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# The Computational Complexity of Multi-Level Linear Programs

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Dedicated to Robert Jeroslow (1942-1988)

ABSTRACT

We show that (L+1)-level linear programs are as difficult as Level L of the polynomial-time hierarchy, even if one only considers problems with unique optimal solutions.

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# The computational complexity of multi-level linear programs

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*Dedicated to Robert Jeroslow* (1942–1988)

**Abstract**

We show that $(L+1)$-level linear programs are as difficult as level $L$ of the polynomial-time hierarchy, even if one only considers problems with unique optimal solutions.

## 1 Introduction

Computer scientists have constructed a hierarchy [4] of degrees of difficulty for problems. At the bottom are problems solvable in polynomial time, a class now known to include linear programs. The next level is the class of NP-complete problems, which includes integer programming, travelling salesman problems, and many others [2]. These first two classes of problem are denoted by $\sum_0^p$ and $\sum_1^p$, with an infinite sequence of problem classes $\sum_L^p$ believed to be of increasing difficulty.

R. Jeroslow [3] shows that the problem of finding the optimal objective value in an $(L+1)$-level linear program is at least as hard as solving problems in the class $\sum_L^p$ of the polynomial-time hierarchy. The version of multi-level LPs in [3] assumes that, when a player has alternatives that are equally favorable in terms of his own objective, he will make the choice most favorable to the player whose move immediately preceded his. This is an essential feature of the construction [3, page 149 and formulas (4.2) and (4.3)].

The possibility arises that the behavior of players in multiple optima situations increases the complexity of the problem. We show this is not the case. As in [3], we construct programs whose difficulty is as great as the different levels of the polynomial-time hierarchy. However, in our programs, the optimal solution at each level will be unique.

We think the programs here are simpler than previous constructions. The special case of bilevel LPs that are NP-complete has been given in [1].

## 2 A Problem at Level $L$ of the Polynomial-time Hierarchy

Since the satisfiability problem is NP-complete, a common way of showing a problem is as hard as any NP-complete problem is to show that a procedure for solving the new problem could be used to solve the satisfiability problem.

We will follow a similar proof strategy for level $L$ of the polynomial-time hierarchy. [4, Theorem 4.1 (2)] shows that the following problem is as hard as any problem at level $L$.

> **THE SATISFIABILITY GAME:** Given a propositional logic formula
>
> $$D_1 \& D_2 \& \ldots \& D_N \qquad [D_n \equiv y_{n1} \vee y_{n2} \vee y_{n3}] \tag{1}$$
>
> where each $y_{ni}$ is either a variable or a negation of a variable. Each of $L$ players chooses values (T or F) for some subset of the variables. Player $j$ makes his choices immediately after player $j+1$, for $1 \leq j \leq L-1$. The odd numbered players are on a team which wants the value of the formula to be true, the others want the formula to be false. Which team will win?

We will show this problem can be solved if we can solve $(L+1)$-level linear programs. Corollary 2 shows that the $L$-player Satisfiability Game may be replaced by an $L$-player "Knapsack Game" in which the two teams are trying to control the sum of a set of numbers. The main result, Theorem 4, shows that the Knapsack Game may be converted to an $(L+1)$-level linear program.

**Lemma 1** *Let $G$ be a formula of the type (1) with variables $x_m, m \in M$. There are natural numbers $u$; $a_m, m \in M$; and $b_h, h \in S$ such that for every truth-value assignment $v : M \rightarrow \{T, F\}$, $v$ makes $G$ true if and only if there is $S_v \subset S$ with*

$$\sum_{\{m|v(m)=T\}} a_m + \sum_{h \in S_v} b_h = u \tag{2}$$

Proof: Results of this kind have been used to prove NP-completeness of PARTITION and similar problems in [2, pages 60–62] and elsewhere. We give the construction below for completeness.

For each $1 \leq n \leq N$, $m \in M$ let $c_{nm} = 0$ if neither $x_m$ nor its negation appears in $D_n$. For the three $x_m$ which appear in $D_n$, assign the values 1, 2, 4 each to one of the $c_{nm}$. Let

$$a_m \equiv \sum_{n=1}^{N} 22^n c_{nm} \qquad u \equiv \sum_{n=1}^{N} 22^n (8)$$

For each $0 \leq j \leq 7$, there are numbers $s_{jt}$, $1 \leq t \leq 4$ such that any number from 1 to 8 *except* $8-j$ may be written as a sum of a subset of the $s_{jt}$. We choose the numbers so that for any $j$, $\sum_t s_{jt} \leq 14$. For example, we may take $s_{11} = 1$, $s_{12} = 2$, $s_{13} = 3$, $s_{14} = 8$.

