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FACTORING MULTIVARIATE POLYNOMIALS OVER
ALGEBRAIC NUMBER FIELDS

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Factoring multivariate polynomials over algebraic number fields *)

by

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ABSTRACT

We present an algorithm to factor multivariate polynomials over algebraic number fields that is polynomial-time in the degrees of the polynomial to be factored. The algorithm is an immediate generalization of the polynomial-time algorithm to factor univariate polynomials with rational coefficients.

KEY WORDS & PHRASES: *polynomial algorithm, polynomial factorization*

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*) This report will be submitted for publication elsewhere.

1. Introduction.

We show that the algorithm from [7] to factor univariate polynomials with rational coefficients can be generalized to multivariate polynomials with coefficients in an algebraic number field. As a result we get an algorithm that is polynomial-time in the degrees and the coefficient-size of the polynomial to be factored.

An outline of the algorithm is as follows. First the polynomial $f \in \mathbb{Q}(\alpha)[X_1, X_2, \ldots, X_t]$ is evaluated in a suitably chosen integer point $(X_2 = s_2, X_3 = s_3, \ldots, X_t = s_t)$. Next, for some prime number p, a p-adic irreducible factor $\bar{h}$ of the resulting polynomial $\bar{f} \in \mathbb{Q}(\alpha)[X_1]$ is determined up to a certain precision. We then show that the irreducible factor h_0 of f for which $\bar{h}$ is a p-adic factor of $\bar{h}_0$, belongs to a certain integral lattice, and that h_0 is relatively short in this lattice. This enables us to compute this factor h_0 by means of the so-called *basis reduction algorithm* (cf. [7: Section 1]).

As [7] is easily available, we do not consider it to be necessary to recall the basis reduction algorithm here; we will assume the reader to be familiar with this algorithm and its properties.

Although the algorithm presented in this paper is polynomial-time, we do not think it is a useful method for practical purposes. Like the other generalizations of the algorithm from [7], which can be found in [8;9;10; 11], the algorithm will be slow, because the basis reduction algorithm has to be applied to huge dimensional lattices with large entries. In practice, a combination of the methods from [6], [14], and [15] can be recommended (cf. [6]).

2. Preliminaries.

In this section we introduce some notation, and we derive an upper bound for the coefficients of factors of multivariate polynomials over algebraic number fields.

Let the algebraic number field $\mathbb{Q}(\alpha)$ be given as the field of rational numbers $\mathbb{Q}$ extended by a root α of a prescribed *minimal polynomial* $F \in \mathbb{Z}[T]$ with leading coefficient equal to one; i.e. $\mathbb{Q}(\alpha) \simeq \mathbb{Q}[T]/(F)$. Similarly, we define $\mathbb{Z}[\alpha] = \mathbb{Z}[T]/(F)$ as a ring of polynomials in α over $\mathbb{Z}$ of degree $< I$, where I denotes the degree δF of F.

Let $f \in \mathbb{Q}(\alpha)[X_1, X_2, \ldots, X_t]$ be the polynomial to be factored, with the number of variables $t \geq 2$. By $\delta_i f = n_i$ we denote the degree of f in X_i, for $1 \leq i \leq t$. We often use n instead of n_1. Let $\ell c_0(f) = f$. For $1 \leq i \leq t$ we define $\ell c_i(f) \in \mathbb{Q}(\alpha)[X_{i+1}, X_{i+2}, \ldots, X_t]$ as the leading coefficient with respect to X_i of $\ell c_{i-1}(f)$, and we put $\ell c(f) = \ell c_t(f)$. Finally, we define the *content* $\mathrm{cont}(f) \in \mathbb{Q}(\alpha)[X_2, X_3, \ldots, X_t]$ of f as the greatest common divisor of the coefficients of f with respect to X_1.

Without loss of generality we may assume that $2 \leq n_i \leq n_{i+1}$ for $1 \leq i < t$, that f is *monic* (i.e. $\ell c(f) = 1$), and that $\delta_i \mathrm{cont}(f) = 0$ for $2 \leq i \leq t$.

