Topological Errors and Optimal Chamfer Distance Coefficients*

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Abstract. In this paper, a theoretical characterisation of the topological errors which arise during the approximation of Euclidean distances from discrete ones is presented. The continuous distance considered is the widely used Euclidean distance whereas we consider as discrete distance the chamfer distance based on 3×3 masks. The objective is to obtain formal results from which algorithms for the exact solution of the Euclidean Distance Transformation using integer arithmetic can be derived. We conclude this study by presenting a global upper bound for a topologically-correct distance mapping, irrespective of the chamfer distance coefficients, and identify the smallest coefficients associated with this bound.

1 Introduction

The main motivation of this work is to analyse the mapping between continuous (Euclidean) and discrete (chamfer) distances on the unit square grid [7].

Section 2 first recalls general digital image processing terminology and the principles behind approximating continuous distances by discrete ones. In [3, 4], empirical results were presented to point out that the pixel ordering induced by the chamfer (discrete) Distance Transformation (DT) only matches with the ordering induced by the Euclidean (continuous) Distance Transformation up to some upper bound. In Section 3, we establish this in mathematical terms and obtain a closed form solution for such upper bounds for any given DT coefficients. The objective is first to characterise the topological errors in the mapping between continuous and discrete distances, and then to derive distance bounds which guarantee an error-free transformation. Finally, Section 3.3 details the characterisation of integer DT coefficients which are proved to be optimal for such a criterion.

While studying in depth the calculation of Euclidean distance values using discrete distance functions, we will derive results concerning the decomposition of integer values which can then form the basis for the development of exact Euclidean Distance Transformation algorithms (see *e.g.*, [8]).

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2 Definitions and Notations

We consider throughout these pages that the continuous distance used is the Euclidean distance d_{E} defined as $d_{\mathrm{E}}(p,q) = \sqrt{(x_p - x_q)^2 + (y_p - y_q)^2}$, where $p = (x_p, y_p)$ and $q = (x_q, y_q)$. We introduce the notation for some standard functions. $\lceil x \rceil$ is the smallest integer greater or equal to than $x \in \mathbb{R}$ and $\lfloor x \rfloor$ is greatest integer smaller or equal to than $x \in \mathbb{R}$. Then, $\mathrm{round}(x) = \lfloor x \rfloor$ if $\lfloor x \rfloor \leq x \leq \lfloor x \rfloor + \frac{1}{2}$. Otherwise, $\mathrm{round}(x) = \lceil x \rceil$.

In approximating the Euclidean distance on the discrete grid, chamfer distances were introduced in [5,6] and studied in [1,2]. Such discrete distances typically rely on the definition of local distances within a mask centred at each pixel. We will consider chamfer distances in relation to 3×3 masks. We define a as the length of the unit horizontal/vertical move (a-move) on the grid, and b as the length of the unit diagonal move (b-move) on the grid. Given two points p and q, the chamfer distance $d_{a,b}(p,q)$ between p and q can be computed as follows.

$$d_{a,b}(p,q) = k_{\mathrm{a}}a + k_{\mathrm{b}}b \tag{1}$$

where k_{a} and k_{b} represent the number of a- and b-moves on the shortest path from p to q on the grid. The conditions on a and b for $d_{a,b}$ to be a distance are given in (2) below (see [9] and [12] for more details).

$$0 < a < b < 2a \tag{2}$$

The number of a- and b-moves $(k_{\mathrm{a}}, k_{\mathrm{b}})$ on the shortest path from p to q can also be used to compute the Euclidean distance between p and q.

$$d_{\mathrm{E}}(p,q) = \sqrt{(k_{\mathrm{a}} + k_{\mathrm{b}})^2 + k_{\mathrm{b}}^2} \tag{3}$$

Without loss of generality, we restrict this study to a and b values such that a and b are relatively prime (*i.e.*, the Greater Common Divisor of a and b, $\gcd(a,b)$, is such that $\gcd(a,b) = 1$). This corresponds to normalising the a and b coefficients to their minimal configuration.

Forchhammer [3,4] pointed out that topological errors occurred when deducing a Euclidean Distance Map from a Discrete Distance Map. This work was based on the distance inequalities to be satisfied during the generation of the Distance Map. From this study, he derived empirical results concerning the limitations of discrete distances in approximating continuous ones. In the next section, we formally detail these topological errors induced by the approximation of continuous distances by discrete ones and present distance limits as upper bounds for the correctness of the Distance Maps.

3 Topological Errors

In [3,4], Forchhammer introduced the topological inconsistencies induced by the discrete distances when used as an approximation of the Euclidean distance.

