November 14, 2018

Aging and Lévy distributions in sandpiles.

Oscar Sotolongo-Costa,[1,2] A. Vazquez[2] and J.C. Antoranz[1]

[1]Depto. Física-Matemática y Fluidos, Fac. Ciencias.
UNED, 28080 Madrid
[2]Depto. de Física Teórica, Fac. Física, Univ. Habana.
Habana 10400, Cuba.

Abstract

Aging in complex systems is studied *via* the sandpile model. Relaxation of avalanches in sandpiles is observed to depend on the time elapsed since the beginning of the relaxation. Lévy behavior is observed in the distribution of characteristic times. In this way, aging and self-organized criticality appear to be closely related.

1

I. Introduction

In the current literature of complex systems, aging is frequently reported as a phenomenon observed in spin glasses and other disordered systems, rather than in biological systems,[1,2,3,4,5,6] though some works have been devoted to biological aging of populations.[7,8] Relevant contributions have been brought presenting this phenomenon from a general perspective, revealing that temporal autocorrelation functions in the Bak-Sneppen model display aging behavior similar to glassy systems.[9,10] To our knowledge those papers give, up to now, the more relevant contribution to the relation between SOC and aging.

Up to our knowledge, a detailed study of individual aging from the viewpoint of self-organized critical systems has not yet been performed.

It is well known that biological aging manifests itself in an individual as the slowing down of many processes as, *e.g.*, slower growing of tumors and manifestation of senescence in the slowing down of reflex behavior. Due to its characteristics, relaxation phenomena in complex systems are good candidates to the study of aging.

About 150 years ago, Weber and Gauss carried out a simple experiment demonstrating that relaxation in complex systems is not exponential. Investigating the contraction of a silk thread they found that it does not contract so quickly but it relaxes slowly following a power law $t^{-\alpha}$. This behavior is not particular of mechanical relaxation, it has been also observed in experiments of magnetic relaxation in spin glasses and high critical temperature superconductors, transient current measurements in amorphous semiconductors, dielectric relaxation, and more.[11]

In relaxation dynamics, aging means that the properties of a system depends on its age.[12] For instance, consider a glass quenched at time $t = 0$ below its glass transition temperature under an external stress. At time $t = t_w$ the stress is released. If the system were near equilibrium, then its response measured at certain time $t > t_w$ will be a function of the difference $t - t_w$ and, therefore, normalized responses taken at different initial times t_w will collapse in a single curve. However, this is not the behavior in glass materials, and, as was pointed above, in live beings.

Aging is a consequence of the nonequilibrium dynamics and may be considered as a characteristic of the dynamics of complex systems far from equilibrium.

In the following sections we demonstrate that these manifestations of individual aging are present in sandpile avalanches modeled with a Bethe lattice, particularly, the process of relaxation. Contrary to,[9,10] we do not consider deterministic "toppling" (like that present in the Bak-Sneppen model) but a probabilistic one, and avalanches are not regarded as infinite, as is,[9] but the finite size of the system (sandpile) is explicitly included.

II. Sandpile model

Sandpiles seem to be simplest systems which lead to complex behavior and are a paradigm for the study of all phenomena manifested in complex systems out of equilibrium, like relaxation under external perturbations. They have been taken as the paradigm of self-organization since the introduction of these ideas by Bak, Tang, and Wiesenfeld.[13] The evolution of a sandpile under an external perturbation has been extensively studied experimentally,[14,15] theoretically,[16,17] and by computer simulations.[18] In general, those works analyze the sandpile dynamics close to the critical state, i.e., close to the critical angle.

