Synthesis from Knowledge-Based Specifications^{*}

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Abstract. In program synthesis, we transform a specification into a program that is guaranteed to satisfy the specification. In synthesis of reactive systems, the environment in which the program operates may behave nondeterministically, e.g., by generating different sequences of inputs in different runs of the system. To satisfy the specification, the program needs to act so that the specification holds in every computation generated by its interaction with the environment. Often, the program cannot observe all attributes of its environment. In this case, we should transform a specification into a program whose behavior depends only on the observable history of the computation. This is called synthesis with incomplete information. In such a setting, it is desirable to have a knowledge-based specification, which can refer to the uncertainty the program has about the environment's behavior. In this work we solve the problem of synthesis with incomplete information with respect to specifications in the logic of knowledge and time. We show that the problem has the same worst-case complexity as synthesis with complete information.

1 Introduction

One of the most significant developments in the area of design verification is the development of of algorithmic methods for verifying temporal specifications of *finite-state* designs [11,29,45,53]. The significance of this follows from the fact that a considerable

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number of the communication and synchronization protocols studied in the literature are in essence finite-state programs or can be abstracted as finite-state programs. A frequent criticism against this approach, however, is that verification is done *after* substantial resources have already been invested in the development of the design. Since designs invariably contain errors, verification simply becomes part of the debugging process. The critics argue that the desired goal is to use the specification in the design development process in order to guarantee the design of correct programs. This is called *program synthesis*.

The classical approach to program synthesis is to extract a program from a proof that the specification is satisfiable. For reactive programs, the specification is typically a temporal formula describing the allowable behaviors of the program [30]. Emerson and Clarke [14] and Manna and Wolper [31] showed how to extract programs from (finite representations of) models of the formula. In the late 1980s, several researchers realized that the classical approach is well suited to *closed* systems, but not to *open* systems [12,42,1]. In open systems the program interacts with the environment. A correct program should be able to handle arbitrary actions of the environment. If one applies the techniques of [14,31] to open systems, one obtains programs that can handle only some actions of the environment.

Pnueli and Rosner [42], Abadi, Lamport and Wolper [1], and Dill [12] argued that the right way to approach synthesis of open systems is to consider the situation as a (possibly infinite) game between the environment and the program. A correct program can then be viewed as a winning strategy in this game. It turns out that satisfiability of the specification is not sufficient to guarantee the existence of such a strategy. Abadi et al. called specifications for which winning strategies exist realizable. A winning strategy can be viewed as an infinite tree. In those papers it is shown how the specification can be transformed into a tree automaton such that a program is realizable precisely when this tree automaton is nonempty, i.e., it accepts some infinite tree. This yields a decision procedure for realizability. (This is closely related to the approach taken by Büchi and Landweber [8] and Rabin [46] to solve Church's solvability problem [10].)

The works discussed so far deal with situations in which the program has complete information about the actions taken by the environment. This is called synthesis with complete information. Often, the program does not have complete information about its environment. Thus, the actions of the program can depend only on the "visible" part of the computation. Synthesizing such programs is called synthesis with incomplete information. The difficulty of synthesis with incomplete information follows from the fact that while in the complete-information case the strategy tree and the computation tree coincide, this is no longer the case when we have incomplete information. Algorithms for synthesis were extended to handle incomplete information in [43,54,3,24,51,25].

It is important to note that temporal logic specifications cannot refer to the uncertainty of the program about the environment, since the logic has no construct for referring to such uncertainty. It has been observed, however, that designers of open systems often reason explicitly in terms of uncertainty [19]. A typical example is a rule of the form "send an acknowledgement as soon as you know that the message has been received". For this reason, it has been proposed in [21] to use epistemic logic as a specification language for open systems with incomplete information. When dealing with ongoing behavior in systems with incomplete information, a combination of temporal and epistemic logic can refer to both behavior and uncertainty [28,27]. In such a logic the above rule can be formalized by the formula $\Box(Kreceived \to ack)$, where \Box is the temporal connective "always", K is the epistemic modality indicating knowledge, and received and ack are atomic propositions.

Reasoning about open systems at the *knowledge level* allows us to abstract away from many concrete details of the systems we are considering. It is often more intuitive to think in terms of the high-level concepts when we design a protocol, and then translate these intuitions into a concrete program, based on the particular properties of the setting we are considering. This style of program development will generally allow us to modify the program more easily when considering a setting with different properties, such as different communication topologies, different guarantees about the reliability of various components of the system, and the like. See [2,6,9,13,18,22,23,36,39,40,47] for examples of knowledge-level analysis of open systems with incomplete information. To be able to translate, however, these high-level intuitions into a concrete program one has to be able to check that the given specification is realizable in the sense described above.

Our goal in this paper is to extend the program synthesis framework to temporal-epistemic specification. The difficulty that we face is that all previous program-synthesis algorithms attempt to construct strategy trees that realize the given specification. Such trees, however, refer to temporal behavior only and they do not contain enough information to interpret the epistemic constructs. (We note that this difficulty is different than the difficulty faced when one attempts to extend synthesis with incomplete information to branching-time specification [25], and the solution described there cannot be applied to knowledge-based specifications.) Our key technical tool is the definition of finitely labelled trees that contain information about both temporal behavior and epistemic uncertainty. Our main result is that we can extend the program synthesis framework to handle knowledge-based specification with no increase in worst-case computational complexity.

In an earlier, extended abstract of the present work [32] we stated this result for specifications in the logic of knowledge and *linear* time, and required the protocols synthesized to be deterministic. The present paper differs from the earlier work in giving full proofs of all results, as well as in the fact that we generalize the specification language to encompass branching as well as linear time logical operators. We also liberalize the class of solutions to encompass nondeterministic protocols. (Our previous result on deterministic protocols is easily recovered, by noting that the branching time specification language can express determinism of the solutions.) These generalizations allow us to give an application of the results to the synthesis of implementations of knowledge-based programs [16], a type of programs in an agent's actions may depend in its knowledge.

The structure of the paper is as follows. Section 2 defines the syntax and semantics of the temporal-epistemic specification language and defines the synthesis problem for this language. In Section 3 we give a characterization of realizability that forms the basis for our synthesis result. Section 4 describes an automaton-theoretic algorithm for deciding whether a specification is realizable, and for extracting a solution in case it is. Section 5 discusses two aspects of this result: a subtlety concerning the knowledge encoded in the states of the solutions, and an application of our result to knowledge-based program implementation. The paper concludes in Section 6 with a discussion of extensions and open problems.

2 Definitions

In this section we define the formal framework within which we will study the problem of synthesis from knowledge-based specifications, provide semantics for the logic of knowledge and time in this framework, and define the realizability problem.

Systems will be decomposed in our framework into two components: the program, or protocol being run, and the remainder of the system, which we call the environment within which this protocol operates. We begin by presenting a model, from [34], for the environment. This model is an adaption of the notion of context of Fagin et al. [16]. Our main result in this paper is restricted to the case of a single agent, but as we will state a result in Section 5.1 that applies in a more general setting, we define the model assuming a finite number of agents.

Intuitively, we model the environment as a finite-state transition system, with the transitions labelled by the agents' actions. For each agent $i=1\ldots n$ let ACT_i be a set of actions associated with agent i. We will also consider the environment as able to perform actions, so assume additionally a set ACT_e of actions for the environment. A joint action will consist of an action for each agent and an action for the environment, i.e., the set of joint actions is the cartesian product $ACT = ACT_e \times ACT_1 \times ... \times ACT_n$.

Suppose we are given such a set of actions, together with a set of *Prop* of atomic propositions. Define a *finite interpreted environment for n agents* to be a tuple E of the form $\langle S_e, I_e, P_e, \tau, O_1, \ldots, O_n, \pi_e \rangle$ where the components are as follows:

- 1. S_e is a finite set of states of the environment. Intuitively, states of the environment may encode such information as messages in transit, failure of components, etc. and possibly the values of certain local variables maintained by the agents.
- 2. I_e is a subset of S_e , representing the possible *initial states* of the environment.
- 3. $P_e: S_e \to \mathcal{P}(ACT_e)$ is a function, called the protocol of the environment, mapping states to subsets of the set ACT_e of actions performable by the environment. Intuitively, $P_e(s)$ represents the set of actions that may be performed by the environment when the system is in state s. We assume that this set is nonempty for all $s \in S_e$.
- 4. τ is a function mapping joint actions $\mathbf{a} \in ACT$ to state transition functions $\tau(\mathbf{a})$: $S_e \to S_e$. Intuitively, when the joint action \mathbf{a} is performed in the state s, the resulting state of the environment is $\tau(\mathbf{a})(s)$.
- 5. For each $i = 1 \dots n$, the component O_i is a function, called the *observation function* of agent i, mapping the set of states S_e to some set \mathcal{O} . If s is a global state then $O_i(s)$ will be called the *observation* of agent i in the state s.
- 6. $\pi_e: S_e \to \{0,1\}^{Prop}$ is an *interpretation*, mapping each state to an assignment of truth values to the atomic propositions in Prop.

A run r of an environment E is an infinite sequence s_0, s_1, \ldots of states such that $s_0 \in I_e$ and for all $m \geq 0$ there exists a joint action $\mathbf{a} = \langle a_e, a_1, \ldots, a_n \rangle$ such that $s_{m+1} = \tau(\mathbf{a})(s_m)$ and $a_e \in P_e(s_m)$. For $m \geq 0$ we write r(m) for s_m . For $k \leq m$ we also write r[k..m] for the sequence $s_k \ldots s_m$ and r[m..] for $s_m s_{m+1} \ldots$

A point is a tuple (r, m), where r is a run and m a natural number. Intuitively, a point identifies a particular instant of time along the history described by the run. A run r' will be said to be a run through a point (r, m) if r[0..m] = r'[0..m]. Intuitively, this is the case when the two runs r and r' describe the same sequence of events up to time m.

