The Path–Connectedness in Z^2 and Z^3 and Classical Topologies

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Abstract. The main problem we persue in this paper is the question of when a given path–connectedness in Z^2 and Z^3 coincides with a topological connectedness. We answer this question provided the path–connectedness is induced by a homogeneous and symmetric neighbourhood structure. On the way we make a study of topological structures, arguing that the point–neighbourhood formalism can be well applied in the digital picture investigations.

1 Introduction

The structure of digital images is strongly related to the topology of the underlying grids, Z^2 or Z^3. Topological properties are often insensitive to (small) geometric distortions that occur in images and disturb the reliable recognition of objects. Connectivity and region adjacencies are typically derived from grid adjacencies after segmentation and constitute a major component of the image structure. Moreover, 3D sensors and image sequences introduce further discrete dimensions which need topological structure.

Following [12] and the motivation therein, we ask when the graph–theoretic path–connectedness of a given digital picture, which was originally induced by an adjacency relation, is determined by the classical topological connectedness. The topological formalism we choose is the one based on the point neighbourhoods. We first show how this formalism coincides, up to a categorical equivalence, with the open–set formalism and closure–operator formalism. This is useful in order to establish a link to the previous results where other formalisms were used – see e. g. [10], [12] – and, also, to prepare the stage for potential applications elsewhere. We then use the point–neighbourhood formalism to find a simple classification of all homogeneous and symmetric adjacency neighbourhood structures in Z^2 (resp. Z^3) so that their path–connectedness be topological. This extends the results of [4] and [10], and supplements [11]. As an initial step in studying nonhomogeneous adjacencies we consider products of topologies which induce typical adjacencies. We obtain rather interesting topologies in this way.

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2 The Point–Neighbourhood Definition of Topology

In this note we will exclusively use the point–neighbourhood definition of topological space. This approach is less standard but it seems quite handy in the context of digital picture studies. Before we go on in justifying the latter statement, let us present formal definitions and establish (categorical) equivalences with other approaches. We do not need more of the theory of categories than the mere definition of category and functor (see e. g. [1] and [8]). (We will generally follow [2], adjusting the exposition to our purpose and to a potential use in image processing. In the notation we will write $\exp P$, for a set P, to denote the set of all subsets of P.)

Definition 2.1: Let P be a nonempty set. Let us assign, to each $x \in P$, a set $\mathscr{N}(x)$, $\mathscr{N}(x) \subset \exp P$ such that

(N1) if $U \in \mathscr{N}(x)$, then $x \in U$,
(N2) if $U \in \mathscr{N}(x)$ and $U \subset V$ $(V \subset P)$, then $V \in \mathscr{N}(x)$,
(N3) if $U, V \in \mathscr{N}(x)$, then $U \cap V \in \mathscr{N}(x)$,
(N4) if $U \in \mathscr{N}(x)$, then there is a set $V \in \mathscr{N}(x)$ such that, for each $y \in V$, $U \in \mathscr{N}(y)$.

Then the set P together with the assignement $x \to \mathscr{N}(x)$ is called a **topological space**. The set $U \in \mathscr{N}(x)$ is called a (topological) neighbourhood of x.

If we stand n for the assignment $n\colon x \to \mathscr{N}(x)$, we can (and shall) refer to the couple (P, n) as the corresponding topological space.

Let us now define the appropriate morphisms between topological spaces.

Definition 2.2: Let (P, n) and (Q, m) be two topological spaces, and let $f\colon P \to Q$ be a mapping. We say that f is **continuous** if the following condition is satisfied: For each $x \in P$ and for each $V \in \mathscr{N}(f(x))$ (taken in m) the set $f^{-1}(V) = \{y \in X | f(y) \in V\}$ belongs to $\mathscr{N}(x)$ (in n). In other words, it is required that the preimages of the neighbourhoods of $f(x)$, for each $x \in P$, are neighbourhoods of x.