If a sandpile is subjected to an external perturbation, like, *e.g.*, small amplitude vibrations, large amplitude grain motions are rare events but from time to time, a grain may jump from its quasiequilibrium position. If a surface grain jumps, it will fall through the slope of the pile until it collides with another sand grains down the slope. After this collision the initial grain may be trapped with those grains or some of them may fall through the slope. If the last possibility happens then each grain "surviving" the collision will fall through the slope as the initial grain did. This image of an avalanche as an initial object that consecutively drags another resembles a branching process for which the Bethe lattice representation seems to be natural.[17,19] The representation of the avalanche dynamics as a Bethe lattice is a mean field approach to the problem. It neglects correlations between branches. Notwithstanding, if the pile angle is below is critical value the avalanches will be rare events and, in case of occurrence, will be very sparse. Thus, such an approximation will be acceptable for analyzing the long time relaxation of the pile angle below its critical value, which is the subject of this paper. Away from the critical point the mean field approximation works quite well.

Let us represent the avalanche as a cascade in the Bethe lattice as follows. Firstly, we start with a single node, which could represent in this case a grain. In a further step F will emerge with probability $p(\theta)$, depending on the pile angle θ. This operation of generation of F identical particles starting from one is repeated in the next step to each node of the new group, and so on. If the percolation process overcomes a given length (that of the border of the pile) those nodes beyond the limit constitute the avalanche. If it does not, there would be no avalanche since the cascade was stopped before reaching the base of the pile (frustrated avalanche). By avalanche size we take the number of nodes, of the corresponding Bethe lattice, in the last step.

After an avalanche the sandpile *autoorganizes itself* with the new number of grains (i.e., a new slope is calculated with the remaining grains). The occurrence of avalanches will carry as consequence a decrease in the number of grains in the pile and, therefore, a decrease of $p(\theta)$. Each time an avalanche occurs, the occurrence of a new avalanche is less probable. Thus, to characterize this feedback mechanism a relation between p and the number of grains in the pile

N is needed.

The drag probability $p(\theta)$ is a function of the slope. Its value is determined by the competition of two contrary forces: the gravity, which conspires against the stability of the slope, and the friction which favors the slope stability. Since the slope forms an angle θ with the horizontal plane the component of the gravity force in the slope direction will be larger with the increase of this angle, varying from zero to a maximum value when θ goes from zero to $\pi/2$. Therefore it is plausible to assume that the contribution of the gravity and, therefore, the tendency of falling down the slope is proportional to $\sin\theta$. On the contrary, the resistance to this tendency given by the static friction decreases with decreasing the pile angle, varying from a maximum value to zero when θ goes from zero to $\pi/2$. Hence, it is also plausible to assume that the resistance to the falling down is proportional to $\cos\theta$.

Thus the slope dependence of p can be expressed through the ratio of both tendencies $\sin\theta/\cos\theta = \tan\theta$. Based on this hypothesis we may propose the exponential relation

$$p(\theta) = \exp(-A/\tan\theta) \; , \tag{1}$$

where A is a parameter determined by the gravitational field, the friction, and vibration intensity. Notice that $p(0) = 0$ and $p(\pi/2) = 1$. This selection for $p(\theta)$ can not be regarded as a sophisticated trick or something imposed *ad hoc* to the model in order to obtain the desired results, since it can be verified that *any* function satisfying the very general conditions already stated is adequate to our model. Incidentally, it must be said that this reveals the robustness of the model.

On the other hand, the number of grains in a pile with slope angle θ is given by

$$N = Bx^3 \tan\theta, \tag{2}$$

where B is a geometrical factor and $x = D/d$ is the ration between a characteristic size of the pile base D and sand grain d. Combining equations (1) and (2) it is obtained

$$p(N) = \exp(-cx^3/N) \; . \tag{3}$$

where $c = AB$. Thus Equation 1 relates the dragging probability with the number of grains in the pile.

The parameter c should not depend on the size of the system. It must be a function of parameters describing the vibrations, amplitude a and frequency ω, and of the gravitational field acceleration g. The only nondimensional combination of these magnitudes is given by the ratio between the vibration acceleration $a\omega^2$ and g and, therefore, $c = c(a\omega^2/g)$. This conclusion, obtained from dimensional analysis, is corroborated by experiments on sandpiles under vibrations[21] which show that the ratio $a\omega^2/g$ is the relevant parameter.

