B \sqsubseteq B + A \tag{7}$$

$$(A + B) + C \sqsubseteq A + (B + C) \tag{8}$$

Substituting $A := \sim A$ in rule (1) we get that relation "$\sqsubseteq$" is transitive:

$$A \sqsubseteq B \;\wedge\; B \sqsubseteq C \;\Rightarrow\; A \sqsubseteq C \tag{9}$$

By (6) and (9), relation "$\sqsubseteq$" is a preorder. Therefore, it induces an equivalence relation "$\approx$", to be called *linear equivalence,* on the set of formulae by

$$A \approx B \;\equiv\; A \sqsubseteq B \;\wedge\; B \sqsubseteq A \tag{10}$$

Using the symmetry in A and B in (7) we get

$$A + B \approx B + A \tag{11}$$

As for associativity, by repeated application of (7), (8) and (9) one can prove that

$$A + (B + C) \sqsubseteq (A + B) + C$$

By (8) and (10), this implies

$$(A + B) + C \approx A + (B + C) \tag{12}$$

It follows from (11) and (12) that up to linear equivalence the sum of a list of formulae is independent of the parentheses and of the order of the summands.

Now rule (0) comes into the play. We have

$$\begin{aligned} & \models A + B \\ \equiv\; & \{(0),(3)\} \models 0 + (A + B) \;\wedge\; \models \sim(A + B) + (B + A) \\ \Rightarrow\; & \{(1),(0)\} \models B + A \end{aligned}$$

By symmetry this proves

$$\models A+B \;\equiv\; \models B+A \tag{13}$$

We claim that

$$\models A+B \;\equiv\; {\sim}A \sqsubseteq B \tag{14}$$

This follows from the implication

$$\begin{aligned} & {\sim}A \sqsubseteq B \\ \equiv\; & \{(5),(2)\} \models {\sim}{\sim}A+B \;\wedge\; \models {\sim}A+A \\ \Rightarrow\; & \{(13),(1)\} \models A+B \end{aligned}$$

and the converse implication

$$\begin{aligned} & \models A+B \\ \equiv\; & \{(2)\} \models A+B \;\wedge\; \models {\sim}{\sim}A+{\sim}A \\ \Rightarrow\; & \{(13),(1)\} \models {\sim}{\sim}A+B \\ \equiv\; & \{(5)\} \; {\sim}A \sqsubseteq B \end{aligned}$$

By rule (0), the substitution $A, B := 0, A$ in (14) yields

$$\models A \;\equiv\; {\sim}0 \sqsubseteq A. \tag{15}$$

For later use we notice that (14) and (15) combine to

$$\sim A \sqsubseteq B \;\equiv\; {\sim}0 \sqsubseteq A+B \tag{16}$$

With (9) and (10), it follows from (15) that

$$A \approx B \;\Rightarrow\; (\models A \equiv \models B) \tag{17}$$

By the sentence after (12), this implies that

Theorem 0. The validity of the sum of a list of formulae is independent of the parentheses and of the order of the summands.

We observe

$$\begin{aligned} & B \sqsubseteq 0+A \\ \equiv\; & \{(5)\} \models {\sim}B+(0+A) \\ \equiv\; & \{\text{theorem } 0\} \models 0+({\sim}B+A) \\ \equiv\; & \{(0) \text{ and } (5)\} \; B \sqsubseteq A \end{aligned}$$

By the substitutions $B := A$ and $B := 0+A$, this implies

$$0+A \approx A \tag{18}$$

The formulae (11), (12), (18) can be summarised by saying that up to linear equivalence the addition is commutative, associative, and has 0 as neutral element.

We now turn to the duality. In order to show that it is an involution we observe that

$$
\begin{aligned}
& \sim{\sim}A \sqsubseteq B \\
\equiv\ & \{(14)\}\ \models {\sim}A + B \\
\equiv\ & \{(5)\}\ A \sqsubseteq B
\end{aligned}
$$

By (6) and (10) this implies that up to "$\approx$" the duality is an involution:

$$\sim{\sim}A \approx A \tag{19}$$

We also have that the duality inverts the preorder

$$
\begin{aligned}
& ({\sim}A \sqsubseteq {\sim}B \ \equiv\ B \sqsubseteq A) \\
\equiv\ & \{(14),(5)\}\ (\models A + {\sim}B \ \equiv\ \models {\sim}B + A) \\
\equiv\ & \{(13)\}\ \mathit{true}
\end{aligned}
\tag{20}
$$

It follows from (10) and (20) that the duality operator respects linear equivalence:

$$A \approx B \ \equiv\ {\sim}A \approx {\sim}B \tag{21}$$

With regard to the addition, we observe that

$$
\begin{aligned}
& A + B \sqsubseteq A + C \\
\equiv\ & \{(5)\}\ \models {\sim}(A+B) + (A+C) \\
\equiv\ & \{\text{theorem } 0\}\ \models ({\sim}(A+B)+A)+C \\
\Leftarrow\ & \{(1)\}\ \models ({\sim}(A+B)+A)+B \ \wedge\ \models {\sim}B + C \\
\equiv\ & \{\text{theorem } 0\}\ \models {\sim}(A+B)+(A+B) \ \wedge\ \models {\sim}B + C \\
\equiv\ & \{(2),(5)\}\ B \sqsubseteq C
\end{aligned}
$$

This proves that the addition is monotone in its second argument:

$$B \sqsubseteq C \ \Rightarrow\ A + B \sqsubseteq A + C. \tag{22}$$

Using (9), (10) and (11), one can easily conclude from (22) that the addition is monotone in both arguments:

$$A \sqsubseteq B \wedge C \sqsubseteq D \ \Rightarrow\ A + C \sqsubseteq B + D \tag{23}$$

Just as in the proof of (21), it follows from (10) and (23) that the addition preserves linear equivalence:

$$A \approx B \wedge C \approx D \ \Rightarrow\ A + C \approx B + D \tag{24}$$

For any formula $A \in H$, let $[A]$ denote the equivalence class of A with respect to "$\approx$". Let $H/{\approx}$ denote the set of the equivalence classes. It follows from (24) that $H/{\approx}$ can be equipped with an operation "+" given by $[A]+[B] = [A+B]$. The formulae (11), (12), (18) imply that $H/{\approx}$ is a commutative monoid with neutral element [0]. By (21), $H/{\approx}$ has an induced unary operator "$\sim$" with ${\sim}[A] = [{\sim}A]$. By (19), function "$\sim$" on $H/{\approx}$ is an involution. Using (6), (9) and (10), one can prove that relation $\sqsubseteq$ induces a partial order on

$H/\approx$. It follows from (16) that for all $x, y \in H/\approx$

$$\sim x \sqsubseteq y \;\equiv\; \sim 0 \sqsubseteq x + y$$

In Section 3.1, these results will be summarised by saying that $H/\approx$ is a reduced Girard monoid.

1.2. The Operator of Modal Weakening "?"

In this section we present the five remaining postulates, which all concern the operator "?".

