OPTICAL FLOW

On the motion of 3D curves and its relationship to optical flow *

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Abstract

I establish fundamental equations that relate the three dimensional motion of a curve to its observed image motion. I introduce the notion of spatio-temporal surface and study its differential properties up to the second order. In order to do this, I only make the assumption that the 3D motion of the curve preserves arc-length, a more general assumption than that of rigid motion. I show that, contrarily to what is commonly believed, the full optical flow of the curve can never be recovered from this surface. I nonetheless then show that the hypothesis of a rigid 3D motion allows in general to recover the structure and the motion of the curve, in fact without explicitely computing the tangential optical flow.

1 Introduction

This article is a condensed version of a longer version which will appear elsewhere [7]. In particular, I have omitted all the proofs of the theorems. It presents a mathematical formalism for dealing with the motion of curved objects, specifically curves. In our previous work on stereo [1] and motion [9], we have limited ourselves to primitives such as points and lines. I attempt here to lay the ground for extending this work to general curvilinear features. More specifically, I study the image motion of 3D curves moving in a "non-elastic" way (to be defined later), such as ropes. I show that under this weak assumption the full apparent optical flow (to be defined later) can be recovered. I also show that recovering the full real optical flow (i.e the projection of the 3D velocity field) is impossible. If rigid motion is hypothesized, then I show that, in general, the full 3D structure and motion of the curve can be recovered without explicitely computing the full flow real. I assume that pixels along curves have been extracted by some standard edge detection techniques [3,4].

This is related and inspired by the work of Koenderink [14], the work of Horn and Schunk [13] as well as that of Longuet-Higgins and Prazdny [15] who pioneered the analysis

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of motion in computer vision, that of Nagel [16] who showed first that at grey level corners the full optical flow could be recovered, as well as by the work of Hildreth [12] who proposed a scheme for recovering the full flow along image intensity edges from the normal flow by using a smoothness constraint. This is also related to the work of D'Hayer [5] who studied a differential equation satisfied by the optical flow but who did not relate it to the actual 3D motion and to that of Gong and Brady [11] who recently extended Nagel's result and showed that it also held along intensity gradient edges. All assume, though, that the standard motion constrain equation:

$$\frac{dI}{d\tau} = \nabla I \cdot \mathbf{v} + I_\tau = 0 \tag{1}$$

is true, where I is the image intensity, $\mathbf{v}$ the optical flow (a mysterious quantity which is in fact defined by this equation), and τ the time. It is known that this equation is, in general, far from being true [19]. In my approach, I do not make this assumption and, instead, keep explicit the relation between image and 3D velocities.

In fact there is a big confusion in the Computer Vision literature about the exact meaning of the optical flow. I define it precisely in this paper and show that two flows, the "apparent" and the "real" one must be distinguished. I show that only the apparent one can be recovered from the image for a large class of 3D motions.

My work is also related to that of Baker and Bolles [2] in the sense that I also work with spatio-temporal surfaces for which I provide a beginning of quantitative description through Differential Geometry in the case where they are generated by curves.

It is also motivated by the work of Girosi, Torre and Verri [18] who have investigated various ways of replacing equation (1) by several equations to remove the inherent ambiguity in the determination of the optical flow $\mathbf{v}$.

2 Definitions and notations

I use some elementary notions from Differential Geometry of curves and surfaces. I summarize these notions in the next sections and introduce my notations.

2.1 Camera model

I assume the standard pinhole model for the camera. The retina plane $\mathcal{R}$ is perpendicular to the optical axis Oz, O is the optical center. The focal distance is assumed to be 1. Those hypothesis are quite reasonable and it is always possible, up to a good approximation, to transform a real camera into such an ideal model [17,10].

