Direct Reconstruction of Single-Photon Emission Computed Tomography Images Using Retained Matrix Elements

Jonathan P. Pratt and James L. Lear

In clinical applications, two methods of single-photon emission computed tomography (SPECT) reconstruction are widely used. These are filtered backprojection and iterative reconstruction. Filtered backprojection is fast and produces acceptable images. Iterative reconstruction is slow, but produces images of greater accuracy than backprojection. The authors sought to develop a method of SPECT reconstruction that would have the advantages of both established methods: close in speed to backprojection and with the accuracy of iterative reconstruction. This was accomplished by computing a direct solution to the set of linear equations governing SPECT reconstruction. We tested this method of SPECT reconstruction using a set of projections from a cold rod and sphere phantom. Direct reconstruction produced images having equivalent resolution to backprojected images, but with double the contrast ratio. The direct method required 10 seconds of computation per slice on a Macintosh Quadra 950 (Apple Computer; Cupertin, CA), significantly faster than most iterative methods.

KEY WORDS: single-photon emission computed tomography (SPECT), computer processing, image reconstruction.

IN SINGLE-PHOTON emission computed tomography (SPECT), the spatial distribution of a radioisotope in a medium is estimated by acquiring a set of planar projection images at varying angles from the subject. The projection images are processed with any one of a number of different methods for determining the source distribution. These methods include filtered backprojection,[1] iterative techniques,[2-5] and direct reconstruction. These methods vary in the amount of computation required and the quality of their results.

Filtered backprojection is the fastest method for obtaining SPECT reconstructions from projection data. Backprojection is implemented relatively easily, and it is the most widely used reconstruction technique in clinical image processing. A drawback of backprojection is that it only solves a straightforward theoretical problem.[6] Real-life issues must be addressed separately. For example, the point-spread function is typically handled with prefiltering, whereas attenuation correction is usually applied after only a first reconstruction of the data.[7]

Iterative methods of reconstruction are based on solving a set of linear equations that models the system of image acquisition. Iterative methods are generally slow and require the incorporation of tests to avoid divergence.[5] The model on which these methods are based can account for such a priori information as attenuation and point spread.[8] In addition, greater contrast resolution can be achieved in images compared with backprojection.

Direction reconstruction is similar to iterative methods in some aspects. Like iterative methods, direct reconstruction uses a set of linear equations that models the acquisition system. It can, therefore, incorporate some nonidealities, such as camera resolution. However, direct reconstruction differs from iterative methods in other aspects. For example, nonidealites such as attenuation that require knowledge of the reconstructed object's boundary are difficult to address. Another difference is that although iterative methods are geared toward finding a solution for a particular set of projection data, the method of direct reconstruction consists of precomputing a general solution, or matrix inverse. This matrix inverse can be applied to any subsequent vector of projection data emanating from the specified system.

Interest in direct reconstruction has increased as computer technology has improved. This is because solution matrices are large and difficult to compute. For example, in the case of 64 projections that are 64 pixels wide, the inverse matrix has more than 16 million elements. Also, the linear equations are typically ill conditioned, so just as iterative solutions can diverge, a straightforward matrix inversion is unlikely to produce a useful result. In performing this work, we sought to demonstrate that these difficulties can be overcome, and that direct reconstruction verges on being practical for

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From the Department of Radiology, University of Colorado Health Sciences Center, Denver, CO.

Supported by National Institutes of Health grant 26657 and a Scholar grant of the Radiological Society of North America.

Address reprint requests to Jonathan P. Pratt, PhD, Division of Nuclear Medicine, Box A034, University of Colorado Health Sciences Cntr, 4200 E 9th Ave, Denver, CO 80262.

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clinical use, having some of the advantages of both backprojection and iterative methods.

METHODS

Neglecting noise, the linear equations modeling the projection-acquisition system can be represented by the equation

$$\mathbf{p} = \mathbf{A}\mathbf{x} \tag{1}$$

where $\mathbf{p}$ is the vector of measured projection data for a given slice, $\mathbf{A}$ is the system transition matrix based on geometry and system response, and $\mathbf{x}$ is the reconstructed-slice vector, which contains the image approximating the source distribution. $\mathbf{p}$ has $n \cdot m$ elements, where n is the horizontal size in pixels of the projections, and m is the number of views. If the reconstructed slice has the same resolution as the projection data, then $\mathbf{A}$ has dimensions $n \cdot m$ by n^2, and $\mathbf{x}$ has n^2 elements.