It is easily seen that topological spaces (taken for the objects) and continuous mappings (taken for the morphisms) form a category (see e. g. [1] or [8]). Let us denote this category by $\mathscr{T}_n$.

3 Topological Spaces Defined by Open Sets and by Closure Operation. The Equivalence with the Point–Neighbourhood Approach.

In this paragraph we want to show how one can pass from one definition of topology to another whenever there is a need for that. Other commonly used definitions of topological space are either based on the specification of open sets – an approach previously used in digital pictures (see [3], [4], [6], [7], [10], [12], [13], etc.) – or on the closure operation in the sense of the classical definition.

Definition 3.1: Let P be a nonempty set and let $\mathcal{O} \subset \exp P$ such that

($\mathcal{O}1$) $\emptyset \in \mathcal{O}, P \in \mathcal{O}$,

($\mathcal{O}2$) if $A_1, A_2, \ldots, A_n \in \mathcal{O}$, then $\bigcap_{i=1}^{n} A_i \in \mathcal{O}$,

($\mathcal{O}3$) if $A_i \in \mathcal{O}$ ($i \in I$, I is an arbitrary set), then $\bigcup_{i \in I} A_i \in \mathcal{O}$.

The set P together with the collection $\mathcal{O}$ (resp. the couple $(P, \mathcal{O})$) is called **a topological space with given open sets.** The morphisms are then defined as follows.

Definition 3.2: Let $(P, \mathcal{O})$, $(Q, \mathcal{V})$ be topological spaces in the sense of Def. 2.1. Let $f \colon P \to Q$ be a mapping. We say that f is continuous if $f^{-1}(A) \in \mathcal{O}$ for any $A \in \mathcal{V}$.

Again, the topological spaces in the sense of Def. 3.1 together with continuous mappings in the sense of Definition 3.2 constitute a category. Let us denote this category by $\mathcal{T}_{op}$.

The third frequently used approach to general topologies is the approach based on the closure operation of sets. This definition reads as follows.

Definition 3.3: Let P be a nonempty set. Let $^- \colon \exp P \to \exp P$ be a mapping such that the following conditions are satisfied:

($^-$ 1) $\bar{\emptyset} = \emptyset$,

($^-$ 2) $X \subset \bar{X}$ for each $X \in \exp P$,

($^-$ 3) $\overline{X \cup Y} = \bar{X} \cup \bar{Y}$ for each $X, Y \in \exp P$,

($^-$ 4) $\bar{\bar{X}} = \bar{X}$ for each $X \in \exp P$.

Then the set P together with the mapping $^- \colon \exp P \to \exp P$ is called **a topological space given by closure operation.**

Definition 3.4: Let $(P, ^-)$, $(Q, \tilde{})$ be two topological spaces in the sense of Def. 3.3. Let $f \colon P \to Q$ be a mapping. We say that f is continuous if $f(\bar{X}) \subset \widetilde{f(X)}$ for each $X \in \exp P$.

Once again, the topological spaces in the sense of Def. 3.3 together with continuous mappings in the sense of Def. 3.4 constitute a category. Let us denote this category by $\mathcal{T}_-$.

Thus, we have introduced three "topological" categories. We will use the category $\mathcal{T}_n$ (later on denoted simply by $\mathcal{TOP}$) for the topological considerations of digital pictures which follow. We would like to advocate this approach since the topological phenomena of image processing seem to be best expressed in this language. Since the previous considerations mostly used the approach via $\mathcal{T}_{op}$ or $\mathcal{T}_-$ and since sometimes it may be practical to pass from one formalism to the other, let us see why (how) these three categories are equivalent. This question will be addressed in the next paragraph.

4 Equivalence of the Topological Categories $\mathcal{T}_n$, $\mathcal{T}_{op}$, and $\mathcal{T}_-$.

Let us recall when two categories are said to be equivalent (see e. g. [8] or [1]). This notion just precises the intuitive feeling of when two structures possess the same intrinsic quality.