Essentially, the ordering of the discrete distance does not match the ordering of the Euclidean distance. Consider the following example (see Fig. 1). Let the DT coefficients be $a = 2$, $b = 3$, and consider the three integer points (pixels), $p = (0,0)$, $q = (10,1)$ and $r = (9,4)$. The shortest path on the grid from p to q is given by $k_\mathrm{a} = 9$ and $k_\mathrm{b} = 1$ and that from p to r is given by $k_\mathrm{a} = 5$ and $k_\mathrm{b} = 4$. Using Equations (1) and (3), we have $d_{a,b}(p,q) = 21$, $d_\mathrm{E}(p,q) = \sqrt{101}$, $d_{a,b}(p,r) = 22$ and $d_\mathrm{E}(p,r) = \sqrt{97}$. If q and r are border pixels, the discrete DT will lead to consider q as the nearest border pixel to p (by the chamfer distance measure) giving an approximate Euclidean distance of $\sqrt{101}$. This is clearly incorrect since there is a smaller Euclidean distance between p and another border pixel (namely r) giving a Euclidean distance of $\sqrt{97}$. In other words, since, $d_{a,b}(p,q) < d_{a,b}(p,r)$ and $d_\mathrm{E}(p,q) > d_\mathrm{E}(p,r)$, the ordering of $d_{a,b}$ differs from the ordering of d_E.

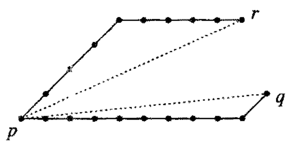

Fig. 1. An example of a topological error.

Given a pair of DT coefficients (a,b), we characterise the configurations for which this problem occurs precisely. First, we introduce how restrictions for the decomposition of a given discrete distance value D into a- and b-moves can be given by the solution to the Frobenius problem (see [11]).

Theorem 1. *[11]. Given* $0 < a < b$ *such that* $(a,b) \in \mathbb{N}^2$ *and* $\gcd(a,b) = 1$. *Consider the equation:* $k_\mathrm{a} a + k_\mathrm{b} b = D \quad (k_\mathrm{a}, k_\mathrm{b}) \in \mathbb{N}^2$ *If* $\chi = (a-1)(b-1)$, *then we have the following instances.*

(i) If $D \geq \chi$, *there is always at least one solution* $(k_\mathrm{a}, k_\mathrm{b})$.
(ii) If $D = \chi - 1$, *there is no solution.*
(iii) There is exactly $\frac{1}{2}\chi$ *values of* D *that have no solution.*

We will use the solution to this classical problem to characterise the topological error introduced earlier. Two types of errors are distinguished and presented in Sections 3.1 and 3.2 respectively.

3.1 Type 1 error

Given a pair of DT coefficients (a,b) and three integer points p, q and r, a Type 1 error occurs between q and r relative to p if $d_{a,b}(p,q) = d_{a,b}(p,r)$ and $d_\mathrm{E}(p,q) \neq d_\mathrm{E}(p,r)$. More formally, we make the following definition.

Definition 2. Type 1 error. Given a pair of DT coefficients (a,b) and a discrete distance value D, a Type 1 topological error occurs if there exist two integer pairs (k_{a_1}, k_{b_1}) and (k_{a_2}, k_{b_2}) such that $k_{a_1}a + k_{b_1}b = k_{a_2}a + k_{b_2}b = D$ and $\sqrt{(k_{a_1}+k_{b_1})^2 + k_{b_1}^2} \neq \sqrt{(k_{a_2}+k_{b_2})^2 + k_{b_2}^2}$.

An example for Type 1 error is illustrated in Fig. 2 where, (k_{a_1}, k_{b_1}) represents the shortest path from p to q, and (k_{a_2}, k_{b_2}) the shortest path from p to r. In this example, the DT coefficients are $a = 2$ and $b = 3$, and the three integer points are $p = (0,0)$, $q = (3,0)$ and $r = (2,2)$. We obtain $d_{a,b}(p,q) = d_{a,b}(p,r) = D = 6$, since $k_{a_1} = 3$, $k_{b_1} = 0$ and $k_{a_2} = 3$, $k_{b_2} = 0$. On the other hand, we have, $d_E(p,q) = 3$ and $d_E(p,r) = \sqrt{8}$.

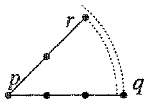

Fig. 2. The first instance of Type 1 topological error for $(a,b) = (2,3)$.

Lemma 3. *[7] Given a pair of DT coefficients (a,b) and a discrete distance value D, we assume that the existence condition (i) in Theorem 1 holds. Then, the maximal value $k_{b_{max}}$ of k_b such that $k_a a + k_{b_{max}} b = D$ with $k_a \geq 0$ and $k_{b_{max}} \geq 0$ is given by:*

$$k_{b_{max}} = \left\lfloor \frac{D}{b} \right\rfloor - \psi((D \bmod b) \bmod a) \tag{4}$$

where ψ is the implicit integer function such that:
$\psi : \{0, 1, \cdots, a-1\} \mapsto \{0, 1, \cdots, a-1\}$ *and* $\psi((x.(2a-b)) \bmod a) = x$.