Runs of an environment provide sufficient structure for the interpretation of formulae of linear temporal logic. To interpret formulae involving knowledge, we need

additional structure. Knowledge arises not from a single run, but from the position a run occupies within the collection of all possible runs of the system under study. Following [16], define a system to be a set \mathcal{R} of runs and an interpreted system to be a tuple $\mathcal{I} = (\mathcal{R}, \pi)$ consisting of a system \mathcal{R} together with an interpretation function π mapping the points of runs in \mathcal{R} to assignments of truth value to the propositions in Prop. As we will show below, interpreted systems also provide enough structure to interpret branching time logics [11], by means of a slight modification of the usual semantics for such logics.

All the interpreted systems we deal with in this paper will have all runs drawn from the same environment, and the interpretation π derived from the interpretation of the environment by means of the equation $\pi(r,m)(p) = \pi_e(r(m))(p)$, where (r,m) is a point and p an atomic proposition. That is, the value of a proposition at a point of a run is determined from the state of the environment at that point, as described by the environment generating the run.

The definition of run presented above is a slight modification of the definitions of Fagin et al. [16]. Roughly corresponding to our notion of state of the environment is their notion of a *global state*, which has additional structure. Specifically, a global state identifies a *local state* for each agent, which plays a crucial role in the semantics of knowledge. We have avoided the use of such extra structure in our states because we focus on just one particular definition of local states that may be represented in the general framework of [16].

In particular, we will work with respect to a synchronous perfect-recall semantics of knowledge. Given a run $r = s_0, s_1 \dots$ of an environment with observation functions O_i , we define the local state of agent i at time $m \geq 0$ to be the sequence $r_i(m) = O_i(s_0) \dots O_i(s_m)$. That is, the local state of an agent at a point in a run consists of a complete record of the observations the agent has made up to that point.

These local states may be used to define for each agent i a relation \sim_i of indistinguishability on points, by $(r,m) \sim_i (r',m')$ if $r_i(m) = r_i'(m')$. Intuititively, when $(r,m) \sim_i (r',m')$, agent i has failed to receive enough information to time m in run r and time m' in run r' to determine whether it is on one situation or the other. Clearly, each \sim_i is an equivalence relation. The use of the term "synchronous" above is due to the fact that an agent is able to determine the time simply by counting the number of observations in its local state. This is reflected in the fact that if $(r,m) \sim_i (r',m')$, we must have m=m'. (There also exists an asynchronous version of perfect recall [16], which will not concern us in the present paper.)

To specify systems, we will use a propositional multimodal language for knowledge and time based on a set Prop of atomic propositions, with formulae generated by the modalities \bigcirc (next time), U (until), and a knowledge operator K_i for each agent $i=1\ldots n$. Time may be either branching or linear, so we also consider the branching time quantifier \exists . More precisely, the set of formulae of the language is defined as follows: each atomic proposition $p \in Prop$ is a formula, and if φ and ψ are formulae, then so are $\neg \varphi$, $\varphi \wedge \psi$, $\bigcirc \varphi$, $\varphi U \psi$, $\exists \varphi$ and $K_i \varphi$ for each $i=1\ldots n$. As usual, we use the abbrevations $\Diamond \varphi$ for $\mathbf{true} U \varphi$, and $\Box \varphi$ for $\neg \Diamond \neg \varphi$.

The semantics of this language is defined as follows. Suppose we are given an interpreted system $\mathcal{I} = (\mathcal{R}, \pi)$, where \mathcal{R} is a set of runs of environment E and π is determined from the environment as described above. We define satisfaction of a formula φ at a point (r, m) of a run in \mathcal{R} , denoted $\mathcal{I}, (r, m) \models \varphi$, inductively on the structure of φ . The cases for the temporal fragment of the language are standard:

1. $\mathcal{I}_{r}(r,m) \models p$, where p is an atomic proposition, if $\pi(r,m)(p) = 1$,

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2. \mathcal{I}, (r, m) \models \varphi_1 \land \varphi_2, if \mathcal{I}, (r, m) \models \varphi_1 and \mathcal{I}, (r, m) \models \varphi_2,
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- 3. $\mathcal{I}, (r, m) \models \neg \varphi$, if not $\mathcal{I}, (r, m) \models \varphi$,
- 4. $\mathcal{I}, (r, m) \models \bigcirc \varphi$, if $\mathcal{I}, (r, m + 1) \models \varphi$,
- 5. $\mathcal{I}, (r, m) \models \varphi_1 U \varphi_2$, if there exists $k \geq m$ such that $\mathcal{I}, (r, k) \models \varphi_2$ and $\mathcal{I}, (r, l) \models \varphi_1$ for all l with m < l < k.
- 6. $\mathcal{I}, (r, m) \models \exists \varphi$ if there exists a run r' in \mathcal{R} through (r, m) such that $\mathcal{I}, (r', m) \models \varphi$.

The semantics of the knowledge operators is defined by

7.
$$\mathcal{I}, (r,m) \models K_i \varphi$$
, if $\mathcal{I}, (r',m') \models \varphi$ for all points (r',m') of \mathcal{I} satisfying $(r',m') \sim_i (r,m)$

That is, an agent knows a formula to be true if this formula holds at all points that it is unable to distinguish from the actual point. This definition follows the general framework for the semantics of knowledge proposed by Halpern and Moses [21]. We use the particular equivalence relations \sim_i obtained from the assumption of synchronous perfect recall, but the same semantics for knowledge applies for other ways of defining local states, and hence the relations \sim_i . We refer the reader to [21,16] for further background on this topic.

The systems \mathcal{I} we will be interested in will not have completely arbitrary sets of runs, but rather will have sets of runs that arise from the agents running some program, or protocol, within a given environment. Intuitively, an agent's choice of actions in such a program should depend on the information it has been able to obtain about the environment, but no more. We have used observations to model the agent's source of information about the environment. The maximum information that an agent has about the environment at a point (r,m) is given by the local state $r_i(m)$. Thus, it is natural to model an agent's program as assigning to each local state of the agent a nonempty set of actions for that agent. We define a protocol for agent i to be a function $P_i: \mathcal{O}^+ \to \mathcal{P}(ACT_i) \setminus \{\emptyset\}$. A joint protocol \mathbf{P} is a tuple $\langle P_1, \ldots, P_n \rangle$, where each P_i is a protocol for agent i. We say that \mathbf{P} is deterministic if $\mathcal{P}_i(v)$ is a singleton for all agents i and local states v.

The systems we consider will consist of all the runs in which at each point of time each agent behaves as required by its protocol. As usual, we also require that the environment follows its own protocol. Formally, the system generated by a joint protocol \mathbf{P} in environment E is the set $\mathcal{R}(\mathbf{P}, E)$ of all runs r of E such that for all $m \geq 0$ we have $r(m+1) = \tau(\mathbf{a})(r(m))$, where \mathbf{a} is a joint action in $P_e(r(m)) \times P_1(r_1(m)) \times \ldots \times P_n(r_n(m))$. The interpreted system generated by a joint protocol \mathbf{P} in environment E is the interpreted system $\mathcal{I}(\mathbf{P}, E) = (\mathcal{R}(\mathbf{P}, E), \pi)$, where π is the interpretation derived from the environment E as described above.

Finally, we may define the relation between specifications and implementations that is our main topic of study. We say that a joint protocol \mathbf{P} realizes a specification φ in an environment E if for all runs r of $\mathcal{I}(\mathbf{P},E)$ we have $\mathcal{I}(\mathbf{P},E), (r,0) \models \varphi$. A specification φ is realizable in environment E if there exists a joint protocol \mathbf{P} that realizes φ in E. The following example illustrates the framework and provides examples of realizable and unrealizable formulae.

Example 1. Consider a timed toggle switch with two positions (on, off), with a light intended to indicate the position. If the light is on, then the switch must be in the on position. However, the light is faulty, so it might be off when the switch is on. Suppose that there is a single agent that has two actions: "toggle" and "do nothing". If the agent toggles, the switch changes position. If the agent does nothing, the toggle either

stays in the same position or, if it is on, may timeout and switch to off automatically. The timer is unreliable, so the timeout may happen any time the switch is on, or never, even if the switch remains on forever. The agent observes only the light, not the toggle position.

This system may be represented as an environment with states consisting of pairs $\langle t, l \rangle$, where t is a boolean variable indicating the toggle position and l is a boolean variable representing the light, subject to the constraint that l=0 if t=0. The agent's observation function is given by $O_1(\langle t, l \rangle) = l$. To represent the effect of the agent's actions on the state, write T for the toggle action and Λ for the agent's null action. The environment's actions may be taken to be pairs (u,v) where u and v are boolean variables indicating, respectively, that the environment times out the toggle, and that it switches the light on (provided the switch is on). Thus, the transition function is given by $\tau(\langle (u,v),a_1\rangle)(\langle t,l\rangle)=\langle t',l'\rangle$ where (i) $t'=\overline{t}$ if either $a_1=T$ or t=u=1, else t'=t, and (ii) l'=1 iff t'=1 and v=1.

If "toggle-on" is the proposition true in states $\langle t, l \rangle$ where t = 1, then the formula $\Box(K_1 \text{toggle-on}) \lor K_1 \neg \text{toggle-on})$ expresses that the agent knows at all times whether or not the toggle is on. This formula is realizable when the initial states of the environment are those in which the toggle is on (and the light is either on or off). The protocol by which the agent realizes this formula is that in which it performs T at all steps. Since it has perfect recall it can determine whether the toggle is on or off by checking if it has made (respectively) an odd or an even number of observations.

However, the same formula is not realizable if all states are initial. In this case, if the light is off at time 0, the agent cannot know whether the switch is on. As it has had at time 0 no opportunity to influence the state of the environment through its actions, this is the case whatever the agent's protocol. \Box

3 A Characterization of Realizability

In this section we characterize realizability in environments for a single agent in terms of the existence of a certain type of labelled tree. Intuitively, the nodes of this tree correspond to the local states of the agent, and the label at a node is intended to express (i) the relevant knowledge of the agent and (ii) the action the agent performs when in the corresponding local state.