For a given value of c and x, there is a critical number of grains in the pile N_c. This value can be found recalling that in the Bethe lattice the critical value of p for percolation exists is $1/F$, resulting

$$N_c = cx^3/\ln F .\tag{4}$$

In a static sandpile (i.e., no vibrations) this value corresponds with the number of grains in the pile at the angle of repose.[17]

Another magnitude of interest is the penetration length, the number of steps $n(\theta)$ in the Bethe lattice for an avalanche to take place. This magnitude must be proportional to the length of the pile slope and, therefore, should be given by the expression

$$n(\theta) = x/\cos\theta\tag{5}$$

Notice that the possible existence of a geometrical factor in equation 5 may be absorbed in x, redefining B in Equation 2. Moreover, the relation between n and N may be easily obtained using Equation 2.

Equations 3 and 5 relate the parameters of the pile with those of its Bethe lattice representation. They were obtained here in a different way than that proposed by Zaperi *et al.*[17] Other dependencies between the dragging probability and the number of grains in the sandpile may be proposed. Notwithstanding, as it is discussed below, the precise form of this functional dependence is not relevant.

III. Simulations and results

The numerical experiment of relaxation is performed as follows. We start with a certain number of N. In a first step, we test if an avalanche takes place using the Bethe lattice representation. If it does, then we simulate the process and then recalculate the value of $p(N)$ by simply substracting the size of the avalanche to N and using Equation 3. The size of the avalanche is the number of nodes generated in the Bethe lattice which surpass the size of the pile whose measure is given by (5). Then this step is repeated again and again, whereas avalanche sizes and times are registered.

If avalanches are considered as instantaneous the number of steps is a measure of time. This approximation is valid for low vibration intensities. In this case grain jumps which trigger avalanches are rare events and, therefore, the time between two successive grain jumps will be much larger than the duration of avalanches.

To describe the behavior of our system, we define the relaxation function $\phi(t + t_w, t_w)$ of the sandpile as:

$$\phi(t + t_w, t_w) = \frac{N(t + t_w)}{N(t_w)},\tag{6}$$

Figure 1: Aging. Normalized relaxation function $\phi(t + t_w) = N(t + t_w)/N(t_w)$ taken at three different initial times t_w. The long time tail exhibited in the figure has an exponent $\alpha \simeq 0.089$.

where we include the time dependence of the number of grains in the sand-pile. t_w is the "waiting time", i.e., the instant at which we begin the counting of the number of grains since the start of the relaxation. (avalanche), as in.[9,10]

The existence of aging in our model is illustrated in Figure 1. We have plotted the normalized relaxation $\phi(t + t_w, t_w)$ at different ages (steps) of the system (simulation) by taking as initial time $t = t_w$ for $x = 10$ and $c = 1$. As it can be seen the relaxation becomes slower with increasing the age of the system, in agreement with experiments in structural and spin glasses[12] (and, unfortunately, with human life). The system exhibits a delay in the relaxation, after which the relaxation function decreases as a power law (long time tail) with an exponent $\alpha \simeq 0.089$. Thus, since the angle relaxation becomes slower with the age of the pile then it will never reach an equilibrium angle and, therefore, properties like translational invariance and the fluctuation dissipation theorem do not hold.[12]

Associated with these slow relaxation dynamics and aging phenomena we expect to observe a wide distribution of time between avalanches. As time increases the time between two consecutive avalanches Δt becomes larger, because after an avalanche the occurrence of a new avalanche becomes smaller. Therefore, it is expected that the mean time between avalanches diverges as $t \to \infty$. Hence, the distribution of time between avalanches $n(\Delta t)$ should satisfy the

Figure 2: Distribution of time between avalanches. The dashed line is a power tail with exponent $-1-\alpha$, α being the exponent of the long time tail shown in Figure 1.

asymptotic behavior for large Δt

$$n(\Delta t) \sim \Delta t^{-1-\beta}, \tag{7}$$

with $0 < \beta < 1$.

