Type Introduction for Equational Rewriting*

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Abstract

Type introduction is a useful technique for simplifying the task of proving properties of rewrite systems by restricting the set of terms that have to be considered to the well-typed terms according to any many-sorted type discipline which is compatible with the rewrite system under consideration. A property of rewrite systems for which type introduction is correct is called persistent. Zantema showed that termination is a persistent property of non-collapsing rewrite systems and non-duplicating rewrite systems. We extend his result to the more complicated case of equational rewriting. As a simple application we prove the undecidability of AC-termination for terminating rewrite systems. We also present sufficient conditions for the persistence of acyclicity and non-loopingness, two properties which guarantee the absence of certain kinds of infinite rewrite sequences. In the final part of the paper we show how our results on persistence give rise to new modularity results.

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1 Introduction

Term rewriting is an important method for equational reasoning. In term rewriting the axioms of the equational system under consideration are used in one direction only. Since in the presence of axioms like commutativity, a common situation in equational reasoning, rewriting is non-terminating, the framework of equational term rewriting has been proposed. Equational term rewriting is an extension of rewriting in which certain axioms are used bidirectionally, more precisely, an equational rewrite system $\mathcal{R}/\mathcal{E}$ consists of a term rewriting system $\mathcal{R}$ and an equational system $\mathcal{E}$ and a term s rewrites in one step to a term t if there exists a rewrite rule $l \to r$ in $\mathcal{R}$ and a substitution σ such that s is equivalent (in the equational theory generated by $\mathcal{E}$) to a term s' which contains $l\sigma$ and t is equivalent to the term t' obtained from s' by replacing $l\sigma$ by $r\sigma$.

Here we are interested in termination of equational rewrite systems. An early paper on termination of equational rewriting is Jouannaud and Muñoz [11]. In that paper sufficient conditions are given for reducing (equational) termination of $\mathcal{R}/\mathcal{E}$ to termination of $\mathcal{R}$. In another early paper (Ben Cherifa and Lescanne [4]) a characterization is given of the polynomials that can be used in a polynomial interpretation proof of AC-termination, i.e., termination of equational rewrite systems $\mathcal{R}/\mathcal{E}$ where $\mathcal{E}$ consists of the associativity and commutativity axioms $f(f(x,y),z) \approx f(x,f(y,z))$ and $f(x,y) \approx f(y,x)$ for (some of) the binary function symbols in $\mathcal{R}$. In more recent papers [12, 19, 20, 21] syntactic methods like the well-known recursive path order for proving termination of rewriting are extended to AC-termination. Another recent paper is Ferreira [8] where the dummy elimination technique of [9] for proving termination is extended to equational rewriting.

In this paper we extend the type introduction technique of Zantema [22] for proving properties of rewriting to equational rewriting. More precisely, we show that termination is a persistent property of equational rewrite systems $\mathcal{R}/\mathcal{E}$ such that $\mathcal{R}$ does not contain both collapsing and duplicating rules and $\mathcal{E}$ is variable preserving and does not contain collapsing axioms. Type introduction is known to be useful for proving undecidability results for termination of rewriting [15], and in this paper we give a simple proof of the undecidability of AC-termination for terminating rewrite systems using type introduction. This result clearly shows that equational termination is a much harder problem than termination. We also show that, under the same conditions as for termination, acyclicity and non-loopingness are persistent properties of equational rewrite systems. The last result enables us to simplify several proofs of non-loopingness that can be found in the literature.

This paper is organized as follows. In the next section we briefly define equational rewriting and we recall the results of Zantema [22] on type introduction.

In Section 3 we generalize these results to equational rewriting. In Section 4 the usefulness of the results of Section 3 is illustrated by showing the undecidability of AC-termination for terminating rewrite systems and in Section 5 we address persistence of acyclicity and non-loopingness. Persistence is closely related ([18, 22]) to modularity, a property which has been thoroughly investigated in the term rewriting literature. Along this line we obtain several new modularity results. These are described in Section 6. In particular, we give a simple proof to an extension of a recent result of Aoto and Toyama [1] concerning the preservation of termination under non-disjoint combinations of term rewrite systems.

2 Preliminaries

Familiarity with the basic notions of term rewriting (as expounded in e.g. [3, 6, 13]) will be helpful in the following. We start this preliminary section with a very brief introduction to *many-sorted* equational reasoning and term rewriting.