Postulates. There is one derivation rule

$$\models \sim A + ?B \;\equiv\; \models \sim ?A + ?B \tag{25}$$

and four axiom schemes

$$\models \sim 0 + ?A \tag{26}$$

$$\models \sim (?A + ?A) + ?A \tag{27}$$

$$\models \sim ?0 \tag{28}$$

$$\models \sim ?(?A + ?B) + (?A + ?B) \tag{29}$$

□

As a sequel to the investigation in Section 1.1, we prove that "?" induces an operation on $H/\approx$ and that the equivalence classes $[?A]$ form a submonoid T of $H/\approx$. It turns out that T is a topology in $H/\approx$ in the sense of Section 3.7 below. This result will be the second part of the completeness theorem of the model theory.

By (5), rule (25) is equivalent to

$$A \sqsubseteq ?B \;\equiv\; ?A \sqsubseteq ?B \tag{30}$$

Substituting $B := A$ in (30), we get that "?" is weakening:

$$A \sqsubseteq ?A \tag{31}$$

It follows that function "?" is monotone:

$$A \sqsubseteq B \;\Rightarrow\; ?A \sqsubseteq ?B \tag{32}$$

This is proved in

$$\begin{array}{ll} & A \sqsubseteq B \\ \Rightarrow & \{(31),(9)\}\; A \sqsubseteq ?B \\ \equiv & \{(30)\}\; ?A \sqsubseteq ?B \end{array}$$

By the same arguments as in the proof of (21), it follows that "?" preserves equivalence:

$$A \approx B \;\Rightarrow\; ?A \approx ?B \tag{33}$$

We now come back to the axioms (26)–(29). By (0) and (13), axiom (28) is equivalent to $\models \sim ?0 + 0$. Therefore, by (5), these four axioms are equivalent to

$$\begin{aligned}
&0 \sqsubseteq ?A \\
&?A + ?A \sqsubseteq ?A \\
&?0 \sqsubseteq 0 \\
&?(?A + ?B) \sqsubseteq ?A + ?B && (34)
\end{aligned}$$

By (10) and (31) the last two rules imply

$$?0 \approx 0, \tag{35}$$

$$?(?A + ?B) \approx ?A + ?B \tag{36}$$

It follows from (33) that the operator "?" induces an operator "?" on the set $H/\!\approx$ of linear equivalence classes, which is given by $?[A] = [?A]$. Let T be the set of equivalence classes in $H/\!\approx$ that consists of the classes $[?A]$. By (35) and (36), T is a submonoid of $H/\!\approx$, i.e. $[0] \in T$ and T is closed under "+". The remaining assertions of (34) imply that

$$(\forall t \in T :: [0] \sqsubseteq t \quad \wedge \quad t + t \sqsubseteq t)$$

Finally, rule (30) can be rephrased by saying that for any $x \in H/\!\approx$ and any $t \in T$

$$x \sqsubseteq t \quad \equiv \quad ?x \sqsubseteq t$$

1.3. The Equivalence Theorem

We can now show that the equivalence relation "$\approx$" is characterized by the fact that validity of a formula is not changed when some part of the formula is replaced by an equivalent part.

Theorem 1 (the Leibniz principle). Formulae A and B satisfy $A \approx B$ if and only if for every expression C in a variable x it holds that

$$\models C[A/x] \quad \equiv \quad \models C[B/x] \tag{37}$$

(i.e. validity of C with x replaced by A or B is independent of the choice).

Proof. For any expression C in x we have

$$\begin{aligned}
&\quad A \approx B \\
&\Rightarrow \{\text{structural induction and (21), (24), (33)}\} \\
&\quad C[A/x] \approx C[B/x] \\
&\Rightarrow \{(17)\} \\
&\quad \models C[A/x] \quad \equiv \quad \models C[B/x].
\end{aligned}$$

Sufficiency of (37) is proved in

$$\begin{aligned}
&\quad A \approx B \\
&\equiv \{(10), (5)\} \\
&\quad \models \sim A + B \quad \wedge \quad \models \sim B + A \\
&\equiv \{\text{use (37) with } C := \sim x + B \text{ and with } C := \sim B + x\} \\
&\quad \models \sim B + B \\
&\equiv \{(2)\} \\
&\quad \mathit{true}.
\end{aligned}$$

□

1.4. The Implication

Girard's development of linear logic originates from his observation that in the qualitative domains of [Gir86a], the function type (i.e. the implication) can be decomposed. In the present framework this decomposition corresponds to the definition of the implication operator "$\rightarrow$" by

$$A \rightarrow B \;=\; B + ?(\sim A) \tag{38}$$

The implication "$\rightarrow$" is weaker than the entailment "$\multimap$", but as we shall see in Section 1.6 it is the implication that is needed for our version of the deduction theorem. As a preparation for that theorem, we prove that the implication has some of the expected properties.

A first expected property is a version of the "positive paradox":

$$\models B \;\Rightarrow\; \models A \rightarrow B \tag{39}$$

This is proved in

$$\begin{aligned} & \models A \rightarrow B \\ \equiv\; & \{(38)\} \models B + ?(\sim A) \\ \Leftarrow\; & \{(1)\} \models B + 0 \;\wedge\; \models \sim 0 + ?(\sim A) \\ \equiv\; & \{(0), (13), (26)\} \models B \end{aligned}$$

Another expected property is

$$\begin{aligned} & \models A \rightarrow A \\ \equiv\; & \{(38)\} \models A + ?(\sim A) \\ \Leftarrow\; & \{(1)\} \models A + \sim A \;\wedge\; \models \sim\sim A + ?(\sim A) \\ \equiv\; & \{(2), (13), (25)\} \models \sim ?(\sim A) + ?(\sim A) \\ \equiv\; & \{(2)\}\; \mathit{true} \end{aligned} \tag{40}$$

A third expected property is the detachment rule (cf. [Sho67] 3.1):

$$\models A \rightarrow B \;\wedge\; \models A \;\Rightarrow\; \models B \tag{41}$$

To prove this rule, we first observe that

$$\begin{aligned} & (\models A \;\equiv\; \models \sim ?(\sim A)) \\ \equiv\; & \{(0), (13)\}\; (\models A + 0 \;\equiv\; \models \sim ?(\sim A) + 0) \\ \equiv\; & \{(19), (35), \text{theorem } 1\} \\ & (\models \sim\sim A + ?0 \;\equiv\; \models \sim ?(\sim A) + ?0) \\ \equiv\; & \{(25) \text{ with } A := \sim A \text{ and } B := 0\}\; \mathit{true} \end{aligned} \tag{42}$$

Now, (41) is proved in

$$\begin{aligned} & \models A \rightarrow B \;\wedge\; \models A \\ \equiv\; & \{(38), (42)\} \models B + ?(\sim A) \;\wedge\; \models \sim ?(\sim A) \\ \Rightarrow\; & \{(0), (1), (13)\} \models B \end{aligned}$$