2.2 Two-dimensional curves

A planar curve (c) (usually in the retina plane) is defined as a C^2 mapping $u \rightarrow \mathbf{m}(u)$ from an interval of R into R^2. We will assume that the parameter u is the arclength s of (c). We then have the well known two-dimensional Frenet formulas:

$$\frac{d\mathbf{m}}{ds} = \mathbf{t} \quad \frac{d\mathbf{t}}{ds} = \kappa\mathbf{n} \quad \frac{d\mathbf{n}}{ds} = -\kappa\mathbf{t} \tag{2}$$

where $\mathbf{t}$ and $\mathbf{n}$ are the tangent and normal unit vectors to (c) at the point under consideration, and κ is the curvature of (c), the inverse of the radius of curvature r.

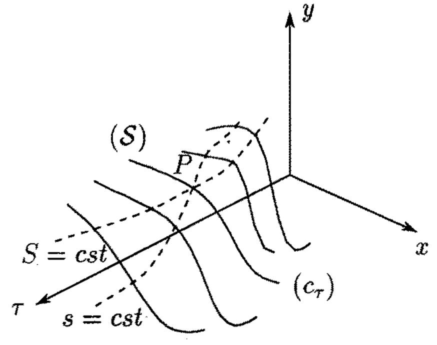

Figure 1: Definition of the spatio-temporal surface $(\mathcal{S})$

2.3 Surface patches

A surface patch $(\mathcal{S})$ is defined as a C^2 mapping $(u, v) \rightarrow \mathbf{P}(u, v)$ from an open set of R^2 into R^3. Such a patch is intrinsically characterized, up to a rigid motion, by two quadratic forms, called the two fundamental forms [6], which are defined at every point of the patch. The first quadratic form Φ_1 defines the length of a vector in the tangent plane T_P. More precisely, the two vectors $\mathbf{P}_u = \frac{\partial \mathbf{P}}{\partial u}$ and $\mathbf{P}_v = \frac{\partial \mathbf{P}}{\partial v}$ are parallel to this plane and define therein a system of coordinates. Each vector in the tangent plane can be defined as a linear combination $\lambda \mathbf{P}_u + \mu \mathbf{P}_v$. Its squared length is given by the value of the first fundamental form Φ_1. Moreover, the normal $\mathbf{N}_P$ to $(\mathcal{S})$ is parallel to the cross-product $\mathbf{P}_u \times \mathbf{P}_v$.

The second fundamental quadratic from Φ_2 is related to curvature. For a vector $\mathbf{x} = \lambda \mathbf{P}_u + \mu \mathbf{P}_v$ in the tangent plane, we can consider all curves drawn on $(\mathcal{S})$ tangent to $\mathbf{x}$ at P. These curves have all the same normal curvature, the ratio $\frac{\Phi_2(\mathbf{x})}{\Phi_1(\mathbf{x})}$.

It is important to study the invariants of $\mathbf{\Phi}_2$, ie. quantities which do not depend upon the parametrization $(u,\ v)$ of $(\mathcal{S})$. $\mathbf{\Phi}_2$ defines a linear mapping $T_P \rightarrow T_P$ by $\mathbf{\Phi}_2(\mathbf{x}) = \psi(\mathbf{x}) \cdot \mathbf{x}$. The invariants of $\mathbf{\Phi}_2$ are those of ψ. Those of interest to us are the principal directions, the principal curvatures from which the mean and gaussian curvatures can be computed.

3 Setting the stage: real and apparent optical flows

We now assume that we observe in a sequence of images a family (c_τ) of curves, where τ denotes the time, which we assume to be the perspective projection in the retina of a 3D curve (C) that moves in space. If we consider the three-dimensional space $(x,\ y,\ \tau)$, this family of curves sweeps in that space a surface $(\mathcal{S})$ defined as the set of points $((c_\tau),\ \tau)$ (see figure 1).

At a given time instant τ, let us consider the observed curve (c_τ). Its arclength s can be computed and (c_τ) can be parameterized by s and τ: it is the set of points $m_\tau(s)$. The corresponding points P on $(\mathcal{S})$ are represented by the vector $\mathbf{P} = (\mathbf{m}_\tau^T(s),\ \tau)^T$. The key observation is that the arclength s of (c_τ) is a function $s(S,\ \tau)$ of the arclength S

of the 3D curve (C) and the time τ, and that the two parameters (S, τ) can be used to parameterize $(\mathcal{S})$ in a neighborhood of P. Of course, the function $s(S, \tau)$ is unknown.