Let $\mathcal{S}$ be a set of sorts. An $\mathcal{S}$-sorted signature is a set $\mathcal{F}$ of function symbols together with a sort declaration $\alpha_1 \times \cdots \times \alpha_n \to \alpha$ for every $f \in \mathcal{F}$. Here $\alpha_1, \ldots, \alpha_n, \alpha \in \mathcal{S}$ and n is called the arity of f. Function symbols of arity 0 are called constants. We assume the existence of countably infinite sets of variables $\mathcal{V}_\alpha$ for every sort $\alpha \in \mathcal{S}$. The union of all $\mathcal{V}_\alpha$ is denoted by $\mathcal{V}$. The set $\mathcal{T}(\mathcal{F}, \mathcal{V})$ of well-typed terms is the union of the sets $\mathcal{T}_\alpha(\mathcal{F}, \mathcal{V})$ for $\alpha \in \mathcal{S}$ that are inductively defined as follows: $\mathcal{V}_\alpha \subseteq \mathcal{T}_\alpha(\mathcal{F}, \mathcal{V})$ and $f(t_1, \ldots, t_n) \in \mathcal{T}_\alpha(\mathcal{F}, \mathcal{V})$ whenever $f \in \mathcal{F}$ has sort declaration $\alpha_1 \times \cdots \times \alpha_n \to \alpha$ and $t_i \in \mathcal{T}_{\alpha_i}(\mathcal{F}, \mathcal{V})$ for all $1 \leqslant i \leqslant n$. If $t \in \mathcal{T}_\alpha(\mathcal{F}, \mathcal{V})$ for some $\alpha \in \mathcal{S}$ then we say that t has sort α and we write $\mathsf{sort}(t) = \alpha$. The set of variables appearing in a term t is denoted by $\mathsf{var}(t)$. For every $\alpha \in \mathcal{S}$, let $\Box_\alpha$ be a fresh constant, named *hole*, of sort α. Elements of $\mathcal{T}(\mathcal{F} \cup \{\Box_\alpha \mid \alpha \in \mathcal{S}\}, \mathcal{V})$ are called *contexts*. So contexts are well-typed terms over the extended signature $\mathcal{F} \cup \{\Box_\alpha \mid \alpha \in \mathcal{S}\}$. An empty context is a hole. If C is a context with n holes $\Box_{\alpha_1}, \ldots, \Box_{\alpha_n}$ (from left to right) and $t_1, \ldots, t_n$ are terms with $\mathsf{sort}(t_i) = \alpha_i$ then $C[t_1, \ldots, t_n]$ denotes the term obtained from C by replacing the holes by $t_1, \ldots, t_n$. A *substitution* is a mapping σ from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$ such that $\mathsf{sort}(\sigma(x)) = \alpha$ if $x \in \mathcal{V}_\alpha$ and $\{x \in \mathcal{V} \mid \sigma(x) \neq x\}$ is finite. This latter set is called the *domain* of σ and denoted by $\mathsf{dom}(\sigma)$. We write $t\sigma$ for the result of applying σ to a term t. The set $\{\sigma(x) \mid x \in \mathsf{dom}(\sigma)\}$ is denoted by $\mathsf{ran}(\sigma)$. The restriction of σ to a subset $V \subseteq \mathcal{V}$ is denoted by $\sigma\restriction_V$ and we write $\sigma = \tau\ [V]$ if $\sigma\restriction_V = \tau\restriction_V$.

An $\mathcal{S}$-sorted *equational system* (ES for short) consists of an $\mathcal{S}$-sorted signature $\mathcal{F}$ and a set $\mathcal{E}$ of equations between well-typed terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$ such that $\mathsf{sort}(l) = \mathsf{sort}(r)$ for every equation $l \approx r \in \mathcal{E}$. We write $s \to_\mathcal{E} t$ if there exist an equation $l \approx r$ in $\mathcal{E}$, a substitution σ, and a context C such that $s = C[l\sigma]$ and $t = C[r\sigma]$.

The symmetric closure of $\rightarrow_{\mathcal{E}}$ is denoted by $\vdash\dashv_{\mathcal{E}}$ and the transitive reflexive closure of $\vdash\dashv_{\mathcal{E}}$ by $\sim_{\mathcal{E}}$. Note that $\mathsf{sort}(s) = \mathsf{sort}(t)$ whenever $s \sim_{\mathcal{E}} t$. An equation $l \approx r$ is called *non-erasing* if the sets of variables in l and r are the same. We say that $l \approx r$ is *variable preserving* if the multisets of variable occurrences in l and r are the same. The equation $l \approx r$ is called *collapsing* if l or r is a variable. An ($\mathcal{S}$-sorted) ES is non-erasing (variable preserving) if all its equations are so and collapsing if it has a collapsing equation. We denote the ESs $\{f(x,y) \approx f(y,x)\}$ and $\{f(f(x,y),z) \approx f(x,f(y,z))\}$ by $\mathrm{C}(f)$ and $\mathrm{A}(f)$ respectively. The union of $\mathrm{A}(f)$ and $\mathrm{C}(f)$ is denoted by $\mathrm{AC}(f)$.