Definition 4.1: Let $\mathcal{K}$ and $\mathcal{L}$ be categories. We say that they are equivalent if there are two functors $F\colon \mathcal{K} \to \mathcal{L}$ and $G\colon \mathcal{L} \to \mathcal{K}$ such that $F \circ G = Id_{\mathcal{K}}$ and $G \circ F = Id_{\mathcal{L}}$ (the symbols $Id_{\mathcal{K}}$, $Id_{\mathcal{L}}$ mean the respective identity functors).

Theorem 4.2: The categories $\mathcal{T}_n$, $\mathcal{T}_{op}$ and $\mathcal{T}_-$ are mutually equivalent.
Proof: We will only indicate the basic ideas of the proof, the details being a routine verification. The manner how we proceed is clearly seen from the proof of the first equivalence. We therefore allow ourselves to define the respective functors and leave the verification as simple excercise.

1. Let us show first that the categories $\mathcal{T}_n$ and $\mathcal{T}_{op}$ are equivalent. Define the functors $F\colon \mathcal{T}_n \to \mathcal{T}_{op}$ and $G\colon \mathcal{T}_{op} \to \mathcal{T}_n$ as follows:

(i) If $(P, n) \in \mathcal{T}_n$, then $F(P, n) = (P, \mathcal{O})$, where

$$\mathcal{O} = \{A \subset P |\ \text{for each } x \in A \text{ there is } \mathcal{N}(x) \in n \text{ such that } \mathcal{N}(x) \subset A)\};$$

(ii) If $(P, \mathcal{O}) \in \mathcal{T}_{op}$, then $G(P, \mathcal{O}) = (P, n)$, where

$$\mathcal{N}(x) = \{V \subset P |\ \text{there is a set } A \in \mathcal{O} \text{ such that } x \in A \text{ and } A \subset V\}.$$

As regards the morphisms, let us define both F and G to be identities (i.e., $F(f) = f$, $G(f) = f$). We claim that both F and G are functors and $F \circ G = Id_{\mathcal{T}_n}$ and $G \circ F = Id_{\mathcal{T}_{op}}$. In the proof of this statement, we first need to verify that the definition is correct (i.e., F and G are well defined on the objects of $\mathcal{T}_n$ and $\mathcal{T}_{op}$). But this is straightforward. What requires a more careful checking is the fact that $F \circ G = Id_{\mathcal{T}_n}$ (resp. $G \circ F = Id_{\mathcal{T}_{op}}$). Let us consider the composition $F \circ G$ ($G \circ F$ can be treated similarly). Take a space (P, n). We have to show that if $x \in P$ and $V \in \mathcal{N}(x)$, then there is an open set $A \in F(P, n)$ such that $x \in A \subset P$. By our definition, the open sets in $F(P, n)$ are exactly those sets which contain each point together with a neighbourhood of the space (P, n). We need find an $A \in F(P, n)$ with this property. Take the set $P - V$ and put $\overline{P - V} = \{p \in P |$ for each $U \in \mathcal{N}(p)$ we have $U \cap (P - V) \neq \emptyset\}$. Then it can be shown that if we put $A = P - (\overline{P - V})$, we have $x \in A$ and $A \in F(P, n)$. Indeed, $x \in A$ since V is a neighbourhood of x and therefore $x \notin (\overline{P - V})$. Further, take an arbitrary point $a \in A$. Thus, $a \in P - (\overline{P - V})$. It follows that $a \notin (\overline{P - V})$ and therefore there is a neighbourhood, $U \in \mathcal{N}(a)$, such that $U \cap (P - V) = \emptyset$. According to axiom (N4), there is a neighbourhood, $W \in \mathcal{N}(a)$, such that U is a neighbourhood of each $w \in W$. If $W \cap (\overline{P - V}) \neq \emptyset$, then there is a point $z \in W \cap (\overline{P - V})$. But U is a neighbourhood of z and $U \cap (P - V) = \emptyset$. Thus, $z \notin \overline{P - V}$ which is a contradiction. We have proved that for each $a \in A$ there is a neighbourhood $W \in \mathcal{N}(a)$ so that $W \cap (\overline{P - V}) = \emptyset$. This means that each $a \in A$ has a neighbourhood which is a subset of A. In other words, the set $A = P - (\overline{P - V})$ is an "open" set containing x. We see that for each point $x \in P$ and each neighbourhood $V \in (P, n)$ there exists an "open set" $A \in F(P, n)$ such that $x \in A$ and $A \subset V$. As a consequence, $F \circ G = Id_{\mathcal{T}_n}$. The identity $G \circ F = Id_{\mathcal{T}_{op}}$ can be derived analogously.