1.5. The Implied Versions of the Derivation Rules

As additional preparation for the deduction theorem, we prove in this section the rules that are obtained from the derivation rules (0), (1), (25) by replacing "$\models$" by "$\models E\rightarrow$". For notational convenience, the operator "$\rightarrow$" is given lower priority than the operator "+". The implied version of formula (0) is

$$\models E\rightarrow A \quad \equiv \quad \models E\rightarrow 0+A \tag{43}$$

It is proved in

$$\begin{aligned}
&\models E\rightarrow 0+A\\
\equiv\ &\{(38)\}\ \models (0+A)+?(\sim E)\\
\equiv\ &\{(0),\text{ theorem } 0\}\ \models A+?(\sim E)\\
\equiv\ &\{(38)\}\ \models E\rightarrow A
\end{aligned}$$

The implied version of the cut-rule (1) is

$$\models E\rightarrow A+B \ \wedge\ \models E\rightarrow \sim B+C \ \Rrightarrow\ \models E\rightarrow A+C \tag{44}$$

It is proved in

$$\begin{aligned}
&\models E\rightarrow A+B \ \wedge\ \models E\rightarrow \sim B+C\\
\equiv\ &\{(38)\},\text{ theorem } 0\}\\
&\models(?(\sim E)+A)+B \ \wedge\ \models \sim B+(C+?(\sim E))\\
\Rrightarrow\ &\{(1),\text{ theorem } 0\}\\
&\models(A+C)+(?(\sim E)+?(\sim E))\\
\Rrightarrow\ &\{(1),(27),(38)\}\\
&\models E\rightarrow A+C
\end{aligned}$$

Finally, the implied version of (25) is

$$\begin{aligned}
&(\models E\rightarrow \sim A+?B \ \equiv\ \models E\rightarrow \sim ?A+?B)\\
\equiv\ &\{(38),\text{ theorem } 0\}\\
&(\models \sim A+(?B+?(\sim E)) \ \equiv\ \models \sim ?A+(?B+?(\sim E)))\\
\equiv\ &\{(36)\text{ and theorem } 1;\ (25)\}\\
&\mathit{true}
\end{aligned} \tag{45}$$

1.6. Relative Linear Logic

If a mathematician wishes to prove a statement "if P, then Q", he will generally assume P and then prove Q, cf. [Sho67] 3.3. This mathematician may miss many elegant proofs, but that is not our concern here. In this section, we prove that this method is sound and complete in linear logic.

Definition. We define *derivability* of an expression A from a set of expressions

S (notation $S \vdash A$) by the following rules:

(i) for all $A \in S$ it holds that $S \vdash A$,

(ii) the postulates hold which are obtained from the postulates (0) $\cdots$ (4) and (25) $\cdots$ (29) by replacing the operator "$\models$" by "$S\vdash$".

(iii) $S \vdash A$ only holds if that can be derived with the rules (i) and (ii). □

In the preceding sections, we were only concerned with validity as expressed by "$\models$", not with the derivability expressed by "$S\vdash$". Since derivability satisfies all rules postulated for validity, all results of the previous sections apply with "$\models$" replaced by "$S\vdash$".

If S is a set of expressions and E is an expression, we use "S, E" to denote the union of S with the singleton set consisting of E. Using structural induction one proves that

$$S \vdash A \Rightarrow S, E \vdash A \tag{46}$$

The next result can be compared with the deduction theorem of [Sho67] 3.3.

Theorem 2 (deduction theorem). $S \vdash E \rightarrow A \;\equiv\; S, E \vdash A$

Proof. The direction "$\Rightarrow$" is proved in

$$\begin{aligned} & S \vdash E \rightarrow A \\ \Rightarrow\ & \{(46) \text{ and rule (i)}\}\ S, E \vdash E \rightarrow A \ \wedge\ S, E \vdash E \\ \Rightarrow\ & \{(41)\}\ S, E \vdash A \end{aligned}$$

The other implication ($\Leftarrow$) is proved by induction on the length of the derivation of

$$S, E \vdash A. \tag{47}$$

For the basis of the induction, we assume that (47) is proved in one step from rule (i) or rule (ii). If (47) follows from rule (i), we have

$$\begin{aligned} & A \in S, E \\ \equiv\ & A \in S \ \vee\ A = E \\ \Rightarrow\ & \{\text{rule (i)}\}\ S \vdash A \vee E = A \\ \Rightarrow\ & \{(39), (40)\}\ S \vdash E \rightarrow A \end{aligned}$$

Otherwise, if (47) follows in one step from rule (ii), then $\models A$ is an axiom, so that we also have $S \vdash A$, and hence by (39)

$$S \vdash E \rightarrow A$$

Now assume that (47) is proved from one or two previous results of the form $S, E \vdash B$ (and $S, E \vdash C$) by appeal to one of the derivation rules postulated in (ii). By induction we have that

$$S \vdash E \rightarrow B \text{ (and } S \vdash E \rightarrow C)$$

It suffices to prove that the rule which is of the form

$$S,\, E \vdash B \text{ (and } S,\, E \vdash C) \;\Rightarrow\; S,\, E \vdash A$$

has a variation of the form

$$S \vdash E \rightarrow B \text{ (and } S \vdash E \rightarrow C) \;\Rightarrow\; S \vdash E \rightarrow A$$

As the applied rule is one of the postulates (0), (1) or (25) the theorem follows from the implied versions (43), (44) or (45), proved in the previous section. □

2. The Correspondence with Girard's Sequent Calculus

2.0. In this section, we investigate the equivalence of the system defined above with Girard's system. Girard's sequent calculus without the lattice type operators can be summarised as follows ([Gir87] chapter 1). We use the constant 0 and the operators $+$, $\otimes$, $\sim$ instead of $\perp$, and $\wp$, $\otimes$, $^{\perp}$, respectively (see Fig. 1).

Atoms are the letters $0, a, b, c, \ldots$ and their duals $\sim 0, \sim a, \sim b, \sim c, \ldots$. The dual of the dual of a letter is the letter itself. Formulae are formed from the atoms by means of the binary connectives $+$, $\otimes$, and the unary function symbols "?" and "!". Formulae are denoted by capital letters $A, B, C, \ldots$. The dual of a formula is defined recursively by

$$\begin{aligned}&\sim(A+B) = \sim A \otimes \sim B \;\wedge\; \sim(A \otimes B) = \sim A + \sim B\\ &\wedge\; \sim ?A = !(\sim A) \;\wedge\; \sim !A = ?(\sim A)\end{aligned} \tag{0}$$

By structural induction, one can easily verify that for every formula A:

$$\sim\sim A = A \tag{1}$$

A *sequent* is defined to be a list of formulae, separated by commas. Sequents are denoted by capitals $X, Y, Z, \ldots$. The empty sequent is denoted by $\varnothing$. Sequents are derived by means of axioms and derivation rules. Derivability of a sequent X is denoted by "$\vdash X$".