A rewrite rule is an equation $l \approx r$ such that l is not a variable and variables which occur in r also occur in l. Rewrite rules $l \approx r$ are written as $l \rightarrow r$. An $\mathcal{S}$-sorted *term rewriting system* (TRS for short) is an $\mathcal{S}$-sorted ES all of whose equations are rewrite rules. A rewrite rule $l \rightarrow r$ is *duplicating* if some variable occurs more often in r than in l. An $\mathcal{S}$-sorted TRS is duplicating if it has a duplicating rewrite rule. An $\mathcal{S}$-sorted *equational term rewriting system* (ETRS for short) $\mathcal{R}/\mathcal{E}$ consists of an $\mathcal{S}$-sorted TRS $\mathcal{R}$ and an $\mathcal{S}$-sorted ES $\mathcal{E}$ over the same signature. We write $s \rightarrow_{\mathcal{R}/\mathcal{E}} t$ if there exist terms s' and t' such that $s \sim_{\mathcal{E}} s' \rightarrow_{\mathcal{R}} t' \sim_{\mathcal{E}} t$.

An ES (TRS, ETRS) is an $\mathcal{S}$-sorted ES (TRS, ETRS) with $\mathcal{S}$ a singleton set. This is equivalent to the usual (unsorted) definition found in the literature. The underlying ES $\Theta(\mathcal{E})$ of an $\mathcal{S}$-sorted ES $\mathcal{E}$ is obtained by simply dropping all sort declarations; likewise for TRSs and ETRSs. The term rewriting literature is mainly concerned with unsorted (E)TRSs. In this paper we show how many-sorted ETRSs can help to simplify the task of proving properties of unsorted ETRSs. A property P of (many-sorted) ETRS is called *persistent* if the following equivalence holds for every many-sorted ETRS $\mathcal{R}/\mathcal{E}$: $\mathcal{R}/\mathcal{E}$ has the property P if and only if $\Theta(\mathcal{R}/\mathcal{E})$ has the property P. For most properties the "if" direction is trivial; we are interested in the "only if" direction. In order to show that a given ETRS $\mathcal{R}/\mathcal{E}$ has a certain property P, which is known to be persistent, it is sufficient to find suitable $\mathcal{S}$ and sort declarations such that the $\mathcal{S}$-sorted ETRS $\mathcal{R}/\mathcal{E}$ has the property P. The latter is often easier to prove since only well-typed terms have to be considered. Hence persistence facilitates proving properties of ETRSs by type introduction. In this paper we are mainly concerned with the termination property. An ETRS $\mathcal{R}/\mathcal{E}$ is called *terminating* if there are no infinite $\mathcal{R}/\mathcal{E}$-rewrite sequences.

Zantema [22] obtained the following result. In the next section we generalize it to ETRSs.

Theorem 2.1 *Termination is persistent for TRSs that do not contain both collapsing and duplicating rules.* $\square$

3 Persistence of Termination for Equational Rewriting

In the following few definitions and lemmata $\mathcal{R}$ is an $\mathcal{S}$-sorted TRS and $\mathcal{E}$ an $\mathcal{S}$-sorted ES. Terms in $\Theta(\mathcal{R})$ need not be well-typed (with respect to $\mathcal{R}$), but they can be partitioned into well-typed components. This yields a natural layered structure, which is formalized below.

Definition 3.1 We write $t = C[\![t_1, \ldots, t_n]\!]$ if $t = C[t_1, \ldots, t_n]$ such that the context C is non-empty and maximal well-typed. Note that every term can be uniquely written as $C[\![t_1, \ldots, t_n]\!]$. We write $\mathsf{top}(t) = C$. The subterms $t_1, \ldots, t_n$ of t are called *aliens* and we denote the multiset $\{t_1, \ldots, t_n\}$ by $\mathsf{alien}(t)$. The *rank* of a term is the maximum number of type-clashes along any of its paths:

$$\mathsf{rank}(t) = \begin{cases} 0 & \text{if } t \text{ is well-typed,} \\ 1 + \max\{\mathsf{rank}(s) \mid s \in \mathsf{alien}(t)\} & \text{otherwise.} \end{cases}$$

The rank of a $\Theta(\mathcal{R}/\mathcal{E})$-rewrite sequence is the rank of its initial term. We extend the definition of sort in Section 2 to arbitrary (non-well-typed) terms by letting $\mathsf{sort}(t) = \alpha$ if $t = f(t_1, \ldots, t_n)$ with $f\colon \alpha_1 \times \cdots \times \alpha_n \to \alpha$.

Let us illustrate these concepts on a small example. Consider $\mathcal{S} = \{\alpha, \beta, \gamma\}$ with sort declarations $f\colon \alpha \times \gamma \to \beta$, $g\colon \beta \to \gamma$, $h\colon \gamma \to \gamma$, $a\colon \alpha$, $b\colon \beta$, and $c\colon \gamma$. For the term $t_1 = f(f(b, g(c)), h(f(a, g(h(g(a))))))$ we have $\mathsf{sort}(t_1) = \beta$, $\mathsf{top}(t_1) = f(\Box_\alpha, h(\Box_\gamma))$, $\mathsf{alien}(t_1) = \{f(b, g(c)), f(a, g(h(g(a))))\}$, and $\mathsf{rank}(t_1) = 3$. Figure 1 shows the decomposition of t_1 in maximal well-typed parts.