Let $d \in \mathbb{Z}_{>0}$ be such that $f \in \frac{1}{d}\mathbb{Z}[\alpha][X_1, X_2, \ldots, X_t]$, and let $\mathrm{discr}(F)$ denote the discriminant of F. It is well-known (cf. [15]) that if we take $D = d|\mathrm{discr}(F)|$, then all monic factors of f are in $\frac{1}{D}\mathbb{Z}[\alpha][X_1, X_2, \ldots, X_t]$ (in fact it is sufficient to take $D = ds$, where s is the largest integer such that s^2 divides $\mathrm{discr}(F)$, but this integer s might be too difficult to compute).

We now introduce some notation, similar to [8: Section 1]. Suppose that we are given a prime number p such that

(2.1) p does not divide D.

For $G = \Sigma_i a_i T^i \in \mathbb{Z}[T]$ we denote by G_ℓ or $G \bmod p^\ell$ the polynomial $\Sigma_i (a_i \bmod p^\ell) T^i \in (\mathbb{Z}/p^\ell \mathbb{Z})[T]$, for any positive integer ℓ. Suppose furthermore that we are given some positive integer k, and that p is chosen in such a way that a polynomial $H \in \mathbb{Z}[T]$ exists such that

(2.2) H has leading coefficient equal to one,

(2.3) H_k divides F_k in $(\mathbb{Z}/p^k \mathbb{Z})[T]$,

(2.4) H_1 is irreducible in $(\mathbb{Z}/p\mathbb{Z})[T]$,

(2.5) $(H_1)^2$ does not divide F_1 in $(\mathbb{Z}/p\mathbb{Z})[T]$.

Clearly H_1 divides F_1 in $(\mathbb{Z}/p\mathbb{Z})[T]$, and $0 < \delta H \le I$. In the sequel we will assume that conditions (2.1), (2.2), (2.3), (2.4), and (2.5) are satisfied.

By $\mathbb{F}_q$ we denote the finite field containing $q = p^{\delta H}$ elements. From (2.4) we have $\mathbb{F}_q \simeq (\mathbb{Z}/p\mathbb{Z})[T]/(H_1) \simeq \{\Sigma_{i=0}^{\delta H-1} a_i \alpha_1^i : a_i \in \mathbb{Z}/p\mathbb{Z}\}$, where $\alpha_1 = T \bmod (H_1)$ is a zero of H_1. Furthermore we put $W_k(\mathbb{F}_q) = (\mathbb{Z}/p^k\mathbb{Z})[T]/(H_k) = \{\Sigma_{i=0}^{\delta H-1} a_i \alpha_k^i : a_i \in \mathbb{Z}/p^k\mathbb{Z}\}$, where $\alpha_k = T \bmod (H_k)$ is a zero of H_k. Notice that $W_k(\mathbb{F}_q)$ is a ring containing q^k elements, and that $W_1(\mathbb{F}_q) \simeq \mathbb{F}_q$. For $a \in \mathbb{Z}[\alpha]$ we denote by $a \bmod (p^\ell, H_\ell) \in W_\ell(\mathbb{F}_q)$ the result of the canonical mapping from $\mathbb{Z}[\alpha] = \mathbb{Z}[T]/(F)$ to $W_\ell(\mathbb{F}_q) = (\mathbb{Z}/p^\ell\mathbb{Z})[T]/(H_\ell)$ applied to a, for $\ell = 1, k$. For $\tilde{g} = \Sigma_i \frac{a_i}{D} X_1^i \in \frac{1}{D}\mathbb{Z}[\alpha][X_1]$ we denote by $\tilde{g} \bmod (p^\ell, H_\ell)$ the polynomial $\Sigma_i (((D^{-1} \bmod p^\ell) a_i) \bmod (p^\ell, H_\ell)) X_1^i \in W_\ell(\mathbb{F}_q)[X_1]$ (notice that $D^{-1} \bmod p^\ell$ exists due to (2.1)).

