ON INDUCTION VS. *-CONTINUITY

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Abstract. In this paper we study the relative expressibility of the infinitary **-continuity condition*

(*-cont) $\quad <\alpha^*>X \equiv \vee_n <\alpha^n>X$

and the equational but weaker *induction axiom*

(ind) $\quad X \wedge [\alpha^*](X \supset [\alpha]X) \equiv [\alpha^*]X$

in Propositional Dynamic Logic. We show: (1) under **ind** only, there is a first-order sentence distinguishing separable dynamic algebras from standard Kripke models; whereas (2) under the stronger axiom ***-cont**, the class of separable dynamic algebras and the class of standard Kripke models are indistinguishable by any sentence of infinitary first-order logic.

1. Introduction

Propositional Dynamic Logic (PDL), introduced by Fischer and Ladner [FL], is the propositional version of Dynamic Logic [Pr1, see also H]. It is a maximally succinct vehicle for the illustration of fundamental principles of program/assertion interaction, since all but the absolutely essential structure is excluded (in particular any structure on the domain of computation). The theory is captured axiomatically in the deductive system of Segerberg [Se], proved complete by Parikh [Pa, see also KP]. PDL combines and generalizes classical propositional logic (for the assertions), the calculus of regular events (for the programs), and modal logic (for their interaction). The three components fit together neatly into a simple but mathematically rich system. Results of a fundamental nature have been established which perhaps would not have been apparent in a more powerful system [K1-4,Pr3-5,KP].

This paper deals with a fundamental principle of looping, namely that *looping is inherently infinitary.* Simpler programming language constructs, such as composition and conditional tests, are captured up to isomorphism by their equations [K1], whereas looping cannot be so captured [RT,K3,K4]. This principle is quite evident in programming language semantics and data type specification (see for example [Sa]). In this paper we illustrate this principle in the context of PDL with two results comparing the expressive power of the familiar PDL induction axiom

(ind) $\quad X \wedge [\alpha^*](X \supset [\alpha]X) \equiv [\alpha^*]X$

and the stronger **-continuity condition*

(*-cont) $\quad <\alpha^*>X \equiv \vee_n <\alpha^n>X$.

The *-continuity condition says that $<\alpha^*>X$ is the join or least upper bound of the propositions $<\alpha^n>X$ with respect to implication. A proof that ***-cont** implies **ind** can be found in [K1]. The axiom ***-cont** appeared in the original definition of dynamic algebras [K1], but later V. Pratt recommended dropping it in favor of **ind**, allowing more models. We shall adopt Pratt's more general definition and call dynamic algebras **-continuous* if they satisfy ***-cont**. All dynamic algebras arising in practice, including and especially the standard Kripke models, are *-continuous.

In [K1] it was shown that any separable dynamic algebra is represented by a (possibly nonstandard) Kripke model. (A dynamic algebra is called *separable* [K1] if $<\alpha>X = <\beta>X$ for all X implies $\alpha = \beta$. A Kripke model is *standard* if α^* is the reflexive transitive closure of binary relation α, otherwise it is *nonstandard*.) In [K3,RT,K4] it was shown that there exist separable *-continuous dynamic algebras that are not represented by any standard Kripke model.

Pratt [Pr1] used universal algebraic techniques to show that dynamic algebras and standard Kripke models share the same equational theory, giving an alternative proof to the completeness of the Segerberg axioms. In this paper we prove the following two results, which compare the expressive power of the two axioms **ind** and ***-cont**: (1) *there is a first-order sentence that distinguishes separable dynamic algebras from standard Kripke models;* but (2) *the class of separable *-continuous dynamic algebras and the class of standard Kripke models agree on all sentences of the infinitary language* $L_{\omega_1\omega}$. These two results are proved in sections 2 and 3, respectively. In section 4 we discuss the effect of allowing an equality symbol between elements of the Kleene (or regular) sort of a dynamic algebra. We show in that section that the infinitary condition $\alpha\beta^*\gamma = \bigvee_n \alpha\beta^n\gamma$ allows a natural axiomatization of the equational theory of regular events. It is known that no purely equational axiomatization exists [R].