Definition 3.2 A rewrite step $s \to_{\Theta(\mathcal{R})} t$ is called *inner* if it takes place in one of the aliens of s. Non-inner steps are called *outer*. An outer step $s \to_{\Theta(\mathcal{R})} t$ is called *collapsing* if $\mathsf{sort}(s) \neq \mathsf{sort}(t)$. An inner step $s \to_{\Theta(\mathcal{R})} t$ is called *collapsing* if $\mathsf{top}(s) \neq \mathsf{top}(t)$.

Note that collapsing rewrite steps necessarily employ collapsing rewrite rules, but not every (outer) step using a collapsing rewrite rule is collapsing. In particular, terms of rank 0 do not admit collapsing steps. Let us continue the above example by considering the $\mathcal{S}$-sorted ETRS $\mathcal{R}/\mathcal{E}$ with

$$\mathcal{R} = \left\{ \begin{array}{rcl} f(x, g(y)) & \to & y \\ h(h(x)) & \to & x \end{array} \right\}$$

and $\mathcal{E} = \varnothing$. The rewrite step $t_1 \to_{\Theta(\mathcal{R})} t_2$ with $t_2 = f(c, h(f(a, g(h(g(a))))))$ is inner non-collapsing, even though $f(b, g(c)) \to_{\Theta(\mathcal{R})} c$ is outer collapsing. The

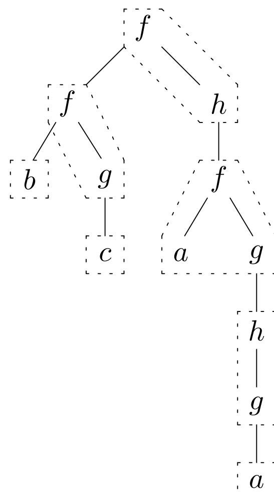

Figure 1: The maximal well-typed parts of t_1.

rewrite step $t_2 \to_{\Theta(\mathcal{R})} t_3$ with $t_3 = f(c, h(h(g(a))))$ is inner collapsing. The outer rewrite step $t_3 \to_{\Theta(\mathcal{R})} t_4$ with $t_4 = f(c, g(a))$ is non-collapsing despite the fact that it uses a collapsing rewrite rule. The rewrite step $t_4 \to_{\Theta(\mathcal{R})} t_5$ with $t_5 = a$ is outer collapsing. Figure 2 shows how the maximal well-typed parts are affected during the rewrite sequence from t_1 to t_5; the contracted redexes are indicated by boxes around the function symbols of the left-hand sides of the corresponding rewrite rules.

The next three lemmata express well-known facts in the context of modularity.

Lemma 3.3 *If* $s \to_{\Theta(\mathcal{R})} t$ *then* $\mathsf{rank}(s) \geqslant \mathsf{rank}(t)$. *If* $s \to_{\Theta(\mathcal{R})} t$ *is outer collapsing then* $\mathsf{rank}(s) > \mathsf{rank}(t)$. $\square$

Lemma 3.4 *Suppose* $s \to_{\Theta(\mathcal{R})} t$ *is non-collapsing. If* $s \to_{\Theta(\mathcal{R})} t$ *is outer then* $\mathsf{top}(s) \to_{\mathcal{R}} \mathsf{top}(t)$, *otherwise* $\mathsf{top}(s) = \mathsf{top}(t)$. $\square$

Lemma 3.5 *If* $s \to_{\Theta(\mathcal{R})} t$ *is outer, non-collapsing, and non-duplicating then* $\mathsf{alien}(t) \subseteq \mathsf{alien}(s)$. *If* $s = C[\![s_1, \ldots, s_i, \ldots, s_n]\!] \to_{\Theta(\mathcal{R})} C[s_1, \ldots, t_i, \ldots, s_n] = t$ *with* $s_i \to_{\Theta(\mathcal{R})} t_i$ *is collapsing then* $\mathsf{alien}(t) = (\mathsf{alien}(s) - \{s_i\}) \uplus \mathsf{alien}(t_i)$, *otherwise* $\mathsf{alien}(t) = (\mathsf{alien}(s) - \{s_i\}) \uplus \{t_i\}$. $\square$

Next we consider how $\vdash\!\dashv_{\Theta(\mathcal{E})}$ and $\sim_{\Theta(\mathcal{E})}$-steps affect the layered structure of terms. Because $\mathcal{E}$ is assumed to be non-collapsing and non-erasing, $\mathcal{E} \cup \mathcal{E}^{-1}$ is a TRS and the relation $\vdash\!\dashv_{\Theta(\mathcal{E})}$ coincides with $\to_{\Theta(\mathcal{E} \cup \mathcal{E}^{-1})}$ and $\leftarrow_{\Theta(\mathcal{E} \cup \mathcal{E}^{-1})}$. (Here