ψ can be easily calculated as a one-to-one mapping of the set $\{0, 1, \cdots, a-1\}$ onto itself. For example (see Fig. 3), if $a = 5$ and $b = 7$, the mapping is given by $\{0,1,2,3,4\} \overset{\psi}{\mapsto} \{0,2,4,1,3\}$. Therefore, if $D = 86$, say, we have $\lfloor \frac{D}{b} \rfloor = 12$ and $(D \bmod b) \bmod a = 2$. Hence, from Equation (4), $k_{b_{max}} = 12 - \psi(2) = 12 - 4 = 8$. We can also easily compute $k_a = \frac{D - k_{b_{max}} b}{a} = 6$. Lemma 3 would allow us to have all the values of k_b since these can be given by $(k_{b_{max}} - ia)$ with $i = 0, \cdots, \left\lfloor \frac{k_{b_{max}}}{a} \right\rfloor$. Therefore, the pairs (k_a, k_b) for the decomposition of $D = 86$ are (6,8) and (13,3).

Thus, Lemma 3 readily gives an exhaustive list of (k_a, k_b) pairs for decomposing any discrete distance value D for any DT coefficients a and b. Note that, if Condition *(i)* in Theorem 1 is not matched (*i.e.*, no possible decomposition), we obtain $k_{b_{max}} < 0$ (*e.g.*, $a = 3$, $b = 4$, $\chi = 6$, if $D = 5 = \chi - 1$, we obtain $k_{b_{max}} = -1$). This simple test can prove useful when developing the mapping algorithms.

Definition 4. Given a pair of DT coefficients (a,b) and a discrete distance value D which can be decomposed in at least one manner, we define:

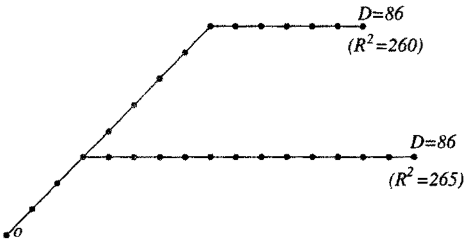

Fig. 3. All decompositions for $D = 86$.

- The set $\Theta = \{(k_{\mathrm{a}_i}, k_{\mathrm{b}_i}), i = 0, \cdots, n\}$ as the exhaustive list of all possible decomposition pairs (*i.e.*, $D = k_{\mathrm{a}_i} a + k_{\mathrm{b}_i} b$, $k_{\mathrm{a}_i} \geq 0$, $k_{\mathrm{b}_i} \geq 0$ $\forall i = 0, \cdots, n$ with $n = \left\lfloor \frac{k_{\mathrm{b}_{\max}}}{a} \right\rfloor$). Note that $k_{\mathrm{b}_{\max}} = \max_{i=0,\cdots,n} k_{\mathrm{b}_i}$ and, $k_{\mathrm{b}_i} = k_{\mathrm{b}_{\max}} - ia$. Therefore, the set Θ can be fully computed using Lemma 3.
- $R_i(D)$ as the Euclidean distance associated with the pair $(k_{\mathrm{a}_i}, k_{\mathrm{b}_i})$. From Equation (3), we have,

$$R_i(D) = \sqrt{(k_{\mathrm{a}_i} + k_{\mathrm{b}_i})^2 + k_{\mathrm{b}_i}^2} \tag{5}$$

- $R_{\max}(D)$ (*resp.* $R_{\min}(D)$) as the maximal (*resp.* minimal) Euclidean distance over all $n + 1$ possible decompositions.
- l (*resp.* m) as the index in the set Θ of the decomposition $(k_{\mathrm{a}_l}, k_{\mathrm{b}_l})$ (*resp.* $(k_{\mathrm{a}_m}, k_{\mathrm{b}_m})$) leading to $R_{\max}(D)$ (*resp.* $R_{\min}(D)$).

Continuing with the example in Fig. 3, we had $a = 5$, $b = 7$ and $D = 86$. We obtained $k_{\mathrm{b}_{\max}} = 8$, $n = 1$ and $\Theta = \{(6, 8), (13, 3)\}$ (*i.e.*, $R_0(86) = \sqrt{260}$ and $R_1(86) = \sqrt{265}$). Hence, $m = 0$, $l = 1$ (*i.e.*, $R_{\min}(D) = R_0(86) = \sqrt{260}$ and $R_{\max}(D) = R_1(86) = \sqrt{265}$). Note that in [7], formulae for calculating m and l without the need of enumeration were derived.