It is assumed the reader is familiar with PDL and dynamic algebra. PDL was first defined in [FL], and this reference remains the best introduction. Definitions, basic properties, and examples of dynamic algebras can be found in [K1-4,Pr1-3].

Let L be the usual two-sorted language for PDL and dynamic algebra, consisting of primitive symbols a, b,... (for the Kleene or program sort) and P, Q,... (for the Boolean sort). Terms α, β,... for the Kleene sort and X, Y,... for the Boolean sort are built up using the usual Boolean operators $\wedge$, $\vee$, $\neg$, 0, and 1, the binary Kleene operators $\cup$ (choice) and ; (composition), the unary operators $^-$ (reverse) and * (iteration), and the nullary operators λ (identity) and 0. In addition there are the modal operators $< >$ and $[ ]$ by which the two sorts interact.

If the defined Boolean operator $\equiv$ is considered an equality, then L can be considered an equational language. Any PDL formula X has an equivalent equational formula $X \equiv 1$, and each equation $X \equiv Y$ is a PDL formula. Thus with no loss of generality we can assume L contains an explicit symbol $=$ for $\equiv$ and insist that all atomic formulas are equations. L then extends naturally to the first-order language $L_{\omega\omega}$ by adding propositional connectives, countably many variables ranging over Kleene elements, countably many variables ranging over Boolean elements, and quantifiers $\forall$, $\exists$ which can be applied to variables of either sort. $L_{\omega\omega}$ can be extended to the infinitary language $L_{\omega_1\omega}$ by allowing countable conjunctions and disjunctions.

The symbols $\vee$, $\wedge$, and $\neg$ will refer to both the Boolean algebra operators and the first-order logical connectives; the intent will always be clear from context.

Since well-formed expressions allow the equality symbol between Boolean elements only, there is no direct way to express identity between Kleene elements. The closest $L_{\omega\omega}$ can come to this is the functional equivalence of α and β, via the relation $\approx$ of *inseparability*:

$$\alpha \approx \beta \text{ iff } \forall X \; <\alpha>X = <\beta>X \,.$$

Thus to say that the dynamic algebra $(\mathbf{K},\mathbf{B},<>)$ is separable is the same as saying that $\mathbf{K}$ does not contain two distinct inseparable elements. The property of separability is not first-order expressible, as Lemma 3.1 below shows, but it would be if there were an equality symbol for Kleene elements.

2. A first-order sentence that distinguishes separable dynamic algebras from standard Kripke models

In this section we show that, in the absence of the *-continuity condition, there is a first-order sentence that distinguishes separable dynamic algebras from standard Kripke models. Thus, without ***-cont**, standard Kripke models and separable dynamic algebras can agree only on first-order sentences involving at most a few alternations of quantifiers. The entire construction is an implementation of the following idea: An *atom* of a Boolean algebra is a minimal nonzero element. An element X of a Boolean algebra is said to be *atomless* if there does not exist an atom $Y \leq X$. An element X is said to be *atomic* if no nonzero $Y \leq X$ is atomless, or in other words, if every nonzero $Y \leq X$ has an atom $Z \leq Y$. The properties of being an atom, atomless, or atomic are first-order expressible. We construct an dynamic algebra $(\mathbf{K},\mathbf{B},<>)$ whose Boolean algebra $\mathbf{B}$ is a subalgebra of the direct product of an atomic Boolean algebra and an atomless Boolean algebra. $\mathbf{K}$ has a program δ such that both the atomic part and the atomless part of $\mathbf{B}$ are preserved under application of $<\delta>$, but the neither part is preserved under $<\delta^*>$. The structure $(\mathbf{K},\mathbf{B},<>)$ therefore violates the first-order property "for any α, if $<\alpha>X$ is atomless whenever X is, then $<\alpha^*>X$ is atomless whenever X is." On the other hand, any standard Kripke model has this property, since $<\alpha^*>X = \bigcup_n <\alpha^n>X$, and if all elements of a family of sets are atomless, then their union is.