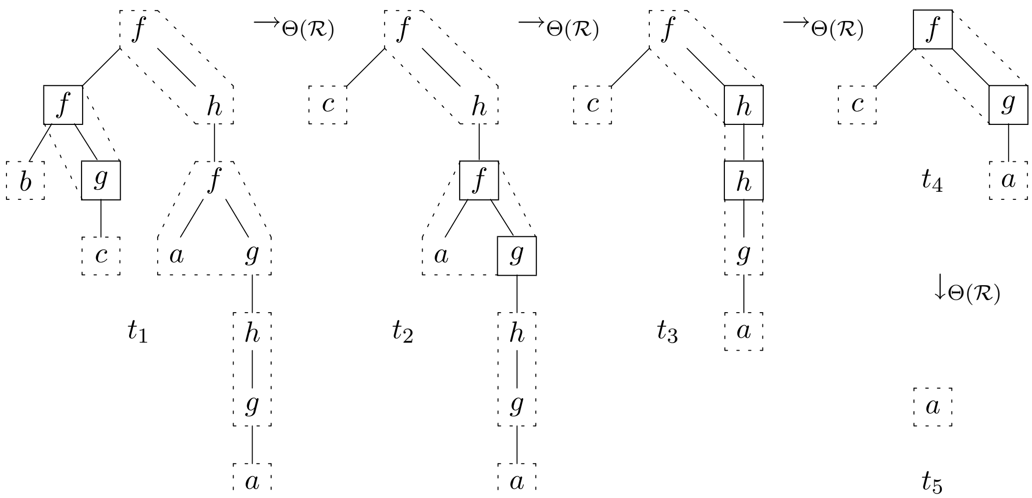

Figure 2: The maximal well-typed parts along a rewrite sequence.

$\mathcal{E}^{-1}$ denotes the ES $\{r \approx l \mid l \approx r \in \mathcal{E}\}$.) Hence we can reuse the above results when reasoning about $\vdash\!\dashv_{\Theta(\mathcal{E})}$. Note that $\mathcal{E} \cup \mathcal{E}^{-1}$ is non-duplicating for variable preserving $\mathcal{E}$.

Lemma 3.6 *Let $\mathcal{E}$ be non-erasing and non-collapsing. If $s \vdash\!\dashv_{\Theta(\mathcal{E})} t$ then $\mathsf{rank}(s) = \mathsf{rank}(t)$.*

Proof We have $s \rightarrow_{\Theta(\mathcal{E}\cup\mathcal{E}^{-1})} t$ and thus $\mathsf{rank}(s) \geqslant \mathsf{rank}(t)$ by Lemma 3.3. Symmetry yields $\mathsf{rank}(t) \geqslant \mathsf{rank}(s)$ and hence $\mathsf{rank}(s) = \mathsf{rank}(t)$. $\square$

The next lemma expresses that the layered structure of terms is essentially preserved by $\Theta(\mathcal{E})$-steps. For the second part it is essential that $\mathcal{E}$ is variable preserving.

Lemma 3.7 *Let $\mathcal{E}$ be variable preserving and non-collapsing. Suppose $\mathsf{alien}(s) = \{s_1, \ldots, s_n\}$ and $\mathsf{alien}(t) = \{t_1, \ldots t_m\}$. If $s \sim_{\Theta(\mathcal{E})} t$ then $\mathsf{top}(s) \sim_{\mathcal{E}} \mathsf{top}(t)$, $m = n$, and there exists a permutation π such that $s_i \sim_{\Theta(\mathcal{E})} t_{\pi(i)}$ for all $1 \leqslant i \leqslant n$.*

Proof We use induction on the number of $\vdash\!\dashv_{\Theta(\mathcal{E})}$-steps in $s \sim_{\Theta(\mathcal{E})} t$. The base case is trivial. Let $s \vdash\!\dashv_{\Theta(\mathcal{E})} u \sim_{\Theta(\mathcal{E})} t$ with $\mathsf{alien}(u) = \{u_1, \ldots, u_k\}$. From the induction hypothesis we obtain $\mathsf{top}(u) \sim_{\mathcal{E}} \mathsf{top}(t)$, $k = m$, and a permutation π' such that $u_i \sim_{\Theta(\mathcal{E})} t_{\pi'(i)}$ for all $1 \leqslant i \leqslant m$. Consider $s \rightarrow_{\Theta(\mathcal{E}\cup\mathcal{E}^{-1})} u$.

Suppose this step is inner. Without loss of generality we assume that $s = C[\![s_1, \ldots, s_n]\!]$ and $u = C'[\![u_1, \ldots, u_m]\!]$ for some contexts C and C'. We obtain

$\mathsf{top}(s) = \mathsf{top}(u)$ and $\mathsf{alien}(u) = (\mathsf{alien}(s) - \{s_i\}) \uplus \{u_i\}$ for some i with $s_i \to_{\Theta(\mathcal{E} \cup \mathcal{E}^{-1})} u_i$ from Lemmata 3.4 and 3.5. Hence $m = n$ and $s_j = u_j$ for $j \neq i$ and thus we can take $\pi = \pi'$.