The assumption that s is a function of S and τ implies that S itself is not a function of time; in other words we do not consider here elastic motions but only motions for which S is preserved, i.e non-elastic motions such as the motion of a rope or the motion of a curve attached to a moving rigid object. We could call such motions *isometric* motions.

As shown in figure 1, we can consider on $(\mathcal{S})$ the curves defined by $s = cst$ or $S = cst$. These curves are in general different, and their projections, parallel to the τ-axis, in the (x, y)- or retina plane have an important physical interpretation, related to our upcoming definition of the optical flow.

Indeed, suppose we choose a point M_0 on (C) and fix its arclength S_0. When (C) moves, this point follows a trajectory (C_{M_0}) in 3-space and its image m_0 follows a trajectory $(c^r_{m_0})$ in the retina plane. This last curve is the projection in the retina plane, parallel to the τ-axis, of the curve defined by $S = S_0$ on the surface $(\mathcal{S})$. We call it the "real" trajectory of m_0.

We can also consider the same projection of another curve defined on $(\mathcal{S})$ by $s = s_0$. The corresponding curve $(c^a_{m_0})$ in the retina plane is the trajectory of the image point m_0 of arclength s_0 on (c_τ). We call this curve the "apparent" trajectory of m_0.

The mathematical reason why those two curves are different is that the first one is defined by $S = S_0$ while the second is defined by $s(S, \tau) = s_0$.

Let me now define precisely what I mean by optical flow. If we consider figure 2, point m on (c_τ) is the image of point M on (C). This point has a $3D$ velocity $\mathbf{V}_M$ whose projection in the retina is the *real optical flow* v_r (r for *real*); mathematically speaking:

- v_r is the partial derivative of m with respect to time when S is kept constant, or its total time derivative.

- The *apparent optical flow* v_a (a for *apparent*) of m is the partial derivative with respect to time when s is kept constant.

Those two quantities are in general distinct. To relate this to the previous discussion about the curves $S = S_0$ and $s = s_0$ of $(\mathcal{S})$, the vector v_a is tangent to the "apparent" trajectory of m, while v_r is tangent to the "real" one.

I now make the following fundamental remark. All the information about the motion of points of (c_τ) (and of the 3D points of (C) which project onto them) is entirely contained in the surface $(\mathcal{S})$. Since $(\mathcal{S})$ is intrinsically characterized, up to a rigid motion, by its first and second fundamental forms [6], they are all we need to characterize the optical flow of (c_τ) and the motion of (C).

4 Characterization of the spatio-temporal surface $(\mathcal{S})$

In this section, we compute the first and second fundamental forms of the spatio-temporal surface $(\mathcal{S})$. We will be using over and over again the following result.

Given a function f of the variables s and τ, it is also a function f' of S and τ. We will have to compute $\frac{\partial f'}{\partial S}$ and $\frac{\partial f'}{\partial \tau}$, also called the total time derivative of f with respect

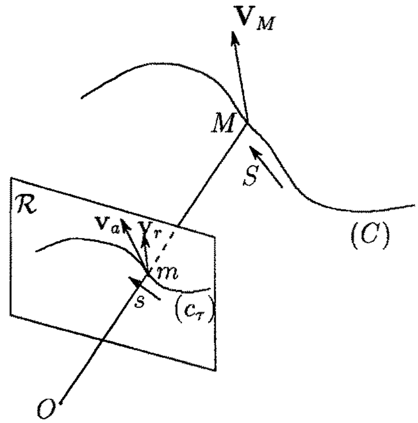

Figure 2: Definition of the two optical flows: the real and the apparent

to time, $\dot{f}$; introducing $u = \frac{\partial s}{\partial S}$ and $v = \frac{\partial s}{\partial \tau}$, we have the following equations:

$$\frac{\partial f'}{\partial S} = u\frac{\partial f}{\partial s} \quad \dot{f} = \frac{\partial f'}{\partial \tau} = v\frac{\partial f}{\partial s} + \frac{\partial f}{\partial \tau} \tag{3}$$