Now we give the explicit construction of the dynamic algebra $(\mathbf{K},\mathbf{B},<>)$. Let ω be a copy of the natural numbers and let $\mathbf{R}^+$ be a copy of the nonnegative real numbers disjoint from ω. Let S be the disjoint union $\omega \cup \mathbf{R}^+$ Points of S will be denoted x,y,... .

Let $\mathbf{B}_\omega$ be the Boolean algebra of finite and cofinite subsets of ω, and let $\mathbf{B}_{\mathbf{R}^+}$ be the Boolean algebra of subsets of $\mathbf{R}^+$ consisting of finite unions of intervals $[x,y)$ or $[x,\infty)$. Note that $\mathbf{B}_\omega$ is atomic and $\mathbf{B}_{\mathbf{R}^+}$ is atomless. The Boolean algebra $\mathbf{B}$ is the following family of subsets of S:

$$\mathbf{B} = \{\, U \cup V \mid U \in \mathbf{B}_\omega,\ V \in \mathbf{B}_{\mathbf{R}^+}, \text{ and U is bounded iff V is bounded} \,\} \,.$$

The atoms of $\mathbf{B}$ are the singleton subsets of ω. Thus if $X \in \mathbf{B}$, then X is atomic iff $X \subseteq \omega$, and X is atomless iff $X \subseteq \mathbf{R}^+$. Note that neither ω nor $\mathbf{R}^+$ is an element of $\mathbf{B}$.

Now we define a Kleene algebra $\mathbf{K}$ of binary relations on S. Let δ be the following binary relation:

$$\begin{aligned}\delta = {} & \{\, (x,y) \mid x,y \in \omega \text{ and } |y-x| \leq 1 \,\} \\ & \cup \{\, (x,y) \mid x,y \in \mathbf{R}^+ \text{ and } |y-x| \leq 1 \,\} \,.\end{aligned}$$

Note that $\delta = \delta^-$, since the definition is symmetric in x and y.

Let **K** be the set of binary relations generated by δ, the zero relation 0, the identity relation λ, and the total relation $S^2 = S \times S$ under the standard operations $\cup$ (set union), ; (relational composition), and $^-$ (reverse).

Lemma 2.1. $\mathbf{K} = \{ 0, \delta^0, \delta^1, \delta^2, \ldots, S^2 \}$.

Proof. Clearly everything on the right side of the equation is in **K**. For the reverse inclusion, since the set on the right contains the generators 0, δ, $\lambda = \delta^0$, and S^2 of **K**, it remains to show that it is closed under the operations $\cup$, ; , and $^-$. Suppose α, β are of the form 0, S^2, or δ^n. Then so are $\alpha;\beta$ and α^- (recall $\delta = \delta^-$ and therefore $\delta^n = (\delta^n)^-$). Also, $\alpha \cup \beta$ is easily seen to be of this form if either α or β is either 0 or S^2. Finally, if $\alpha = \delta^m$, $\beta = \delta^n$ for some m, n, then since δ is reflexive (i.e. $\lambda \subseteq \delta$), $\delta^m \cup \delta^n$ is either δ^m if $m \geq n$, or δ^n if $m \leq n$. □

In order to make (**K**,**B**,<>) into an dynamic algebra, we need to define the Kleene algebra operations $\cup$, ; , $^-$, and $*$ on **K** and the scalar multiplication <> on $\mathbf{K} \times \mathbf{B}$. The operations $\cup$, ; , and $^-$ will have their standard interpretations. For 0 and λ, define $\lambda^* = 0^* = \lambda$, and for any other $\alpha \in \mathbf{K}$, define α^* to be the total relation S^2. We can give <> its standard interpretation, since in light of Lemma 2.1 it is easy to see that if $X \in \mathbf{B}$ then $<\alpha>X \in \mathbf{B}$ for any $\alpha \in \mathbf{K}$.

