Deciding Reachability for Planar Multi-polynomial Systems *

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Abstract. In this paper we investigate the decidability of the reachability problem for planar non-linear hybrid systems. A planar hybrid system has the property that its state space corresponds to the standard Euclidean plane, which is partitioned into a finite number of (polyhedral) regions. To each of these regions is assigned some vector field which governs the dynamical behaviour of the system within this region. We prove the decidability of point to point and region to region reachability problems for planar hybrid systems for the case when trajectories within the regions can be described by polynomials of arbitrary degree.

1 Introduction

During recent years intensive research has been devoted to the problem of automated analysis of various classes of hybrid systems (HS). The difficulty of this problem is due to the presence of a continuos projection of the system state space (every system state of a HS typically consists of control location, which is chosen from some finite domain, and the value vector for some continuous variables), this usually makes the system state space (wildly) infinite. However, there has already been much progress in the area, starting from the region graph based methods for Timed Automata [2, 3, 6], and leading to recent more general results and systematic investigations on what is decidable about hybrid systems (see, for instance, [1, 9, 8]).

Still, most of these results are concerned with the analysis automation for *linear* hybrid systems, where the continuous variables are allowed to change the value during the course of time at some fixed rate (or the value change can be non-deterministic, with any rate from a certain fixed interval).

It can be noted, however, that for the full class of linear HS even the simpliest verification problems are undecidable (see, for instance [6, 5, 1]), so any decidability result in this area is bound to indentify a certain subclass of systems to which it does apply.

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On the other hand, the behaviour of practical hybrid systems is most often governed by some non-linear laws. Therefore it is natural to ask, whether there are natural classes of non-linear HS, which do admit automated verification. An important study of this problem is already [7], where the possibility of verification on nonlinear hybrid systems via the reduction to linear clock and rate hybrid automata is discussed, and corresponding at least sound verification methods are presented.

In this paper we investigate the decidability of the reachability problem for planar non-linear hybrid systems. A planar hybrid system has the property that its state space corresponds to the standard Euclid plane, which is partitioned into a finite number of (polyhedral) regions. To each of these regions is assigned some vector field which governs the dynamic behaviour of the system within this region. We prove the decidability of point to point, edge to edge and region to region reachability problems for planar hybrid systems for the case when trajectories within the regions can be described by polynomials of arbitrary degree.

Our results are a generalization of those of [10], where the subcase of our problem with the vector fields within the regions being constant (the so-called multi-linear model) was considered. We are able to reuse also a part of the proof from [10] to show that every infinite trajectory of the system either intersects only with a finite number of the region boundaries, or starting from some point will repeatingly intersect certain fixed sequence of region boundaries (in fact this result holds even for much wider classes of systems, its demonstration relies essentially on the fact that trajectories within the regions do not intersect).

The main problem to be dealt with in our "multi-polynomial" case essentially consists in showing the decidability of the "abandonment" of an edge (region boundary, or some its part): given some repeating sequence of edges intersected by a trajectory, decide, whether this repetition will last forever, or after some finite number of edge-to-edge steps some other edge intersection sequence will appear.

This problem is solved, in essence, by explicating the polynomial dependencies of future region border intersection points from the previous ones on the trajectories, and characterizing the "limit points" of the intersection point sequences in the terms of fixed points and roots of appropriate polynomials.

Our results can be viewed as showing the non-essentiality of the *linearity* requirement for the decidability results in the setting of [10]. However, as it has been shown in [4], the 2-dimensionality requirement is essential for the decidability of the reachability even for the case of multi-linear systems (in [4] a construction modeling Turing machines by 3-dimensional multi-linear systems is presented, thus proving the undecidability of any nontrivial verification problem for that class of systems).

The organization of the rest of the paper is, as follows. In the next section we give main definitions and notation used throughout the paper. Section 3 reminds already known results about general planar hybrid systems. Section 4 contains our results about decidability results for multi-polynomial systems.

Finally, Section 5 contains some conclusions and indicates possible directions for future work.

2 Main definitions

Definitions and notations in this paper are more or less standard and similar to [10].

Symbols $\mathbf{R}$ and $\mathbf{R}_+$ will stand for the sets of real and real positive numbers, $\mathbf{Q}$ and $\mathbf{Q}_+$ for the sets of rational and rational positive numbers. With $\mathbf{N}$ we denote the set of natural numbers, with $\mathbf{N}_+$ – the set of positive natural numbers.