We derive an upper bound for the height of a monic factor g of f. As usual, for $g=\sum_{i_1}\sum_{i_2}\dots\sum_{i_t}\sum_{j} a_{i_1 i_2 \dots i_t j}\alpha^j X_1^{i_1} X_2^{i_2}\dots X_t^{i_t} \in \mathbb{Q}(\alpha)[X_1, X_2, \dots, X_t]$, the *height* g_{max} is defined as $\max|a_{i_1 i_2 \dots i_t j}|$, and the *length* $|g|$ as $(\Sigma a^2_{i_1 i_2 \dots i_t j})^{\frac{1}{2}}$. Similarly, for a polynomial h with complex coefficients, we define its height h_{max} as the maximum of the absolute values of its complex coefficients.

For any choice of $\alpha \in \{\alpha_1, \alpha_2, \dots, \alpha_I\}$, where $\alpha_1, \alpha_2, \dots, \alpha_I$ are the conjugates of α, we can regard g as a polynomial g_α with complex coefficients. We define $\|g\|$ as $\max_{1\le i\le I}(g_{\alpha_i})_{max}$. From [3] we have

$$\|g\| \le e^{\Sigma_{i=1}^t n_i}\|f\|.$$

In [8: Section 4] we have shown that this leads to

$$g_{max} \le e^{\Sigma_{i=1}^t n_i}\|f\|\, I\,(I-1)^{(I-1)/2}|F|^{I-1}|\mathrm{discr}(F)|^{-\frac{1}{2}}. \tag{2.6}$$

From [13] we know that the length $|F|$ of F is an upper bound for the absolute value of the conjugates of α, so that

$$\|f\| \le f_{max}\Sigma_{i=0}^{I-1}|F|^i,$$

which yields, combined with (2.6),

$$g_{max} \le e^{\Sigma_{i=1}^t n_i} f_{max}\, I\,(I-1)^{(I-1)/2}|F|^{I-1}|\mathrm{discr}(F)|^{-\frac{1}{2}}\,\Sigma_{i=0}^{I-1}|F|^i. \tag{2.7}$$

The upper bound for the height of monic factors of f, as given by the right hand side of (2.7), will be denoted by B_f. Because $|\mathrm{discr}(F)| \ge 1$, we find

$$\log B_f = O(\Sigma_{i=1}^t n_i + \log f_{max} + I\log(I|F|)). \tag{2.8}$$

3. Factoring multivariate polynomials over algebraic number fields.

We describe an algorithm to compute the irreducible factorization of f in $\mathbb{Q}(\alpha)[X_1, X_2, \ldots, X_t]$.

Let $s_2, s_3, \ldots, s_t \in \mathbb{Z}_{>0}$ be a $(t-1)$-tuple of integers. For $g \in \mathbb{Q}(\alpha)[X_1, X_2, \ldots, X_t]$ we denote by $\tilde{g}_j$ the polynomial g modulo $((X_2-s_2), (X_3-s_3), \ldots, (X_j-s_j)) \in \mathbb{Q}(\alpha)[X_1, X_{j+1}, X_{j+2}, \ldots, X_t]$; i.e. $\tilde{g}_j$ is g with s_i substituted for X_i, for $2 \le i \le j$. Notice that $\tilde{g}_1 = g$ and that $\tilde{g}_j = \tilde{g}_{j-1}$ modulo $(X_j - s_j)$. We put $\tilde{g} = \tilde{g}_t$.

Suppose that a polynomial $\tilde{h} \in \mathbb{Z}[\alpha][X_1]$ is given such that

(3.1) $\quad\tilde{h}$ is monic,

(3.2) $\quad\tilde{h} \bmod (p^k, H_k)$ divides $\tilde{f} \bmod (p^k, H_k)$ in $W_k(\mathbb{F}_q)[X_1]$,

(3.3) $\quad\tilde{h} \bmod (p, H_1)$ is irreducible in $\mathbb{F}_q[X_1]$,

(3.4) $\quad(\tilde{h} \bmod (p, H_1))^2$ does not divide $\tilde{f} \bmod (p, H_1)$ in $\mathbb{F}_q[X_1]$.