We claim now that (**K**,**B**,<>) is a separable dynamic algebra. It is certainly separable, since it is clear from Lemma 2.1 that if $\alpha \neq \beta$ then $<\alpha>\{0\} \neq <\beta>\{0\}$. All axioms for dynamic algebras not involving * must hold, since all operators other than * have their standard interpretation. Therefore it remains to show

$$<\alpha^*>X = X \vee <\alpha><\alpha^*>X ,$$
$$<\alpha^*>X = X \vee <\alpha^*>(\neg X \wedge <\alpha>X) .$$

A simple calculation suffices for each case: If $X = 0$ then both sides of both equations are 0. If $\alpha = 0$ or λ, then $\alpha^* = \lambda$, so both sides of both equations are X. Finally, if $X \neq 0$ and $\alpha = S^2$ or $\alpha = \delta^n$, $n \geq 1$, then both sides of the first equation and the left side of the second are S, thus it remains to show that the right side of the second is S. This is true if $X = S$; if $X \neq S$, then $<\alpha>X$ is strictly larger than X, so $\neg X \wedge <\alpha>X$ is nonempty, and therefore $<\alpha^*>(\neg X \wedge <\alpha>X) = S$. We have proved

Lemma 2.2. (**K**,**B**,<>) is a separable dynamic algebra. □

Now we construct a sentence σ of $L_{\omega\omega}$ satisfied by every standard Kripke model but violated by (**K**,**B**,<>). A Kleene element α is said to *preserve atomless elements* if $<\alpha>X$ is atomless whenever X is. Define

$$\mathbf{atom}(X) = X \neq 0 \wedge \forall Y \ (0 \leq Y \leq X \rightarrow (0 = Y \vee Y = X))$$
$$\mathbf{atomless}(X) = \forall Y \leq X \ \neg\mathbf{atom}(Y)$$
$$\mathbf{pres}(\alpha) = \forall X \ \mathbf{atomless}(X) \rightarrow \mathbf{atomless}(<\alpha>X)$$
$$\sigma = \forall \alpha \ \mathbf{pres}(\alpha) \rightarrow \mathbf{pres}(\alpha^*) .$$

The formulas **atom**(X) and **atomless**(X) say that X is an atom and atomless, respectively; **pres**(α) says that α preserves atomless elements; and the sentence σ says that for any α, if α preserves atomless elements then so does α^*.

Theorem 2.3. Let $(\mathbf{K},\mathbf{B},<>)$ be the dynamic algebra constructed in Lemma 2.2. Then $(\mathbf{K},\mathbf{B},<>) \models \neg\sigma$ but $A \models \sigma$ for all standard Kripke models A.

Proof. $(\mathbf{K},\mathbf{B},<>)$ violates σ since $X \in \mathbf{B}$ is atomless iff $X \subseteq \mathbf{R}^+$, and δ preserves such sets, whereas δ^* does not, since $<\delta^*>X = S$ for any nonzero X.

On the other hand, for any standard Kripke model A, if β preserves atomless elements, then for any atomless X, $<\beta^n>X$ is atomless for all n. Since A is standard, $<\beta^*>X = \cup_n <\beta^n>X$, thus if $<\beta^*>X$ were to contain an atom Y, then Y must intersect some $<\beta^n>X$, and thus $Y \leq <\beta^n>X$ since Y is an atom, contradicting the fact that $<\beta^n>X$ is atomless. Therefore $<\beta^*>X$ must be atomless. Since X was arbitrary, β^* preserves atomless elements. □

3. The power of *-continuity

In this section we show that the class of *-continuous dynamic algebras and the class of standard Kripke models share the same $L_{\omega_1\omega}$ theory. The proof uses the Löwenheim-Skolem theorem for infinitary logic [Ke] in conjuction with results obtained in [K4].

Let $A = (\mathbf{K},\mathbf{B},<>)$ be a *-continuous dynamic algebra. Recall the definition that $\alpha \approx \beta$ iff $<\alpha>X = <\beta>X$ for all X, and that A is called *separable* if $\alpha \approx \beta$ implies $\alpha = \beta$ for any α, β. This property cannot be expressed by any infinitary sentence over the language L, as Lemma 3.1 below shows.