By $\mathbf{A}$ we denote the set of algebraic numbers - i.e. the set of numbers which are roots of polynomials $p(x)$ with rational coefficients. We represent each such number a as a pair $\langle p(x), i\rangle$, where $p(x)$ is some polynomial with coefficients from $\mathbf{Q}$, such that $p(a) = 0$, and $i \in \mathbf{N}_+$ is the index of a in the increasingly ordered sequence of p roots, i.e., $i = card(\{x \in \mathbf{R} \mid p(x) = 0 \ \&\ x < a\}) + 1$. (Of course, such representation is not unique.) We say that an algebraic number a is computable (from some subset A of natural numbers), if there exists an algorithm that on input A computes coefficients of some $p(x)$ and number i, such that $a = \langle p(x), i\rangle$.

We consider the Euclid plane $\mathbf{R}^2$ with standard metric d, i.e. such that for any two points $a = (x_1, y_1), b = (x_2, y_2) \in \mathbf{R}^2$ the distance $d(a, b)$ between a and b is defined by the equality $d(a, b) = \sqrt{((x_1 - x_2)^2 + (y_1 - y_2)^2)}$.

A closed half-plane is defined as a set H in the form $H = \{(x, y) \in \mathbf{R}^2 \mid Ax + By + C \geq 0\}$ for some constants $A, B, C \in \mathbf{R}$.

For an arbitrary set $S \subseteq \mathbf{R}^2$ we define the set of interior points of S as $int(S) = \{a \in S \mid \exists\varepsilon \in \mathbf{R}_+ : \ U(a, \varepsilon) \subseteq S\}$, where $U(a, \varepsilon) = \{b \in \mathbf{R} \mid d(a, b) < \varepsilon\}$. We define the closure of S as $cl(S) = \{a \in \mathbf{R}^2 \mid \forall\delta \in \mathbf{R}_+\ \exists b \in S : \ d(a, b) < \delta\}$, and the boundary of S as $bd(S) = cl(S) - int(S)$.

A (closed) polyhedral set P is an intersection of finitely many closed half-planes, such that $int(P) \neq \emptyset$.

Definition 1. A finite polyhedral partition of $\mathbf{R}^2$ is a family of polyhedral sets $\mathcal{P} = \{P_1, \ldots, P_n\}$ with disjoint sets of interior points, such that $\bigcup_{i=1}^{n} P_i = \mathbf{R}^2$.$\diamond$

By $bd(\mathcal{P})$ we denote the set of all points which belong to $bd(P_i)$ for some $P_i \in \mathcal{P}$.

Let P be some polyhedral set. We say that a vector field is defined in the set P, if a system of following differential equations is assigned to the set P

$$\begin{cases} \dot{x} = f_1(x, y), \\ \dot{y} = f_2(x, y), \end{cases} \tag{1}$$

where $f_1(x, y)$ and $f_2(x, y)$ are continuous functions jointly in x and y.

Such equation allows to split the region P into set of disjoint trajectories, i.e. into set of curves with equations

$$\begin{cases} x(t) = g_x(t), \\ y(t) = g_y(t), \end{cases} \tag{2}$$

which satisfy the given system (1) and are defined for values of $t \in \mathbf{R}$, such that $(x(t), y(t)) \in P$. We assume that these curves are oriented in the direction in which the value of t is increasing.

In this paper we restrict our attention to the case when all trajectories in P are polynomial and without singularities, i.e. such that for some polynomial $p(x, y)$ all trajectories satisfy equation $p(x, y) = C$ for some $C \in \mathbf{R}$, and such that all trajectories that enter the region P also leave it and vice versa. It is known that this case corresponds to the situation given in the next definition.

Definition 2. We say that in a polyhedral region P a vector field is defined by polynomial $p(x, y)$, if it is defined by system

$$\begin{cases} \dot{x} = p'_y(x, y), \\ \dot{y} = -p'_x(x, y), \end{cases} \tag{3}$$

and $p(x, y)$ is a polynomial, such that for all points $a = (x, y) \in int(P)$ either $p'_y(x, y) \neq 0$, or $p'_x(x, y) \neq 0$.$\diamond$

All trajectories in P in this case will satisfy equation $p(x, y) = C$ for some $C \in \mathbf{R}$. We shall assume that in parametrical form these trajectories can be described by equation

$$\begin{cases} x(t) = g_{p,x}(t, C), \\ y(t) = g_{p,y}(t, C). \end{cases} \tag{4}$$

It is known that functions $g_{p,x}$ and $g_{p,y}$ can be effectively approximated from $p(x, y)$ (i.e. it is possible to compute $g_{p,x}(t, C)$ and $g_{p,y}(t, C)$ up to an arbitrary degree of approximation). However, in general case they are neither polynomial, nor can be effectively found.

If P is a polyhedral set in which a polynomial vector field satisfying the system (3) is defined, then any point $A \in bd(P)$ belongs to one of three different types with respect to the region P (see Fig. 1).