We put $\ell = \delta_1 \tilde{h}$, so $0 < \ell \le n$. By $h_0 \in \frac{1}{D}\mathbb{Z}[\alpha][X_1, X_2, \ldots, X_t]$ we denote the unique, monic, irreducible factor of f such that $\tilde{h} \bmod (p^k, H_k)$ divides $\tilde{h}_0 \bmod (p^k, H_k)$ in $W_k(\mathbb{F}_q)[X_1]$ (cf. (3.2), (3.3), (3.4)).

(3.5) Let $m = m_1, m_2, m_3, \ldots, m_t$ be a t-tuple of integers satisfying $\ell \le m < n$ and $0 \le m_i \le \delta_i \ell c_{i-1}(f)$ for $2 \le i \le t$, and let $M = 1 + I\sum_{i=1}^{t} m_i N_{i+1}$ (where of course $N_{t+1} = 1$). We define $L \subset (\frac{\mathbb{Z}}{D})^M$ as the lattice of rank M, consisting of the polynomials $g \in \frac{1}{D}\mathbb{Z}[\alpha][X_1, X_2, \ldots, X_t]$ for which

(i) $\quad\delta_1 g \le m$ and $\delta_i g \le n_i$ for $2 \le i \le t$;

(ii) If $\delta_j \ell c_{j-1}(g) = m_j$ for $1 \le j \le i$, then $\delta_{i+1} \ell c_i(g) \le m_{i+1}$ for $1 \le i < t$;

(iii) If $\delta_i \ell c_{i-1}(g) = m_i$ for $1 \le i \le t$, then $\ell c(g) \in \mathbb{Z}$;

(iv) $\tilde{h} \bmod (p^k, H_k)$ divides $\tilde{g} \bmod (p^k, H_k)$ in $W_k(\mathbb{F}_q)[X_1]$.

Here M-dimensional vectors and polynomials satisfying conditions (i), (ii), and (iii), are identified in the usual way (cf. [8: (2.6); 11: (2.2)]). For notational convenience we only give a basis for L in the case that $m_i = n_i$ for $2 \le i \le t$; the general case can easily be derived from this:

$$\{\tfrac{1}{D} p^k \alpha^j X_1^i:\ 0 \le j < \delta H,\ \ 0 \le i < \ell\}$$

$$\cup\ \{\tfrac{1}{D} \alpha^{j-\delta H} H(\alpha) X_1^i:\ \delta H \le j < I,\ \ 0 \le i < \ell\}$$

$$\cup\ \{\tfrac{1}{D} \alpha^j \tilde{h} X_1^{i-\ell}:\ 0 \le j < I,\ \ \ell \le i \le m\}$$

$$\cup\ \{\tfrac{1}{D} \alpha^j X_1^{i_1} \Pi_{r=2}^t (X_r - s_r)^{i_r}:\ 0 \le j < I,\ \ 0 \le i_1 \le m,\ \ 0 \le i_r \le n_r \text{ for } 2 \le r \le t,\ (i_2, i_3, \ldots, i_t) \ne (0, 0, \ldots, 0), \text{ and } (i_1, i_2, i_3, \ldots, i_t) \ne (m, n_2, n_3, \ldots, n_t)\}$$

$$\cup\ \{X_1^m \Pi_{r=2}^t (X_r - s_r)^{n_r}\}$$

(cf. [8: (2.6); 11: (2.19)], (2.2), and (3.1)).

<u>(3.6) Proposition.</u> *Let*

(3.7) $$\tilde{B}_j = f_{max}^m b_{max}^n (n+m)! \left(D N_2 (1+F_{max})^{I-1} \Pi_{i=2}^j s_i^{n_i} \right)^{n+m},$$

for $1 \le j \le t$, *where* f_{max}^m *denotes* $(f_{max})^m$. *Suppose that* b *is a non-zero element of* L *such that*

(3.8) $$s_j \ge ((n+m) n_j + 1)^{\frac{1}{2}} \tilde{B}_{j-1}$$

for $2 \le j \le t$, *and*

$$(3.9) \qquad p^{k\delta H} \ge |F|^{I-1}(I^{\frac{1}{2}}\tilde{B}_t)^I.$$

Then $\gcd(f,b) \ne 1$ *in* $\mathbb{Q}(\alpha)[X_1, X_2, \ldots, X_t]$.