2. Let us show that the categories $\mathcal{T}_n$ and $\mathcal{T}_-$ are equivalent. We will only determine the functors $F\colon \mathcal{T}_n \to \mathcal{T}_-$ and $G\colon \mathcal{T}_- \to \mathcal{T}_n$ so that $F \circ G = Id_{\mathcal{T}_n}$ and $G \circ F = Id_{\mathcal{T}_-}$.

(i) Suppose that $(P, n) \in \mathcal{T}_n$. If $X \subset P$, take $\bar{X} = \{p \in P |$ for each $U \in \mathcal{N}(x), U \cap X \neq \emptyset\}$. Then $^-\colon \exp P \to \exp P$ is a closure operation on P, and we set $F(P, n) = (P,^-)$.
(ii) Suppose that $\underline{(P,^-) \in \mathcal{T}_-}$. If $x \in P$, take $\mathcal{N}(x) = \{U \subset P|$ $x \in U$ and $x \in P - (\overline{P - U})\}$. Then the collection $\mathcal{N}(x)$, for each $x \in P$, forms the sets of all neighbourhods in the sense of Def. 1.1. Let us denote by (P, n) the corresponding topological structure. Then it suffices to set $G(P,^-) = (P, n)$.

3. Let us finally show that categories $\mathcal{T}_{op}$ and $\mathcal{T}_-$ are equivalent. Let us construct the functors $F\colon \mathcal{T}_{op} \to \mathcal{T}_-$ and $G\colon \mathcal{T}_- \to \mathcal{T}_{op}$ such that $F \circ G = Id_{\mathcal{T}_{op}}$ and $G \circ F = Id_{\mathcal{T}_-}$.

(i) Suppose that $(P, \mathcal{O}) \in \mathcal{T}_{op}$. If $X \subset P$, take $\bar{X} = \{p \in P|$ for each $A \in \mathcal{O}, A \cap X \neq \emptyset\}$. Then $^-\colon \exp P \to \exp P$ is a closure operation on P, and it suffices to set $F(P, \mathcal{O}) = (P,^-)$.
(ii) Suppose that $(P,^-) \in \mathcal{T}_-$. Put $\mathcal{O} = \{A \subset P \mid \overline{P - A} = P - A\}$. Then $\mathcal{O}$ fulfils the axioms $\mathcal{O}1 - \mathcal{O}3$ of Def. 1.3. Thus, $(P, \mathcal{O})$ is a topological space given by open sets, and it suffices to set $G(P,^-) = (P, \mathcal{O})$. One easily checks that $F \circ G = Id_{\mathcal{T}op}$ and $G \circ F = Id_{\mathcal{T}_-}$.

The previous result establishes the equivalence of the definitions of topologies. Thus, we can use the same symbol, say $\mathcal{T}OP$ for each of them (though we shall exclusively use the point–neighbourhood approach). The "translation" into other formalisms, when needed here or elsewhere, can be easily done by the procedures we outlined above.