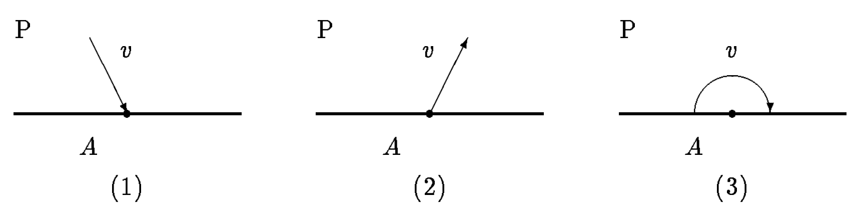

Fig. 1. *Three different types of boundary points for a region with polynomial trajectories. In case 1 point A is an exit from P, in case 2 point A is an entry to P and in case 3 point A is neutral with respect to P.*

Each point on border of P can belong either to one of the types 1–3, or simultaneously to types 1 and 2.

These three types can be described more formally in the following way.

If the vector field in P is given by the system (3), then point $A = (x_0, y_0)$ belongs to some trajectory T which satisfies equation (4) for some functions $g_{p,x}$ and $g_{p,y}$. We assume that $g_{p,x}(t_0, C) = x_0$ and $g_{p,y}(t_0, C) = y_0$ for some $t_0 \in \mathbf{R}$.

We say that point $A \in bd(P)$ is an entry to P, if there exists $t' > t_0$, such that $(x(t), y(t)) \in P$ for all $t \in [t_0, t']$, where $x(t) = g_{p,x}(t, C)$ and $y(t) = g_{p,y}(t, C)$.

We say that point $A \in bd(P)$ is an exit from P, if there exists $t' < t_0$, such that $(x(t), y(t)) \in P$ for all $t' \in [t', t_0]$, where $x(t) = g_{p,x}(t, C)$ and $y(t) = g_{p,y}(t, C)$.

Otherwise we say that point A is neutral with respect to P. In this case the trajectory T consists of a single point A.

The set of all entry points to P we shall denote by $In(P)$, the set of all exit points from P – by $Out(P)$ and the set of all neutral points – by $Neut(P)$.

Definition 3. A multi-polynomial hybrid system on $\mathbf{R}^2$ is $\mathcal{H} = (\mathcal{P}, \varphi)$, where $\mathcal{P}$ is a polyhedral partition of $\mathbf{R}^2$ and φ is a function which assigns to each region of $\mathcal{P}$ a vector field defined by some polynomial $p(x, y)$.$\diamond$

Thus, a multi-polynomial hybrid system gives a partition of the Euclid plane into polyhedral sets in each of which some polynomial vector field without singularities is defined.

Since we are interested in the decidability of reachability problems in such systems, we have to ensure that multi-polynomial hybrid system is represented in some effective way. Therefore, throughout this paper we shall further assume (if not explicitly stated otherwise) that for a given hybrid system $\mathcal{H} = (\mathcal{P}, \varphi)$ all regions in $\mathcal{P}$ are defined as intersections of half-planes $Ax + By + C \geq 0$ with algebraic coefficients A, B and C (i.e. such that $A, B, C \in \mathbf{A}$). Similarly, we shall further assume that function φ to each $P \in \mathcal{P}$ will assign a polynomial with algebraic coefficients.

Definition 4. A step of a multi-polynomial hybrid system $\mathcal{H} = (\mathcal{P}, \varphi)$ is a pair (a, a') of boundary points $a = (a_x, a_y), a' = (a'_x, a'_y) \in bd(P)$, for some $P \in \mathcal{P}$, such that for the polynomial $p(x, y) = \varphi(P)$ the equalities $a_x = g_{p,x}(t_0, C)$, $a_y = g_{p,y}(t_0, C)$, $a'_x = g_{p,y}(t', C)$ and $a'_y = g_{p,y}(t', C)$ hold for some $t_0, t', C \in \mathbf{R}$, with $t' > t_0$, and for all $t \in [t_0, t']$ we have inclusion $(g_{p,x}(t, C), g_{p,y}(t, C)) \in P$.$\diamond$

In such case we shall also say that the step (a, a') defines trajectory $p(x, y) = C$.

Definition 5. A path of a hybrid system $\mathcal{H} = (\mathcal{P}, \varphi)$ is a sequence (finite or infinite) $s = a_1, a_2, \ldots$ of points $a_1, a_2, \ldots \in bd(\mathcal{P})$, such that either $s = a_1$, or for every $i > 0$, and a_{i+1} from s:

1. the pair (a_i, a_{i+1}) is a step, and
2. there does not exist $a'_{i+1} \in bd(\mathcal{P})$, such that $a_{i+1} \neq a'_{i+1}$, and (a_i, a'_{i+1}) is a step.$\diamond$

The second condition in the definition of the path allows to eliminate non determinism in hybrid system – i.e. if in some situation we can proceed by two different trajectories we are not allowed to choose either of them.