For each $n$ let $j(n)$ be the sum of the $c_{nm}$ such that the negation of the corresponding $x_m$ appears in $D_n$. Thus $D_n$ is assigned the value T by $v$ unless

$$\sum_{\{m|v(m)=T\}} c_{nm} = j(n)$$

To obtain the $b_h$, let $S \equiv \{(n,t) | 1 \leq n \leq N, 1 \leq t \leq 4\}$ and

$$b_{(n,t)} \equiv 22^n s_{j(n)t}, 1 \leq t \leq 4$$

There is an $S_v$ such that (2) holds if and only if, for each $n$, there is a $W \subset \{1,2,3,4\}$ with

$$\sum_{\{m|v(m)=T\}} c_{nm} + \sum_{k \in W} s_{j(n)k} = 8$$

By construction of the of the $s_{jt}$, such a $W$ exists unless $\sum_m c_{nm} = j(n)$. This happens only if $v$ makes $D_n$ false. □

**Corollary 2** *If we can solve*

> **THE KNAPSACK GAME:** *Given natural numbers $u$ and $a_i, i \in I$. Let $I = \cup I_j$, with the sets disjoint. Each of the $L$ players chooses $P_j \subset I_j$, with player $j$ making his choice immediately after $j+1$. The odd-numbered players want*
>
> $$\sum_{i \in \cup P_j} a_i = u$$
>
> *while the even-numbered players are trying to stop this. Which team will win?*

*then we can solve the Satisfiability Game. Thus the L-player Knapsack Game is as hard as any problem at level L of the polynomial-time hierarchy.*

Proof: We construct a Knapsack Game corresponding to each Satisfiability Game. Given a formula (1), construct $u$, $a_m$, $b_h$ as in Lemma 1. Let $I \equiv M \cup S$. For $2 \leq j \leq L$, $I_j$ corresponds to the $x_m$ controlled by player $j$ in the Satisfiability Game. Player 1 controls the $b_h$ as well as the $a_m$ corresponding to his Satisfiability counterpart.

The team that wins in one game will also win in the other. □

# 3 A multi-level LP as hard as level $L$ of the hierarchy

By Corollary 2, it is sufficient to constuct, for each Knapsack Game, a multi-level LP whose solution tells which team wins. As in [3], we will have to add an additional player (player 0) to deal with the difficulty of simulating a discrete problem using continuous variables. Consistent with our earlier conventions, this player moves last.

The constraints of our problem will be:

$$\begin{array}{ll} \multicolumn{2}{c}{A - B + \sum_{i \in I} a_i X_i = u} \\ Q \leq A + B & 0 \leq X_i \leq 1 \\ Q \leq .6 & D_i \leq X_i \\ A, B \geq 0 & D_i \leq 1 - X_i \end{array} \tag{3}$$

Player 0 controls the variables $A, B, Q, D_i$. For $1 \leq j \leq n$, player $j$ controls $X_i$, $i \in I_j$.

The objective for player 0 is

$$\max\ Q - A - B + \sum_{i \in I} D_i \tag{4}$$

It is easy to see that the unique optimal solution to the problem for player 0 will make $D_i = \delta(X_i)$, where $\delta(t)$ is the distance from $t$ to the nearest integer and $Q = \min\{.6, |u - \sum a_i X_i|\}$.

Let $c \geq \max\{a_i\}, i \in I$. For $1 \leq j \leq L$ the objective for player $j$ is

$$\max\ (-1)^j Q - 2^{j+2} c \sum_{i \in I_j} D_i - \sum_{i \in I_j} \epsilon_i X_i \tag{5}$$

The $D_i$ are penalties which encourage players 1 to $L$ to choose integers. If the $X_i$ are close to integer, $Q$ will be .6 if $\sum a_i X_i \neq u$ (remember the $a_i$ are integers), close to 0 otherwise.

The $\epsilon_i$ are only used to guarantee that the optimal solutions will be unique. It will be sufficient to choose $\epsilon_i > 0$ such that $\sum \epsilon_i < .3$ and for any $H, H' \subset I_j$,

$$\sum_{i \in H} \epsilon_i \neq \sum_{i \in H'} \epsilon_i \text{ if } H \neq H' \tag{6}$$

Our main task is to prove that the optimal solution to the multi-level has all $X_i$ integer (thus either 0 or 1). Once player $L$ has made his choices, we have a linear program with one fewer level in which $u$ has been replaced by $u - \sum_{i \in I_L} a_i X_i$. Since the variables are continuous, we must consider the possibility that this quantity is not an integer, thus we will prove results about $u$ that are close to integer, as well as integer.

We begin with a technical result whose main content is that the value of $Q$ dominates the part of the objective due to the $\epsilon_i$.