The relation $\approx$ is a dynamic algebra congruence. Moreover, it is easily checked that $\approx$ respects *-continuity. This allows us to construct the quotient algebra $A/\approx\ = (\mathbf{K}/\approx,\mathbf{B},<>)$, where

$$\mathbf{K}/\approx\ = \{\ \alpha/\approx\ \mid\ \alpha \in \mathbf{K}\ \}$$

and $\alpha/\approx$ is the $\approx$-class of α. Thus $A/\approx$ is a *-continuous and separable, and A is separable iff A and $A/\approx$ are isomorphic.

Lemma 3.1. A and $A/\approx$ are equivalent with respect to all $L_{\omega_1\omega}$ sentences.

Proof. Let $f{:}A \to A/\approx$ be the canonical homomorphism which takes α to $\alpha/\approx$ and X to X. We show by induction on formula structure that for any $L_{\omega_1\omega}$ formula $\phi(\alpha_1,\ldots,\alpha_k,X_1,\ldots,X_m)$ with parameters $\alpha_1,\ldots,\alpha_k \in \mathbf{K}$, $X_1,\ldots,X_m \in \mathbf{B}$,

$$\begin{aligned} &A \models \phi(\alpha_1,\ldots,\alpha_k,X_1,\ldots,X_m) \\ &\qquad \text{iff}\ \ A/\approx\ \models \phi(f(\alpha_1),\ldots,f(\alpha_k),X_1,\ldots,X_m)\ . \end{aligned}$$

If ϕ is atomic, then it is an equation between elements of $\mathbf{B}$; since $<\alpha>X = <f(\alpha)>X$ for any α and X, the two statements $\phi(\alpha_1,\ldots,\alpha_k,X_1,\ldots,X_m)$ and $\phi(f(\alpha_1),\ldots,f(\alpha_k),X_1,\ldots,X_m)$ express the same property of $\mathbf{B}$. If ϕ is a negation or a finite or countable join or meet, then the induction step is immediate. If ϕ is of the form $\exists$X

$\psi(\alpha_1,\ldots,\alpha_k,X_1,\ldots,X_m,X)$, then

$$\begin{aligned} A \models \phi \text{ iff } & A \models \psi(\alpha_1,\ldots,\alpha_k,X_1,\ldots,X_m,X) \text{ for some } X \in \mathbf{B} \\ & \text{iff (by the induction hypothesis) there is an } X \in \mathbf{B} \text{ such that} \\ & \qquad A/\approx\ \models \psi(f(\alpha_1),\ldots,f(\alpha_k),X_1,\ldots,X_m,X) \\ & \text{iff } A/\approx\ \models \exists X\ \psi(f(\alpha_1),\ldots,f(\alpha_k),X_1,\ldots,X_m,X)\ . \end{aligned}$$

Finally, suppose ϕ is of the form $\exists\alpha\ \psi(\alpha_1,\ldots,\alpha_k,\alpha,X_1,\ldots,X_m)$. Then

$$\begin{aligned} A \models \phi \text{ iff } & \text{for some } \alpha \in \mathbf{K},\ A \models \psi(\alpha_1,\ldots,\alpha_k,\alpha,X_1,\ldots,X_m) \\ & \text{iff (by induction hypothesis) for some } f(\alpha) \in \mathbf{K}/\approx, \\ & \qquad A/\approx\ \models \psi(f(\alpha_1),\ldots,f(\alpha_k),f(\alpha),X_1,\ldots,X_m) \\ & \text{iff } A/\approx\ \models \exists\alpha\ \psi(f(\alpha_1),\ldots,f(\alpha_k),\alpha,X_1,\ldots,X_m)\ .\ \square \end{aligned}$$

Lemma 3.2. Any countable separable *-continuous dynamic algebra is isomorphic to $A/\approx$ for some standard Kripke model A.

Proof. This was proved in detail in [K4, Theorem 5]. We outline the proof here for the sake of completeness, and to give an idea of the techniques involved.