A finite path $s = a_1, \dots, a_k$ is extendable, if there exists a path $s' = a_1, \dots, a_k, a_{k+1}$.

For a polyhedral set of a given hybrid system $\mathcal{H} = (\mathcal{P}, \varphi)$ we partition the boundary of P into edges in such a way that each edge (if we do not consider its end points)

- intersects with boundaries of exactly two polyhedral sets $P, Q \in \mathcal{P}$, and
- contains only entry, or only exit, or only neutral points with respect to both regions P and Q.

A line segment S in $\mathbf{R}^2$ is a set in the one of the forms $S = \{(x, y) \in \mathbf{R}^2 \mid Ax + By + C = 0,\ u_1 \le x \le u_2\}$ or $S = \{(x, y) \in \mathbf{R}^2 \mid Ax + By + C = 0,\ w_1 \le y \le w_2\}$ for some constants $A, B, C \in \mathbf{R}$, $u_1, u_2, w_1, w_2 \in \mathbf{R} \cup \{\infty\}$, with $u_1 < u_2$ and $w_1 < w_2$. Point $A(x_0, y_0) \in S$ is an end point of S, if $x_0 \in \{u_1, u_2\} \cap \mathbf{R}$ or $y_0 \in \{w_1, w_2\} \cap \mathbf{R}$. The set of all (i.e. of one or two) end points of S we denote by $B(S)$. By $I(S)$ we denote the set $S - B(S)$.

Definition 6. The line segment e is an edge of $P \in \mathcal{P}$, if

1. $e \subseteq P \cap Q$ for some $Q \in \mathcal{P}$,
2. each point of $I(e)$ belongs to only one type with respect to region P and to only one type with respect to region Q, and
3. if $A \in B(e)$, then either $A \in R$ for some $R \in \mathcal{P}$, $R \neq P$, $R \neq Q$, or A is of different type than points in $I(e)$ with respect to one of the regions P or Q.$\diamond$

Set of all edges of $P \in \mathcal{P}$ we shall denote by $E(P)$. By $E(\mathcal{P})$ we shall denote $\bigcup_{P \in \mathcal{P}} E(P)$.

Definition 7. Let $s = a_1, a_2, \dots$ be a path in $\mathcal{H}$. We say that $X(s) = S_1, S_2, \dots$ is a signature of s, if for each $i \ge 1$ we have $S_i = \{e \in E(\mathcal{P}) \mid a_i \in e\}$.$\diamond$

Finally, we define the notion of the reachability from point a to point b. We begin with a more technical notion of 1-reachability which covers the case when b is reachable from a by a trajectory within one region.

Definition 8. Let $\mathcal{H} = (\mathcal{P}, \varphi)$ be a (non singular) hybrid system. Let $a = (a_x, a_y), b = (b_x, b_y) \in \mathbf{R}^2$. We say that b is 1-reachable from a, if $a, b \in P$ for some $P \in \mathcal{P}$ and, if there exist $a_1, a_2 \in bd(P)$, such that (a_1, a_2) is a step and both a and b lie on the trajectory $p(x, y) = C$ defined by the step (a_1, a_2), and, besides that, there exist $t_1, t_2 \in \mathbf{R}$, such that $g_{p,x}(t_1, C) = a_x$, $g_{p,y}(t_1, C) = a_y$, $g_{p,x}(t_2, C) = b_x$, $g_{p,y}(t_2, C) = b_y$ and $t_1 \le t_2$.$\diamond$

Definition 9. Let $\mathcal{H} = (\mathcal{P}, \varphi)$ be a (non-singular) hybrid system. Let $a, b \in \mathbf{R}^2$. We say that b is reachable from a, if there exists a finite path $s = a_1, \dots, a_n$, such that a_1 is 1-reachable from a and b is 1-reachable from a_n.$\diamond$

We say that an edge e_2 is reachable from an edge e_1, if there exist $a_1 \in e_1$ and $a_2 \in e_2$, such that point a_2 is reachable from point a_1. Similarly, we say that a region P_2 is reachable from a region P_1, if there exist $a_1 \in P_1$ and $a_2 \in P_2$, such that point a_2 is reachable from point a_1.

3 Some properties of planar deterministic systems

In this section we are going to remind some general properties of planar hybrid systems, which are proved (or can be proved similarly as) in [10] and which in particular hold also for multi-polynomial systems.