Proof. Denote by $R = R(Df,Db) \in \mathbb{Z}[\alpha][X_2, X_3, \ldots, X_t]$ the resultant of Df and Db (with respect to the variable X_1). An outline of the proof is as follows. First we prove that an upper bound for $(\tilde{R}_j)_{max}$ is given by $\tilde{B}_j$. Combining this with (3.8), we then see that $X_j = s_j$ cannot be a zero of $\tilde{R}_{j-1}$ if $\tilde{R}_{j-1} \ne 0$, for $2 \le j \le t$. This implies that the assumption that $R \ne 0$ (i.e. $\gcd(f,b) = 1$) leads to $\tilde{R} \ne 0$. We then apply a result from [6], and we find with (3.9) that $\tilde{R} \bmod (p^k, H_k) \ne 0$. But this is a contradiction, because $\tilde{h} \bmod (p^k, H_k)$ divides both $\tilde{f} \bmod (p^k, H_k)$ and $\tilde{b} \bmod (p^k, H_k)$ in $W_k(\mathbb{F}_q)[X_1]$. We conclude that $R = 0$, so that $\gcd(f,b) \ne 1$ in $\mathbb{Q}(\alpha)[X_1, X_2, \ldots, X_t]$.

If a and b are two polynomials in any number of variables over $\mathbb{Q}(\alpha)$, having ℓ_a and ℓ_b terms respectively, then

$$(3.10) \qquad (ab)_{max} \le a_{max}\, b_{max} \min(\ell_a, \ell_b)\, (1 + F_{max})^{I-1}.$$

From (3.10) we easily derive an upper bound for $(\tilde{R}_j)_{max}$, because $\tilde{R}_j \in \mathbb{Z}[\alpha][X_{j+1}, X_{j+2}, \ldots, X_t]$ is the resultant of $D\tilde{f}_j$ and $D\tilde{b}_j$:

$$(3.11) \qquad (\tilde{R}_j)_{max} \le (D\tilde{f}_j)^m_{max} (D\tilde{b}_j)^n_{max} (n+m)!\, N_{j+1}^{n+m-1} (1 + F_{max})^{(I-1)(n+m-1)}.$$

It follows from $\tilde{f}_j = \tilde{f}_{j-1}$ modulo $(X_j - s_j)$, that $(\tilde{f}_j)_{max} \le (\tilde{f}_{j-1})_{max}(n_j+1)s_j^{n_j}$, so that

$$(3.12) \qquad (\tilde{f}_j)_{max} \le f_{max} \prod_{i=2}^{j} (n_i+1)s_i^{n_i}.$$

Combining (3.11), (3.12), and a similar bound for $(\tilde{b}_j)_{max}$, we obtain

(3.13) $$(\tilde{R}_j)_{max} < f_{max}^m b_{max}^n (n+m)!(D N_2 \Pi_{i=2}^j s_i^{n_i})^{n+m} (1+F_{max})^{(I-1)(n+m-1)},$$

for $1 \le j < t$. (Remark that (3.13) with "<" replaced by "≤" holds for $j = t$.)

Now assume, for some j with $2 \le j \le t$, that $\tilde{R}_{j-1}$ is unequal to zero. We prove that $\tilde{R}_j \ne 0$. Because $\tilde{R}_j = \tilde{R}_{j-1}$ modulo $(X_j - s_j)$, the condition $\tilde{R}_j = 0$ would imply that all polynomials in $\mathbb{Z}[X_j]$ that result from $\tilde{R}_{j-1}$ by grouping together all terms with identical exponents in α and X_{j+1} up to X_t, have $(X_j - s_j)$ as a factor. These polynomials have degree (in X_j) at most $(n+m)n_j$, so that we get, with the result from [12], that

$$|s_j| \le ((n+m)n_j+1)^{\frac{1}{2}}(\tilde{R}_{j-1})_{max}.$$

Combined with (3.13) and (3.7) this is a contradiction with (3.8). We conclude that $\tilde{R}_j \ne 0$ if $\tilde{R}_{j-1} \ne 0$ for any j with $2 \le j \le t$, so that the assumption $\gcd(f,b) = 1$ (i.e. $R \ne 0$) leads to $\tilde{R} \ne 0$.