Following these notations, we denote by $\mathbf{P}(s, \tau) = (\mathbf{m}^T(s, \tau), \tau)^T$ the generic point of $(\mathcal{S})$ and by $\mathbf{P}'(S, \tau) = (\mathbf{m}'^T(S, \tau), \tau)^T$ the same point considered as a function of S and τ.

4.1 Computation of the first fundamental form

Using equations (3), we write immediatly:

$$\mathbf{P}'_\tau = v\mathbf{P}_s + \mathbf{P}_\tau = [v\mathbf{t}^T + \mathbf{v}_a^T,\, 1]^T \tag{4}$$

We now write the apparent optical flow $\mathbf{v}_a$ in the reference frame defined by $\mathbf{t}$ and $\mathbf{n}$:

$$\mathbf{v}_a = \alpha\mathbf{t} + \beta\mathbf{n} \tag{5}$$

We see from equations (4) and (5) that $\mathbf{P}'_\tau = [(v + \alpha)\mathbf{t}^T + \beta\mathbf{n}^T,\, 1]^T$; but by definition, $\mathbf{P}'_\tau = [\mathbf{v}_r^T,\, 1]^T$. Therefore $w = v + \alpha$ is the real tangential optical flow and β the normal real optical flow. Therefore, the real and apparent optical flows have the same component along $\mathbf{n}$, we call it the normal optical flow. The real optical flow is given by:

$$\mathbf{v}_r = w\mathbf{t} + \beta\mathbf{n} \tag{6}$$

Without entering the details and referring the reader to [7], we can write simple formulas for the time derivatives of a function f from $(\mathcal{S})$ into R, for example.

$$\frac{\partial f}{\partial \tau} = \alpha\frac{\partial f}{\partial s} + \partial_{\mathbf{n}_\beta} f \tag{7}$$

$$\dot{f} = w\frac{\partial f}{\partial s} + \partial_{\mathbf{n}_\beta} f \tag{8}$$

where $\partial_{\mathbf{n}_\beta} f$ means the partial derivative of f in the direction of $\mathbf{n}_\beta = [\beta\mathbf{n}, 1]^T$ of T_P, the tangent plane to $(\mathcal{S})$ at P.

These relations also hold for functions f from $(\mathcal{S})$ into R^p. We will be using heavily the case $p = 2$ in what follows.

From equations (4), and the one giving $\mathbf{P}'_S$, we can compute the coefficients of the first fundamental form. I skip the details of the computation and state the main result:

Given the normal $\mathbf{N}_P$ to the spatio-temporal surface $(\mathcal{S})$ whose coordinates in the coordinate system $(\mathbf{t}, \mathbf{n}, \tau)$ (τ is the unit vector defining the τ-axis) are denoted by N_1, N_2, N_3, we have:

$$\beta = -\tfrac{N_3}{N_2} \quad N_1 = 0$$

We have thus the following theorem:

Theorem 1 *The normal to the spatio-temporal surface $(\mathcal{S})$ yields an estimate of the* normal optical flow β.

4.2 Computation of the second fundamental form

Again, we skip the details and state only the results:

Theorem 2 *The tangential apparent optical flow α satisfies:*

$$\frac{\partial \alpha}{\partial s} = \kappa\beta \tag{9}$$

where κ is the curvature of (c_τ).