Let $\mathcal{H} = (\mathcal{P}, \varphi)$ be some hybrid system. We shall assume that with an arbitrary edge $e \in E(\mathcal{P})$ there is associated an ordering of the points of e, namely, that to some point $a_0 = (a_{0,x}, a_{0,y}) \in e$, such that $a_{0,x}, a_{0,y} \in \mathbf{A}$, there is assigned coordinate 0, and to any other point $a \in e$ there is assigned some coordinate $c(a, e) \in \mathbf{R}$, such that for any two points $a_1, a_2 \in e$ the equality $(c(a_1, e) - c(a_2, e))^2 = d(a_1, a_2)^2$ holds. By $a \preceq b$, where $a, b \in E$, we shall denote the fact that $c(a, e) \leq c(b, e)$. (It is not important, exactly which systems of coordinates for edges we chose, we only assume that such systems of coordinates are fixed for a given hybrid system $\mathcal{H}$ and that the coordinates are effectively computable from $\mathcal{H}$.)

The following result can be proved similarly as in [10].

Theorem 10 (Maler, Pnueli). *Let $s = a_1, a_2, \ldots$ be a path that intersects $e \in E(\mathcal{P})$ in three points $b_1 = a_i, b_2 = a_j, b_3 = a_k$, such that $i < j < k$. Then, $b_1 \preceq b_2$ implies $b_2 \preceq b_3$ and $b_1 \succeq b_2$ implies $b_2 \succeq b_3$.*$\diamond$

Let $e_1, \ldots, e_n \in E(\mathcal{P})$. We say that the sequence $e_1, \ldots, e_n$ forms a cycle, if $e_1 = e_n$ and the edges $e_2, \ldots, e_{n-1}$ are mutually distinct.

An edge e is said to be abandoned by a path $s = a_1, a_2, \ldots$ with signature $X(s) = S_1, S_2, \ldots$ after position i, if $e \in S_i$ and either s is finite and $e \notin S_j$ for $j > i$, or s is infinite and for some j, k, with $i < j < k$, there is a cycle $e_j \in S_j, \ldots, e_k \in S_k$.

Theorem 11 (Maler, Pnueli). *If and edge e is abandoned by a path $s = a_1, a_2, \ldots$ with signature $X(s) = S_1, S_2, \ldots$ after position i, then $e \notin S_j$ for an arbitrary $j > i$.*$\diamond$

As a corollary it is possible to obtain the following result.

Corollary 12 (Maler, Pnueli). *Every infinite path $s = a_1, a_2, \ldots$ has a signature in the form $X(s) = S_1, S_2, \ldots, S_i, (S_{i+1}, \ldots, S_{i+j})^\star$ for some $i, j \in \mathbf{N}_+$. Besides, the number j does not exceed the number of regions in hybrid system $\mathcal{H} = (\mathcal{P}, \varphi)$.*$\diamond$

Thus, for a path $s = a_1, a_2, \ldots$ with $X(s) = S_1, S_2, \ldots$ and for an edge $e \in S_i$ for some $i \in \mathbf{N}_+$ we have $e \in S_j$ for some $j > i$ if and only if edge e will not be abandoned.

4 Reachability results

In this section we shall show that for a given multi-polynomial hybrid system $\mathcal{H}$ the reachability problems between points, edges or regions are decidable. Our results to a large extent are based on the following three theorems about algebraic numbers. For the sake of brevity we are giving them here without proofs. We also do not expect the novelty of these results.

Theorem 13. *Let $p(x) = p_n x^n + p_{n-1}x^{n-1} + \cdots + p_1 x + p_0$ be a polynomial with algebraic coefficients $p_n, \ldots, p_0 \in \mathbf{A}$. Then all real roots of $p(x)$ are algebraic and computable from $p_n, \ldots, p_0$.*$\diamond$

Theorem 14. *It is decidable, whether two algebraic numbers $a, b \in \mathbf{A}$ are equal or not.*$\diamond$

For an arbitrary function $f : \mathbf{R} \to \mathbf{R}$ we iteratively define functions $f^{(1)}(x) = f(x), f^{(2)}(x) = f(f^{(1)}(x)), \ldots, f^{(n)}(x) = f(f^{(n-1)}(x)), \ldots$.

Theorem 15. *Let $p(x) : \mathbf{R} \to \mathbf{R}$ be a polynomial that is monotonous in some interval $[a, b] \subseteq \mathbf{R}$ (i.e. either, for all $x, y \in [a, b]$, with $x > y$, we have inequality $p(x) \leq p(y)$, or, for all $x, y \in [a, b]$ with $x > y$, we have inequality $p(x) \geq p(y)$). Let $x_0 \in [a, b]$ be such that $l(x_0) = \lim_{n\to\infty} p^{(n)}(x_0) \in [a, b]$. Then $l(x_0)$ is the first root of the polynomial $p(x) - x$, larger than x_0, if $p(x_0) > x_0$. Similarly, $l(x_0)$ is the first root of the polynomial $p(x) - x$, smaller than x_0, if $p(x_0) < x_0$.*$\diamond$

We shall also use the following relatively simple propositions. For the sake of brevity their proofs are omitted.