Assume that $H_k(T)$ divides $\tilde{R}(T) \in \mathbb{Z}[T]$ in $(\mathbb{Z}/p^k\mathbb{Z})[T]$, i.e. $\tilde{R} \bmod (p^k, H_k) = 0$. The polynomial $H_k(T)$ is also a divisor of $F(T)$ in $(\mathbb{Z}/p^k\mathbb{Z})[T]$, so that $\gcd(F(T), \tilde{R}(T)) = 1$ and [6: Theorem 2] lead to

$$p^{k\delta H} \le |F|^{I-1}(I^{\frac{1}{2}}\tilde{R}_{max})^I.$$

With the remark after (3.13) and (3.7) this is a contradiction with (3.9), so that $\tilde{R} \bmod (p^k, H_k) \ne 0$. This concludes the proof of (3.6). □

<u>(3.14) Proposition</u>. *Let* $b_1, b_2, \ldots, b_M$ *be a reduced basis for* L *(cf.* [7: Section 1]*), where* L *and* M *are as in* (3.5)*, and let*

$$(3.15)\qquad B_j = (n+m)!\,(M2^{M-1})^{n/2}\left(B_f\, D\, N_2 (1+F_{max})^{I-1} \Pi_{i=2}^{j} s_i^{n_i}\right)^{n+m},$$

for $2 \le j \le t$, *where* B_f *is as in Section 2. Suppose that*

$$(3.16)\qquad s_j \ge ((n+m)n_j+1)^{\frac{1}{2}} B_{j-1}$$

for $2 \le j \le t$, *that*

$$(3.17)\qquad p^{k\delta H} \ge |F|^{I-1} (I^{\frac{1}{2}} B_t)^I,$$

and that f *does not contain multiple factors. Then*

$$(3.18)\qquad (b_1)_{max} \le (M2^{M-1})^{\frac{1}{2}} B_f$$

and h_0 *divides* b_1, *if and only if* $h_0 \in L$.

Proof. If h_0 divides b_1, then $h_0 \in L$, because $b_1 \in L$; this proves the "if"-part.

To prove the "only if"-part, suppose that $h_0 \in L$. Because h_0 is a monic factor of f, we have from (2.7) that $(h_0)_{max} \le B_f$. With [7: (1.11)] and $h_0 \in L$ this gives $|b_1| \le (M2^{M-1})^{\frac{1}{2}} B_f$ so that (3.18) holds, because $(b_1)_{max} \le |b_1|$. Because of (3.18), (3.16), (3.17), (3.15), and the definition of B_f, we can apply (3.6), which yields $\gcd(f,b_1) \ne 1$.

Now suppose that h_0 does not divide b_1. This implies that h_0 also does not divide $r = \gcd(f,b_1)$, where r can be assumed to be monic. But then $\tilde{h} \bmod (p^k, H_k)$ divides $(\tilde{f}/\tilde{r}) \bmod (p^k, H_k)$, so that Proposition (3.6) can be applied with f replaced by f/r. Conditions (3.8) and (3.9) are satisfied because $(f/r)_{max} \le B_f$ (cf. (2.7)) and because of (3.16), (3.17), and (3.15). It follows that $\gcd(f/r,b_1) \ne 1$, which contradicts $r = \gcd(f,b_1)$ because f does not contain multiple factors. □

(3.19) We describe how to compute the irreducible factor h_0 of f. Suppose that f does not contain multiple factors, and that the polynomial $\tilde{h}$, the $(t-1)$-tuple $s_2, s_3, \ldots, s_t$, and the prime power p^k are chosen such that (3.1), (3.2), (3.3), (3.4), (3.16), and (3.17) are satisfied with, for (3.16) and (3.17), m replaced by $n-1$. Remember that we also have to take care that conditions (2.1), (2.2), (2.3), (2.4), and (2.5) on p and H are satisfied.