This hypothesis is confirmed in our simulations. Figure 2 shows the distribution of time between avalanches for $x = 10$ and $c = 1$. It is approximately constant for small values and then it decays following a power law in more than two decades. The plateau at small values of Δt is associated to the rapid decay observed at short times while the tail for large Δt should be related to the long time tail. Here we observe that $\beta \simeq \alpha \simeq 0.089$ Thus, there should be some connection between the distribution of time between avalanches and the long time relaxation.

Finally we want to mention that these simulations were also carried out assuming other functional dependences between the dragging probability and the number of grains in the pile $p(N)$. In all cases the results where qualitatively similar to those presented here using equation 3, reflecting that the precise dependence is not relevant.

IV.   Conclusions

A sandpile model for aging was presented revealing that this phenomenon is manifested in relaxation of sandpile avalanches. Biological systems and individuals show similar behavior. Phenomena related to some kind of relaxation in live beings deserve more quantitative research. The introduction of a Bethe lattice representation for the avalanches and a feedback mechanism describes quite well the principal features of the relaxation in sandpiles under low intensity vibrations. The proposed representation leads to long time tails relaxation, aging and Lévy (fractal) distributions of time constants, which are characteristic properties of the dynamics of complex systems out of equilibrium.

A.   Acknowledgments:

References

[1] K. Binder, A. P. Young, Rev. Mod. Phys **58**, 801 (1986).

[2] J. Jackle, Rep. Prog. Phys. **49**, 171 (1986).

[3] B. Derrida, H. Spohn, J. Stat. Phys. **51**, 817 (1988).

[4] M. Mézard, G. Parisi, J. Phys. A **23**, L1229 (1990).

[5] S. John, T. C. Lubenski Phys. Rev. B **34**, 4815 (1986).

[6] G. Kriza, G. Mahály Phys. Rev. Lett **56**, 2529 (1986).

[7] T. J. P. Penna J. Stat. Phys. **78**, 1629 (1995), Preprint cond-mat/**9503099**

[8] R. M. C. de Almeida, C. Moukarzel Physica A **257**, 10 (1998).

[9] S. Boettcher, M. Paczuski Phys. Rev. Lett **79**, 889 (1997).

[10] S. Boettcher, Phys. Rev. E **56**, 6466 (1997).

[11] *Relaxation in Complex Systems and Related Topics*, edited by I. A. Cambell and C. Giovanella, NATO ASI series B222 (Plenum Press, New York, 1990).

[12] J. P. Bouchaud, L. F. Cugliandolo, J. Kurchan, and M. Mézard, *Out of Equilibrium Dynamics in Spin-Glasses and Other Glassy Systems*, cond-mat/**9702070**.

[13] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987).

[14] H. M. Jaeger, C. Liu, and S. R. Nagel, Phys. Rev. Lett. **62**, 40 (1989).

[15] G. A. Held, D. H. Solina, D. T. Keane, W. J. Haag, and G. Grinstein, Phys. Rev. Lett. **65**, 1120 (1990).

[16] C. Tang and P. Bak, J. Stat. Phys. **51**, 797 (1988).

[17] S. Zapperi, K. B. Lauritsen, and H. E. Stanley, Phys. Rev. Lett. **75**, 4071 (1995); K. B. Lauritsen, S. Zapperi, and H. E. Stanley, Phys. Rev. E **54**, 2483 (1996).

[18] L. P. Kadanoff, S. R. Nagel, L. Wu, and S. Zhou, Phys. Rev A **39**, 6524 (1989).

[19] O. Sotolongo-Costa, A. Vazquez, J. C. Antoranz, Preprint cond-mat/**9806176.**

[20] G. C. Barker and A. Metha, Phys. Rev. E **47**, 184 (1993);

[21] P. Evesque and J. Rajchenbach, Phys. Rev. Lett **62**, 44 (1989).