If $s \to_{\Theta(\mathcal{E} \cup \mathcal{E}^{-1})} u$ is outer then $\mathsf{top}(s) \to_{\mathcal{E} \cup \mathcal{E}^{-1}} \mathsf{top}(u)$ by Lemma 3.4 and $\mathsf{alien}(s) = \mathsf{alien}(u)$ by Lemma 3.5. Hence $m = n$ and thus there exists a permutation π'' such that $s_i = u_{\pi''(i)}$ for all $1 \leqslant i \leqslant n$. In this case we clearly have $\mathsf{top}(s) \sim_{\mathcal{E}} \mathsf{top}(t)$ and $s_i \sim_{\Theta(\mathcal{E})} t_{\pi(i)}$ for all $1 \leqslant i \leqslant n$ with π defined as $\pi' \circ \pi''$. $\square$

Using all of the preceding lemmata, the following result can now be proved by a routine 'minimal counterexample' argument (cf. Ohlebusch [16]).

Lemma 3.8 *Let $\mathcal{R}/\mathcal{E}$ be a terminating $\mathcal{S}$-sorted ETRS with $\mathcal{E}$ variable preserving and non-collapsing. If $\Theta(\mathcal{R}/\mathcal{E})$ is not terminating then there exists an infinite rewrite sequence in which all terms have the same rank and which contains an outer duplicating and an inner collapsing $\Theta(\mathcal{R})$-step.*

Proof Let $\mathcal{A}$ be an infinite rewrite sequence of minimal rank. According to Lemmata 3.3 and 3.6 this implies that all terms in $\mathcal{A}$ have the same rank and thus $\mathcal{A}$ contains no outer collapsing $\Theta(\mathcal{R})$-steps. For a proof by contradiction suppose that $\mathcal{A}$ lacks outer duplicating or lacks inner collapsing $\Theta(\mathcal{R})$-steps.

First we show that there exists an infinite tail $\mathcal{B}$ of $\mathcal{A}$ with the property that all $\Theta(\mathcal{R})$-steps in $\mathcal{B}$ are either outer or inner non-collapsing. If $\mathcal{A}$ lacks inner collapsing $\Theta(\mathcal{R})$-steps then we can take $\mathcal{B} = \mathcal{A}$. If $\mathcal{A}$ lacks outer duplicating $\Theta(\mathcal{R})$-steps, we reason as follows. Associate with every term t the multiset $\sharp(t) = \{\mathsf{rank}(t') \mid t' \in \mathsf{alien}(t)\}$. As a consequence of Lemmata 3.6 and 3.7 we have $\sharp(u) = \sharp(v)$ for every $\Theta(\mathcal{E})$-step $u \sim v$ in $\mathcal{A}$. Let $u \to v$ be a $\Theta(\mathcal{R})$-step in $\mathcal{A}$. From Lemma 3.5 we infer that

1. $\sharp(u) \geqslant_{\mathrm{mul}} \sharp(v)$ if $u \to v$ is outer (and thus non-duplicating),

2. $\sharp(u) >_{\mathrm{mul}} \sharp(v)$ if $u \to v$ is inner collapsing, and

3. $\sharp(u) \geqslant_{\mathrm{mul}} \sharp(v)$ if $u \to v$ is inner non-collapsing.

(In the example of Figure 2 we have $\sharp(t_1) = \{1, 3\}$, $\sharp(t_2) = \{0, 3\}$, $\sharp(t_3) = \sharp(t_4) = \{0, 0\}$, and $\sharp(t_5) = \varnothing$.) Since $>_{\mathrm{mul}}$ is a well-founded order on multisets (Dershowitz and Manna [7]), the second alternative occurs finitely often. Hence $\mathcal{A}$ has an infinite tail $\mathcal{B}$ in which all $\Theta(\mathcal{R})$-steps are outer or are inner non-collapsing.

If $\mathcal{B}$ contains infinitely many outer $\Theta(\mathcal{R})$-steps then by applying top to every term in $\mathcal{B}$ we obtain an infinite $\mathcal{R}/\mathcal{E}$-rewrite sequence as a consequence of Lemmata 3.4 and 3.7, contradicting the termination of $\mathcal{R}/\mathcal{E}$ (over the extended signature $\mathcal{F} \cup \{\square_\alpha \mid \alpha \in \mathcal{S}\}$, which is an easy consequence of the termination of

$\mathcal{R}/\mathcal{E}$ over the original signature $\mathcal{F}$). Hence there exists an infinite tail $\mathcal{C}$ of $\mathcal{B}$ such that all $\Theta(\mathcal{R})$-steps in $\mathcal{C}$ are inner non-collapsing. With help of Lemma 3.7 and the pigeon-hole principle, we obtain an infinite $\Theta(\mathcal{R}/\mathcal{E})$-rewrite sequence starting from one of the aliens of $\mathcal{C}$. This contradicts the minimality of $\mathcal{A}$. □

The proof of the above lemma can easily be massaged to yield an infinite $\Theta(\mathcal{R}/\mathcal{E})$-rewrite sequence that contains *infinitely* many outer duplicating and *infinitely* many inner collapsing $\Theta(\mathcal{R})$-steps. This observation will be used in the proof of Lemma 6.5.