Lemma 5. *Characterisation of a Type 1 error. Given a pair of DT coefficients (a, b), a Type 1 error occurs for any discrete distance value D for which $R_{\max}(D) \neq R_{\min}(D)$.*

Clearly, the first instance of D for which $R_{\max}(D) \neq R_{\min}(D)$ is $D = ab$. In this case, $k_{\mathrm{b}_{\max}} = a$, $n = 1$, $\Theta = \{(0, a), (b, 0)\}$, (*i.e.*, $R_0 = a\sqrt{2}$ and $R_1 = b$). Hence, $R_{\max}(ab) \neq R_{\min}(ab)$. Therefore, we define the following Euclidean distance limit when considering Type 1 errors only.

Definition 6. Euclidean distance limit induced by Type 1 error, $\mathcal{R}_1(a, b)$. Given a pair of DT coefficients (a, b), and $D = ab$ as the minimum discrete distance value for which $R_{\min}(D) \neq R_{\max}(D)$, we define the Euclidean distance limit $\mathcal{R}_1(a, b)$ for Type 1 errors as follows.

$\mathcal{R}_1(a, b)$ is the maximal Euclidean distance value deduced from a discrete distance value (*i.e.*, using Equation (3)) up to which both discrete and continuous distance ordering match. More formally, $\mathcal{R}_1(a, b)$ is the maximal Euclidean distance value R such that $\exists\ k_{\mathrm{a}},\ k_{\mathrm{b}} \in \mathbb{N}$ such that $R = \sqrt{(k_{\mathrm{a}} + k_{\mathrm{b}})^2 + k_{\mathrm{b}}^2}$ and $R < R_{\max}(ab)$.

In other words, $R_{\max}(ab)$ can be considered as a strict (*i.e.*, non-feasible) Euclidean distance limit. In order to obtain a feasible limit, we search D' the maximal discrete distance value such that $R_{\min}(D') < R_{\max}(ab)$ and consider $\mathcal{R}_1(a,b) = R_{\min}(D')$. Using the previous study, we can easily design an algorithm to compute, for any pair of DT coefficients (a,b), the value of $\mathcal{R}_1(a,b)$. In Fig. 4, $(\mathcal{R}_1(a,b))^2$ is plotted for each pair of valid DT coefficients such that $a \leq 10$.

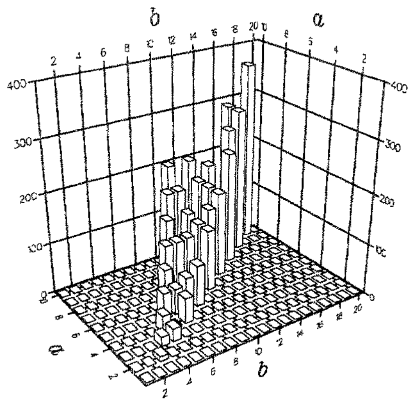

Fig. 4. Euclidean distance limit induced by Type 1 of topological error $(\mathcal{R}_1(a,b))^2$.

The following table gives the corresponding discrete distance values $D = ab$ and D' for some instances of DT coefficients (a,b).

a	b	$D = ab$	D'	$R_{\max}(ab)$	$\mathcal{R}_1(a,b) = R_{\min}(D')$
2	3	6	6	$\sqrt{9}$	$\sqrt{8}$
3	4	12	13	$\sqrt{18}$	$\sqrt{17}$
3	5	15	16	$\sqrt{25}$	$\sqrt{20}$
5	7	35	36	$\sqrt{50}$	$\sqrt{45}$

3.2 Type 2 error

Given a pair of DT coefficients (a,b) and three integer points p, q and r, a Type 2 error occurs between q and r, relative to p, if $d_{a,b}(p,q) < d_{a,b}(p,r)$ and $d_{\mathrm{E}}(p,q) > d_{\mathrm{E}}(p,r)$. More formally, we make the following definition.

Definition 7. Type 2 error. Given a pair of DT coefficients (a,b) and two discrete distance values D_1 and D_2, we assume that $\exists\ (k_{\mathrm{a}_1}, k_{\mathrm{b}_1})$and $(k_{\mathrm{a}_2}, k_{\mathrm{b}_2})$ such that $k_{a_1}a + k_{b_1}b = D_1$ and $k_{a_2}a + k_{b_2}b = D_2$ (see Theorem 1). A Type 2 error occurs if, $D_1 < D_2$ and $\sqrt{(k_{a_1} + k_{b_1})^2 + k_{b_1}^2} > \sqrt{(k_{a_2} + k_{b_2})^2 + k_{b_2}^2}$.