Consider \mathcal{O}^* , the set of all finite sequences of observations of agent 1, including the empty sequence. This set may be viewed as an infinite tree, where the root is the null sequence and the successors of a vertex $v \in \mathcal{O}^*$ are the vertices $v \cdot o$, where $o \in \mathcal{O}$ is an observation. A labelling of \mathcal{O}^* is a function $\mathcal{T}: \mathcal{O}^* \to L$ for some set L. We call \mathcal{T} a labelled tree. We will work with trees in which the labels are constructed from the states of the environment, a formula ψ and the actions of the agent. Define an atom for a formula ψ to be a mapping X from the set of all subformulae of ψ to $\{0,1\}$. A knowledge set for ψ in E is a set of pairs of the form (X,s), where X is an atom of ψ and s is a state of E. Take $L_{\psi,E}$ to be the set of all pairs of the form (K,B) where K is a knowledge set for ψ in E and E is an anomempty set of actions of agent 1. We will consider trees that are labellings of \mathcal{O}^* by $L_{\psi,E}$. We will call such a tree a labelled tree for ψ and E.

Given such a labelled tree \mathcal{T} , we may define the functions K, mapping \mathcal{O}^* to knowledge sets, and P, mapping \mathcal{O}^* to nonempty sets of actions of agent 1, such that for all $v \in \mathcal{O}^*$ we have $\mathcal{T}(v) = (K(v), P(v))$. Note that P is a protocol for agent 1. This protocol generates an interpreted system $\mathcal{I}(P, E)$ in the given environment E.

Intuitively, we are interested in trees in which the K(v) describe the states of knowledge of the agent in this system. We now set about stating some constraints on the labels in the tree \mathcal{T} that are intended to ensure this is the case.

Suppose we are given a sequence of states $r = s_0 s_1 \dots$ and a vertex v of \mathcal{T} with $v = w \cdot O_1(s_0)$ for some w. Then we obtain a branch $v_0 v_1 \dots$ of \mathcal{T} , where $v_0 = v$ and $v_m = v_{m-1} \cdot O_1(s_m)$ for m > 0. We say that r is a run of \mathcal{T} from v if there exists an atom X such that $(X, s_0) \in K(v)$, and for each $m \geq 0$ there exists a joint action $\mathbf{a} \in P_e(s_m) \times P(v_m)$ such that $s_{m+1} = \tau(\mathbf{a})(s_m)$. That is, the actions of agent 1 labelling the branch corresponding to r, together with some choice of the environment's actions, generate the sequence of states in the run.

We now define a relation \models^* on points of the runs from vertices of \mathcal{T} . This relation interprets subformulae of ψ by treating the linear temporal operators as usual, but referring to the knowledge sets to interpret formulae involving knowledge or the branching time operator. Intuitively, $\mathcal{T}, v, (r, m) \models^* \varphi$ asserts that the formula φ "holds" at the mth vertex v_m reached from v along r, as described above. More formally, this relation is defined by means of the following recursion:

- 1. $\mathcal{T}, v, (r, m) \models^* p \text{ if } \pi_e(r(m), p) = 1$
- 2. $\mathcal{T}, v, (r, m) \models^* \neg \varphi \text{ if not } \mathcal{T}, v, (r, m) \models^* \varphi.$
- 3. $\mathcal{T}, v, (r, m) \models^* \varphi_1 \wedge \varphi_2 \text{ if } \mathcal{T}, v, (r, m) \models^* \varphi_1 \text{ and } \mathcal{T}, v, (r, m) \models^* \varphi_2.$
- 4. $\mathcal{T}, v, (r, m) \models^* \bigcirc \varphi \text{ if } \mathcal{T}, v, (r, m+1) \models^* \varphi$
- 5. $\mathcal{T}, v, (r, m) \models^* \varphi_1 U \varphi_2$ if there exists $k \geq m$ such that $\mathcal{T}, v, (r, l) \models^* \varphi_1$ for $m \leq l < k$ and $\mathcal{T}, v, (r, k) \models^* \varphi_2$.
- 6. $\mathcal{T}, v, (r, m) \models^* K_1 \varphi$ if $X(\varphi) = 1$ for all $(X, s) \in K(v_m)$, where v_m is determined as above.
- 7. $\mathcal{T}, v, (r, m) \models^* \exists \varphi \text{ if } X(\varphi) = 1 \text{ for some } (X, s) \in K(v_m), \text{ with } r(m) = s, \text{ where } v_m \text{ is determined as above.}$

We use the abbreviation $\mathcal{T}, (r, m) \models^* \varphi$ for $\mathcal{T}, r_1(0), (r, m) \models^* \varphi$. (The choice of the vertex $r_1(0)$ here is not really significant: it is not difficult to show that for all $k \leq m$ we have $\mathcal{T}, (r, m) \models^* \varphi$ iff $\mathcal{T}, r_1(k), (r[k..], m - k) \models^* \varphi$.)

Define a labelled tree $\mathcal T$ for ψ and E to be acceptable if it satisfies the following conditions:

- (Real) For all observations o, and for all $(X,s) \in K(o)$, we have $X(\psi) = 1$ and $s \in I_e$. (Init) For all initial states $s \in I_e$, there exists an atom X for ψ such that (X,s) is in $K(O_1(s))$.
- (Obs) For all observations o and all vertices v of \mathcal{T} , we have $O_1(s) = o$ for all $(X, s) \in K(v \cdot o)$.
- (Pred) For all observations o, for all vertices v other than the root, and for all $(X, s) \in K(v \cdot o)$, there exists $(Y, t) \in K(v)$ and a joint action $\mathbf{a} \in P_e(t) \times P(v)$ such that $s = \tau(\mathbf{a})(t)$.
- (Succ) For all vertices v other than the root, for all $(X, s) \in K(v)$ and for all $\mathbf{a} \in P_e(s) \times P(v)$, if $t = \tau(\mathbf{a})(s)$ then there exists an atom Y such that $(Y, t) \in K(v \cdot O_1(t))$.
- (\exists sound) For all vertices v, and $(X,s) \in K(v)$, if $X(\exists \varphi) = 1$ then there exists Y such that $(Y,s) \in K(v)$ and $Y(\varphi) = 1$.
- ($\exists \mathsf{comp}$) For all vertices v, and $(X, s), (Y, s) \in K(v)$, if $X(\varphi) = 1$ then $Y(\exists \varphi) = 1$.
- (Ksound) For all vertices v (other than the root) and all $(X,s) \in K(v)$, there exists a run r from v such that r(0) = s and for all subformulae φ of ψ we have $\mathcal{T}, v, (r,0) \models^* \varphi$ iff $X(\varphi) = 1$.

(Kcomp) For all vertices v and all runs r from v there exists $(X, s) \in K(v)$ such that r(0) = s and for all subformulae φ of ψ we have $\mathcal{T}, v, (r, 0) \models^* \varphi$ iff $X(\varphi) = 1$.

The following theorem provides the characterization of realizability of knowledgebased specifications that forms the basis for our synthesis procedure.

Theorem 1. A specification ψ for a single agent is realizable in the environment E iff there exists an acceptable labelled tree for ψ in E.

Proof: We first show that if there exists an acceptable tree then the specification is realizable. Suppose \mathcal{T} is an acceptable tree for ψ in E. We show that the protocol P for agent 1 derived from this tree realizes ψ . Let \mathcal{I} be the system generated by P in E.

We claim that for all points (r,m) of \mathcal{I} and all subformulae φ of ψ we have $\mathcal{T}, (r,m) \models^* \varphi$ iff $\mathcal{I}, (r,m) \models \varphi$. It follows from this that P realizes ψ in E. For, let r be a run of \mathcal{I} . Take v to be the vertex $r_1(0)$. By Init, there exists a pair $(Y,s) \in K(v)$ with s = r(0). Thus, r is a run of \mathcal{T} from the vertex v. By Kcomp, there exists an atom X such that (X,s) is in K(v) and for all subformulae φ of ψ , we have $\mathcal{T}, (r,0) \models^* \varphi$ iff $X(\varphi) = 1$. By the claim, we obtain in particular that $\mathcal{I}, (r,0) \models \psi$ iff $X(\psi) = 1$. But by Real, we have that $X(\psi) = 1$, so $\mathcal{I}, (r,0) \models \psi$ also holds. This shows that P realizes ψ in E.

The proof of the claim is by induction on the complexity of φ . The base case, when φ is an atomic proposition, is straightforward, as are the cases where φ is built using boolean or temporal operators from subformulae satisfying the claim. We establish the cases where φ is of the form $K_1\varphi'$ or $\exists \varphi'$.

We first assume that $\mathcal{I}, (r, m) \models K_1\varphi'$, and show $\mathcal{T}, (r, m) \models^* K_1(\varphi')$. That is, for all $(X, s) \in K(r_1(m))$ we show $X(\varphi') = 1$. By Ksound, for each $(X, s) \in K(r_1(m))$ there exists a run r' of \mathcal{T} from $r_1(m)$ with $r'_e(0) = s$ and $\mathcal{T}, r_1(m), r' \models^* \varphi'$ iff $X(\varphi') = 1$. Applying Pred and Init, this run may be extended backwards to a run r'' of \mathcal{I} with $(r'', m) \sim_1 (r, m)$ and r''[m..] = r'. By the assumption, we have that $\mathcal{I}, (r'', m) \models \varphi'$. It follows using the induction hypothesis that $\mathcal{T}, (r'', m) \models^* \varphi'$. Note that this implies $\mathcal{T}, r_1(m), r' \models^* \varphi'$, hence $X(\varphi') = 1$.