Let $(\mathbf{K},\mathbf{B},<>)$ be a separable *-continuous dynamic algebra. If the construction of the representation theorem of [K1] is carried out, the result is a (possibly nonstandard) Kripke model with the same dynamic algebra $(\mathbf{K},\mathbf{B},<>)$. Elements of $\mathbf{B}$ are now subsets of a set S of states, elements of $\mathbf{K}$ are binary relations on S, and all the operations have their standard Kripke model interpretations with the possible exception of *.

In spite of the fact that $<\alpha^*>X$ need not be $\bigcup_n<\alpha^n>X$, the *-continuity condition guarantees that $<\alpha^*>X$ is the least element of $\mathbf{B}$ containing $\bigcup_n<\alpha^n>X$. In the topology on S generated by the elements of $\mathbf{B}$, this says that sets of the form $<\alpha^*>X - \bigcup_n<\alpha^n>X$ are nowhere dense. Therefore, if $\mathbf{K}$ and $\mathbf{B}$ are both countable, then the union of all such sets, call it M, is meager. The Baire Category Theorem then implies that every nonnull $X \in \mathbf{B}$ intersects $S - M$; using this fact, it can be shown [K4, Theorem 4] that all points of M can be dropped from the Kripke model without changing the dynamic algebra.

The resulting Kripke model B may still be nonstandard, for although now $<\alpha^*>X = \bigcup_n<\alpha^n>X$, it is still not necessary that α^* be the reflexive transitive closure of α. However, the elements of $\mathbf{K}$, taken as primitive, generate a standard Kripke model A, using reflexive transitive closure instead of *. Since $<\alpha^*>X = \bigcup_n<\alpha^n>X$, this process introduces no new Boolean elements. Using this and the fact that B is separable, it is then easy to show that $B \cong A/\approx$, thus A is the desired standard model. $\square$

We are now ready to prove the main theorem of this section.

Theorem 3.3. The class of standard Kripke models and the class of *-continuous dynamic algebras share the same $L_{\omega_1\omega}$ theory.

Proof. Let ϕ be any sentence of $L_{\omega_1\omega}$. We wish to show that ϕ is satisfied by some standard Kripke model iff ϕ is satisfied by some *-continuous dynamic algebra.

($\rightarrow$) This direction is trivial, since every standard Kripke model is a *-continuous dynamic algebra.

($\leftarrow$) Suppose ϕ is satisfied by some *-continuous dynamic algebra. By the downward Löwenheim-Skolem theorem for infinitary logic [Ke], ϕ is satisfied by a countable *-continuous dynamic algebra B. By Lemma 3.1, ϕ is also satisfied by the countable *-continuous dynamic algebra $B/\approx$, and $B/\approx$ is separable, thus by Lemma 3.2, $B/\approx \cong A/\approx$ for some standard Kripke model A. Again by Lemma 3.1, $A \models \phi$. $\square$

4. Equality between Kleene elements

The results of the previous section depend heavily on the fact that equality between Kleene elements cannot be expressed. Thus a natural question at this point is how the $L^=$, $L^=_{\omega\omega}$, and $L^=_{\omega_1\omega}$ theories of dynamic algebras, *-continuous dynamic algebras, and standard Kripke models relate, where $L^=$ is L is augmented with an equality symbol $=$ for Kleene elements.

Separability is expressible in $L^=_{\omega\omega}$, so the analog of Lemma 3.1 fails, since non-separable standard Kripke models exist. However, this condition can be weakened without affecting the main results of [K1-4,Pr1-3]. Let us call a Kleene algebra **K** *inherently separable* if there exists a separable dynamic algebra over **K**. We shall call a dynamic algebra (**K**,**B**,<>) *inherently separable* if its Kleene algebra **K** is. Then every standard Kripke model is inherently separable, since the Boolean algebra can be extended to the full power set. This says that inherent separability is necessary for representation by a standard Kripke model; in [K1] it was shown to be sufficient for representation by a nonstandard Kripke model. Non-inherently separable dynamic algebras have been shown to exist [K1, ex. 2.5]. A problem posed in [K1], still open, is whether every nonstandard Kripke model is inherently separable; this problem is interesting because a positive answer would say that inherent separability is necessary and sufficient for representation by a nonstandard Kripke model.