Fig. 5 illustrates an instance of Type 2 error between q and r, relative to p. In this example, the DT coefficients are $a = 2$ and $b = 3$. We consider $p = (0,0)$, $q = (6,0)$ and $r = (5,3)$. Therefore, $D_1 = d_{a,b}(p,q) = 12$, since $k_{a_1} = 6$ and $k_{b_1} = 0$. We also obtain $D_2 = d_{a,b}(p,r) = 13$, since $k_{a_2} = 2$ and $k_{b_2} = 3$. On the other hand, we have $d_E(p,q) = \sqrt{36}$ and $d_E(p,r) = \sqrt{34}$. Therefore, $d_{a,b}(p,q) < d_{a,b}(p,r)$ and $d_E(p,q) > d_E(p,r)$.

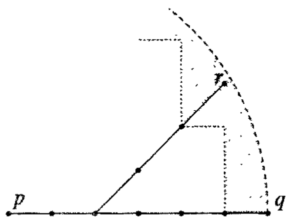

Fig. 5. The first instance of Type 2 topological error for $(a,b) = (2,3)$.

From a geometric viewpoint, given three integer points p, q and r such that $D_1 = d_{a,b}(p,q) < D_2 = d_{a,b}(p,r)$, a Type 2 error occurs between q and r, relative to p, if there exists at least one integer point (namely, r) included in the area between the discrete disc of radius D_1 centred at p and the Euclidean disc that contains this discrete disc. In Fig. 5, this area is illustrated by the shaded surface. A Type 2 error occurs between q and r, relative to p, since r lies in this shaded surface.

The geometrical characterisation of Type 2 errors will, therefore, be investigated through the characterisation of the radius of the smallest Euclidean disc that contains the discrete disc of radius D for any given values of the DT coefficients (a,b) and for any discrete distance value D. The radius of such a Euclidean disc was noted $R_{\max}(D)$ in Section 3.1 (see Definition 4). Therefore, the geometrical characterisation of Type 2 error can be formally written as follows.

Lemma 8. *Given a pair of DT coefficients (a,b) and a discrete distance value D, a Type 2 error occurs in the discrete disc of radius D if there exists a discrete distance value $D' > D$ such that $R_{\min}(D') < R_{\max}(D)$.*

Using this characterisation, the Euclidean distance limit $\mathcal{R}_2(a,b)$ induced by Type 2 error can be defined as follows.

Definition 9. Euclidean distance limit induced by Type 2 error, $\mathcal{R}_2(a,b)$ Given a pair of DT coefficients (a,b), and D, the minimum discrete distance value for which there exists a discrete distance value D' such that $R_{\min}(D') < R_{\max}(D)$, we define the Euclidean distance limit $\mathcal{R}_2(a,b)$ for Type 2 errors as follows.

$\mathcal{R}_2(a,b) = R_{\min}(D')$ where D' is the smallest discrete distance value such that $R_{\min}(D') < R_{\max}(D)$.

In other words, if a Type 2 error occurs for the discrete distance value D (*e.g.*, at point q in Fig. 5, with $D = 12$), we consider the Euclidean distance limit as the value $R_{\min}(D')$ where D' is the discrete distance value at the second point for which Type 2 error occurred (*e.g.*, point r in Fig. 5, and $D' = 13$).

Using the previous study, we can also design an algorithm to compute, for any pair of DT coefficients (a, b), the value of $\mathcal{R}_2(a, b)$. In Fig. 6(A), $(\mathcal{R}_2(a, b))^2$ is plotted for each DT coefficients pair such that $a \leq 10$. The table below gives the corresponding discrete distance values D and D' for some instances of DT coefficients (a, b).

a	b	D	D'	$R_{\max}(D)$	$\mathcal{R}_2(a,b) = R_{\min}(D')$
2	3	12	13	$\sqrt{36}$	$\sqrt{34}$
3	4	12	13	$\sqrt{18}$	$\sqrt{17}$
3	5	9	10	$\sqrt{9}$	$\sqrt{8}$
5	7	21	22	$\sqrt{18}$	$\sqrt{17}$

Using the results of Sections 3.1 and 3.2, we can now define a combined Euclidean distance limit where no topological error of any type can occur.

Definition 10. Global Euclidean distance limit, $\mathcal{R}(a, b)$. Given a pair of DT coefficients (a, b), we define the global Euclidean distance limit $\mathcal{R}(a, b)$ as the minimum between the distance limits induced by both Type 1 and 2 errors. Therefore, $\mathcal{R}(a, b) = \min(\mathcal{R}_1(a, b), \mathcal{R}_2(a, b))$.