Corollary 3.9 *Termination is persistent for ETRSs $\mathcal{R}/\mathcal{E}$ with $\mathcal{R}$ non-collapsing or non-duplicating and $\mathcal{E}$ non-collapsing and variable preserving.* □

Variable-preservingness of $\mathcal{E}$ cannot be weakened to non-erasingness. Consider for instance the $\{\alpha, \beta\}$-sorted ETRS $\mathcal{R}/\mathcal{E}$ with $\mathcal{R} = \{a \to b\}$, $\mathcal{E} = \{f(x,x,y) \approx f(y,x,y)\}$, and sort declarations $f\colon \alpha \times \alpha \times \alpha \to \beta$ and $a, b\colon \beta$. The ETRS $\mathcal{R}/\mathcal{E}$ is terminating since the only reducible well-typed term is a, but in $\Theta(\mathcal{R}/\mathcal{E})$ we have the following infinite sequence: $f(a,b,a) \to_{\Theta(\mathcal{R})} f(b,b,a) \vdash\!\dashv_{\Theta(\mathcal{E})} f(a,b,a) \to_{\Theta(\mathcal{R})} \cdots$. Note that $f(a,b,a)$ is not well-typed.

At present it is unclear whether Corollary 3.9 holds for collapsing $\mathcal{E}$. Our proof does not allow for collapsing $\mathcal{E}$ since Lemmata 3.6 and 3.7 no longer hold. Note that $\Theta(\mathcal{R}/\mathcal{E})$ (with non-empty $\mathcal{R}$) cannot be terminating if $\mathcal{E}$ contains a collapsing equation $l \approx x$ such that x has more than one occurrence in l (cf. [11, p. 181]).

4 Undecidability of AC-Termination

We start this section by showing the undecidability of termination modulo commutativity for terminating TRSs. To this end we make use of the following well-known result (e.g. [15]).

Lemma 4.1 *It is undecidable whether a TRS admits an infinite rewrite sequence in which all steps take place at the root position.* □

Theorem 4.2 *It is undecidable whether a terminating TRS is C-terminating.*

Proof Let $\mathcal{R}$ be an arbitrary TRS. Define

$$\mathcal{R}' = \{f(l,a) \to f(a,r) \mid l \to r \in \mathcal{R}\}$$

with f and a are fresh symbols. Termination of $\mathcal{R}'$ can be shown by the lexicographic path order with any total precedence in which f is maximum and a minimum. We show that $\mathcal{R}'$ is $\mathrm{C}(f)$-terminating (i.e., $\mathcal{R}'/\mathrm{C}(f)$ is terminating) if and only if $\mathcal{R}$ does not admit an infinite rewrite sequence in which all steps take place at the root position.

Let $l_1\sigma_1 \to r_1\sigma_1 = l_2\sigma_2 \to r_2\sigma_2 = \cdots$ with $l_i \to r_i \in \mathcal{R}$ for $i \geqslant 1$ be an infinite $\mathcal{R}$-rewrite sequence in which all steps take place at the root position. This sequence can be transformed into the following infinite $\mathcal{R}'/\mathrm{C}(f)$-rewrite sequence: $f(l_1\sigma_1, a) \to_{\mathcal{R}'} f(a, r_1\sigma_1) \vdash\!\dashv_{\mathrm{C}(f)} f(r_1\sigma_1, a) = f(l_2\sigma_2, a) \to_{\mathcal{R}'} f(a, r_2\sigma_2) \vdash\!\dashv_{\mathrm{C}(f)} \cdots$.

For the other direction we reason as follows. Since $\mathcal{R}'$ is non-collapsing and $\mathrm{C}(f)$ trivially non-collapsing and variable preserving, we can apply Corollary 3.9. To this end we consider the sort declarations $a\colon \alpha$, $f\colon \alpha \times \alpha \to \beta$, and $g\colon \alpha \times \cdots \times \alpha \to \alpha$ for all function symbols g of $\mathcal{R}$. In order to show that $\mathcal{R}'$ is $\mathrm{C}(f)$-terminating, it is sufficient to prove termination of all well-typed terms. Terms of sort α are trivially terminating. An infinite $\mathcal{R}'/\mathrm{C}(f)$-rewrite sequence starting from a well-typed term of sort β must have the form $f(l_1\sigma_1, a) \to_{\mathcal{R}'} f(a, r_1\sigma_1) \vdash\!\dashv_{\mathrm{C}(f)} f(r_1\sigma_1, a) = f(l_2\sigma_2, a) \to_{\mathcal{R}'} f(a, r_2\sigma_2) \vdash\!\dashv_{\mathrm{C}(f)} \cdots$ with $l_i \to r_i \in \mathcal{R}$ for $i \geqslant 1$. This gives rise to an infinite rewrite sequence $l_1\sigma_1 \to_{\mathcal{R}} r_1\sigma_1 = l_2\sigma_2 \to_{\mathcal{R}} r_2\sigma_2 = \cdots$ in which all steps take place at the root position, contradicting the assumption. Hence $\mathcal{R}'$ is $\mathrm{C}(f)$-terminating.