5 Path–Connectedness and Topological Connectedness

Let us briefly recall the formulation of our problem (see [4], [12], etc.). Let Z be the set of all integers. Consider a digital picture in the Cartesian product Z^2 (resp. Z^3). This can be understand as a subset of Z^2 (resp. Z^3) together with a given adjacency neighbourhood assigned to each point. Thus, for instance, we can talk on the standard 6-neighbourhood structure in Z^3 if each point $(x, y, z) \in Z^3$ is given the following adjacency neighbourhood structure:

$$(x,y,z), (x-1,y,z), (x+1,y,z), (x,y-1,z), (x,y+1,z), (x,y,z-1), (x,y,z+1).$$

The expression of 6-neighbourhood structure refers here to the fact that the adjacency of a given point introduces 6 new points. We will use this expression in case of **homogeneous and symmetric adjacencies** (homogeneous means that the adjacencies neighbourhoods are geometrically identical at each point, and symmetric means that the adjacency neighbourhoods are symmetric with respect to each "centre" point). We will be exclusively interested in these adjacencies. We will also assume for the sake of a lucid formulation of our results that **no adjacency in Z^2 exceeds the standard 8-adjacency** and **no adjacency in Z^3 exceeds the standard 26-adjacency**.

Suppose that we are given an adjacency neighbourhood structure. Let us call it $\mathscr{S}$. Thus, to each point x we are given an adjacency neighbourhood, some set S_x. Suppose $p, q \in Z^2$ (resp. Z^3). Let us say that p, q are $\mathscr{S}$-related if $q \in S_p$. Let us say that p, q are $\mathscr{S}$-path-related if there is a finite sequence $p_1, \ldots, p_n$ such that $p_1 = p, p_n = q$ and each points p_i, p_{i+1} $(i \leq n)$ are $\mathscr{S}$-related.

Let X be a subset of Z^2 (resp. Z^3). Restrict the relation of $\mathscr{S}$-path-connectedness to X. Since $\mathscr{S}$ is obviously an equivalence relation, it follows that $\mathscr{S}$ decomposes X into the corresponding equivalence classes.

We are in a position to formulate the question dealt with in this paper. **Let us suppose that $\mathscr{S}$ is an adjacency neighbourhood structure on Z^2 (resp. Z^3). We ask if there is a topology on Z^2 (resp. Z^3) so that, for each $X \subset Z^2$ (resp. Z^3), the $\mathscr{S}$-path-connectedness on X coincides with the topological connectedness given by t.** This topology, if it exists, may (and will) be called the **topology compatible with $\mathscr{S}$**.

The above formulated question seems to be first posed in [12] and [4]. The results obtained so far to which we want to contribute here, can be found in [5], [9], [10], [3], [13] and [6]).

Let us recall the topological notions we need and let us also state the formulation of our question in the point–neighbourhood setup. Let $(P, t) \in \mathscr{T}OP$ be understood in the sense of point–neighbourhoods. We say that P is **disconnected** if there is a partition of P into two open (or, equivalently, closed) sets. Thus, P is disconnected if it allows for a decomposition $P = R \cup S$, where $R \cap S = \emptyset$ and both R, S are open in P. If P is not disconnected it is called connected. Finally, a set $X \subset P$ is called connected if it is connected in the topology of the topological subspace of P. (Each set $X, X \subset P$, can be naturally given a topology inherited from (P, t) – one assigns $x \to U \cap X, U \in \mathscr{N}(x)$. With this topology, X is called a topological subspace of P.)

6 Characterizing Homogeneous and Symmetric Adjacencies in Z^2 and Z^3.

In [11] the authors applied the topological point–neighbourhood formalism to study homogeneous and symmetric adjacencies in Z^2. Reproving first known results in a unified and simpler way and adding to them, they then found a characterization of homogeneous and symmetric neighbourhood adjacencies which allow for compatible topologies. The highlights of the results are as follows.

Proposition 6.1: Let $\mathscr{S}$ be a homogeneous and symmetric adjacency on Z^2. Then $\mathscr{S}$ allows for a topology compatible with $\mathscr{S}$ if and only if $\mathscr{S}$ does not exceed a 4-adjacency. For any 4-adjacency, there are exactly two topologies compatible with $\mathscr{S}$.