It follows from the completeness of the Segerberg axioms for PDL that the class of all dynamic algebras and the class of standard Kripke models have the same L equational theory. Pratt proved that separable dynamic algebras and standard Kripke models have the same $L^=$ equational theory [Pr1]. It is an easy observation that this theory is shared by the inherently separable dynamic algebras as well. However, as Pratt observed, the class of all dynamic algebras satisfies strictly fewer equations $\alpha = \beta$ than the class of standard Kripke models. In fact, since there is no finite equational axiomatization of the equational theory of regular events [R], it follows that even with the addition of finitely many equational axioms $\alpha = \beta$, there is always an equation true in all standard Kripke models and false in some (non-inherently separable) dynamic algebra. Thus pure equational logic, although adequate for the L theory of dynamic algebras, fails in $L^=$.

In [K1] a finite set of axiom schemata for Kleene algebras was given, all of which were equations of the form $\alpha = \beta$, except for the infinitary **-continuity condition*

$$\alpha\beta^*\gamma = \bigvee_n \alpha\beta^n\gamma \; .$$

([K1] omitted one equational axiom for the reverse operator $^-$, which we postulate here: $\alpha \leq \alpha\alpha^-\alpha$.) In contrast to the failure of pure equational logic, this simple infinitary extension completely characterizes the $L^=$ equational theory of the standard Kripke models, as Theorem 4.1 below shows. Moreover, it does so in a very natural and intuitive way, since no reference is made to the Boolean part of dynamic algebra.

Theorem 4.1. The *-continuous Kleene algebras and the algebras of binary relations satisfy the same set of equations $\alpha = \beta$.

Proof. One direction is trivial. For the other direction, let X be the set of primitive symbols and let $X^- = \{ a^- \mid a \in X \}$. Strings $x, y \in (X \cup X^-)^*$ are just terms α without $\cup$ or $*$ and with $^-$ applied only to primitive symbols. For $y \in (X \cup X^-)^*$, let $|y|$ denote the length of y, and let y^- denote the string obtained by reversing the order of the symbols in y and changing all the signs. Write $y \twoheadrightarrow x$ if x can be obtained from y via repeated application of the rule $\alpha\alpha^-\alpha \twoheadrightarrow \alpha$. For example, $ab^-c^-cbb^-c^-a \twoheadrightarrow ab^-c^-a$ in one step. For any $x \in (X \cup X^-)^*$, let M_x be the binary relation algebra consisting of $|x|+1$ states $s_0,\ldots,s_{|x|}$ and relations $(s_{i-1},s_i) \in a$ iff the i^{th} symbol of x is a, and $(s_i,s_{i-1}) \in a$ iff the i^{th} symbol of x is a^-. Certainly $(s_0,s_{|x|}) \in x$ in M_x.

We claim that the following four statements are equivalent:

(i) $x \leq y$ in all *-continuous Kleene algebras

(ii) $x \leq y$ in all binary relation algebras

(iii) $(s_0,s_{|x|}) \in y$ in M_x

(iv) $y \twoheadrightarrow x$.

The implications (i) $\twoheadrightarrow$ (ii) $\twoheadrightarrow$ (iii) are trivial. (iii) $\twoheadrightarrow$ (iv) follows from the observation that if y describes a path from s_0 to $s_{|x|}$, and if $|y| > |x|$, then there must be a zigzag in y of the form zz^-z for some substring z of x. (iv) $\twoheadrightarrow$ (i) is proved by repeated application of the Kleene algebra axiom $\alpha \leq \alpha\alpha^-\alpha$.

Let α be a Kleene term with k occurrences of *. The *-continuity condition implies that in all *-continuous Kleene algebras,

$$\alpha = \vee\, \alpha(m_1,\ldots,m_k) \ ,$$

where $\alpha(m_1,\ldots,m_k)$ denotes the *-free term obtained by replacing the i^{th} occurrence of * in α by $m_i \in \omega$, and the join is taken over all k-tuples $(m_1,\ldots,m_k) \in \omega^k$. But the Kleene algebra axioms allow any *-free term to be written as a finite join of strings in $(X \cup X^-)^*$, thus there is a countable set $I_\alpha \subseteq (X \cup X^-)^*$ such that $\alpha = \vee\, I_\alpha$ in any *-continuous Kleene algebra.