Conversely, we suppose that $\mathcal{T}, (r,m) \models^* K_1(\varphi')$ and show that $\mathcal{I}, (r,m) \models K_1\varphi'$. Suppose that r' is a run of \mathcal{I} with $(r',m) \sim_1 (r,m)$. We need to prove that $\mathcal{I}, (r',m) \models \varphi'$. Using Init, r' is a run of \mathcal{T} from $r'_1(0)$. By Succ and induction, r'[m..] is a run of \mathcal{T} from $r'_1(m) = r_1(m)$. Thus, by Kcomp, there exists $(X,s) \in K(r_1(m))$ such that $r'_e(m) = s$ and for all subformulae φ of ψ we have $\mathcal{T}, (r',m) \models^* \varphi$ iff $X(\varphi) = 1$. By the assumption that $\mathcal{T}, (r,m) \models^* K_1(\varphi')$, we have that $X(\varphi') = 1$ for all $(X,s) \in K(r_1(m))$. Thus, $\mathcal{T}, (r',m) \models^* \varphi'$. By the induction hypothesis it follows that $\mathcal{I}, (r',m) \models \varphi'$, which is what we set out to establish. This completes the proof of the claim, and also the proof that the existence of an acceptable tree implies the existence of a realization.

For the case where $\varphi = \exists \varphi'$, we argue as follows. First, we assume that $\mathcal{T}, (r, m) \models^* \exists (\varphi')$ and show that $\mathcal{T}, (r, m) \models \exists \varphi'$. From the assumption, there exists X such that $(X, r(m)) \in K(r_1(m))$ and $X(\varphi') = 1$. By Ksound, there exists a run r' from $r_1(m)$ in T such that r'(0) = r(m) and $\mathcal{T}, r_1(m), (r', 0) \models^* \exists (\varphi')$. Let $r'' = r[0..m-1] \cdot r'$. This is a run of P, and we have $\mathcal{T}, (r'', m) \models^* \varphi'$. By the induction hypothesis, it follows that $\mathcal{T}, (r'', m) \models \varphi'$. Since r''[0..m] = r[0..m], it follows that $\mathcal{T}, (r, m) \models \exists \varphi'$.

Conversely, assume that $\mathcal{I}, (r, m) \models \exists \varphi'$. We show that $\mathcal{T}, (r, m) \models^* \exists (\varphi')$. From assumption, there exists a run r' such that r[0..m] = r'[0..m] and $\mathcal{I}, (r', m) \models \varphi'$. By induction, we have $\mathcal{T}, (r', m) \models^* \varphi'$. This is equivalent to $\mathcal{T}, r_1(m), (r'[m..], 0) \models^* \varphi'$. By Kcomp, there exists X such that $(X, r(m)) \in K(r_1(m))$ and $X(\varphi') = 1$. It

follows that $\mathcal{T}, (r, m) \models^* \exists (\varphi')$. This completes the argument from the existence of an acceptable tree for ψ in E to realizability of ψ in E.

Next, we show that if ψ is realizable in E then there exists an acceptable tree for ψ and E. Suppose that the protocol P for agent 1 realizes ψ in E. We construct a labelled tree \mathcal{T} as follows. Let \mathcal{I} be the system generated by P in E. If (r,m) is a point of \mathcal{I} , define the atom X(r,m) by $X(r,m)(\varphi)=1$ iff $\mathcal{I},(r,m)\models\varphi$. Define the function f to map the point (r,m) of \mathcal{I} to the pair (X(r,m),r(m)). For all v in \mathcal{O}^+ , define K(v) to be the set of all f(r,m), where (r,m) is a point of \mathcal{I} with $r_1(m)=v$. Define \mathcal{T} by $\mathcal{T}(v)=(K(v),P(v))$ for each $v\in\mathcal{O}^+$. (The label of the root can be chosen arbitrarily.) We claim that \mathcal{T} is an acceptable tree for ψ and E.

For Real, let v = o for an observation o, and suppose that $(X, s) \in K(v)$. Then there exists a run r of \mathcal{I} such that X = X(r, 0) and s = r(s). Since \mathcal{I} realizes ψ , we have that $\mathcal{I}, (r, 0) \models \psi$ and it is immediate that $X(\psi) = 1$.

For Init, let s be an initial state. Take r to be any run of \mathcal{I} with r(0) = s and let X = X(r, 0). Then $(X, s) \in K(O_1(s))$.

For Obs, Let o be an observation and v a vertex of \mathcal{T} . If $(X, s) \in K(v \cdot o)$, then there exists a run r such that $r_1(m) = v \cdot o$, X = X(r, m) and s = r(m), where $m = |v \cdot o|$. This implies that $o = O_i(r(m)) = O_i(s)$, as required.

For Pred, let o be an observation, v a vertex of \mathcal{T} other than the root, and $(X,s) \in K(v \cdot o)$. Then there exists a run r of \mathcal{I} such that $r_1(m) = v \cdot o$ and X = X(r,m) and s = r(m). Let Y = X(r,m-1) and t = r(m-1). Since $r_1(m) = v \cdot o$ we have $r_1(m-1) = v$. It follows that $(Y,s) \in K(v)$. Moreover, since r is a run of \mathcal{I} , there exists an action $\mathbf{a} \in P_e(t) \times P(v)$ such that $s = \tau(\mathbf{a})(t)$. This gives the conditions required for the consequent of Pred.

For Succ, let v be a vertex other than the root, and consider $(X,s) \in K(v)$ and $\mathbf{a} \in \mathcal{P}_e(s) \times P(v)$. Let $t = \tau(\mathbf{a})(s)$. By construction, there exists a run r such that $r_1(m) = v$ and X = X(r,m) and s = r(m), where m = |v|. Let r' be any run extending $r[0..m] \cdot t$. Take Y = X(r', m+1). Then $r'_1(m+1) = v \cdot O_i(t)$ and r(m+1) = t, so $(Y,t) \in K(v \cdot O_1(t))$, as required for Succ.

For \exists sound, suppose that $(X,s) \in K(v)$ and $X(\exists \varphi) = 1$. We need to show that there exists Y such that $(Y,s) \in K(v)$ and $Y(\varphi) = 1$. By construction, there exists a run r of \mathcal{I} such that X = X(r,m) and s = r(m), where m = |v|. Moreover, we have $\mathcal{I}, (r,m) \models \exists \varphi$. Thus, there exists a run r' of \mathcal{I} such that r'[0..m] = r[0..m] and $\mathcal{I}, (r',m) \models \varphi$. Let Y = X(r',m); plainly, this satisfies $Y(\varphi) = 1$. Note that Y'(m) = Y(m) = s. It follows from Y'[0..m] = Y(m) = r[0..m] that Y(m) = r(m) = r(m)

For $\exists \mathsf{comp}$, suppose $(X,s), (Y,s) \in K(v)$ and $X(\varphi) = 1$. We show that $Y(\exists \varphi) = 1$. By construction, there exist runs r, r' such that X = X(r,m) and Y = Y(r',m) and $r_1(m) = r_i'(m) = v$ and r(m) = r'(m) = s. Since $X(\varphi) = 1$ we have $\mathcal{I}, (r,m) \models \varphi$. Consider the sequence r'' = r[0..m-1]r'[m..]. This is a run of \mathcal{I} with r''[0..m] = r[0..m]. Thus, $\mathcal{I}, (r'', m) \models \exists \varphi$. Since satisfaction of formulas depends only on the future and r''[m..] = r'[m..], we obtain that $\mathcal{I}, (r', m) \models \exists \varphi$. This yields that $Y(\exists \varphi) = 1$, as required.

We next prove Ksound and Kcomp. For this, we first prove that for all points (r,m) of \mathcal{I} we have $\mathcal{I}, (r,m) \models \varphi$ iff $\mathcal{T}, (r,m) \models^* \varphi$. The proof is by induction on the complexity of φ . As above, the cases not involving knowledge are straightforward, so we focus on the case where φ is of the form $K_1\varphi'$. By definition, $\mathcal{I}, (r,m) \models K_1\varphi'$ iff $\mathcal{I}, (r',m) \models \varphi'$ for all $(r',m) \sim_1 (r,m)$. By definition of \mathcal{T} and the induction hypothesis, this holds just when $X(\varphi') = 1$ for all $(X,s) \in K(r_1(m))$. This latter condition is equivalent to $\mathcal{T}, (r,m) \models^* K_1\varphi'$, so we are done.

For Ksound, suppose that v is a vertex not equal to the root and that (X,s) is in K(v). Then there exists a point (r,m) of \mathcal{I} such that $v=r_1(m)$ and (X,s)=f(r,m). The sequence r[m..] is a run of \mathcal{T} from v with initial state s. To establish Ksound, we need to show that for all subformulae φ of ψ we have $X(\varphi)=1$ iff $\mathcal{T},v,r[m..]\models^*\varphi$. This holds because $\mathcal{I},(r,m)\models\varphi$ iff $\mathcal{T},(r,m)\models^*\varphi$.

We now prove that \mathcal{T} satisfies Kcomp. Let v be a vertex of \mathcal{T} not equal to the root and let r be a run of \mathcal{T} from v. We need to show that there exists a pair (X,s) in K(v) such that s=r(0) and, for all subformulae φ of ψ , $X(\varphi)=1$ iff $\mathcal{T},v,r\models^*\varphi$. By definition of a run from v, there exists $(Y,s)\in K(v)$ with r(0)=s. By construction of \mathcal{T} , there exists a point (r',m) of \mathcal{I} such that (Y,s)=f(r',m). Clearly, the sequence $r''=r'[0..m-1]\cdot r$ is a run of \mathcal{I} . Thus, we have that f(r'',m)=(X(r'',m),s) is in K(v). By definition we have $X(r'',m)(\varphi)=1$ iff $\mathcal{I},(r'',m)\models\varphi$. As shown above, the latter holds just when $\mathcal{T},(r'',m)\models^*\varphi$. But this last condition is equivalent to $\mathcal{T},v,r\models^*\varphi$, since r''[m..]=r. This shows that f(r'',m) is the required pair (X,s).