Equation (1) typically does not have a unique solution. The linear equations may present an underdetermined or overdetermined solution, and noise in the projection data makes the overdetermined case inconsistent. The preferred course of action with such linear systems is to accept as a solution a vector that minimizes some error function. Mathematically and practically, a good choice for the error function is

$$e(\mathrm{x}) = \|\mathbf{p} - \mathbf{A}\mathbf{x}\| \tag{2}$$

where $\|\mathbf{v}\|$ indicates the L-2 norm of a vector $\mathbf{v}$. This is the well-known method of least squares. The value of $\mathbf{x}$ that minimizes $e(\mathbf{x})$ in Equation (2) is

$$\mathbf{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{p} \tag{3}$$

If we define

$$\mathbf{B} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T \tag{4}$$

where $\mathbf{B}$ has dimensions n^2 by $n \cdot m$, then

$$\mathbf{x} = \mathbf{B}\mathbf{p} \tag{5}$$

Equation (5) represents the theoretical basis for a direct solution. It suggests that if the matrix $\mathbf{B}$ can be computed and retained, then any number of slices can be computed, each requiring only a simple matrix-vector multiply. In theory, $\mathbf{B}$ can be computed directly as in Equation (4). In practice, going from Equations (1) to (3) essentially squares the ill-conditioning present in the problem. Therefore, alternative methods are used to find a useful approximation to $\mathbf{B}$.

We used the following method: First, Equation (1) is modified to achieve regularization.[9]

$$\mathbf{p}' = \mathbf{A}'\mathbf{x}' \tag{6}$$

where

$$\mathbf{A}' = \begin{bmatrix} \mathbf{A} \\ \mathbf{I} \end{bmatrix} \tag{7}$$

and

$$\mathbf{p}' = \begin{bmatrix} \mathbf{A} \\ \mathbf{1} \end{bmatrix} \tag{8}$$

$\mathbf{I}$ is the identity matrix, and $\mathbf{1}$ is a column vector of n^2 ones. This modification effectively stabilizes the problem and removes negative values from $\mathbf{x}'$. The one complication to this is that $\mathbf{p}$ must be normalized to have an average element value of one.

The next step is to solve for $\mathbf{B}'$, as in Equation (4).

$$\mathbf{B}' = (\mathbf{A}'^T\mathbf{A}')^{-1}\mathbf{A}'^T \tag{9}$$

Note that this can also be written in terms of $\mathbf{A}$ as

$$\mathbf{B}' = [\mathbf{A}^T\mathbf{A} + \mathbf{I}]^{-1}[\mathbf{A}^T\mathbf{1}] \tag{10}$$

The $\mathbf{A}$ matrix in this calculation included only the fractional areas projected by source pixels. We chose to compute the QR decomposition of $\mathbf{A}'$ and then use this result to determine the individual columns of $\mathbf{B}'$. Once obtained, $\mathbf{B}'$ is applied to individual projection vectors to reconstruct the corresponding slices. The regularization technique and high-precision floating-point arithmetic were sufficient to limit roundoff and truncation errors.

We implemented the method of direct reconstruction on a Macintosh Quadra 950 (Apple Computer, Cupertine, CA) with 256 Mbytes of random access memory (RAM). The standard format for real numbers on this computer requires 12 bytes for each number. Projection data were obtained from a cold rod and sphere phantom. They consisted of 64 views with 48 bins per view. The reconstructed slices were 48×48 pixels.

RESULTS

Computation of the $\mathbf{B}'$ matrix took 5 days, but once $\mathbf{B}'$ was saved on the computer's hard disk, individual reconstructed slices could be computed in under 10 seconds. Overall, computation of $\mathbf{B}'$ required about 240 Mbytes of RAM and permanent storage of $\mathbf{B}'$ required about 80 Mbytes of hard disk space.

Figure 1 shows two reconstructed slices obtained by different methods from the same filtered projection data. Backprojection is used in the first column, and direct reconstruction is used in the second column. We observe equivalent resolution in the two image sets, but direct reconstruction has produced double the contrast ratio of the backprojected images. The difference in contrast is quantified in the graph of Fig 2.

CONCLUSION

Direct SPECT reconstruction still has several drawbacks. First, there is a large RAM requirement for the computer that precomputes the solution matrix $\mathbf{B}'$. Second, the time to compute $\mathbf{B}'$ can be substantial. Both time and computer memory requirements increase in direct proportion to the number of projections and to the cube of the spatial