$\mathcal{R}(a, b)$ represents the maximal achievable Euclidean distance when growing topologically correct discrete discs. Equivalently, given a pair of DT coefficients (a, b), for any discrete distance value D such that $R_{\max}(D) \leq \mathcal{R}(a, b)$, no topological error (of Type 1 or Type 2) occurs in the discrete disc of radius D. In Fig. 6(B), $(\mathcal{R}(a, b))^2$ is plotted for each DT coefficients pair such that $a \leq 10$.

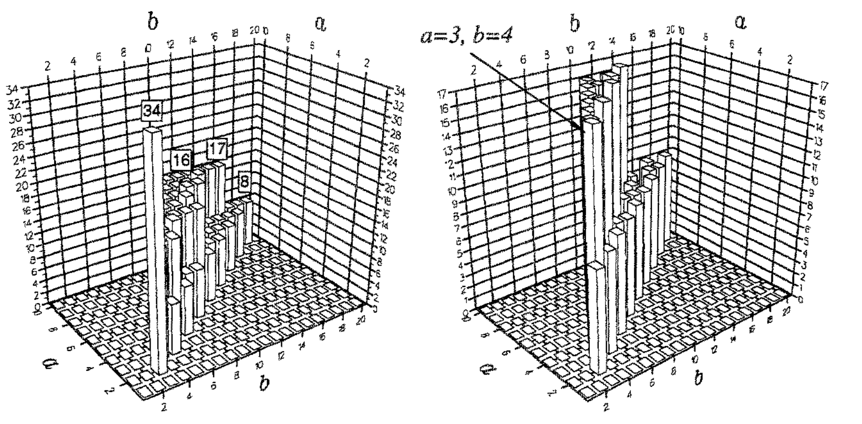

Fig. 6. A:(left) Distance limit induced by Type 2 error. B:(right) Global Euclidean distance limit for the correctness of the EDT.

Note that, for large values of a and b, the limit induced by the Type 2 error dominates. Type 1 error only dominates for the smallest possible values of (a, b) (*i.e.*, a=2, b=3). For any other pair, Type 2 error dominates. The following table summarises the distance limits obtained for some values of the DT coefficients (a, b).

a	b	D	D'	$R_{\max}(D)$	$\mathcal{R}(a,b)$	Type
2	3	6	6	$\sqrt{9}$	$\sqrt{8}$	1
3	4	12	13	$\sqrt{18}$	$\sqrt{17}$	1 and 2
3	5	9	10	$\sqrt{9}$	$\sqrt{8}$	2
5	7	21	22	$\sqrt{18}$	$\sqrt{17}$	2
9	11	33	36	$\sqrt{18}$	$\sqrt{16}$	2

In summary, the results in Sections 3.1 and 3.2 lead us to the characterisation of a Euclidean distance limit $\mathcal{R}(a, b)$ for any pair of DT coefficients (a, b). The Euclidean distance limit can readily give a discrete distance limit via the definitions of $R_{\max}$ and $R_{\min}$ (see Definition 4).

3.3 Global Euclidean distance limit and optimal DT coefficients

Our aim now is to determine, whether an optimal pair exists among all valid pairs of DT coefficients. We define optimality here as the smallest integer pair of DT coefficients which guarantees the maximum achievable Euclidean distance limit. Using the results plotted in Fig. 6(B), we could say that, for all pair of DT coefficients such that $a \leq 10$, the pair $(3, 4)$ is a local optimum in the sense that it is the smallest pair of DT coefficients that leads to a (local) maximum Euclidean distance limit (*i.e.*, $\mathcal{R}(3, 4) = \sqrt{17}$). In order to extend this result to any pair of DT coefficients, we will use an analytical approach rather than the geometric approach which was used previously.

As suggested in Lemma 5 and Definition 6, an analytical Euclidean distance limit induced by Type 1 errors can be estimated by $R_{\max}(ab)$. Since this limit increases with the values of (a, b), we pointed out earlier that Type 2 error dominates for greater values of DT coefficients. Hence, we will mainly concentrate on an analytical study of Type 2 errors and finally combine the result with those of the previous study of Type 1 errors. The result of this study can be stated as follows.

Theorem 11. *Euclidean distance limit and optimal DT coefficients Considering the chamfer distance $d_{a,b}$ as a discrete distance, the maximal error-free Euclidean distance achievable is $\sqrt{17}$ and the smallest integer pair of DT coefficients that achieves this limit is $(a, b) = (3, 4)$.*

We introduce the idea behind the proof of Theorem 11 (full details of the proof of this result can be found in reference [7]). According to the conditions in (2) given by $0 < a < b < 2a$, a pair of DT coefficients (a, b) is valid if and only if the integer coordinates (a, b) in the plane (x, y) lie in the positive quadrant of the plane (*i.e.*, $x \geq 0$ and $y \geq 0$) and between the lines $y = x$ and $y = 2x$. Moreover, given a pair of integer values (a, b) satisfying (2), the only integer pair of DT coefficients which is exactly on the line $y = \frac{b}{a}x$ is the pair (a, b) itself since, by definition, $\gcd(a, b) = 1$. In Fig. 7, the valid pairs (a, b) such that $a \leq 10$ and $b \leq 11$ are represented as dots (o) in the plane (x, y).