Proposition 16. *Let $\mathcal{H} = (\mathcal{P}, \varphi)$ be a hybrid system (with algebraic coefficients) and let $a \in \mathbf{A}^2$, $P \in \mathcal{P}$. Then it is decidable, whether or not $a \in P$.*$\diamond$

Proposition 17. *Let $\mathcal{H} = (\mathcal{P}, \varphi)$ be a multi-polynomial hybrid system (with algebraic coefficients). Then the set of edges $E(\mathcal{P})$ is finite and for each $e \in E(\mathcal{P})$ the elements of $B(e)$ are algebraic and computable from $\mathcal{H}$.*$\diamond$

Proposition 18. *Let $\mathcal{H} = (\mathcal{P}, \varphi)$ be a multi-polynomial hybrid system and let $a = (a_x, a_y) \in e \cap \mathbf{A}^2$ for some $e \in E(\mathcal{P})$. Then the maximal (i.e. non-extendable) path $s = (a_1 = a), a_2, \ldots$ containing a is computable (i.e. all numbers a_i are algebraic and computable from a and i, and if s is finite of length n, then there is an algorithm which for $i > n$ produces the answer that a_i is undefined).*$\diamond$

Proposition 19. *Let $\mathcal{H} = (\mathcal{P}, \varphi)$ be a multi-polynomial hybrid system. Let $s = a_1, a_2, \ldots$ be a path, such that $a_1 \in \mathbf{A}^2$. Then signature $X(s)$ is computable, i.e. for each $i \in \mathbf{N}_+$ we can compute points in $B(e)$ for all edges $e \in S_i$.*$\diamond$

Proposition 20. *Let $\mathcal{H} = (\mathcal{P}, \varphi)$ be a multi-polynomial hybrid system. Let $a = (a_x, a_y) \in int(P) \cap \mathbf{A}^2$ for some $P \in \mathcal{P}$ and let $p(x, y) = \varphi(P)$. Then there exists a step $(a_1 = (a_{1,x}, a_{1,y}), a_2 = (a_{2,x}, a_{2,y}))$, such that $p(a_x, a_y) = p(a_{1,x}, a_{1,y}) = p(a_{2,x}, a_{2,y})$, and the numbers a_1, a_2 are computable from a (i.e. there exists a trajectory through a, end points of which can be computed).*$\diamond$

From Proposition 17 it easy follows that for a given hybrid system $\mathcal{H} = (\mathcal{P}, \varphi)$ and for any edge $e \in E(\mathcal{P})$ it is decidable whether signature $X(s)$ for some path $s = a_1, a_2, \ldots$ will contain an edge e in the i-th position. The following principal lemma shows that it is also decidable whether the edge e will be eventually abandoned.

Lemma 21. *Let $\mathcal{H} = (\mathcal{P}, \varphi)$ be a multi-polynomial hybrid system. Let $s = a_1, a_2, \ldots$ be a path in $\mathcal{H}$ with signature $X(s) = S_1, S_2, \ldots$ and with $a_1 \in \mathbf{A}^2$. Let $e \in S_i$ for some $i \in \mathbf{N}_+$. Then it is decidable, whether or not the edge e will be abandoned.*$\diamond$

Proof. From the definition of abandonment and Theorem 12 it follows that e will not be abandoned if and only if $X(s) = S_1, S_2, \ldots, S_i, (S_{i+1}, \ldots, S_{i+j})^\star$ for some $i, j \in \mathbf{N}_+$ and $e \in S_{i+k}$ for some $k \in \mathbf{N}_+$, with $1 \leq k \leq j$.

Thus, the abandonment of e is decidable, if for arbitrary edges $e_1 \in S_{m+1}, \ldots, e_n = e_1 \in S_{m+n}$ it is decidable whether in the cycle $e_1, \ldots, e_n$ some edge will be eventually abandoned. It is not hard to see that it is sufficient to show that it is decidable whether edge e_k, with $1 \leq k \leq n$, will be abandoned, if either e_k is the first edge from the cycle $e_1, \ldots, e_n$, which actually will be abandoned, or there are no edges in $e_1, \ldots, e_n$, which will be abandoned.

Without loss of generality we can assume that e_1 is an edge with such a property. We have to show that it is decidable, whether e_1 will be abandoned or not. Let a_{m+1} be the point in the path s that corresponds to the set S_{m+1} from the signature $X(s)$. We denote $b_1 = a_{m+1}$, $b_2 = a_{m+n}$, $b_3 = a_{m+2n-1}, \ldots$. Clearly the edge e_1 will not be abandoned if and only if for all $i \in \mathbf{N}_+$ we have $b_i \in e_1$.