Now suppose that $\alpha = \beta$ in all binary relation algebras. Then $\vee\, I_\alpha = \vee\, I_\beta$ in all binary relation algebras, and we need only show that this implies that $\vee\, I_\alpha = \vee\, I_\beta$ in all *-continuous Kleene algebras as well. For any $x \in I_\alpha$, since $x \leq \vee\, I_\beta$ in all binary relation algebras, it certainly holds in the algebra M_x constructed above. Since $(s_0,s_{|x|}) \in x$ and since join is set union in M_x, $(s_0,s_{|x|}) \in y$ for some $y \in I_\beta$. By (iii) $\twoheadrightarrow$ (i) above, $x \leq y$ and thus $x \leq \vee\, I_\beta$ in all *-continuous Kleene algebras. Since $x \in I_\alpha$ was arbitrary, $\vee\, I_\alpha \leq \vee\, I_\beta$ in all *-continuous Kleene algebras. The reverse inequality holds by a symmetric argument. □

Without the assumption of inherent separability, *-continuity does not go much farther:

Theorem 4.2. There is a universal Horn sentence of $L^{=}_{\omega\omega}$ true in all standard Kripke models but violated in a (non-inherently separable) *-continuous dynamic algebra.

Proof. The property "if $\alpha \leq \lambda$ then $\alpha^2 = \alpha$" is clearly valid in all standard Kripke models. In [K1] an example was given of a Kleene algebra violating this property [K1, ex. 2.5]. This Kleene algebra can be made into a dynamic algebra over the two-element Boolean algebra in a straightforward way. □

Even with the assumption of inherent separability, the $L^{=}_{\omega_1\omega}$ analogy of Theorem 3.3 fails:

Theorem 4.3. There is an $L^{=}_{\omega_1\omega}$ sentence true in all standard Kripke models but false in some inherently separable *-continuous dynamic algebra.

Proof. In [K4], a countable separable *-continuous dynamic algebra (**K**,**B**,<>) was constructed such that (**K**,**B**,<>) is not isomorphic to any standard Kripke model. By Scott's theorem [Ke], there is a sentence σ of $L^{=}_{\omega_1\omega}$ that characterizes (**K**,**B**,<>) up to isomorphism on countable models, thus (**K**,**B**,<>) $\models \sigma$ but no countable standard Kripke model satisfies σ. Therefore no standard Kripke model of any cardinality can satisfy σ, since the downward Löwenheim-Skolem theorem would give a countable subalgebra satisfying σ, and such a subalgebra would still be representable as a standard Kripke model. □

Thus the question remains: for what fragments of $L^{=}_{\omega_1\omega}$ do inherently separable *-continuous dynamic algebras and standard Kripke models agree? In particular, do they agree on all sentences of $L^{=}_{\omega\omega}$?

5. Conclusion

A disadvantage of the *-continuity axiom is that, unlike the induction axiom, it is not equational, and therefore is not expressible within the language of PDL. However the emphasis on equational specifications and finitary deductive systems is in a way unrealistic. Looping is inherently infinitary and nonequational; simpler programming language constructs, such as composition and conditional tests, are captured up to isomorphism by their equations [K1], whereas looping cannot be so captured [K3,RT,K4]. Thus the equational approach must eventually be given up if we are ever to bridge the gap between algebraic and operational semantics. The *-continuity condition is an example of how to do this without sacrificing algebraic elegance.

Besides the theoretical advantage of descriptive precision, the *-continuity condition has a practical advantage as well: it is easier to use, since it is simpler in form than the PDL induction axiom. We have found that it is often easier to start a PDL proof with ***-cont**, using induction informally on the n appearing in the definition of ***-cont**, and then later massage the proof to replace applications of ***-cont** with applications of **ind**.

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