We apply the basis reduction algorithm (cf. [7: Section 1]) to a sequence of M_j-dimensional lattices as in (3.5), where the $M_j = 1 + I\sum_{i=1}^{t} m_i N_{i+1}$ run through the range of admissible values for $m_1, m_2, \ldots, m_t$ (cf. (3.5)), in such a way that $M_j < M_{j+1}$. (So, for $m = \ell, \ell+1, \ldots, n-1$, and $m_i = 0, 1, \ldots, \delta_i \ell c_{i-1}(f)$ for $i = t, t-1, \ldots, 2$ in succession.) According to (3.14), the first vector b_1 that we find that satisfies (3.18) equals $\pm h_0$ (remember that b_1 belongs to a basis for the lattice), so that we can stop if such a vector is found. If for none of the lattices a vector satisfying (3.18) is found, then h_0 is not contained in any of these lattices according to (3.14), so that $h_0 = f$.

<u>(3.20) Proposition</u>. *Assume that the conditions in* (3.19) *are satisfied. The polynomial* h_0 *can be computed in* $O((\delta_1 h_0 I N_2)^4 k \log p)$ *arithmetic operations on integers having binary length* $O(I N k \log p)$.

<u>Proof</u>. Observing that $\log(I N p^{2k}) = O(k \log p)$ (cf. (3.17), (3.15), and (2.8)), the proof immediately follows from (3.19), (3.5), and [7: (1.26), (1.37)]. □

(3.21) We now show how $s_2, s_3, \ldots, s_t$ and p can be chosen in such a way that the conditions in (3.19) can be satisfied. The algorithm to factor f then easily follows by repeated application of (3.19).

We assume that f does not contain multiple factors, so that the resultant $R = R(df, df')$ of df and its derivative df' with respect to X_1 is unequal to zero. First we choose $s_2, s_3, \ldots, s_t \in \mathbb{Z}_{>0}$ minimal such that (3.16) is satisfied with m replaced by $n-1$. It follows from (3.16), (3.15), (2.8), and $\log D = O(\log d + I\log(I|F|))$ (because $D = d|\mathrm{discr}(F)|$), that

$$\log s_j = O(\log((n+m)n_j) + \log B_{j-1})$$

$$= O(I\,n\,N + n(\log B_f + \log D + I\log(1+F_{max}) + \Sigma_{i=1}^{j-1} n_i \log s_i))$$

$$= O(n(I\,N + \log(df_{max}) + I\log(I|F|) + \Sigma_{i=1}^{j-1} n_i \log s_i))$$

for $2 \le j \le t$, so that

$$\log s_j = O(n(I\,N + \log(df_{max}) + I\log(I|F|))\,\Pi_{i=2}^{j-1}(1+n\,n_i))$$

and

$$\Sigma_{i=2}^{t} n_i \log s_i = O(n^{t-2} N(I\,N + \log(df_{max}) + I\log(I|F|))). \tag{3.22}$$

From the proof of (3.6) it follows that, for this choice of $s_2, s_3, \ldots, s_t$ the resultant $\tilde{R} \in \mathbb{Z}[\alpha]$ of $d\tilde{f}$ and $d\tilde{f}'$ is unequal to zero.

Next we choose p minimal such that p does not divide D or $\mathrm{discr}(F)$, and such that $\tilde{R} \not\equiv 0$ modulo p. Clearly

$$\Pi_{q \text{ prime},\, q<p}\, q \le d\,\mathrm{discr}(F)\,\tilde{R}_{max}$$

which yields, together with

$$\Pi_{q \text{ prime},\, q<p}\, q > e^{Ap}$$

for all $p > 2$ and some constant $A > 0$ (cf. [4: Section 22.2]), that

(3.23) $$p = O(\log d + I \log(I|F|) + \log \tilde{R}_{max}).$$

Similar to (3.13) we obtain

$$\tilde{R}_{max} \le f_{max}^{2n-1} n^n (2n-1)!(d N_2 \Pi_{i=2}^{t} s_i^{n_i})^{2n-1} (1 + F_{max})^{(I-1)(2n-2)},$$