If X is a sequent, the sequent $?X$ is defined as the list of the formulae $?B$ where B are the consecutive members of sequent X. So we have

$$?\varnothing = \varnothing, \quad ?(A, X) = ?A, ?X \tag{2}$$

2.1. Postulates

The axioms and rules of sequent calculus are

$$\vdash X, A \;\wedge\; \vdash \sim A, Y \;\Rightarrow\; \vdash X, Y \quad \text{(the cut-rule)} \tag{3}$$

$$\text{If sequent } Y \text{ is a permutation of } X, \text{ then } \vdash X \;\Rightarrow\; \vdash Y \tag{4}$$

$$\vdash \sim A, A \tag{5}$$

$$\vdash X \;\Rightarrow\; \vdash X, 0 \tag{6}$$

$$\vdash \sim 0 \tag{7}$$

$$\vdash X, A, B \;\Rightarrow\; \vdash X, A + B \tag{8}$$

$$\vdash A, X \;\wedge\; \vdash B, Y \;\Rightarrow\; \vdash A \otimes B, X, Y \tag{9}$$

$$\vdash X \;\Rightarrow\; \vdash X, ?A \tag{10}$$

$$\vdash X, A \;\Rightarrow\; \vdash X, ?A \tag{11}$$

$$\vdash X, ?A, ?A \;\Rightarrow\; \vdash X, ?A \tag{12}$$

$$\vdash A, ?X \;\Rightarrow\; \vdash !A, ?X \tag{13}$$

□

2.2. Reformulation

We investigate the postulates in order to get an equivalent system of postulates that is closer to the axioms of Section 1.

For the moment we assume the postulates (3), (4) and (5), and we investigate the other postulates. We start with the observation that (3) and (5) imply

$$\vdash \sim A, Y \;\equiv\; (\forall X :: \vdash X, A \;\Rightarrow\; \vdash X, Y) \tag{14}$$

In fact, the implication "$\Rightarrow$" is the cut-rule (3). The implication "$\Leftarrow$" follows from (5) by taking $X := \sim A$ (conversely, (14) implies (3) and (5)). We now apply (14) with $A := 0$ and Y empty, and get

$$\vdash \sim 0 \;\equiv (\forall X :: \vdash X, 0 \;\Rightarrow\; \vdash X)$$

Here, the right-hand side is the converse implication of (6). Therefore, we may replace postulates (6) and (7) together by

$$\vdash X \;\equiv\; \vdash X, 0 \tag{15}$$

Using (14) we can prove that (9) implies a converse of (8):

$$\begin{aligned}
& (\forall X :: \vdash X, A + B \;\Rightarrow\; \vdash X, A, B) \\
\equiv\; & \{(14)\}\; \vdash \sim(A + B), A, B \\
\equiv\; & \{(0)\}\; \vdash (\sim A) \otimes (\sim B), A, B \\
\Leftarrow\; & \{(9)\}\; \vdash \sim A, A \;\wedge\; \vdash \sim B, B \\
\equiv\; & \{(5)\}\; \mathit{true}
\end{aligned} \tag{16}$$

Conversely, however, (9) follows from (16) as is shown in

$$\begin{aligned}
& \vdash A, X \;\wedge\; \vdash B, Y \;\Rightarrow\; \vdash A \otimes B, X, Y \\
\equiv\; & \{(4)\}\; \vdash X, A \;\wedge\; \vdash B, Y \;\Rightarrow\; \vdash X, A \otimes B, Y \\
\Leftarrow\; & \{(3)\text{ twice, and }(1)\}\; \vdash A, A \otimes B, \sim B \\
\Leftarrow\; & \{(4), (16)\}\; \vdash A \otimes B, \sim A + \sim B \\
\equiv\; & \{(5), (0)\}\; \mathit{true}
\end{aligned}$$

This proves that we may replace (8) and (9) together by

$$\vdash X, A, B \;\equiv\; \vdash X, A+B \tag{17}$$

By (14) and (15), postulate (10) is equivalent to

$$\vdash \sim 0, ?A \tag{18}$$

By (14), postulate (11) is equivalent to

$$\vdash \sim A, ?A \tag{19}$$

By (14) and (17), postulate (12) is equivalent to

$$\vdash \sim(?A + ?A), ?A \tag{20}$$

Postulate (13) is disturbing, as it uses the modal weakening of a sequent as defined in (2). For the moment we are content with the observation that A may be replaced by its dual. By (0), this yields that (13) is equivalent to

$$\vdash \sim A, ?X \;\Rightarrow\; \vdash \sim ?A, ?X \tag{21}$$

We summarise by stating that the postulates (6)–(13) may be replaced by (15) and (17)–(21).

2.3. Derivability of Formulae in Sequent Calculus

In view of (15) and (17), we define the sum $\sum X$ of a sequent X by summing from the right, according to

$$\begin{aligned} &\sum \varnothing = 0 \\ &\sum (A, X) = A + \sum X \end{aligned} \tag{22}$$

By induction on the length of X, it follows from (15) and (17) that for any pair of sequents X and Y

$$\vdash Y, X \;\equiv\; \vdash Y, \sum X$$

In particular, taking Y empty, we get that the derivability of a sequent is equivalent to the derivability of its sum

$$\vdash X \;\equiv\; \vdash \sum X \tag{23}$$

In (23), the derivability of a sequent is reduced to the derivability of a formula. Therefore, we are interested in the restriction of the derivability predicate to formulae (even though in the formal derivation of almost every formula longer sequents are needed).

We now prove that the derivability predicate on formulae satisfies the postulates of Section 1. The verification of the first three postulates of Section

1.1 is almost done. Indeed, postulate 1(0) follows from (15), (23) and (4). Postulate 1(1) follows from (3) and (23). Axiom 1(2) follows from (5) and (23). Axiom 1(3) is proved in

$$\begin{aligned}
&\vdash \sim(A+B)+(B+A)\\
\equiv\;& \{(23),(14)\}\ (\forall X :: \vdash X, A+B \;\Rightarrow\; \vdash X, B+A)\\
\equiv\;& \{(17),(4)\}\ \textit{true}.
\end{aligned}$$

In precisely the same way, one can prove axiom 1(4):

$$\vdash \sim((A+B)+C)+(A+(B+C)).$$

The modal rule 1(25) is proved in

$$\begin{aligned}
&\vdash \sim A+?B \;\equiv\; \vdash \sim ?A+?B\\
\equiv\;& \{(23),(21)\}\ \vdash \sim A, ?B \;\Leftarrow\; \vdash \sim ?A+?B\\
\Leftarrow\;& \{(3)\}\ \vdash \sim A, ?A\\
\equiv\;& \{(19)\}\ \textit{true}.
\end{aligned}$$

In this derivation, we used (21) with a singleton sequent. By the definition of $?X$ in (2), rule (21) also has the special cases

$$\vdash \sim A \;\Rightarrow\; \vdash \sim ?A, \tag{24}$$

$$\vdash \sim A, ?B, ?C \;\Rightarrow\; \vdash \sim ?A, ?B, ?C \tag{25}$$

Now the postulates 1(26) and 1(27) follow from (18) and (23), and (20) and (23), respectively. Postulate 1(28) follows from (24) and (7). Finally, postulate 1(29) is proved in

$$\begin{aligned}
&\vdash \sim ?(?A+?B)+(?A+?B)\\
\Leftarrow\;& \{(17),(25)\}\ \vdash \sim(?A+?B), ?A, ?B\\
\equiv\;& \{(17),(5)\}\ \textit{true}
\end{aligned}$$

This proves that Girard's sequent calculus, when restricted to formulae, satisfies the postulates of Section 1.

