Some Closed Form Results for Circuit Switching in a Hypercube Network

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Abstract. We consider circuit switching in a hypercube network where each session has to establish a dedicated connection (or circuit) to a destination node for a random length of time. We first obtain an expression for the steady-state probability that a session successfully establishes a circuit, and then obtain analytic expressions for various delay parameters, such as the time between the arrival of a session and the time it is completed (including input queueing delays). The analytic expressions allow the delay parameters to be calculated without performing costly fixed-point iterations, and show how these parameters depend on the session arrival rate and the hypercube dimension.

1 Introduction

In this paper we consider circuit switching in a hypercube network of processors. In previous analytical and simulation approaches ([GeR88], [GeR89],[BrS91] [CGK91]), queueing at the input played no role and the total delay of a session was not calculated. Furthermore, approximations such as the independence of link acquisitions were used, the solutions obtained were of a numerical or recursive nature, and the delay parameters were not obtained in a closed-form.

In our model, connection requests are generated at each node of a 2^d-node hypercube according to a Poisson process with rate λ independently of other nodes, and session destinations are uniformly distributed over the remaining nodes. In our routing scheme, the setup packet that is sent to the destination to establish a connection crosses the hypercube dimensions in a random order. A source node has d link-input queues, each which has infinite buffer space and uses a FIFO queueing discipline. When all sessions ahead of a session have been served, the session advances to the head of the queue, and a set-up packet is sent to establish a circuit. If the set-up packet is successful in establishing a connection, the links required by the circuit are reserved for the session duration, and the session is served without interruptions. If the circuit cannot be established, a retrial is made after a random time, and this process is repeated until a connection for the session is finally established.

We first obtain an analytic expression for the steady-state probability $P_{success}$ that a new session arriving at a random time successfully establishes a connection. In our case, this calculation, which has so far required the iterative solution of an approximate Markov chain, is simplified through the characterization of a network link by a small number of mutually exclusive states, whose probabilities can be calculated explicitly (under the model assumed, which is more complete than that of others, since it allows for input queueing). We then derive analytical results for the average queueing delay, the average connection delay, the average waiting time, and the average total delay required to serve a connection request in a d-dimensional hypercube.

2 Probability of Successfully Establishing a Circuit

We define a *continuing circuit* at a node s as a circuit for which node s is an intermediate node on the path, and a *starting circuit* at a node s as a circuit for which node s is the origin. A network link L is at any time in one of three states: state 0, which corresponds to L being idle, state 1, which corresponds to L being used by a continuing circuit at node s (and has $d-1$ substates, depending on the dimension from which the circuit turns into node s), and state 3, which corresponds to L being used by a starting circuit at node s. We denote by q_i, $i = 0, 1, 2$, the steady-state probability that link L is in state i, where $q_0 + q_1 + q_2 = 1$.

By using Little's Theorem and taking into account the routing scheme used, we obtain, after some analysis, that

$$q_0 = 1 - \frac{\lambda 2^{d-1}\overline{X}}{2^d - 1}, \tag{1}$$

$$q_1 = \frac{\lambda \overline{X}}{d(2^d - 1)}\left[(d-2)2^{d-1} + 1\right], \tag{2}$$

and

$$q_2 = \frac{\lambda \overline{X}}{d}, \tag{3}$$

where λ is the session arrival rate per node, and $\overline{X}$ is the mean session holding time.

For uniformly distributed destinations, the probability of successfully establishing a connection can be found to be [ShV96]

$$P_{success} = \frac{q_0}{\alpha(2^d - 1)} \left[(1+\alpha)^d - 1 \right], \text{ where} \tag{4}$$

$$\alpha = \Pr(L \text{ available} \mid L \text{ not in a given substate of state } 1) = \frac{q_0}{1 - \frac{q_1}{d-1}}. \tag{5}$$

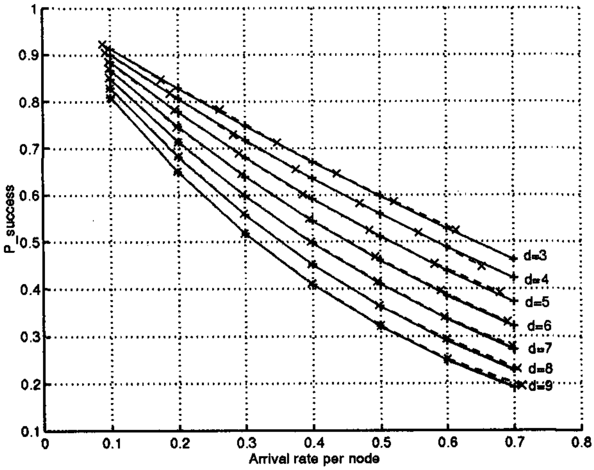

Figure 1: Analytical (solid, +) and simulation (dashed, $\times$) results obtained for $P_{success}$ as a function of the arrival rate λ of sessions, for hypercubes of dimensions ranging from $d = 3$ to $d = 9$. The curves correspond to exponentially distributed holding times with mean $\overline{X} = 1$.