so that we get, using (3.22)

$$\log \tilde{R}_{max} = O(n^{t-1} N(I N + \log(df_{max}) + I \log(I|F|))).$$

Combining this with (3.23) we conclude that

(3.24) $$p = O(n^{t-1} N(I N + \log(df_{max}) + I \log(I|F|))).$$

Notice that (2.1) is now satisfied. In order to compute a polynomial $H \in \mathbb{Z}[T]$ satisfying (2.2), (2.4), (2.5), and (2.3) with k replaced by 1, we factor $F \bmod p$ by means of Berlekamp's algorithm [5: Section 4.6.2] and we choose H as an irreducible factor of $F \bmod p$ for which $\tilde{R} \bmod (p,H_1) \neq 0$; such a polynomial H exists because $\tilde{R} \bmod p \neq 0$. Conditions (2.4) and (2.3) with k replaced by 1 are clear from the construction of H, and because we may assume that H has leading coefficient equal to one, (2.2) also holds. The condition that $\operatorname{discr}(F) \bmod p \neq 0$, finally, guarantees that $F \bmod p$ does not contain multiple factors, so that (2.5) is satisfied.

We choose k minimal such that (3.17) holds, so that

$$k \log p = O(I(I n N + n \log(df_{max}) + I n \log(I|F|) + n \Sigma_{i=2}^{t} n_i \log s_i) + \log p)$$

(cf. (3.15) and (2.8)), which gives, with (3.22) and (3.24)

(3.25) $$k \log p = O(I n^{t-1} N(I N + \log(df_{max}) + I \log(I|F|))).$$

Now we apply Hensel's lemma [5: Exercise 4.6.22] to modify H in such a way that (2.3) holds for this value of k (this is possible because (2.3) already holds for $k=1$), and finally we apply Berlekamp's algorithm as described in [1: Section 5] and Hensel's lemma as in [14] to compute the irreducible factorization of $\tilde{f} \bmod (p^k, H_k)$ in $W_k(\mathbb{F}_q)[X_1]$. Condition (3.4) is satisfied for each irreducible factor $\tilde{h} \bmod (p^k, H_k)$ of $\tilde{f} \bmod (p^k, H_k)$ because $\tilde{R} \bmod (p, H_1) \neq 0$, and (3.1), (3.2), and (3.3) are clear from the construction of $\tilde{h}$.

We have shown how to choose $s_2, s_3, \ldots, s_t$ and p, and how to satisfy the conditions in (3.19). We are now ready for our theorem.

(3.26) Theorem. *Let* f *be a monic polynomial in* $\frac{1}{d}\mathbb{Z}[\alpha][X_1, X_2, \ldots, X_t]$ *with* $t \geq 2$, *of degree* n_i *in* X_i, *and* $2 \leq n = n_1 \leq n_2 \leq \ldots \leq n_t$. *The irreducible factorization of* f *can be found in* $O(n^{t-1}(IN)^5(IN + \log(df_{max}) + I\log(I|F|)))$ *arithmetic operations on integers having binary length* $O(n^{t-1}(IN)^2(IN + \log(df_{max}) + I\log(I|F|)))$, *where* $N = \prod_{i=1}^{t}(n_i+1)$.

Proof. If f does not contain multiple factors, then f can be factored by repeated application of (3.19). In that case (3.26) follows from (3.21). (3.20), (3.25), and the well-known estimates for the applications of Berlekamp's algorithm and Hensel's lemma (cf.[5;1] and [16]).

If f contains multiple factors, then we first have to compute the monic gcd g of f and its derivative with respect to X_1, and the factoring algorithm is then applied to f/g. The cost of factoring f/g satisfies the same estimates as above, because $(f/g)_{max} \leq B_f$ (cf. (2.7)), and this dominates the costs of the computation of g, which can be done by means of the subresultant algorithm (cf. [2]). □

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