Let P be the region containing edges e_1 and e_2. Let $p(x, y) = \varphi(P)$. Let $p(x, y) = C_1$ be the trajectory going through points a_{m+1} and a_{m+2}. Since e_1 is a line segment, we have that C_1 is algebraic and can be computed from a_{m+1}. Also a_{m+2} is algebraic and can be computed from C_1, and thus, by Theorem 13, also from $b_1 = a_{m+1}$. Similarly, we can show that a_{m+3} is algebraic and computable from a_{m+2}, and, thus, by Theorem 13, also from b_1, etc., up to $b_2 = a_{m+n}$.

Thus, b_2 is algebraic and computable from b_1. Let $b_2 = \langle p_1(x, b_1), i_1 \rangle$. Similarly, we can show that $b_3 = \langle p_2(x, y), i_1 \rangle$, etc., while $b_i \in e_1$ holds. Since the polynomials p_i depend only on regions defined by the pairs of edges $(e_1, e_2), (e_2, e_3), \ldots, (e_{n_1}, e_n)$, then $p_1 = p_2 = \cdots$. We denote $p_1(x, y)$ by $g(x, y)$.

However, we can not guarantee that $g(x, y)$ as polynomial on x has the same number of roots for all $y \in \mathbf{A}$. Thus, we shall not necessarily have $i_1 = i_2 = \cdots$. Still, it is not hard to see that the number of roots for $g(x, y)$ can change only on values of y, such that $g'_x(x, y) = 0$ and $g(x, y) = 0$ for some $x \in \mathbf{A}$. The number of such values of y is finite, they are algebraic and computable from $g(x, y)$. Thus, we can split the edge e_1 into finite number of (open or closed) subintervals $I_1, \ldots, I_r$, such that for an arbitrary q, with $1 \leq q \leq r$, for all $y \in I_q$ the polynomial $g(x, y)$ has the same number of roots. Since in each region $P \in \mathcal{P}$ trajectories $p(x, y) = C$ change continuously with respect to C, for all $b_u \in I_q$, the values of i_u must be equal.

By Theorem 10 the sequence $b_1, b_2, \ldots$ is monotonous, thus, by one or more applications of Theorem 15, $b = \lim_{i \to \infty} b_i$ is computable from b_1. Clearly, we shall have $b_i \in e_1$ if and only if $b \in e_1$. Therefore, it is decidable whether the edge e_1 will be abandoned, and, thus, also for an arbitrary $e \in S_i$ it is decidable whether or not the edge e will be abandoned.$\diamond$

Theorem 22 Main result. *Let $\mathcal{H} = (\mathcal{P}, \varphi)$ be a multi-polynomial hybrid system. Let $a, b \in \mathbf{A}^2$. Then it is algorithmically decidable whether point b is reachable from point a.*$\diamond$

Proof. **Case 1.** Let $a = (a_x, a_y) \notin e$ for all $e \in E(\mathcal{P})$. Then $a \in int(P)$ for some $P \in \mathcal{P}$ (and such P is uniquely defined), and due to Proposition 16 region P can be found algorithmically. From Proposition 20 it follows that we can compute algebraic numbers a_1, a_2, such that (a_1, a_2) is a step and the point a lies on a trajectory from a_1 to a_2. Similarly, we can compute region Q and points b_1, b_2, such that $b = (b_x, b_y)$ lies on a trajectory defined by step (b_1, b_2).

If $a_1 = b_1$ (and thus also $a_2 = b_2$), then points a and b lie on the same trajectory given by equations

$$\begin{cases} x(t) = g_{p,x}(t, C), \\ y(t) = g_{p,y}(t, C), \end{cases}$$

and $p(x, y) = C$, for $p(x, y) = \varphi(P)$. Since $C = p(a_{1,x}, a_{1,y})$, the number C is algebraic and computable from a. Up to an arbitrary degree of approximation we can compute t_1 and t_2, such that $a_x = g_{p,x}(t_1, C)$, $a_y = g_{p,y}(t_1, C)$ and $b_x = g_{p,x}(t_2, C)$, $b_y = g_{p,y}(t_2, C)$. By definition b is reachable from a if and only if either $t_1 < t_2$, or $a = b$. Decidability whether $a = b$ follows from Theorem 14. If $a \neq b$, we can eventually decide whether $t_1 < t_2$ or $t_1 > t_2$. Therefore, in this subcase, it is decidable, whether b is reachable from a.

If $a_1 \neq b_1$, then by definition b is reachable from a if and only if b_1 is reachable from a_1. This subcase is covered by Case 2.

Case 2. Let $a = (a_x, a_y) \in e$ for some $e \in E(\mathcal{P})$. Let (b_1, b_2) be the step containing trajectory through b, and let $S = \{e \in E(\mathcal{P}) \mid b_1 \in e\}$.

If $a \neq b$ (case $a = b$ is trivial), then by definition b is reachable from a if and only if b_1 is reachable from a. By Proposition 18 all elements in maximal path $s = (a_1 = a), a_2, \ldots$ are computable. We continue computation of a_i for $i \in \mathbf{N}_+$ until one of the following holds.