In the next section, we show how this result can be used to yield an automatatheoretic procedure for constructing a realization of a specification.

4 An Algorithm for Realizability

We first recall the definitions of the two types of automata we require. Section 4.1 deals with automata on infinite words, and Section 4.2 deals with alternating automata on infinite trees. We apply these to our realizability problem in Section 4.3.

4.1 Automata on Infinite Words

For an introduction to the theory of automata on infinite words and trees see [48].

The types of finite automata on infinite words we consider are those defined by Büchi [7]. A (nondeterministic) automaton on words is a tuple $\mathcal{A} = \langle \Sigma, S, S_0, \rho, \alpha \rangle$, where Σ is a finite alphabet, S is a finite set of states, $S_0 \subseteq S$ is a set of starting states, $\rho: S \times \Sigma \to 2^S$ is a (nondeterministic) transition function, and α is an acceptance condition. A Büchi acceptance condition is a set $F \subseteq S$.

A run r of \mathcal{A} over a infinite word $w = a_0 a_1 \cdots$, is a sequence s_0, s_1, \cdots , where $s_0 \in S_0$ and $s_i \in \rho(s_{i-1}, a_{i-1})$, for all $i \geq 1$. Let inf(r) denote the set of states in Q that appear in $r(\rho)$ infinitely often. The run r satisfies a Büchi condition F if there is some state in F that repeats infinitely often in r, i.e., $F \cap inf(r) \neq \emptyset$. The run r is accepting if it satisfies the acceptance condition, and the infinite word w is accepted by \mathcal{A} if there is an accepting run of A over w. The set of infinite words accepted by \mathcal{A} is denoted $\mathcal{L}(\mathcal{A})$.

The following theorem establishes the correspondence between temporal formulae and Büchi automata.

Proposition 1. [49] Given a temporal formula φ over a set Prop of propositions, one can build a Büchi automaton $\mathcal{A}_{\varphi} = \langle 2^{Prop}, S, S_0, \rho, F \rangle$, where $|S| \leq 2^{O(|\varphi|)}$, such that $\mathcal{L}(\mathcal{A}_{\varphi})$ is exactly the set of computations satisfying the formula φ .

4.2 Alternating Automata on Infinite Trees

Alternating tree automata generalize nondeterministic tree automata and were first introduced in [37]. They have recently found usage in computer-aided verification

[5,50,52]. An alternating tree automaton $\mathcal{A} = \langle \Sigma, Q, q_0, \delta, \alpha \rangle$ runs on Σ -labelled Υ -trees (i.e., mappings from Υ^* to Σ). It consists of a finite set Q of states, an initial state $q_0 \in Q$, a transition function δ , and an acceptance condition α (a condition that defines a subset of Q^{ω}).

For a set D, let $\mathcal{B}^+(D)$ be the set of positive Boolean formulae over D; i.e., Boolean formulae built from elements in D using \wedge and \vee , where we also allow the formulae **true** and **false**. For a set $C \subseteq D$ and a formula $\theta \in \mathcal{B}^+(D)$, we say that C satisfies θ iff assigning **true** to elements in C and assigning **false** to elements in $D \setminus C$ makes θ true.

The transition function $\delta: Q \times \Sigma \to \mathcal{B}^+(\Upsilon \times Q)$ maps a state and an input letter to a formula that suggests a new configuration for the automaton. A run of an alternating automaton \mathcal{A} on an input Σ -labelled Υ -tree \mathcal{T} is a tree $\langle T_r, r \rangle$ in which the root is labelled by q_0 and every other node is labelled by an element of $\Upsilon^* \times Q$. Here T_r is a prefix-closed subset of \mathbf{N}^* and $r: T_r \to \Upsilon^* \times Q$ is the labeling function. Each node of T_r corresponds to a node of Υ^* . A node y in T_r , labelled by r(y) = (x, q), describes a copy of the automaton that reads the node x of Υ^* and visits the state q. Formally, $\langle T_r, r \rangle$ is a Σ_r -labelled tree where $\Sigma_r = \Upsilon^* \times Q$ and $\langle T_r, r \rangle$ satisfies the following:

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1. \epsilon \in T_r and r(\epsilon) = (\epsilon, q_0).
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- 2. Let $y \in T_r$ with r(y) = (x, q) and $\delta(q, \mathcal{T}(x)) = \theta$. Then there is a (possibly empty) set $S = \{(c_1, q_1), \dots, (c_n, q_n)\} \subseteq \Upsilon \times Q$, such that the following hold:
 - S satisfies θ , and
 - for all $1 \le i \le n$, we have $y \cdot i \in T_r$ and $r(y \cdot i) = (x \cdot c_i, q_i)$.

For example, if $\langle T, V \rangle$ is a $\{0, 1\}$ -tree with $V(\epsilon) = a$ and $\delta(q_0, a) = ((0, q_1) \vee (0, q_2)) \wedge ((0, q_3) \vee (1, q_2))$, then the nodes of $\langle T_r, r \rangle$ at level 1 include the label $(0, q_1)$ or $(0, q_2)$, and include the label $(0, q_3)$ or $(1, q_2)$.

Each infinite path ρ in $\langle T_r, r \rangle$ is labelled by a word $r(\rho)$ in Q^{ω} . A run $\langle T_r, r \rangle$ is accepting iff all its infinite paths satisfy the acceptance condition. Let $inf(\rho)$ denote the set of states in Q that appear in $r(\rho)$ infinitely often. In a Büchi acceptance condition, $\alpha \subseteq Q$ and an infinite path ρ satisfies an acceptance condition α if $\alpha \cap inf(\rho) \neq \emptyset$, In a co-Büchi acceptance condition, $\alpha \subseteq Q$ and an infinite path ρ satisfies an acceptance condition α if $\alpha \cap inf(\rho) = \emptyset$, In a Rabin acceptance condition, $\alpha \subseteq 2^Q \times 2^Q$, and an infinite path ρ satisfies an acceptance condition $\alpha = \{\langle G_1, B_1 \rangle, \dots, \langle G_m, B_m \rangle\}$ iff there exists $1 \leq i \leq m$ for which $inf(\rho) \cap G_i \neq \emptyset$ and $inf(\rho) \cap B_i = \emptyset$. As with nondeterministic automata, an automaton accepts a tree iff there exists an accepting run on it. We denote by $\mathcal{L}(A)$ the language of the automaton A; i.e., the set of all labelled trees that A accepts. A is empty if $\mathcal{L}(A) = \emptyset$.

Nondeterministic tree automata are a special case of alternating tree automata. An automaton $\mathcal{A} = \langle \mathcal{D}, Q, q_0, \delta, \alpha \rangle$ is nondeterministic if, for each state $q \in Q$ and letter $a \in \mathcal{D}$, the formula $\delta(q, a)$ does not contain two pairs (c, q_1) and (c, q_2) , where $q_1 \neq q_2$, that are conjunctively related (i.e., both appear in the same disjunct of the disjunctive normal form of $\delta(q, a)$). Intuitively, it means that the automaton cannot send two distinct copies in the same direction [37].

Proposition 2. [38] Given an alternating Rabin automaton with n states and m pairs, we can translate it into an equivalent nondeterministic Rabin automaton with $(mn)^{O(mn)}$ states and mn pairs.

Proposition 3. [15,42,26] Emptiness of a nondeterministic Rabin automaton with n states and m pairs over an alphabet with l letters can be tested in time $(lmn)^{O(m)}$.

4.3 Realizability

We now derive an automata-theoretic algorithm for realizability for knowledge-based specifications involving a single agent.

Theorem 2. There is an algorithm that constructs for a given specification ψ and an environment E an nondeterministic Rabin automaton $\mathcal{A}_{\psi,E}$ such that $\mathcal{A}_{\psi,E}$ accepts precisely the acceptable trees for ψ in E. The automaton $\mathcal{A}_{\psi,E}$ has $2^{||E|| \cdot 2^{O(||\psi||)}}$ states and $||E|| \cdot 2^{O(||\psi||)}$ pairs.

Proof: (sketch) The inputs to the automaton $\mathcal{A}_{\psi,E}$ are $L_{\psi,E}$ -labeled trees. Note that the size of $L_{\psi,E}$ is exponential in the number of states and actions in E and doubly exponential in the length of ψ .

To check that an input tree \mathcal{T} is acceptable, the automaton has to check that it satisfies the properties Real, Init, Obs, Pred, Succ, \exists sound, \exists comp, Ksound, and Kcomp. We describe automata that check these properties; $\mathcal{A}_{\psi,E}$ is obtained as the intersection of these automata. The property Real is a condition on the children of the root of \mathcal{T} that can be checked by a nondeterministic automaton with O(1) states. The property Init is a condition on the children of the root of \mathcal{T} that can be checked by a nondeterministic automaton with O(||E||) states. The property Obs is a condition on the labels of nodes in the tree that can be checked by a nondeterministic automaton with O(||E||) states.

The property Pred is a condition on relationships between labels of nodes and labels of their children that can be checked by a nondeterministic automaton with $2^{O(||E||)}$ states; after visiting a node v, the automaton remembers P(v) and the set $\{t:(Y,t)\in K(v)\}$. It then uses this to check that for all $(X,s)\in K(v\cdot o)$ there exists $(Y,t)\in K(v)$ and $\mathbf{a}\in P_e(t)\times P(v)$ such that $s=\tau(\mathbf{a})(t)$. The property Succ is a condition on relationships between labels of nodes and labels of their children that can be checked by a nondeterministic automaton with $2^{O(||E||)}$ states; after visiting a node v, the automaton remembers P(v) and the set $\{t:(Y,t)\in K(v)\}$. It then uses this to check that for all $(Y,s)\in K(v)$ and $\mathbf{a}\in P_e(s)\times P(v)$ there exists $(Y,t)\in K(v\cdot o)$ such that $t=\tau(\mathbf{a})(s)$.