2.4. The Equivalence of the Two Systems

To conclude the proof of the equivalence of the systems, we first enrich system H of Section 1 with operators "$\otimes$" and "!" and with validity of sequents. The extension is called H^*. For formulae $A, B \in H$ we define

$$A \otimes B = \sim(\sim A + \sim B),\quad !A = \sim ?\sim A \tag{26}$$

Since linear equivalence in H preserves validity, cf. 1(17), we may treat "$\approx$" as equality. By 1(19), we have

$$\sim\sim A \approx A \tag{27}$$

Using (27), one can easily prove the variation of (0):

$$\begin{aligned}
&\sim(A+B) \approx \sim A \otimes \sim B \;\wedge\; \sim(A \otimes B) \approx \sim A + \sim B\\
&\wedge\; \sim ?A \approx !(\sim A) \;\wedge\; \sim !A \approx ?(\sim A)
\end{aligned} \tag{28}$$

A sequent over H is a list of formulae of H. We define the sum of a sequent by (22). In view of (23), validity of a sequent is defined by

$$\models X \;\equiv\; \models \sum X \tag{29}$$

It remains to verify that the extension H^* of system H satisfies the postulates of sequent calculus. In view of the postulates 1(0), 1(1), 1(2), of theorem 0 and of definition (29), it is easy to see that H^* satisfies the postulates (4), (3), (5), (15) and (17). Using (29) and 1(5), one easily recognises the postulates (19), (18), (20) in the results 1(31) and 1(34).

We conclude with the verification of (21). Here, we need that system H^* satisfies

$$\sum ?X \approx ?\sum ?X \tag{30}$$

This is proved by induction on the length of X. The base case, with X empty, follows from (2), (22) and 1(35). The induction step, with $X := A,\ X,$ follows from (2), (22) and 1(36). Now postulate (21) is verified in

$$\models \sim A,\ ?X \;\Rightarrow\; \models \sim ?A,\ ?X$$

$$\equiv\ \{(29)\}\ \models \sim A + \sum ?X \;\Rightarrow\; \models \sim ?A + \sum ?X$$

$$\equiv\ \{(30) \text{ and theorem } 1\}$$

$$\models \sim A + ?\sum ?X \quad \models \sim ?A + ?\sum ?X$$

$$\equiv\ \{1(25) \text{ with } B := \sum ?X\}\ \mathit{true}$$

2.5. Cut-Elimination

One of the corner stones of proof theory in classical logic is Gentzen's cut-elimination theorem, cf. [Tak87] 5.1. This theorem asserts that in a proof of any formula all applications of the cut-rule (which is analogous to (3)) can be eliminated. After these eliminations the proof is as direct as possible in a certain sense. As remarked by one of the referees, cut-free proofs are not necessarily as short as possible.

One of the applications of cut-elimination is a proof of consistency of the theory. Unfortunately, the introduction of new axioms (definitions of constants, or theorems to be applied as axioms) usually destroys the possibility of cut-elimination.

The importance of Girard's sequent calculus presentation of linear logic is that it allows cut-elimination:

Theorem 3 (Girard, [Gir87]). If a sequent is derivable, then it is derivable by means of the postulates (4)–(13), i.e. without the cut-rule.

Example. The formula $A = \sim(\sim a + a) + \sim 0$ is not derivable. Indeed, if A is derivable, then sequent $X = (a \otimes \sim a, \sim 0)$ is derivable without the cut-rule. The only rule that can generate X is postulate (9). If X is derived with (9), than a or $\sim a$ is derivable, but clearly neither a nor $\sim a$ is derivable without the cut-rule. □

2.6. Comparison

The cut-rule is an important syntactic instrument. For the model theory, however, we may want to know the algebraic properties of linear logic. From the algebraic point of view, sequents are merely disguised formulae with rather unexpected properties. The empty sequent is a disguised zero, the comma is a disguised form of the operator "+", but modal weakening of sequents differs from modal weakening of formulae: by (2) we have $?(A, B) = (?A, ?B)$, whereas $?(A + B) \neq ?A + ?B$. In fact, one can prove that the sequent

$$\sim(?A + ?B), ?(A + B)$$

is not derivable, cf. Section 3.9 below.

From the algebraic point of view, the postulates (15) and (17)–(20) are nicer than the corresponding postulates (6)–(12), but, even better, postulate (17) disappears in the system of Section 1. The really difficult postulates are (4) and (13). In a certain sense, both postulates are stronger than necessary. Postulate (4) corresponds to the postulates 1(3) and 1(4) in Section 1.1, but the corresponding assertion is Theorem 0. The sole purpose of definition (2) is to enable a short form of postulate (13). This hides the complexity of postulate (13), as one can see in the concluding lines of Sections 2.3 and 2.4.

In our view, the postulates of Section 1 are simpler and more elegant. The sequent calculus is more powerful, as it allows cut-elimination. It seems likely that the two approaches can play useful complementary rôles.

3. Monoidal Models of Linear Logic

3.0. In this section, we present a general class of models of linear logic. These models are inspired by the algebraic properties of the set of equivalence classes $H/\approx$ as investigated in Sections 1.1 and 1.2.

The intensional kernel (cf. 1.1) gives rise to a theory of "Girard monoids", monoids with a preorder and a kind of duality. By choosing a flexible axiomatisation, we can give an easy presentation of Girard's phase structures and make a start on a theory of quotient monoids.

The incorporation of the modal operator "?" (cf. 1.2) leads to topological concepts, cf. [Gir87] 1.19. We define a topology on a Girard monoid to be a submonoid with certain properties. It turns out that general topology is a special case of this concept, cf. Section 3.8.

3.1. Models of the Intensional Kernel

Recall that a commutative monoid (with additive notation) is a triple $\langle M, +, 0\rangle$ where M is a set, 0 is an element of M and "+" is a commutative and associative binary operator on M with neutral element 0.