1. There exists $i \in \mathbf{N}_+$ with $a_i = b_1$. Then, by definition, b is reachable from a.
2. For some i element a_i becomes undefined (i.e. s turns out to be finite) and for all $j < i$ we have $a_j \neq b_1$. Then, by definition, b is not reachable from a.
3. We have computed the path $s = (a_1 = a), a_2, \ldots, a_i, \ldots, a_{i+j}$, such that in the signature $X(s) = S_1, S_2, \ldots, S_i, (S_{i+1}, \ldots, S_{i+j})^\star$, none of the sets $S_{i+1}, \ldots, S_{i+j}$ contains an edge that will be abandoned (by Lemma 21 this problem is decidable).
   If there is no k, with $1 \leq k \leq j$, such that $S = S_{i+k}$, then due to Corollary 12 b is not reachable from a.
   Otherwise, let $S = S_{i+k'}$. If $card(S) > 1$, then clearly $a_{i+k'} = b_1$, thus b is reachable from a.
   If $card(S) = 1$ (we assume in this case that $S = \{e\}$), then by c_m, where $m \in \mathbf{N}_+$, we shall denote the coordinate $c(a_{i+mk'}, e)$. Let $c = c(b_1, e)$. Numbers $c, c_1, c_2, \ldots, c_m, \ldots$ are algebraic and computable from a, b and

m. Similarly as in proof of Lemma 21 we can show that there exists a polynomial with rational coefficients $p(x, y)$ and number $u \in \mathbf{N}_+$, such that $c_{m+1} = \langle p(x, c_m), u \rangle$. Due to Theorem 10 the sequence $c_1, c_2, \ldots$ is monotonous, and due to Theorem 15 it converges to some computable c'. Therefore, by definition b is unreachable from a, if $c > c_1$ and $c > c'$, or $c < c_1$ and $c < c'$. Otherwise we can eventually find $w \in \mathbf{N}_+$, such that either $c_w \leq c \leq c_{w+1}$, or $c_{w+1} \leq c \leq c_w$. In both cases b is reachable from a, if $c = c_w$ or $c = c_{w+1}$, and b is unreachable from a otherwise.

Thus, we have shown that also in Case 2 reachability from a to b is decidable.$\diamond$

Similarly, as it is done in [10], we can modify the proof of Theorem 22 to show that the reachability problem from edge to edge is also decidable.

Theorem 23. *Let $\mathcal{H} = (\mathcal{P}, \varphi)$ be a multi-polynomial hybrid system. Let $e_1, e_2 \in E(\mathcal{P})$. Then it is algorithmically decidable whether edge e_2 is reachable from edge e_1.*$\diamond$

As an easy corollary we obtain decidability result for regions of $\mathcal{P}$.

Corollary 24. *Let $\mathcal{H} = (\mathcal{P}, \varphi)$ be a multi-polynomial hybrid system. Let $P_1, P_2 \in \mathcal{P}$. Then it is algorithmically decidable whether region P_2 is reachable from region P_1.*$\diamond$

5 Some conclusions and open problems

In this paper we have demonstrated that the reachability problem is decidable for planar multi-polynomial hybrid systems. This shows that the fact that HS state space fits on the topology of the plane and has continuous execution trajectories is quite a strong requirement which makes the algorithmic analysis of HS possible even in the case of rather complicated non-linear behaviour rules.

The model of multi-polynomial systems in our paper contains a technical restriction that the vector fields within the regions do not have singularities, however, it is clear that this is not essential, and the decidability results can be proved also for the case when the singularities are allowed. We conjecture that the same results hold also for systems which allow nondeterministic behaviour on the borders between the regions.

It seems that our results can be generalized also for the case, when the borders of the regions of the partition of the plane are polynomial curves, instead of just being straight line segments.

It also would be interesting to study further the classes of HS for which the decidability of the reachability can be proved by exploiting mainly the planar topological properties of the state space (for instance, this method would apply to most of systems where the vector fields inside the regions are defined by some linear autonomous systems, what would amount to having the trajectories of the form $p(x, y, C) = 0$ instead of just $p(x, y) = C$).

We conjecture that it should be possible to generalize our decidability results also for 2-dimensional systems with continuous trajectories (no reset operations), where more general kinds of non-linearity can be admitted. An interesting future study could be looking also at the systems with 2-dimensional state space which is topologically more complicated than the Euclid plane (this is what could be obtained by relaxing the requirement that the values of continuous variables should uniquely determine the control state).

At some point it would be interesting to compare the classes of non-linear HS for which the decidability of the reachability can be shown using primarily the topological arguments with the classes which can be shown decidable by some other means. This largely remains to be a subject of a future work.

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