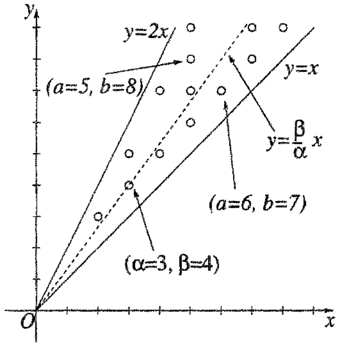

Fig. 7. Representation of the valid pairs of DT coefficients for the proof of Theorem 11.

The approach for deriving an analytical expression of the Euclidean distance limit for Type 2 error (*i.e.*, $\mathcal{R}_2(a, b)$) is made by decomposing the region of the plane (x, y) delimited by the lines $y = x$ and $y = 2x$, by a line $y = \frac{\beta}{\alpha}x$, where (α, β) is an integer pair that matches the conditions for being a pair of DT coefficients (see Fig. 7). For each such pair (α, β), we characterise a Euclidean distance limit in each sub-region of the plane (x, y) delimited by the lines $y = x$, $y = \frac{\beta}{\alpha}x$ and $y = 2x$. Given a valid pair (α, β), and for any pair of DT coefficients (a, b) different from (α, β), two cases are possible, (i): $\frac{b}{a} < \frac{\beta}{\alpha}$ or (ii): $\frac{b}{a} > \frac{\beta}{\alpha}$. Case (i) includes the valid integer points in the sub-region below $y = \frac{\beta}{\alpha}x$ and above $y = x$, whereas Case (ii) includes the valid integer points in the sub-region above $y = \frac{\beta}{\alpha}x$ and below $y = 2x$. The Euclidean distance limit $\mathcal{R}_2(a, b)$ is to be investigated for the two sub-regions separately and we will refer to this as $\mathcal{R}_{\text{inf}}(\alpha, \beta)$ and $\mathcal{R}_{\text{sup}}(\alpha, \beta)$ for cases (i) and (ii) respectively. In the example illustrated in Fig. 7, $\alpha = 3$, $\beta = 4$. Then, for instance, $\mathcal{R}_2(6, 7)$ will include the Euclidean distance limit $\mathcal{R}_{\text{inf}}(3, 4)$, since $\frac{7}{6} < \frac{4}{3}$. Similarly, $\mathcal{R}_2(5, 8)$ will include the Euclidean distance limit $\mathcal{R}_{\text{sup}}(3, 4)$, since $\frac{8}{5} > \frac{4}{3}$.

Hence, given a pair of DT coefficients (a, b), the Euclidean distance limit for Type 2 errors induced by (a, b) (*i.e.*, $\mathcal{R}_2(a, b)$) will result from a combination of all Euclidean distance limits induced by the pairs (α, β) in the following way.

$$\mathcal{R}_2(a, b) = \min\left(\min_{\{(\alpha,\beta) / \frac{b}{a} < \frac{\beta}{\alpha}\}} (\mathcal{R}_{\text{inf}}(\alpha, \beta)), \min_{\{(\alpha,\beta) / \frac{b}{a} > \frac{\beta}{\alpha}\}} (\mathcal{R}_{\text{sup}}(\alpha, \beta)) \right)$$

Proof: Given a pair of DT coefficients (a, b) and an integer pair (α, β) such that $0 < \alpha < \beta < 2\alpha$ and $\gcd(\alpha, \beta) = 1$, we consider two cases: *(i)* $\frac{b}{a} < \frac{\beta}{\alpha}$ and, *(ii)*

$\frac{b}{a} > \frac{\beta}{\alpha}$. It will become apparent that the equality case represents a Type 1 error and, therefore, as noted earlier, will not be studied here.

An investigation of cases *(i)* and *(ii)* leads to analytical expressions for the Euclidean distance limits induced by each case (*i.e.*, $\mathcal{R}_{\inf}(\alpha,\beta)$ and $\mathcal{R}_{\sup}(\alpha,\beta)$, respectively). Details for the development of such expressions can be found in reference [7]. Euclidean distance limits induced by all possible values of (α,β) are summarised in the table below (in this case, $\lceil x \rceil$ is the smallest integer strictly greater than x).