Equation (9) is instructive. Indeed, it shows that α, the tangential component of the apparent optical flow $\mathbf{v}_a$ is entirely determined up to the addition of a function of time by the normal component of the optical flow β and the space curvature κ of (c_τ):

$$\alpha = \int_{s_0}^{s} \kappa(t, \tau)\beta(t, \tau)dt \tag{10}$$

Changing the origin of arclengths from s_0 to s_1 on (c_τ) is equivalent to adding the function $\int_{s_0}^{s_1} \kappa(t, \tau)\beta(t, \tau)dt$ to α, function which is constant on (c_τ). This is the fundamental result of this section. We have proved the following theorem:

Theorem 3 *The tangential apparent optical flow can be recovered from the normal flow up to the addition of a function of time through equation (10).*

We now state an interesting relationship between κ and β which is proved in [7].

Theorem 4 *The curvature κ of (c_τ) and the normal optical flow β satisfy:*

$$\partial_{\mathbf{n}_\beta}\kappa = \frac{\partial^2 \beta}{\partial s^2} + \kappa^2\beta \tag{11}$$

4.3 What information can be extracted from the second fundamental form

The idea now is that after observing (S), we compute an estimate of $\mathbf{\Phi}_2$ from which we attempt to recover the unknowns, for example u or v. We show that it is impossible without making stronger assumptions about the motion of (C).

We have seen that all invariants of $\mathbf{\Phi}_2$ are functions of the principal directions and curvatures. We omit the derivation and only state the result:

Theorem 5 *The invariants of the second fundamental form of the surface (S) are not functions of u, v, w, the real tangential optical flow nor of α, the apparent tangential optical flow.*

4.4 Conclusions

There are three main consequences that we can draw from this analysis. Under the weak assumption of *isometric* motion:

1. The normal optical flow β can be recovered from the normal to the spatio-temporal surface,
2. The tangential apparent optical flow can be recovered from the normal optical flow through equation (10), up to the addition of a function of time,
3. The tangential real optical flow cannot be recovered from the spatio-temporal surface.

Therefore, the full real optical flow is not computable from the observation of the image of a moving curve under the isometric assumption. In order to compute it we must add more hypothesis, for example that the 3D motion is rigid. This makes me wonder what the published algorithms for computing the optical flow are actually computing since they are not making any assumptions about what kind of 3D motion is observed.

I show in the next section that if we assume a 3D rigid motion then the problem is, in general, solvable but that there is no need to compute the full real optical flow.

5 Assuming that (C) is moving rigidly

We are now assuming that (C) is moving rigidly; let $(\mathbf{\Omega}, \mathbf{V})$ be its kinematic screw at the optical center O of the camera. We first derive a fundamental relation between the tangents $\mathbf{t}$ and $\mathbf{T}$ to (c_τ) and (C) and the angular velocity $\mathbf{\Omega}$. In this section, the third coordinate of vectors is a space coordinate (along the z-axis) whereas previously it was a time coordinate (along the τ-axis).

5.1 Stories of tangents

Let us denote by $\mathbf{U_t}$ the vector $\mathbf{Om} \times \begin{bmatrix} \mathbf{t} \\ 0 \end{bmatrix}$. This vector is normal to the plane defined by the optical center of the camera, the point m on (c_τ), and $\mathbf{t}$ (see figure (3)). Since this

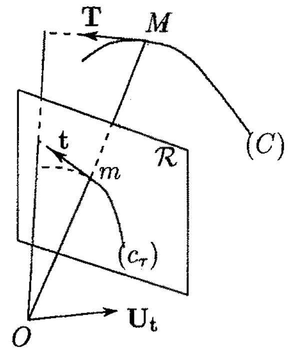

Figure 3: Relation between $\mathbf{t}$ and $\mathbf{T}$

plane contains also the tangent $\mathbf{T}$ to (C) at M, the 3D point whose image is m, we have:

$$\mathbf{U_t} \cdot \mathbf{T} = 0 \tag{12}$$

But, because (C) moves rigidly, $\mathbf{T}$ must also satisfy the following differential equation (this equation is satisfied by any constant length vector attached to (C)):

$$\dot{\mathbf{T}} = \mathbf{\Omega} \times \mathbf{T} \tag{13}$$

Skipping once again the details, we obtain:

$$\mathbf{W} = \mathbf{U_t} \times (\mathbf{U_t} \times \mathbf{\Omega} + \dot{\mathbf{U}}_\mathbf{t}) \tag{14}$$

$$\mathbf{T} = \epsilon \frac{\mathbf{W}}{\|\mathbf{W}\|} \tag{15}$$

Where $\epsilon = \pm 1$. We can assume that $\epsilon = 1$ by orienting correctly (c_τ) and (C).