**Lemma 3** *Suppose that $X_i$ and $\overline{X_i}$ are integer, with $Q$, $\overline{Q}$ determined by player 0. If $\delta(u) \leq .25$ and $Q \neq \overline{Q}$, then $|Q - \overline{Q}| > .35$. Moreover,*

$$(-1)^L Q > (-1)^L \overline{Q} \text{ if and only if } (-1)^L Q - \sum_{i \in I_L} \epsilon_i X_i > (-1)^L \overline{Q} - \sum_{i \in I_L} \epsilon \overline{X_i} \tag{7}$$

Proof: Since the $a_i$ are integer, there is only one value of $\sum a_i X_i$ which produces $Q \neq .6$, and this gives $Q = \delta(u) \leq .25$. The"moreover" follows from $\sum \epsilon_i < .3$ and $0 \leq X_i \leq 1$. □

**Theorem 4** *Let $u'$ be the integer closest to $u$. If $|u - u'| \leq 2^{-L-1}$, the optimal solutions for all players will have all $X_i$ integer and be unique. Moreover, this optimal solution will have the same $X_i$ as for the problem obtained by replacing $u$ by $u'$.*

Proof: We will establish this by induction on $L$. The case $L = 1$ is very similar to the induction step and we will not give the details of it.

Any strategy for player $L$ with

$$\sum_{i \in I_L} \delta(X_i) > 2^{-L-1} c^{-1} \tag{8}$$

will be inferior to making $X_i = 0$ for all $i \in I_L$. Thus we need only consider $X_i$ for which (8) does not hold. Let $X_i'$ be the nearest integer to $X_i$ for $i \in I_L$. Since

$$|(u - \sum_{i \in I_L} a_i X_i) - (u' - \sum_{i \in I_L} a_i X_i')| \leq |u - u'| + c \sum_{i \in I_L} \delta(X_i) \leq 2^{-L}$$

we may apply the induction hypothesis to the problem with one fewer level. This tells us that, once $L$ has made his choices, the optimal solution to the problem for players 1 to $L-1$ with $u$ replaced by $(u-\sum a_i X_i)$ will be integer, and will be the same as the optimal solution if $u$ is replaced by $(u'-\sum a_i X_i')$. It will also be the same solution as if $u$ is replaced by $(u-\sum a_i X_i')$.

Thus if player $L$ replaces $X_i$ by $X_i'$, players 1 to $L-1$ will not change their decisions. Since

$$-\sum_{i\in I_L} a_i\delta(X_i) + 2^{L+2}c\sum_{i\in I_L}\delta(X_i) - .3\sum_{i\in I_L}\delta(X_i) > 0$$

unless all $X_i$ are integer, the gain in the $D_i$ from replacing all $X_i$ by $X_i'$ outweighs adverse effects on the $Q$ and $\epsilon$ parts of the objective for player $L$.

We have established that there will be an optimal solution in which $X_i = 0$ or 1 for all players. To see that it is unique, note that if two all-integer $X_i$ lead to different values for $Q$, the objective function values will be different by (7). If different integer $X_i$ lead to the same value for $Q$, then (6) implies the $\epsilon$ part of the objective will be different. Let $X_i^*$ be the optimal solution to the original problem, $\overline{X}_i$ be any other integer choice for player $L$, with resulting $Q^*$ and $\overline{Q}$.

Finally, we consider what happens when $u$ is replaced by $u'$. Let $X_i$ and $\overline{X}_i$ be two integral possibilities for player $L$, which lead to $Q$ and $\overline{Q}$. When we replace $u$ by $u'$ these lead to $Q^{*\prime}$ and $\overline{Q'}$. Each of $Q^*, \overline{Q}$ is either .6 or $\leq .25$. $Q^* = \overline{Q} = .6$ implies $Q^{*\prime} = \overline{Q'} = .6$. $Q^*, \overline{Q} \leq .25$ implies $Q = \overline{Q}$ and $Q^{*\prime} = \overline{Q'} = 0$. In these cases, the choice for player $L$ between $X^*$ and $\overline{X}$ depends on the $\epsilon$ part of the objective, which is unaffected by the change from $u$ to $u'$. If $Q^* = .6$ and $\overline{Q} \leq .25$, then $Q^{*\prime} = .6$ and $\overline{Q} = 0$. In this case $L$ is trying to maximize $Q$, and .6 is better than 0. The case $Q^* \leq .25$ and $\overline{Q} = .6$ is similar. □

We have established that the solution of the multi-level LP given by (3), (4), (5) corresponds to the Knapsack Game with the subsets $P_j$ chosen by the players corresponding to $\{i \in I_j | x_i = 1\}$. Since the optimal solution to the multi-level LP tells which team wins the Knapsack Game, this is as hard as any problem at level $L$ of the hierarchy. However, the winner of the Knapsack Game can be identified with less information.

**Corollary 5** *Answering any of the following questions about (3)–(5) is as hard as any problem at level L of the hierarchy:*

1. *In the optimal solution, is $Q > .3$?*

2. *Will the objective function value for some odd (even) $j$ be $> -.3$ ($> .3$)?*

3. *For every odd (even) $j$, does the optimal solution have $X_i < .5$ for all $i \in I_j$?*

Proof: $Q$ will be 0 if the even team wins, .6 otherwise. Since the $\epsilon$ part of the objective is $< .3$ and the $D_i = 0$, the second part follows. To establish the third part, note that the optimal solution for all players on the losing team will be to make all $X_i = 0$, to minimize the $\epsilon$ part. □

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