We define a *Girard monoid* to be a quintuple $\langle M, +, 0, \sim, \leq\rangle$ such that $\langle M, +, 0\rangle$ is a commutative monoid, that "$\leq$" is a preorder on M, and that "$\sim$" is a unary operator on M such that for all $x, y \in M$

$$\sim x \leq y \;\equiv\; \sim 0 \leq x + y$$

There are two useful additional conditions on Girard monoids. The Girard monoid is called *strict*, if the duality operator "$\sim$" is an involution, i.e. for all $x \in M$

$$\sim\sim x = x \tag{1}$$

The Girard monoid is called *reduced*, if it is strict and the preorder "$\leq$" is a partial order, i.e. for all $x, y \in M$

$$x \leq y \;\wedge\; y \leq x \;\Rightarrow\; x = y \tag{2}$$

Now the concluding paragraph of Section 1.1 says that $\langle H/\approx, +, [0], \sim, \sqsubseteq\rangle$ is a reduced Girard monoid. It is called the *Lindenbaum algebra* ([Dun86] p. 193, [Avr87] p. 18).

Conversely, any (not necessarily reduced) Girard monoid is a model of the intensional kernel of linear logic. This is shown as follows. Inspired by 1(15), we define a validity predicate "$\models$" on M by

$$\models x \;\equiv\; \sim 0 \leq x \tag{3}$$

This predicate satisfies the postulates of Section 1.1. In fact, rule 1(0) follows from the neutrality of 0 for the addition of M. The cut-rule 1(1) is proved in

$$\begin{aligned}
&\models x + y \;\wedge\; \models \sim y + z \;\Rightarrow\; \models x + z\\
\equiv\;& \{(3)\}\\
&\sim 0 \leq x + y \;\wedge\; \sim 0 \leq \sim y + z \;\Rightarrow\; \sim 0 \leq x + z\\
\equiv\;& \{(0)\}\\
&\sim y \leq x \;\wedge\; \sim z \leq \sim y \;\Rightarrow\; \sim z \leq x\\
\equiv\;& \{\text{transitivity of "}\leq\text{"}\}\; \mathit{true}
\end{aligned}$$

In view of the other axioms of Section 1.1, it is useful to observe

$$\begin{aligned}
&\models \sim x + y \;\equiv\; \sim\sim x \leq y\\
\equiv\;& \{(3)\}\\
&\sim 0 \leq \sim x + y \;\equiv\; \sim\sim x \leq y\\
\equiv\;& \{(0)\}\; \mathit{true}
\end{aligned} \tag{4}$$

On the other hand, we have

$$\begin{aligned}
&\sim\sim x \leq x\\
\equiv\;& \{(0), \text{twice}\}\; \sim x \leq \sim x\\
\equiv\;& \{\text{reflexivity of "}\leq\text{"}\}\; \mathit{true}
\end{aligned}$$

By (4) and the transitivity of "$\le$", this implies

$$x \le y \;\Rightarrow\; \models \sim x + y \tag{5}$$

Now it is easy to see that predicate "$\models$" on M satisfies the axioms 1(2), 1(3) and 1(4). In fact, 1(2) follows from the reflexivity of "$\le$", 1(3) follows from the commutativity of "$+$", and 1(4) from the associativity of "$+$".

Thus, every (not necessarily reduced) Girard monoid is a model of the intensional kernel of linear logic. Therefore, the intensional kernel can be provided with a sound semantics in Girard monoids. As the Lindenbaum algebra $H/\approx$ is a reduced Girard monoid, this semantics is complete, even when restricted to reduced Girard monoids. In other words, everything that holds in all reduced Girard monoids also holds in the intensional kernel.

Remarks. Reduced Girard monoids are slightly more general than the De Morgan monoids of relevance logic (cf. [Dun86] p. 193). There the operator "$\circ$" stands for our operator "$\otimes$". De Morgan monoids have the extra axiom $x \le x \otimes x$, which corresponds to the contraction axiom quoted in 0(1) above. We come back to this in Section 3.6.

We allow non-strict Girard monoids in order to simplify the treatments of phase structures in Section 3.3 and quotient monoids in Section 3.5 below.

3.2. Strictness and Reduction of Girard Monoids

Let M be a not necessarily reduced Girard monoid. By Section 3.1, M yields a model of the intensional kernel of linear logic. Therefore, we can apply the results of Section 1. In particular, there is a second preorder "$\sqsubseteq$" on M, given by 1(5), and an equivalence relation "$\approx$", given by 1(10). By Section 3.1, the corresponding Lindenbaum algebra $M/\approx$ is a reduced Girard monoid. It is called the *reduction* of M and denoted by *red*(M). Now we have

Theorem 4. If M is strict, the relation "$\sqsubseteq$" is equal to "$\le$" and addition in M is monotone with respect to "$\le$".
(b) M is reduced if and only if relation "$\approx$" is the identity relation.

Proof. (a) The first assertion follows from 1(5) and (4). The second assertion follows from the first one and 1(23).

(b) If M is reduced, then "$\le$" is a partial order and "$\sqsubseteq$" equals "$\le$" by part (a), so that "$\approx$" is the identity relation by 1(10). Conversely, if "$\approx$" is the identity, then M is canonically isomorphic to its reduction *red*(M), and therefore reduced. □

3.3. Phase Structures

Here we show that Girard's phase structures form a special case of Girard monoids. A considerable simplification is achieved by allowing non-strict Girard monoids.

Let P be a commutative monoid with additive notation. Let M be the set of the subsets of P. An element x of P is identified with the set $\{x\}$ in M. We define addition in M by

$$A+B=\{x+y \mid x\in A, y\in B\}$$

In this way, M is a commutative monoid with neutral element 0, and P is a submonoid of M. We use the converse of the inclusion order as a partial order "$\leq$" on M, so that

$$A\leq B \;\equiv\; B\subset A \tag{6}$$

Let K be an arbitrary subset of P. Following [Gir87] Section 1.1, the elements of P are called *phases*, those of K *antiphases*. We introduce the operator "$\sim$" on M by

$$x\in{\sim}A \;\equiv\; A+x\subset K \tag{7}$$

The quintuple $\langle M, +, 0, \sim, \leq\rangle$ is a Girard monoid. In fact, in order to prove formula (0), we observe

$$\begin{aligned}
&{\sim}A\leq B \;\equiv\; {\sim}0\leq A+B\\
&\equiv \{(6)\}\; B\subset{\sim}A \;\equiv\; A+B\subset{\sim}0\\
&\equiv \{(7)\}\; A+B\subset K \;\equiv\; A+B+0\subset K\\
&\equiv \mathit{true}
\end{aligned}$$

This proves that the quintuple is a Girard monoid.

Remark. In [Gir87], Girard only considers the elements $A\in M$ with ${\sim}{\sim}A=A$. In this way, he forms a subsystem of M, which is isomorphic to the reduction $red(M)$. The complexity of double duals in his treatment is due to the fact that this subsystem is not a submonoid.

3.4. Ordered Commutative Groups

A completely different class of algebraic examples is obtained as follows.

Let M be an ordered additive commutative group (e.g. the integers, or the reals) and let $k\in M$ be fixed. For elements $x, y\in M$ we have

$$k-x\leq y \;\equiv\; k-0\leq x+y$$

Therefore, operator "$\sim$" defined by ${\sim}x=k-x$ gives M the structure of a (reduced) Girard monoid.