Equations (14) and (15) are important because they relate in a very simple manner the tangent $\mathbf{T}$ to the unknown 3D curve (C) to the known vector $\mathbf{U_t}$, the angular velocity $\mathbf{\Omega}$ and and to $\dot{\mathbf{U}}_\mathbf{t}$. Notice that this last vector contains the unknown tangential real optical flow w.

Furthermore, $\mathbf{W}$ itself satisfies a differential equation. Skipping the details, we state the result:

Theorem 6 *The direction* $\mathbf{W}$ *of the tangent to the 3D curve* (C) *satisfies the following differential equation:*

$$\mathbf{W} \times (\dot{\mathbf{W}} \times \mathbf{W} + (\mathbf{W} \cdot \mathbf{W})\mathbf{\Omega}) = \mathbf{0} \tag{16}$$

Equation (16) is fundamental: it expresses the relationship between the unknown geometry and motion of the 3D curve (C) and the geometry and motion of the 2D curve (c_τ).

In order to exploit equation (16), we have to compute $\mathbf{W}$. This is done in [7] and we find that equation (16) involves w and $\dot{w}$, the real tangential optical flow and its total time derivative, as well as $\dot{\mathbf{\Omega}}$, the angular acceleration.

5.2 Obtaining more equations

We are now going to use the perspective equation:

$$Z\mathbf{Om} = \mathbf{OM} \tag{17}$$

to obtain a number of interesting relations by taking its total time derivative.

Taking the total derivative of equation (17) with respect to time, and projecting it on $\mathbf{t}$ and $\mathbf{n}$, we obtain two scalar equations:

$$Z(w + \mathbf{\Omega} \cdot \mathbf{b}) = V_t - (\mathbf{Om} \cdot \mathbf{t})V_z \tag{18}$$

$$Z(\beta - \mathbf{\Omega} \cdot \mathbf{a}) = V_n - (\mathbf{Om} \cdot \mathbf{n})V_z \tag{19}$$

where $\mathbf{a}$ and $\mathbf{b}$ depend only upon the image geometry.

These equations are the standard flow equations expressed in our formalism. They are fundamental in the sense that they express the relationship between the unknown 3D motion of a point and its observed 2D motion.

Notice that we can eliminate Z between (18) and (19) and obtain the value of the tangential real optical flow w as a function of $\mathbf{\Omega}$ and $\mathbf{V}$.

5.3 Closing the loop or finding the kinematic screw

The basic idea is to combine equation (16) which embeds the local structure of (C) at M (its tangent) and the fact that it moves rigidly, with the equation giving w which is a pure expression of the kinematics of the point M without any reference to the fact that it belongs to a curve.

We take the total time derivative $\dot{w}$ of w. In doing this, we introduce the accelerations $\dot{\mathbf{\Omega}}$ and $\dot{\mathbf{V}}$. If we now replace w and $\dot{w}$ by those values in equation (16), we obtain two polynomial equations in $\mathbf{\Omega}$, $\mathbf{V}$, $\dot{\mathbf{\Omega}}$, and $\dot{\mathbf{V}}$ with coefficients depending on the observed geometry and motion of the 2D curve (the two equations come from the fact that equation (16) is a cross-product). Two such equations are obtained at each point of (c_τ). Those polynomials are of degree 5 in $\mathbf{V}$, 1 in $\dot{\mathbf{V}}$, homogeneous of degree 5 in $(\mathbf{V}, \dot{\mathbf{V}})$, of degree 4 in $\mathbf{\Omega}$, 1 in $\dot{\mathbf{\Omega}}$, and of total degree 9 in all those unknowns.