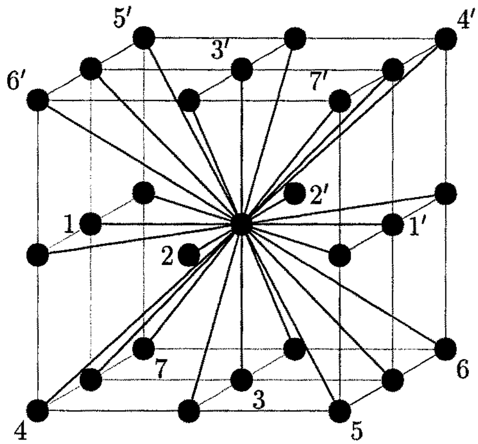

Fig. 1.

When we investigate homogeneous and symmetric adjacencies in Z^3, we can therefore restrict ourselves to those adjacencies which are essentially three dimensional. Let us say that an $\mathscr{S}$-neighbourhood, S, of the point (x, y, z) is **essentially three dimensional** if $S - L \neq \emptyset$ for each (digital) plane L which passes through the point (x, y, z). The previous result in Z^2 can be translated in Z^3 as follows.

Proposition 6.2: Suppose that $\mathscr{S}$ is a homogeneous and symmetric neighbourhood adjacency in Z^3. Suppose that $\mathscr{S}$ is *not* essentially three dimensional. Then there is a topology on Z^3 compatible with $\mathscr{S}$ if and only if $\mathscr{S}$ does not exceed the 4-adjacency.

Let us take up the case of essentially three dimensional adjacencies. The result we obtain is formulated in the following theorem. It should be noted that it supplements and extends known results ([3], [4], [9], [10], [13]). Again, we would like to advocate the point–neighbourhood approach, for the proofs then become simplified.

Theorem 6.3: Suppose that $\mathscr{S}$ is a homogeneous and symmetric neighbourhood adjacency in Z^3. Suppose that $\mathscr{S}$ is essentially three dimensional. Then if $\mathscr{S}$ is a 6-adjacency, then there are exactly two topologies on Z^3 which are compatible with $\mathscr{S}$. If $\mathscr{S}$ exceeds a 6-adjacency, then there is a topology on Z^3 compatible with $\mathscr{S}$ if and only if $\mathscr{S}$ does not exceed the spatial 14-adjacency (see Fig. 1).

Proof: The basic idea of the proof can be formulated as a lemma (the assumptions being those of Th. 6.3):

Lemma: If S is the $\mathscr{S}$–adjacency neighbourhood of a point $(x, y, z) \in Z^3$ and if t is a topology compatible with $\mathscr{S}$, then either S or $\{(x, y, z)\}$ must be a (topological) neighbourhood of the point (x, y, z) in t (i.e., either $S \in \mathscr{N}(x)$ or $\{(x, y, z)\} \in \mathscr{N}(x)$, where $\mathscr{N}(x) \in t$). A corrolary: Each point $(x, y, z) \in Z^3$ has a smallest neighbourhood.

In order to prove this lemma, observe first that each topological neighbourhood in t of (x, y, z) must be a subset of S. Indeed, if we write $Q_1 = \{(x, y, z)\}$ and $Q_2 = Z^3 - S$, then both Q_1 and Q_2 are obviously $\mathscr{S}$-path-connected but the set $Q_1 \cup Q_2$ is not. It follows that there is a topological neighbourhood in t of the point (x, y, z), some set U, which does not intersect Q_2. But then $U \subset Z^3 - (Z^3 - S)$ and therefore $U \subset S$.