Remark. The constant 0 can be regarded as representing *false*, but this case shows that 0 may happen to be valid. In fact, by (3), validity of 0 is equivalent to ${\sim}0\leq 0$ and hence to $k\leq 0$. Since k is arbitrary, this is not excluded.

3.5. Quotient Monoids

Let M be a strict Girard monoid and let P be a submonoid of M. We define relation "$\leq_P$" on M by

$$x\leq_P y \;\equiv\; (\exists p\in P :: x\leq y+p). \tag{8}$$

Relation "$\leq_P$" is a preorder on M. In fact, it is reflexive since $0 \in P$. It is transitive, since $P + P \subset P$ and addition in M is monotone, cf. Theorem 4.

We claim that $M_P = \langle M, +, 0, \leq_P, \sim \rangle$ is a strict Girard monoid. It suffices to verify condition (0) with "$\leq_P$" instead of "$\leq$". This is done in

$$
\begin{aligned}
& \sim x \leq_P y \\
\equiv\ & \{(8)\}\ (\exists p \in P :: \sim x \leq y + p) \\
\equiv\ & \{(0)\}\ (\exists p \in P :: \sim 0 \leq x + y + p) \\
\equiv\ & \{(8)\}\ \sim 0 \leq_P x + y
\end{aligned}
$$

The reduced Girard monoid M/P is defined as the reduction $red(M_P)$. Its underlying set is $M/\approx_P$ where the equivalence relation "$\approx_P$" satisfies

$$x \approx_P y \ \equiv\ (\exists p, q \in P :: x \leq y + p \leq x + q)$$

This result may be regarded as a first little step in the structure theory of Girard monoids.

3.6. Complete Classes of Models for Other Logics

The Lindenbaum algebra approach of the model theory has the advantage that other postulates can be added to the logical system and then be translated into algebraic conditions. Here, we only treat the extra postulates

$$\models A \multimap (B \multimap A) \ \{\text{positive paradox, cf. } 0(0)\} \tag{9}$$

$$\models (A \multimap (A \multimap B)) \multimap (A \multimap B) \ \{\text{contraction, cf. } 0(1)\} \tag{10}$$

where "$\multimap$" is given by

$$A \multimap B \ =\ \sim A + B \ \{\text{cf. } 0(2)\} \tag{11}$$

The system H with either one or both of these additional postulates has a Lindenbaum algebra, which is a strict (even reduced) Girard monoid with some extra properties.

Now it is convenient to go the other way round. Let M be a strict Girard monoid, interpreted as a logic by means of (3). By (11), 1(5) and Theorem 4, we have

$$(\models x \multimap y) \ \equiv\ x \leq y \tag{12}$$

The condition that M satisfies postulate (9), is reformulated in

$$
\begin{aligned}
& (\forall x, y :: \models x \multimap (y \multimap x)) \\
\equiv\ & \{(11), (12)\}\ (\forall x, y :: x \leq \sim y + x) \\
\equiv\ & \{\text{Theorem 4}\}\ (\forall y :: 0 \leq \sim y) \\
\equiv\ & \{M \text{ strict}\}\ (\forall x :: 0 \leq x)
\end{aligned}
$$

Similarly, the condition that M satisfies postulate (10), is reformulated in

$$
\begin{aligned}
& (\forall x, y : \models (x \multimap (x \multimap y)) \multimap (x \multimap y)) \\
\equiv\ & \{(11), (12)\}\ (\forall x, y :: {\sim}x + {\sim}x + y \leq {\sim}x + y) \\
\equiv\ & \{\text{Theorem 4}\}\ (\forall x :: {\sim}x + {\sim}x \leq {\sim}x) \\
\equiv\ & \{M \text{ strict}\}\ (\forall x :: x + x \leq x)
\end{aligned}
$$

To summarise, we state

Theorem 5. Let M be a strict Girard monoid.

(a) M satisfies (9) if and only if $x \geq 0$ for all $x \in M$.
(b) M satisfies (10) if and only if $x + x \leq x$ for all $x \in M$.

3.7. Topological Girard Monoids

We now add the modal operator "?". For simplicity, we restrict our attention to strict Girard monoids.

We define a *topology* in a strict Girard monoid M to be a submonoid T of M that satisfies

$$(\forall t \in T :: 0 \leq t \wedge t + t \leq t) \tag{13}$$

and such that for every $x \in M$ there exists an element $?x \in T$ with

$$(\forall t \in T :: x \leq t \ \equiv\ ?x \leq t) \tag{14}$$

Let a *topological* Girard monoid be defined as a pair $\langle M, T \rangle$ where M is a strict Girard monoid and T is a topology in M.

For reference below, we notice that axiom (14) is equivalent to the conjunction of the following three axioms

$$(\forall x \in M :: x \leq ?x) \tag{15}$$

$$(\forall x, y \in M :: x \leq y \ \Rightarrow\ ?x \leq ?y) \tag{16}$$

$$(\forall t \in T :: ?t \leq t) \tag{17}$$

The verification of this fact is easy and may be left to the reader.

By the final paragraph of Section 1.2, the Lindenbaum algebra $H/{\approx}$ has the set T of the equivalence classes $[?A]$ as a topology. So $\langle H/{\approx}, T \rangle$ is a topological Girard monoid.

Conversely, every topological Girard monoid $\langle M, T \rangle$ is a model of linear logic. In fact, let a function "?" from M to T be chosen such that (14) holds. We have to verify the rules 1(30) and 1(34). Since M is strict, the relations "$\sqsubseteq$" and "$\leq$" on M are equal. Since $?y \in T$ for all $y \in M$, condition (14) implies 1(30). Similarly, condition (13) implies the first two rules of 1(34)

$$0 \leq ?x \ \wedge\ ?x + ?x \leq ?x$$

Since T is a submonoid, formula (17) implies the other two rules of 1(34)

$$?0 \leq 0 \ \wedge\ ?(?x + ?y) \leq ?x + ?y$$

This concludes the proof that $\langle M, T\rangle$ is a model of linear logic. Therefore, the modelling of linear logic by means of topological Girard monoids is sound and complete.

Using the Lindenbaum algebra, we get from Theorem 5:

The topological Girard monoids with $(\forall x :: 0 \leq x)$ form a sound and complete class of models of the system H enriched with postulate (9).

The topological Girard monoids with $(\forall x :: x + x \leq x)$ form a sound and complete class of models of the system H enriched with postulate (10).

3.8. Topology as a Branch of Linear Logic

In order to justify the term topology in the present context, we shall show that the classical concept of topology is a special case.

Recall that a topology on a set X is characterised by the set T of the closed subsets of X. The set T is a subset of the power set of X, subject to the following axioms

$$\emptyset \in T \tag{18}$$

$$A \in T \wedge B \in T \;\Rightarrow\; A \cup B \in T \tag{19}$$

$$U \subset T \;\Rightarrow\; \bigcap U \in T \tag{20}$$

Here, $\bigcap U$ is the intersection of the elements of U. Usually, one adds the axiom $X \in T$, but this axiom is superfluous as it follows from (20) by the convention that $\bigcap U = X$ if U is empty.