3 Queuing Analysis at Link Input-buffer

In our model each hypercube node is a cross-bar switch and has d link entry-buffers, one for each outgoing link. The holding time of session i, denoted by X_i, can be any random variable whose mean $\overline{X}$ and second and third moments $\overline{X^2}$ and $\overline{X^3}$ are known. The delay incurred by the session consists of several components: the *residual time* R_i, which is the time it takes for the the session currently at the head of the queue to finish, and depart from the system; the *connection delay* C_i, which is the time that session i must wait until the connection to its destination is established, since more than one trials (the details of which appear in [ShV96]) may be required to establish the connection; and the *queueing delay*

$$Q_i = R_i + \sum_{j=i-N_i}^{i-1} (X_j + C_j), \tag{6}$$

which is the time it takes to serve the N_i sessions (excluding the session in service) found in queue by session i upon its arrival plus the residual time R_i. The total *waiting time* of session i at the queue includes its own connection delay, and is $W_i = Q_i + C_i$, while the *total delay* is the time that elapses between the arrival of the session and the time it is completed, and is given by $T_i = X_i + W_i$.

We assume that when a session in the input-buffer queue of a link tries to establish a connection, it finds the network in steady-state, except that the first link on its path is not used by a session starting at that link. The probability P_h that a session at the head of a link input-buffer successfully establishes a circuit can then be found to be

$$P_h = \frac{P_{success}}{1 - q_2}. \tag{7}$$

Modulo our approximating assumption A1, the mean connection delay works out to be

$$\overline{C} = \frac{(1 - P_h)}{P_h} \frac{\overline{X^2}}{2\overline{X}} + \frac{\overline{V}}{P_h}. \tag{8}$$

Taking expectations in Eq. (6), and taking the limit as $i \to \infty$, we obtain

$$Q = R + N_Q \left(\overline{X} + \overline{C}\right). \tag{9}$$

where N_Q is the mean number of sessions in queue, and R is the mean residual time. Using Little's Theorem, we have $N_Q = (\lambda/d)Q$. Letting $\rho = \lambda\left(\overline{X} + \overline{C}\right)/d$, and substituting into Eq. (9), we obtain $Q = R + \rho Q$, or equivalently,

$$Q = \frac{R}{1 - \rho}. \tag{10}$$

The mean residual time R can be calculated using well-known graphical arguments (see [BeG92], Chap. 3) to be equal to

$$R = \frac{\lambda}{2d}\overline{(X + C)^2}. \tag{11}$$

Finally, the mean queueing delay is given by

$$\begin{aligned} Q = {} & \frac{\lambda\left(\overline{X^2} + \overline{V^2} + 2\overline{X}\,\overline{V}\right)}{4d(1-\rho)P_h} + 2\frac{\lambda(1 - P_h)}{4d(1-\rho)P_h}\overline{V}\left(\overline{V} + \frac{\overline{X^2}}{\overline{X}}\right) \\ & + \frac{\lambda(1 - P_h)^2}{4d(1-\rho)P_h^2}\left(\frac{\overline{X^2}}{2\overline{X}}\right)^2 + \frac{\lambda(1 - P_h)}{4d(1-\rho)P_h}\left(\frac{\overline{X^3}}{3\overline{X}}\right). \end{aligned} \tag{12}$$

The average waiting time W of a session is given by $W = Q + \overline{C}$, where $\overline{C}$ is given by Eq. (8), and the average total delay is $T = Q + \overline{C} + \overline{X}$.

We simulated both the analytical model and a model of the physical system in which we accounted for the overhead incurred by the setup packets during the reservation phase. Figure 2 presents the main results of our simulations, where we have assumed that the session holding times and times between retrials are both exponential with means $\overline{X} = 1$ and $\overline{V} = 0.5$, respectively.

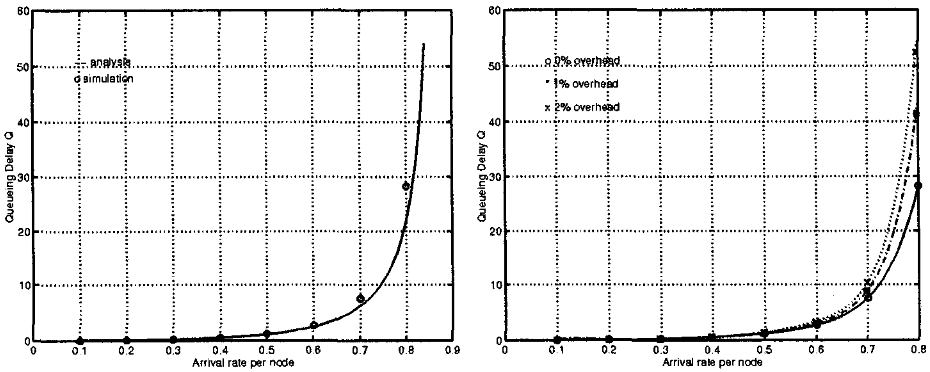

Figure 2: The queueing delay Q versus the arrival rate per node λ, for a hypercube of dimension $d = 8$. The first plot shows the analytically predicted values of the queueing delay Q and the corresponding values obtained through simulations when the setup overhead is equal to zero. The second plot shows the simulation results when the setup overhead is accounted for and is is equal to 0%, 1% and 2%, respectively, of the mean session holding time.

References

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