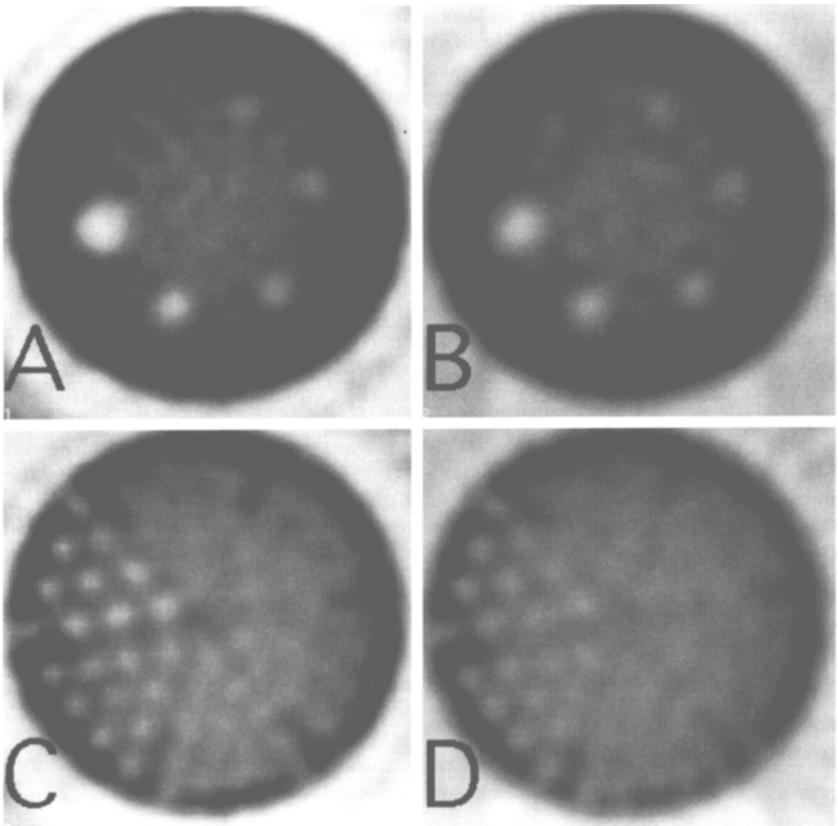

Fig 1. Examples of reconstructions using direct matrix inversion (left column, A and C) and filtered backprojection (right column, B and D). In both cases, data were prefiltered using an optimized two-dimensional Butterworth filter.

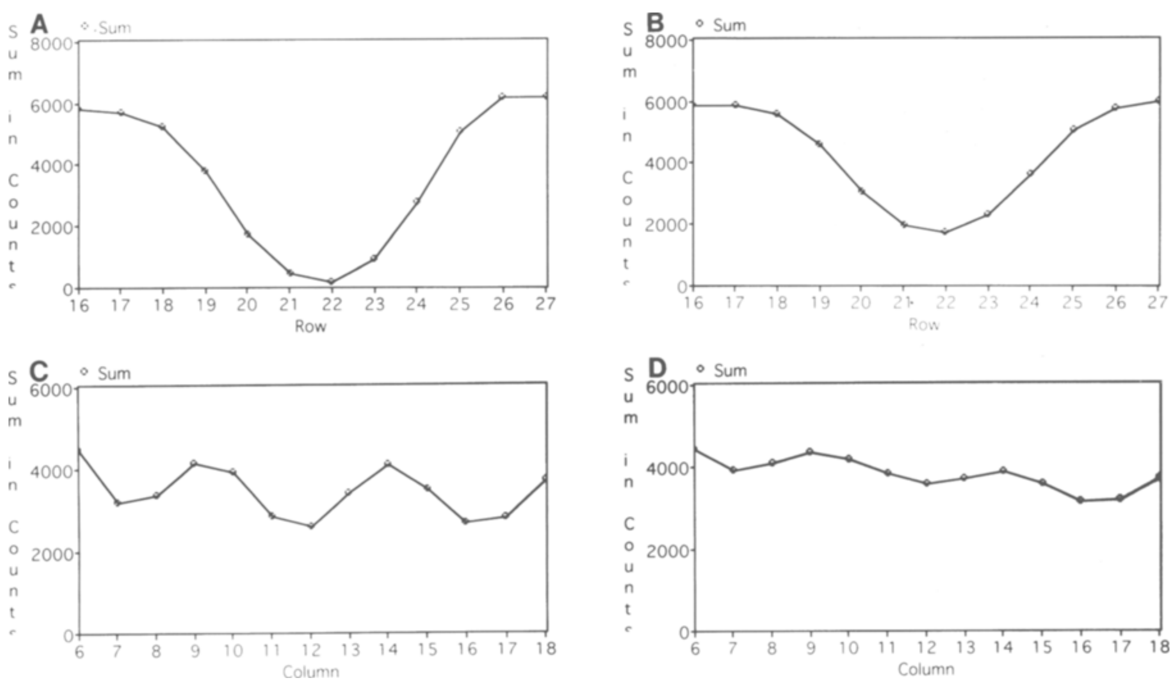

Fig 2. Line activity profiles through the large cold spheres (A, direct reconstruction; B, filtered backprojection) and large cold rods (C, direct reconstruction; D, filtered backprojection). Note the greater contrast with the direct reconstruction.

dimension. Finally, there is a requirement for substantial nonvolatile storage to retain $\mathbf{B}'$. As mentioned earlier, with 64 views and a spatial dimension of 48 pixels, more than 80 Mbytes of hard disk space were used.

There are several favorable points for direct reconstruction. First, most of its drawbacks are technology related, and technology is continually improving, quite visibly even over the course of this work. Second, images produced by direct reconstruction are superior to those made by backprojection. Third, reconstruction speed is better than for iterative methods. Finally, the use of direct reconstruction can be decoupled from intensive computations.

Although solution matrices are large, in that they can occupy more than 100 Mbytes of storage, they can easily be precomputed for common geometries and distributed to clinical sites on hard disk cartridges or compact discs. This work shows that microcomputers with minimal RAM can be used to implement direct reconstruction, once the precomputation has been done. Thus, in addition to making computation of the direct solution feasible, technology now provides means for widespread use of this method.

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