Let M be the power set of X, and let "$\sim$" be the complementation operator with respect to X. It is easy to verify that $\langle M, \cup, \emptyset, \sim, \subset\rangle$ is a reduced Girard monoid. In fact, the only interesting observation is that $X = {\sim}\emptyset$ and that ${\sim}A \subset B$ is equivalent to $X \subset A \cup B$, so that (0) holds.

We now claim that the conjunction of (18), (19) and (20) is equivalent to the condition that T be a topology in the Girard monoid M, cf. Section 3.7. In fact, the conjunction of (18) and (19) says that T is a submonoid. Condition (13) holds trivially, since every set $A \in M$ satisfies

$$\emptyset \subset A \quad \wedge \quad A \cup A \subset A$$

It remains to verify that (14) and (20) are equivalent. Now the operator "?" is a closure operator. Since (14) is equivalent to the conjunction of (15), (16), (17), the equivalence between (14) and (20) is a fairly standard exercise of point set topology. Compare [Hu66] p. 26, or [Kur77] Chapter 10. Notice that, by Theorem 5, general topology satisfies both extra postulates (9) and (10).

3.9. Adding Points at Infinity

Let M be an ordered additive commutative group with the Girard monoid structure constructed in Section 3.4. The only submonoid T of M that satisfies (13), is the singleton set $\{0\}$. Therefore, if M has more than one element, it follows that the Girard monoid M does not admit any topology.

As a kind of remedy, we construct for any strict Girard monoid M an extension M^+ that admits at least one topology. We do this by adding two points at infinity.

Let M be a strict Girard monoid. Let α, ω be not in M and let M^+ be the disjoint union $M \cup \{\alpha, \omega\}$. The set M^+ is equipped with the structure of a strict Girard monoid by extending addition, duality and preorder as follows:

$x + \omega = \omega$ for all x

$x + \alpha = \alpha$ for all $x \neq \omega$

$\sim\omega = \alpha$

$\alpha \leq x \leq \omega$ for all x

Now it is easy to see that $\{0, \omega\}$ is a topology on M^+. The verifications are left to the reader.

By a slight specialisation, this model yields a proof that $\sim(?a + ?b) + ?(a + b)$ is not derivable, cf. 2(6). In fact, let M be as in Section 3.4 with an element $a \in M$ with $a > 0$. Take $b = -a$, and the topology $\{0, \omega\}$ on M^+. Then we get the element

$$\sim(?a + ?b) + ?(a + b) = \sim\omega + ?0 = \alpha$$

which is not valid. Notice that in this model both additional postulates (9) and (10) are invalid, see Theorem 5.

Another interesting special case is a three-valued linear logic. Let M only consist of a zero element with $0 + 0 = 0$ and $\sim 0 = 0$. Then M^+ has three elements α, 0, ω. This model yields a (second) proof that $A = \sim(\sim a + a) + \sim 0$ is not derivable, cf. 2.5. In fact, taking $a = \alpha$ or $a = \omega$, we get $A = \alpha$, which element is not valid. In this model, postulate (10) holds, but (9) does not.

3.10. Probabilistic Logic as a Special Case

Let M be the additive monoid of the real numbers $x \geq 0$, ordered in the usual way. Let "$\sim$" be defined by

$$\sim x = \max(1 - x, 0). \tag{21}$$

The quintuple $\langle M, +, 0, \sim, \leq \rangle$ is a non-strict Girard monoid, because of

$$\begin{aligned} & \sim x \leq y \\ \equiv\ & \{(21)\}\ 1 - x \leq y \ \wedge\ 0 \leq y \\ \equiv\ & \{\text{calculus}\}\ 1 \leq x + y \\ \equiv\ & \{(21)\}\ \sim 0 \leq x + y. \end{aligned}$$

The reduction of the Girard monoid M can be identified with the segment of the numbers x with $0 \leq x \leq 1$, where addition is truncated so as to remain inside of the segment. The resulting logic can be seen as a kind of probabilistic logic. Notice that it satisfies postulate (9), but not (10). The only topology in *red*(M) is the two-point topology $\{0, 1\}$.

We can obtain a linear logic with $n+1$ linearly ordered truth values by taking the submonoid of $red(M)$ that consists of the classes of the fractions i/n with $0 \le i \le n$. For $n > 1$, the logic does not satisfy (10). The case $n = 1$ is ordinary boolean logic. The case $n = 2$ gives a three-valued logic that differs from the one mentioned at the end of Section 3.9. For it violates postulate (10) and satisfies (9).

4. Appendix: The Lattice Properties

In this appendix we realise the afterthought announced in Section 0.3. We add the lattice theoretical constants $\top$ and $\bot$, and the lattice theoretical operators "$\sqcap$" and "$\sqcup$".

In the framework of Girard's sequent calculus, cf. Section 2, the extension is carried out as follows. One adds two atoms $\top$ and $\bot$, which are each others' dual, and two binary operators "$\sqcap$" and "$\sqcup$". The duality function is extended to the new formulae by

$$\sim(A \sqcap B) = \sim A \sqcup \sim B \quad \wedge \quad \sim(A \sqcup B) = \sim A \sqcap \sim B. \tag{0}$$

It is straightforward to verify that formula 2(1) remains valid. In the postulates of Section 2.1 one adds

$$\vdash X, T \tag{1}$$

$$\vdash X, B \ \wedge \ \vdash X, C \ \Rightarrow \ \vdash X, B \sqcap C \tag{2}$$

$$\vdash X, B \ \vee \ \vdash X, C \ \Rightarrow \ \vdash X, B \sqcup C \tag{3}$$

It requires some work to show that under assumption (0) postulate (3) is equivalent to the converse of (2):

$$\vdash X, B \ \wedge \ \vdash X, C \ \Leftarrow \ \vdash X, B \sqcap C \tag{4}$$

Now it is easy to see that the same effect is realised in the system H of Section 1 by adding a constant $\top \in H$ and an operator "$\sqcap$" with the postulates

$$\models \sim A + \top$$

$$\models \sim A + B \ \wedge \ \models \sim A + C \ \equiv \ \models \sim A + (B \sqcap C) \tag{5}$$

In terms of the preorder "$\sqsubseteq$", these postulates are equivalent to

$$A \sqsubseteq \top,$$

$$A \sqsubseteq B \ \wedge \ A \sqsubseteq C \ \equiv \ A \sqsubseteq B \sqcap C. \tag{6}$$

The Theorems 1 and 2 remain valid.

As for the model theory, it is clear that in reduced Girard monoids the postulates equivalent to (6) are the conditions that M has a biggest element and that every pair of elements has a greatest lower bound, in other words that M is an upper semi-lattice. Since, moreover, the duality is an order-reversing involution, M is a lattice.

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