Conditions ∃sound and ∃comp are trivially checked conditions on the labels.

To check Ksound, an alternating automaton guesses, for all vertices v (other than the root) and all $(X,s) \in K(v)$, a run r from v such that r(0) = s and for all subformulae φ of ψ we have $\mathcal{T}, v, (r,0) \models^* \varphi$ iff $X(\varphi) = 1$. A formula ξ can be viewed as a temporal formula by considering every subformula $K\theta$ or $\exists \theta$ as a new proposition. Consider the formula ψ_X that is obtained by taking the conjunction of subformulae of ψ or their negation according to X. We consider ψ_X as a temporal formula in LTL and appeal to Theorem 1 to construct a Büchi automaton A_{ψ_X} that check whether ψ_X is satisfied by a sequence of truth assignments to its extended set of propositions (i.e., atomic propositions and subformulae of the form $K\theta$). Thus, the automaton guesses a sequence v_0, v_1, \ldots of nodes in the tree and a sequence $(X_0, s_0), (X_1, s_1), \ldots$ of atomstate pairs such that $v_0 = v, X_0 = X, s_0 = s, v_{i+1}$ is a child of $v_i, (X_i, s_i) \in K(v_i)$, and $s_{i+1} = \tau(\mathbf{a})(s_i)$ for some $\mathbf{a} \in P_e(s_i) \times P(v_i)$. It then emulates A_{ψ_X} and checks that the sequence X_0, X_1, \ldots is accepted. This automaton has $||E|| \cdot 2^{O(||\psi||)}$ states and a Büchi acceptance condition.

Instead of checking that Kcomp holds, we construct an alternating automaton that checks that Kcomp is violated, since alternating automata can be complemented by dualizing their transition function (i.e., switching \vee and \wedge as well as **true** and **false**) and complementing the acceptance condition [37]. The automaton guesses a vertex v

and a run r from v such that for no $(X,s) \in K(v)$ we have that r(0) = s and for all subformulae φ of ψ we have $\mathcal{T}, v, (r,0) \models^* \varphi$ iff $X(\varphi) = 1$. We already saw how the automaton guesses a run; it guesses a sequence v_0, v_1, \ldots of nodes in the tree and a sequence $(X_0, s_0), (X_1, s_1), \ldots$ of atom-state pairs such that $v_0 = v, (Y_0, s_0) \in K(v)$ for some atom Y_0 , but $(X_0, s_0) \notin K(v), v_{i+1}$ is a child of $v_i, (Y_i, s_i) \in K(v_i)$ for some atom Y_i , and $s_{i+1} = \tau(\mathbf{a})(s_i)$ for some $\mathbf{a} \in P_e(s_i) \times P(v_i)$. It then emulates $A_{\psi_{X_0}}$ and checks that the sequence X_0, X_1, \ldots is accepted. This automaton has $||E|| \cdot 2^{O(||\psi||)}$ states. After complementing it, it has a co-Büchi acceptance condition.

We now apply Proposition 2 to the alternating automata that check Ksound and Kcomp to get nondeterministic Rabin automata with $2^{||E|| \cdot 2^{O(||\psi||)}}$ states and $||E|| \cdot 2^{O(||\psi||)}$ pairs.

Corollary 1. There is an algorithm that decides whether a formula ψ is realizable in an environment E in time $2^{O(||E||)} \cdot 2^{2^{O(||\psi||)}}$.

Proof: By Theorem 2, ψ is realizable in E iff $\mathcal{L}(\mathcal{A}_{\psi,E}) \neq \emptyset$. The claim now follows by Proposition 3, since $\mathcal{A}_{\psi,E}$ has Rabin automata with $2^{||E|| \cdot 2^{O(||\psi||)}}$ states and $||E|| \cdot 2^{O(||\psi||)}$ pairs and the alphabet has $2^{||E|| \cdot 2^{O(||\psi||)}}$ letters.

We note that it is shown in [42] that realizability of temporal formulae with complete information is already 2EXPTIME-hard. Thus, the bound in Corollary 1 is essentially optimal.

So far our focus was on realizability. Recall, however, that if \mathcal{T} is an acceptable tree for ψ in E, then the protocol P for agent 1 derived from this tree realizes ψ in E. The emptiness-testing algorithm used in the realizability test (per Proposition 3) does more than just test emptiness. When the automaton is nonempty the algorithm returns a *finitely-generated* tree, which, as shown in [8], can be viewed as a finite-state protocol. We return to this point in the following section.

5 Discussion

In this section we make a number of remarks concerning realizability of specifications involving knowledge. We first consider, in section 5.1, the question of what a protocol realizing a specification knows. Then, in section 5.2, we relate realizability of specifications involving knowledge to *knowledge-based programs*.

5.1 Knowledge in the Implementation

In this section we remark upon a subtle point concerning the states of knowledge attained in protocols realizing a specification. As these remarks apply equally to the general multi-agent framework we have defined, we return to this context.

We have defined local states, hence the semantics of knowledge, using the assumption of synchronous perfect recall, which involves an infinite space of local states. A protocol realizing a specification is not required to have perfect recall, and could well be represented (like the protocol synthesized by our procedure) using a finite set of states. The sense in which such a protocol satisfies the conditions on knowledge stated by the specification is the following: an agent that follows the actions prescribed by the protocol, but computes its knowledge based on the full record of its observations,

satisfies this specification. Thus, although we may have a finite-state protocol, it appears that we have not in actuality eliminated the need to maintain an unbounded log of all the agent's observations. If this is so, then the system is better characterized as consisting of an infinite state space coupled to a finite-state controller.

Now, there are situations in which we can dispense with the observation logs, leaving just the finite-state controller. This holds when, although we state the specification in knowledge-theoretic terms, we are more concerned with the behavior of the synthesized system than the information encoded in its states. For example, Halpern and Zuck [23] give a knowledge-based specification (in the form of a knowledge-based program) of solutions to a sequence transmission problem. They start with the assumption of perfect recall, but their ultimate interest is to develop implementations for this specification that optimize the memory maintained by agents while preserving their behaviour. One of the implementations they consider, the alternating-bit protocol [4], is a finite-state protocol. (We remark that the definitions of implementation of knowledge-based programs in [17,16] do not admit such optimized protocols as implementations of knowledge-based programs, but the modified approach of [34] does.)

One might wonder whether, if we only wish to specify behaviour, one can state an equivalent specification that makes no use of the knowledge operators. This is not the case: the knowledge operators add expressive power, making it possible to specify that the behavior is *information-theoretically optimal*. For example, the knowledge operators allow one to specify that the agent performs an action as soon as it has information appropriate to that action. A simple example of such a specification is studied by Brafman et al. [6], who consider a robot provided with incomplete information about its location through a noisy position sensor. The robot must satisfy the specification that it halts as soon as it knows that it is inside a goal region. Other examples of behavioral specifications with information-theoretic optimality constraints have been considered in a sequence of papers on agreement protocols in distributed systems [21,13,36,39].

Although in some cases one is concerned only with behavior, in others what one has in mind in writing a knowledge-based specification is to construct an implementation whose states have the information-theoretic property expressed. This is the case when the states of knowledge in question function as an output of the system, or provide inputs to some larger module. For example, we might specify that a controller for a nuclear reactor must keep the reactor temperature below a certain level and must also know of a critical level of radiation whenever this condition holds, with the intention that this information be provided to the operator. In this case it will not do to implement the specification according to its behavioral component alone, since this might lose the attribute, knowledge of radioactivity, that we wish to present as an output.

Clearly, we could always ensure that the knowledge properties specified are available in the implementation by taking the implementation to consist of both the finite-state controller and the log of all the agent's observations. Such an implementation is rather inefficient. Can we do better? One attempt to do so would be simply to take the implementation to consist just of the protocol, and to compute knowledge on the basis of the protocol states.

To make this idea precise, we adopt the following model of a protocol and the knowledge it encodes. We suppose that agent *i*'s protocol is represented as an automaton $A_i = \langle Q_i, q_i, \mu_i, \alpha_i \rangle$, where

- 1. Q_i is the set of protocol states,
- 2. $q_i \in Q$ is the initial state,

- 3. $\mu_i: Q_i \times \mathcal{O} \to Q_i$ is the state transition function, used to update the protocol state given an observation in \mathcal{O} , and
- 4. $\alpha_i: Q_i \to \mathcal{P}(ACT_i)$ is a function mapping each state to a set of actions of the agent.

As usual, we define the state reached after a sequence σ of inputs (i.e., observations of the agent) by $A_i(\epsilon) = q_i$ and $A_i(\sigma \cdot o) = \mu_i(A_i(\sigma), o)$. We may then define the protocol itself, as a function from sequences of observations to sets of actions, by $P_{A_i}(\sigma) = \alpha_i(A_i(\sigma))$.

Suppose we are given a tuple $A = \langle A_1, \ldots, A_n \rangle$ of automata representing the protocols of agents $1 \ldots n$. To interpret the knowledge operators with respect to the states of these automata, we first define, for each agent i, an indistinguishability relation \approx_i^A on points, based on the states of the automata A_i rather than the perfect-recall local states used for the relation \sim_i above. That is, we define $(r,m) \approx_i^A (r',m')$ to hold when $A_i(r_i(m)) = A_i(r_i'(m'))$. We may now define the semantics of knowledge exactly as we did using the relation \sim_i . To distinguish the two interpretations, we introduce new knowledge modalities K_i^A , and define $\mathcal{I}, (r,m) \models K_i^A \varphi$ if $\mathcal{I}, (r',m') \models \varphi$ for all points (r',m') of \mathcal{I} satisfying $(r',m') \approx_i^A (r,m)$. We may now formulate the proposal above as follows. Suppose a specification φ is realized in an environment E by a joint protocol \mathbf{P}_A , represented by the automata A. Is it then the case that this joint protocol realizes in E the specification φ^A obtained from φ by replacing (recursively) each subformula $K_i \psi$ with $K_i^A \psi$? It is not, as the following example shows.