Let U be a smallest topological neighbourhood in t of (x, y, z) (it must exist by the lemma above). We now have to show that if there is $P \in U$, $P \neq (x, y, z)$, then $U = S$. Thus, we have to show that if $Q \in S$ then $Q \in U$. Since the reasoning needed in verifying this statement in general is fully analogous to the standard 6-adjacency case, we will present the proof for this case. Suppose without any loss of generality that $P = (x - 1, y, z)$ and $Q = (x, y + 1, z)$ (otherwise we will just rename the points). Suppose that $Q \notin U$. Since the set $\{P, Q\}$ is not connected and the set $\{(x, y, z), Q\}$ is connected, it follows that the smallest neighbourhood of Q, some set V, must not contain the point P and must contain the point (x, y, z). It follows that $(x, y, z) \in U \cap V$. For an obvious connectedness reason, the set $U \cap V$ cannot contain a point different from (x, y, z). Thus, $U \cap V = (x, y, z)$. Since V is the smallest topological neighbourhood of Q, it must also be a topological neighbourhood of (x, y, z) (Def. 1.1, the property (N4)). Thus, $U \cap V$ must also be a topological neighbourhood of (x, y, z) which is a contradiction. This proves the lemma.

Let us return to the proof of Th. 6.3. If $\mathscr{S}$ is a 6-adjacency, then we necessarily have to assign to each point either the singleton neighbourhood or the corresponding 6-neighbourhood (see Lemma). This correctly induces only two possible topologies t_1, t_2 (generalized Marcus-Wyse topologies, see [13]). A simple inductive argument then shows that these topologies are compatible with the general 6-adjacency (this has already been demonstrated in [13]).

As regards higher adjacencies, the situation is transparently seen in Fig. 1. If we add the diagonals 44', 55', 66' and 77' to the 6-adjacency, we can carry on the Marcus-Wyse construction without arriving to a contradiction. This gives us a topology compatible with the given adjacency. If, however, the adjacency exceeds the spatial 14-adjacency (i.e., if the adjacency neighbourhoods of $\mathscr{S}$ is properly larger than the set $\{1, 1', 2, 2', 3, 3', 4, 4', 5, 5', 6, 6', 7, 7'\}$), then there must be a digital plane in Z^3 the restriction of $\mathscr{S}$ to which properly exceeds a 4-adjacency in Z^2. But this is absurd in view of Prop. 6.1. The proof is finished.

7 Taking Products of Adjacency Topologies

Let us observe a fact which seems interesting in its own right and, also, it is relevant to the investigation of nonhomogeneous adjacencies. Let us ask the following natural question: If adjacency structures $\mathscr{S}_i$ $(i \leq n)$ do allow for compatible topologies, is the same property preserved for the Cartesian products of $\mathscr{S}_i$'s, and which new topologies can come into existence this way? The answer to the former question is unfortunately no — if we take for $\mathscr{S}_i$ $(i = 1, 2, 3)$ the 2-adjacency on Z and for t_i $(i = 1, 2, 3)$ the Marcus-Wyse topologies on Z, then $\mathscr{S} = \prod_{i=1}^{3} \mathscr{S}_i$ is the 26-adjacency on Z^3 and this adjacency does not allow for any topology. Nevertheless, the topological product

$\prod_{i=1}^{3}(Z, t_i)$ constitutes an interesting nontrivial topology (the faces of all dimensions are properly employed for neighbourhoods). Topologies like the latter product topology might have some bearing on theoretical image processing (compare also with [7]).

8 Conclusions

The contribution of this paper is essentially threefold. Firstly, the equivalence of three topological categories previously used in digital topologies is established. The proof presented by the authors is based on simple reasoning and it is thus accessible to non-specialists in topology. Secondly, it is demonstrated how the point–neighbourhood topological formalism is often handier in digital picture studies than the other formalisms. Finally, the characterization of the homogeneous and symmetric path–connectedness which allows for a topological connectedness is found. In Z^2, this is exactly the path–connectedness which does not exceed the 4–adjacency, and in Z^3 this is exactly the path–connectedness which does not exceed the spaciel 14–adjacency. In the last section, an observation on the topological product construction is made.

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