This step is crucial. This is where we combine the structural information about the geometry of (C) embedded in equation (16) with purely kinematic information about the motion of its points embedded in equations (18) and (19). This eliminates the need for the estimation of the real tangential flow w and its time derivative $\dot{w}$. We thus have the following theorem:

Theorem 7 *At each point of (c_τ) we can write two polynomial equations in the coordinates of $\mathbf{\Omega}$, $\mathbf{V}$, $\dot{\mathbf{\Omega}}$ and $\dot{\mathbf{V}}$ with coefficients which are polynomials in quantities that can be measured from the spatio-temporal surface $(\mathcal{S})$:*

$$\begin{array}{ccccc} \beta & \frac{\partial\beta}{\partial s} & \frac{\partial^2\beta}{\partial s^2} & \partial_{\mathbf{n}_\beta}\beta & \partial_{\mathbf{n}_\beta}\frac{\partial\beta}{\partial s} \\ & \kappa & \frac{\partial\kappa}{\partial s} & \partial_{\mathbf{n}_\beta}\kappa\,. & \end{array}$$

Those polynomials are obtained by eliminating w and $\dot{w}$ between equations (16), and the equations giving w and $\dot{w}$ [7]. They are of total degree 9, homogeneous of degree 5 in $(\mathbf{V}, \dot{\mathbf{V}})$, of degree 5 in $\mathbf{V}$, 1 in $\dot{\mathbf{V}}$, 4 in $\mathbf{\Omega}$, 1 in $\dot{\mathbf{\Omega}}$.

Thus, N points on (c_τ) provide $2N$ equations in the 12 unknowns $\mathbf{\Omega}$, $\mathbf{V}$, $\dot{\mathbf{\Omega}}$, and $\dot{\mathbf{V}}$. Therefore, we should expect to be able to find, in some cases, a finite number of solutions. Degenerate cases where such solutions do not exist can be easily found: straight lines, for example [8], are notorious for being degenerate from that standpoint. The problem of studying the cases of degeneracy is left for further research. Ignoring for the moment those difficulties (but not underestimating them), we can state one major conjecture/result:

Conjecture 1 *The kinematic screw $\mathbf{\Omega}$, $\mathbf{V}$, and its time derivative $\dot{\mathbf{\Omega}}$, $\dot{\mathbf{V}}$, of a rigidly moving 3D curve can, in general, be estimated from the observation of the spatio-temporal surface generated by its retinal image, by solving a system of polynomial equations. Depth can then be recovered at each point through equation (19). The tangent to the curve can be recovered at each point through equation (14).*

Notice that we never actually compute the tangential real optical flow w. It is just used as an intermediate unknown and eliminated as quickly as possible, as irrelevant. Of course, if needed, it can be recovered afterwards, from equation (18).

6 Conclusion

I have studied the relationship between the 3D motion of a curve (C) moving isometrically and the motion of its image (c_τ). I have introduced the notion of real and apparent optical flows and shown how they can be interpreted in terms of vector fields defined on the spatio-temporal surface $(\mathcal{S})$ generated by (c_τ).

I have shown that the full apparent flow and the normal real flow can be recovered from the differential properties of that surface, but not the real tangential flow.

I have then shown that if the motion of (C) is rigid, then two polynomial equations in the components of its kinematic screw and its time derivative, with coefficients obtained from geometric properties of the surface $(\mathcal{S})$, can be written for each point of (c_τ). In doing this, the role of the spatio-temporal surface $(\mathcal{S})$ is essential since it is the natural place where all the operations of derivation of the geometric features of the curves (c_τ) take place. Conditions under which those equations yield a finite number of solutions have not been studied. Implementation of those ideas is under way. Some issues related to this implementation are discussed in [7].

I think that the major contribution of this paper is to state what can be computed from the sequence of images, under which assumptions about the observed 3D motions, and how. I also believe that similar ideas can be used to study more general types of motions than rigid ones.

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