Example 2. The protocol in Example 1, which performs the toggle action at all steps, can be represented by an automaton A with a single state. This protocol realizes the specification $\Box(K_1\mathsf{toggle-on} \lor K_1 \neg \mathsf{toggle-on})$. However, with respect to the automaton A, the formula $\Box(K_1^A\mathsf{toggle-on} \lor K_1^A \neg \mathsf{toggle-on})$ is false at time 0 in a run generated by the protocol. For, at even numbered points on these runs the toggle is on and at odd points the toggle is off, and the single state does not suffice to distinguish the two. \Box

Nevertheless, a slight modification of the proposal makes it possible to ensure that the protocols realizing a specification have the desired information theoretic property. All that is required is to reflect an agent's knowledge according to the perfect-recall definition in its behavior. To do so, we first modify the environment so that an agent is provided with actions that allow it to assert what it knows, and then add a constraint to the specification that requires agents to assert their knowledge truthfully.

Suppose that Φ is the set of all the knowledge formulae of the form $K_i(\varphi)$, for some i=i..n, that we wish the implementation to preserve, together with all subformulae of such formulae that have the same form. For each i=1..n, let Φ_i be the set of formulas in Φ of the form $K_i\varphi$. The modification of the environment involves adding to each agent's actions a component in which the agent asserts a subset of Φ_i . That is, we take ACT_i' to be $ACT_i \times \mathcal{P}(\Phi_i)$. We also modify the environment $E = \langle S_e, I_e, P_e, \tau, O_1, \ldots, O_n, \pi_e \rangle$ to the environment $E' = \langle S_e', I_e', P_e', \tau', O_1', \ldots O_n', \pi_e' \rangle$, where

- 1. $S'_e = S_e \times \mathcal{P}(\Phi_1) \times \ldots \times \mathcal{P}(\Phi_n)$; intuitively, a state $(s, \Psi_1, \ldots, \Psi_n) \in S'_e$ represents that the state of the environment is s and each agent i's last assertion is Ψ_i ,
- 2. $I'_e = I_e \times \{\emptyset\} \times \dots \{\emptyset\}$; we take the last assertion to be empty at the initial state,
- 3. $P'_e(s, \Psi_1, \dots, \Psi_n) = P_e(s)$ for all $s \in S_e$ and $\Psi_1, \dots, \Psi_n \subseteq \Phi$; so that the latest assertion has no impact on the environment's protocol,

- 4. $\tau'(\langle a_e, (a_1, \Psi'_1), \dots, (a_n, \Psi'_n) \rangle)[(s, \Psi_1, \dots, \Psi_n)] = (\tau(\langle a_e, a_1, \dots a_n \rangle)(s), \Psi'_1, \dots, \Psi'_n),$ so that state transitions operate as in E, but additionally record each agent's latest assertion.
- 5. $O'_i(s, \Psi_1, \dots, \Psi_n) = O_i(s)$, so that the agent's observations are unchanged, and
- 6. $\pi'_e((s, \Psi_1, \dots, \Psi_n), p) = \pi_e(s, p)$ for all atomic propositions $p \in Prop$.

Additionally, we extend the language by introducing for each formula $\psi \in \Phi_i$ an atomic proposition "said_i(ψ)", with semantics given by $\pi'_e((s, \Psi_1, \dots, \Psi_n), \operatorname{said}_i(\psi)) = 1$ iff $\psi \in \Psi_i$.

Define $Say(\Phi)$ to be the formula

$$\bigwedge_{i=1..n} \bigwedge_{K_i \psi \in \Phi} \Box(K_i \psi \equiv \bigcirc \operatorname{said}_i(K_i \psi)),$$

which asserts that agents say what they know (according to perfect recall.) Additionally, define $\operatorname{Know}(\varPhi,A)$ to be the formula

$$\bigwedge_{i=1..n} \bigwedge_{K_i \psi \in \Phi} \Box(K_i \psi \equiv K_i^A \psi)),$$

which says that each agent knows a fact in Φ according to its protocol just when it knows this fact using perfect recall. We then have the following result.

Proposition 4. The following are equivalent:

- 1. For some (finite state) automaton A, the formula $\varphi \wedge \operatorname{Know}(\Phi, A)$ is realized in $\mathcal{I}(P_A, E)$.
- 2. The formula $\varphi \wedge \operatorname{Say}(\Phi)$ is (finite state) realizable in E'.

Proof: Suppose first that $\varphi \wedge \operatorname{Know}(\Phi, A)$ is realized in E, by the joint protocol represented by the (finite state) automata $A = \langle A_1, \ldots, A_n \rangle$. We construct automata $A' = \langle A'_1, \ldots, A'_n \rangle$ for the environment E' such that the corresponding joint protocol realizes $\varphi \wedge \operatorname{Say}(\Phi)$. For each $A_i = \langle Q_i, q_i, \mu_i, \alpha_i \rangle$, define A'_i to be identical to A_i except that it has action function α'_i , defined by $\alpha'_i(q) = \alpha_i(q) \times \{\Phi_q\}$ where

$$\Phi_q = \{K_i \psi \in \Phi \mid \mathcal{I}(P_A, E), (r, m) \models K_i^A \psi \text{ for some } (r, m) \text{ with } A_i(r_i(m)) = q\}$$

for all $q \in Q_i$. Plainly, A'_i is finite state if A_i is. Observe, moreover, that $\mathcal{I}(P_A, E), (r, m) \models K_i^A \psi$ for some (r, m) with $A_i(r_i(m)) = q$ iff $\mathcal{I}(P_A, E), (r, m) \models K_i^A \psi$ for all (r, m) with $A_i(r_i(m)) = q$.

We note that the runs of $\mathcal{I}(P_A, E)$ and $\mathcal{I}(P_{A'}, E')$ are in one-to-one correspondence, with a run r' of the latter with $r'(m) = (s, \Psi_1, \dots, \Psi_n)$ corresponding to a run r of the former with r(m) = s. It is immediate from this and the fact that $\mathcal{I}(P_A, E)$ realizes φ that $\mathcal{I}(P_{A'}, E')$ realizes φ . It therefore remains to show that $\mathcal{I}(P_{A'}, E')$ realizes $\operatorname{Say}(\Phi)$. Let $K_i\psi \in \Phi$ and let r' be a run of $\mathcal{I}(P_{A'}, E')$ corresponding to run r of $\mathcal{I}(P_A, E)$. Then, by definition of A'_i , and the observation above, we have $\mathcal{I}(P_{A'}, E'), (r', m) \models \mathbb{C}(F_A, E)$ by the fact that $\varphi \land K$ now (Φ, A) is realized in E, we have that $\mathcal{I}(P_A, E), (r, m) \models K_i^A \psi$ iff $\mathcal{I}(P_A, E), (r, m) \models K_i \psi$. By the correspondence noted, we have $\mathcal{I}(P_A, E), (r, m) \models K_i \psi$ iff $\mathcal{I}(P_{A'}, E'), (r', m) \models K_i \psi$. It follows that $\mathcal{I}(P_{A'}, E'), (r', m) \models K_i \psi \Leftrightarrow \bigcirc \operatorname{said}_i(K_i \psi)$, as required for $\mathcal{I}(P_{A'}, E') \models \operatorname{Say}(\Phi)$.

Conversely, suppose first that $\varphi \wedge \operatorname{Say}(\Phi)$ is realized in E', by the joint protocol represented by the (finite state) automata $A' = \langle A'_1, \ldots, A'_n \rangle$. We construct automata

 $A = \langle A_1, \ldots, A_n \rangle$ for the environment E such that the corresponding joint protocol realizes $\varphi \wedge \operatorname{Know}(\Phi, A)$. For each i = 1..n, the automaton A_i is obtained from A_i' by replacing the action function α_i' by the function α_i , such that $a \in \alpha_i(q) = a$ iff there exists Ψ such that $(a, \Psi) \in \alpha'(q)$. Since the state sets are the same, A_i is finite state iff A_i' is.

Note first that for every automaton A, we have that if $(r,m) \sim_i (r',m')$ then $(r,m) \approx_i^A (r',m')$, since the automaton is fed the same sequence of inputs in reaching the point (r,m) and (r',m'). It follows that every automaton A realizes the formula $\bigwedge_{i=1...n} \bigwedge_{K_i\psi \in \Phi} \Box(K_i^A\psi \Rightarrow K_i\psi)$). Thus, we need to establish the converse.

Note that it follows from the fact that $\mathcal{I}(P_{A'}, E')$ realizes $\operatorname{Say}(\Phi)$ that for all points (r,m) of $\mathcal{I}(P_{A'}, E')$, if $A'_i(r_i(m)) = q$ and $(a, \Psi), (a', \Psi') \in \alpha'_i(q)$, then $\Psi = \Psi'$. It follows, by construction of A, A' and E', that the runs of $\mathcal{I}(P_A, E)$ and $\mathcal{I}(P_{A'}, E')$ are in one to one correspondence, the only difference being that in the latter the agents additionally assert some set of formulae. This does not affect satisfaction of formulae in the language based on Prop in any way. Suppose that $\mathcal{I}(P_A, E), (r, m) \models K_i \psi$, where $K_i \psi \in \Phi$. We show that $\mathcal{I}(P_A, E), (r, m) \models K_i^A \psi$. Let (ρ, k) be a point of $\mathcal{I}(P_A, E)$ such that $(\rho, k) \approx_i^A (r, m)$, i.e., $A(\rho_i(k)) = A(r_i(m))$. Since the states and transition functions of A and A' are identical, we also have $A'(\rho_i(k)) = A'(r_i(m))$.