	$\mathcal{R}_{\inf}(\alpha,\beta)$	$\mathcal{R}_{\sup}(\alpha,\beta)$
$\frac{\beta}{\alpha} < \sqrt{2}$	β	$\sqrt{\left(\alpha + \left\lceil \frac{2\alpha^2-\beta^2}{2(\beta-\alpha)} \right\rceil\right)^2 + \alpha^2}$
$\frac{\beta}{\alpha} > \sqrt{2}$	$\sqrt{\left(\beta + \left\lceil \frac{\beta^2-2.\alpha^2}{2(2.\alpha-\beta)} \right\rceil\right)^2 + \left\lceil \frac{\beta^2-2.\alpha^2}{2(2.\alpha-\beta)} \right\rceil^2}$	$\alpha\sqrt{2}$

We can now list the values of the limits for the smallest possible values of α and β. The following table summarises the Euclidean distance limit values obtained when comparing $\frac{b}{a}$ with $\frac{\beta}{\alpha}$ with the first possible values of α and β.

α	β	$\mathcal{R}_{\inf}(\alpha,\beta)$	$\mathcal{R}_{\sup}(\alpha,\beta)$
2	3	$\sqrt{17}$	$\sqrt{8}$
3	4	$\sqrt{16}$	$\sqrt{34}$
3	5	$\sqrt{97}$	$\sqrt{18}$
4	5	$\sqrt{25}$	$\sqrt{80}$
4	7	$\sqrt{337}$	$\sqrt{32}$

Now, each possible combination of a, b, α and β creates an increasing sequence when ordered such that $\alpha^2+\beta^2$ increases (*e.g.*, the expression of $\mathcal{R}_{\inf}(\alpha,\beta)$ when $\frac{\beta}{\alpha} > \sqrt{2}$ leads to the increasing sequence $\sqrt{17}$, $\sqrt{97}$, $\sqrt{337}$, $\cdots$). Therefore, all possible Euclidean distance limits will be obtained as soon as we obtain a limit value for any range of $\frac{b}{a}$. Using the two first lines in the previous table, we deduce that, for $\frac{3}{2} < \frac{b}{a} < 2$, $\mathcal{R}_2(a,b) = \sqrt{8}$; for $\frac{b}{a} = \frac{3}{2}$, $\mathcal{R}_2(a,b) = \sqrt{34}$; for $\frac{4}{3} \leq \frac{b}{a} < \frac{3}{2}$, $\mathcal{R}_2(a,b) = \sqrt{17}$; for $1 < \frac{b}{a} < \frac{4}{3}$, $\mathcal{R}_2(a,b) = \sqrt{16}$, which fits exactly the results shown in Fig. 6(A).

As pointed out earlier, $\mathcal{R}_1(a,b)$ increases with the values of the DT coefficients. Hence, clearly $\mathcal{R}(a,b) = \mathcal{R}_2(a,b)$ for any pair of DT coefficients $(a,b) \neq (2,3)$. Now, $\mathcal{R}_2(2,3) = \mathcal{R}_{\sup}(3,4) = \sqrt{34}$, since $\frac{3}{2} > \frac{4}{3}$. From the result of characterisation of Type 1 error, we obtain $\mathcal{R}_1(2,3) = \sqrt{8}$. Therefore, $\mathcal{R}(2,3) = \min(\mathcal{R}_1(2,3), \mathcal{R}_2(2,3)) = \sqrt{8}$. Hence, the maximal Euclidean distance achievable is $\max_{\{(a,b)\}} \mathcal{R}(a,b) = \sqrt{17}$. Clearly, the first pair (a,b) which realises this maximum is $(a,b) = (3,4)$. Hence, Theorem 11 holds. □

In summary, we have extended the results derived from the geometrical study presented in Section 3. Theorem 11 states that, for any DT coefficients (a,b) such that $\frac{4}{3} \leq \frac{b}{a} < \frac{3}{2}$, the topological order is preserved in any discrete disc of radius D such that $R_{\max}(D) < \sqrt{17}$. In the design of algorithms which require chamfer distances, it is wise to maintain small values of the discrete distances computed. In this context, the minimum DT coefficients for achieving the global upper bound of $\sqrt{17}$ is $(a, b) = (3, 4)$.

4 Conclusion

The problem of approximating continuous distances by discrete ones was considered. We formally characterised topological errors which occur during the mapping of distances from the discrete to the continuous space. Distance limits up to which these errors are guaranteed not to occur were derived for any pair of DT coefficients. Among all DT coefficients, an optimal integer pair was characterised and shown analytically to correspond to a global optimum.

As by-product of this study, we obtained results which give, without the need of enumeration, all possible decompositions of a discrete distance value into a combination of moves on a shortest path on the grid. Among applications, such results can readily be used for the development of exact Euclidean distance mapping algorithms (see *e.g.*, [8]).

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