Let r', ρ' be the runs of $\mathcal{I}(P_{A'}, E')$ corresponding to r, ρ , respectively, and let $(a, \Psi) = A'(r'_i(m))$. Since $\mathcal{I}(P_A, E), (r, m) \models K_i \psi$, we have that $\mathcal{I}(P_{A'}, E'), (r', m) \models K_i \psi$. Since $\mathcal{I}(P_{A'}, E')$ realizes Say(Φ), this implies that $K_i \psi \in \Psi$. Since $A'(\rho_i(k)) = A'(r_i(m))$, we also have $A'(\rho_i(k)) = (a, \Psi)$ and $K_i \psi \in \Psi$. Because $\mathcal{I}(P_{A'}, E')$ realizes Say(Φ), this implies that $\mathcal{I}(P_{A'}, E'), (\rho', k) \models K_i \psi$. By the correspondence, it follows that $\mathcal{I}(P_A, E), (\rho, k) \models K_i \psi$, from which we obtain that $\mathcal{I}(P_A, E), (\rho, k) \models \psi$. Since this holds for all $(\rho, k) \approx_i^A (r, m)$, we have $\mathcal{I}(P_A, E), (r, m) \models K_i^A \psi$. Thus, we have shown that $\mathcal{I}(P_A, E), (r, m) \models K_i^A \psi \Rightarrow K_i \psi$.

Intuitively, this result holds because the implementation can only behave as specified by $\varphi \land \operatorname{Say}(\Phi)$ if the protocol states encode the relevant knowledge. This result shows that, provided some care is taken in writing specifications, the realizability framework we have defined in this paper is capable of handling both the case in which agents are required simply to behave as if they had perfect recall, and the case in which agents are required both to behave in this fashion and encode certain perfect-recall knowledge in their protocol states. (Note that it follows from the fact that $\varphi \land \operatorname{Know}(\Phi, A)$ is realized that φ^A is realized, where the latter is obtained from φ by replacing each occurrence of a knowledge operator K_i by K_i^A .)

In particular, in the single agent case, if we apply the synthesis procedure of the previous section to the specification $\varphi \wedge \operatorname{Say}(\Phi)$, and then project away the "saying" component of the action function, we obtain a protocol that represents knowledge defined according to the perfect-recall semantics, but using only a finite number of states.

5.2 Implementing Knowledge Based Programs

In the literature on knowledge-based specification of distributed systems, many of the examples considered have the form of a description of how an agent determines it next action from its state of knowledge. To formalize this idea, Fagin et al. [17,16] propose a syntax and semantics for what they call *knowledge-based programs*. (The proposal builds on an earlier purely semantic framework of Halpern and Fagin [20].) A

knowledge-based program for an agent i is an expression of the form

$$\begin{array}{c} \text{case of} \\ \text{if } \varphi_1^i \text{ do } a_1^i \\ \vdots \\ \text{if } \varphi_{m_i}^i \text{ do } a_{m_i}^i \\ \text{end case} \end{array} \tag{1}$$

where the φ_j^i are formulae in the logic of knowledge that express some property of agent i's knowledge, and the a_j^i are actions of the agent. A *joint* knowledge-based program consists of such a program for each agent. To ensure that an agent is able to determine whether the formulae in its program hold, these are required to be *local* to the agent. One way to ensure this restriction is to require that the φ_j^i are boolean combinations of formulae of the form $K_i\varphi$.

Informally, such a program represents an infinite loop. At each point of time, the agent determines which of the formulae φ^i_j hold, and nondeterministically selects one of the correspond actions a^i_j for execution. Of course, in order to give semantics to the knowledge formulae we require some interpreted system. This system is required to be one that is obtained by running the program itself.

This description of the semantics of knowledge-based programs appears circular, but it is not viciously so. Fagin et al. show how to eliminate the apparent circularity, by describing what it means for a protocol (in the sense of the present paper), in which the agent's actions are not dependent on it knowledge, but only upon properties of its concrete state, to be an *implementation* of a knowledge-based program. Their definition uses a notion of *context*. Contexts have a structure similar to the environments we have used in the present paper, and the semantics of knowledge-based programs can also be stated in terms of environments, as is done by van der Meyden [34]. We consider the latter approach in the following.

We defined above the interpreted system $\mathcal{I}(\mathbf{P},E)$ generated by a (joint) protocol \mathbf{P} in an environment E. The idea underlying the semantics of knowledge-based programs is that by interpreting the tests for knowledge in a (joint) knowledge-based program $\mathbf{P}\mathbf{g}$ with respect to this system, we obtain for each agent, at each point of the system, the set of actions enabled by the program. Formally, for each point (r,m) of a system \mathcal{I} , and agent i, we define $\mathbf{P}\mathbf{g}_i^{\mathcal{I}}(r,m)$ to be the set of actions a_j^i such that $\mathcal{I}, (r,m) \models \varphi_j^i$. Similarly, the protocol \mathbf{P} prescribes a set of actions at each point of the system, computed from the agents' protocol states at the point. For agent i, this set is $\mathbf{P}_i(r_i(m))$. We may now say that \mathbf{P} implements $\mathbf{P}\mathbf{g}$ in E if these two sets of actions are identical at every point of $\mathcal{I}(\mathbf{P}, E)$, i.e., $\mathbf{P}\mathbf{g}_i^{\mathcal{I}(\mathbf{P}, E)}(r, m) = \mathbf{P}_i(r_i(m))$ for all points (r, m) of $\mathcal{I}(\mathbf{P}, E)$ and agents i.

A consequence of this definition of the semantics of knowledge-based programs is that these may have one, many or no implementations. In this regard, it has been noted that they are more like *specifications* than programs that may be written in a typical imperative programming language. It is possible to make this intuition precise using the definition of realizability of knowledge-based specifications we have proposed in the present paper.

Let $E = \langle S_e, I_e, P_e, \tau, O_1, \ldots, O_n, \pi_e \rangle$ be an environment. We assume that that the set of basic propositions and the set of states are sufficiently rich to express the latest joint action taken in E. More precisely, we assume that for each agent i and for each action $a \in ACT_i$ there exists a proposition " $\operatorname{did}_i(a)$ ", and that for all joint actions $\mathbf{a} = \langle a_e, a_1, \ldots, a_n \rangle$ and states $s \in S_e$, we have that if $\tau(\mathbf{a})(s) = t$ then $\pi_e(t, \operatorname{did}_i(a)) = 1$

iff $a = a_i$. It is not difficult to see that an environment can always be modified to one satisfying this condition, by adding extra components to the states of the environment in a fashion similar to the construction of the previous section. We may then state the connection between knowledge-based programs and realizability as follows:

Proposition 5. A joint protocol \mathbf{P} is an implementation (with respect to the perfect-recall semantics of knowledge) of the joint knowledge-based program, given by (1), in the environment E iff \mathbf{P} realizes the specification

$$\forall \Box \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{m_i} (\varphi_j^i \equiv \exists \bigcirc \operatorname{did}_i(a_j^i))$$

in E.

Proof: The result follows straightforwardly from the facts that $\mathbf{Pg}_{i}^{\mathcal{I}(\mathbf{P},E)}(r,m) = \{a_{j}^{i} \mid \mathcal{I}(\mathbf{P},E),(r,m) \models \varphi_{j}^{i}\}$ and $\mathbf{P}_{i}(r_{i}(m)) = \{a_{j}^{i} \mid \mathcal{I}(\mathbf{P},E),(r,m) \models \exists \bigcirc \operatorname{did}_{i}(a_{j}^{i}))\}$ for all points (r,m) of $\mathcal{I}(\mathbf{P},E)$.

This relates a knowledge-based program for n agents to a CTL*- K_n formula. It follows from this result that the techniques we have developed in this paper may be applied to construct an implementation of a knowledge-based program for a single agent, if one exists.

6 Conclusion

We have been able to treat the case of single agent knowledge-based specifications in this paper. Is it possible to generalize our results to the multi-agent case? In general, it is not. Using ideas from Peterson and Reif's study of the complexity of multi-agent games of incomplete information [41], Pnueli and Rosner [44] show that realizability of linear temporal logic specifications in the context of two agents with incomplete information is undecidable. This result immediately applies to our more expressive specification language.

However, there are limited classes of situations in which realizability of temporal specifications for more than one agent with incomplete information is decidable, and for which one still obtains finite-state implementations. Pnueli and Rosner [44] show that this is the case for hierarchical agents. In our epistemic setting, a related (but not quite equivalent) notion is assumption that the observation functions O_i have the property that for all states s and t of the environment, if $O_i(s) = O_i(t)$ then $O_{i+1}(s) = O_{i+1}(t)$. Intuitively, this means that agent 1 makes more detailed observations than agent 2, which in turn makes more detailed observations than agent 3, etc. On the basis of Pnueli and Rosner's results, we conjectured in an earlier version of this paper that realizability of knowledge-based specifications in hierarchical environments may also be decidable. This has subsequently been decided in the negative for the general case, but under some extra conditions on the formula (that knowledge operators occur only positively), the conjecture turns out to be true [35].

Other restrictions on the environment suggest themselves as candidates for generalization of our results. For example, whereas atemporal knowledge-based programs (in which conditions do not involve temporal operators) do not have finite-state implementations in general [34], in *broadcast* environments this is guaranteed [33]. Again, this suggests that realizability of knowledge specifications in broadcast environments

is worth investigation, particularly as this is a very natural and applicable model. This conjecture has been shown to hold [35].

Finally, one could also consider definitions of knowledge other than the perfect-recall interpretation that we have treated in this paper. We believe that our techniques can be easily adapted to show the decidability of a synthesis for specifications concerning a single agent with asynchronous perfect recall. However, one open question is whether it is decidable to determine the existence of a finite state automaton A realizing a specification stated using the knowledge operator K_1^A . The result of Section 5.1 provides only a sufficient condition for this.

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