The desired result follows from the previous lemma. □

Next we show the undecidability of termination modulo associativity for terminating TRSs.

Theorem 4.3 *It is undecidable whether a terminating TRS is A-terminating.*

Proof Let $\mathcal{R}$ be an arbitrary TRS. Define

$$\mathcal{R}' = \{f(f(e(l), a), a) \to f(e(r), f(a, a)) \mid l \to r \in \mathcal{R}\}.$$

Termination of $\mathcal{R}'$ is easily shown by the lexicographic path order. We can show that $\mathcal{R}'$ is $\mathrm{A}(f)$-terminating if and only if $\mathcal{R}$ does not admit an infinite rewrite sequence in which all steps take place at the root position, similar to the preceding proof. Let $\mathcal{F}$ be the signature of $\mathcal{R}$. For the "if" direction we use sort declarations $a\colon \beta$, $e\colon \alpha \to \beta$, $f\colon \beta \times \beta \to \beta$, and $g\colon \alpha \times \cdots \times \alpha \to \alpha$ for all function symbols $g \in \mathcal{F}$. Every well-typed term t of sort β can be (uniquely) written as $C[t_1, \ldots, t_n]$ such that C contains only f and a symbols and for every $1 \leqslant i \leqslant n$ we have $t_i = e(t_i')$ with $t_i' \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. Let us denote the sequence $(t_1, \ldots, t_n)$ by $\phi(t)$. If $t \to_{\mathcal{R}'} t'$ then there exist an i, a rewrite rule $l \to r \in \mathcal{R}$, and a substitution σ such that $t_i = e(l\sigma)$ and $\phi(t') = (t_1, \ldots, t_i', \ldots t_n)$ with $t_i' = e(r\sigma)$. If $t \sim_{\mathrm{A}(f)} t'$ then $\phi(t) = \phi(t')$. Using the pigeon-hole principle it follows that an infinite $\mathcal{R}'/\mathrm{A}(f)$-rewrite sequence gives rise to infinite $\mathcal{R}$-rewrite sequence in which all steps take place at the root position, contradicting the assumption. □

Note that taking $\mathcal{R}' = \{f(f(l,a),a) \to f(r, f(a,a)) \mid l \to r \in \mathcal{R}\}$ in the above proof precludes the (good) use of type introduction as there can be only one sort.

Theorem 4.4 *It is undecidable whether an A and C-terminating TRS is AC-terminating.*

Proof (sketch) Replace $\mathcal{R}'$ in the previous proof by $\mathcal{R}' = \{f(f(e(l),a),b) \to f(a, f(b, e(r))) \mid l \to r \in \mathcal{R}\}$. The proof that $\mathcal{R}'$ is AC(f)-terminating if and only if $\mathcal{R}$ does not admit an infinite rewrite sequence in which all steps take place at the root is similar to the one above. Here we show that $\mathcal{R}'$ is both A(f) and C(f)-terminating. We use Corollary 3.9. Consider the sort declarations $a, b \colon \beta$, $e \colon \alpha \to \beta$, $f \colon \beta \times \beta \to \beta$, and $g \colon \alpha \times \cdots \times \alpha \to \alpha$ for all $g \in \mathcal{F}$. In order to show the A(f) and C(f)-termination of $\mathcal{R}'$ we only have to consider the well-typed terms. Since well-typed terms of sort α are in normal form, we may restrict our attention to well-typed terms of sort β. Every well-typed term t of sort β can be (uniquely) written as $C[t_1, \ldots, t_n]$ such that the context C consists of f symbols only and for every $1 \leqslant i \leqslant n$ we have either $t_i = e(t_i')$ with $t_i' \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, $t_i = a$, or $t_i = b$. Let us denote the (binary) number $\psi(t_1) \ldots \psi(t_n)$ where ψ is defined as

$$\psi(s) = \begin{cases} 1 & \text{if } \mathsf{root}(s) = e, \\ 0 & \text{otherwise} \end{cases}$$

by $\phi(t)$. It is not difficult to verify that $\phi(s) = \phi(t)$ for $s \vdash\dashv_A t$ and $\phi(s) > \phi(t)$ for $s \to_{\mathcal{R}'} t$. Hence $\mathcal{R}'$ is A(f)-terminating. For proving the C(f)-termination of $\mathcal{R}'$ we associate with every well-typed term t the sum of the depths of all occurrences of a in t, which we denote by $\varphi(t)$. One easily verifies that $\varphi(s) = \varphi(t)$ for $s \vdash\dashv_C t$ and $\varphi(s) = \varphi(t) + 1$ for $s \to_{\mathcal{R}'} t$, implying C(f)-termination. $\square$

Note that identifying the constants a and b in the above $\mathcal{R}'$ would result in a TRS that is not necessarily C(f)-terminating.

5 Persistence of Acyclicity and Non-Loopingness

An ETRS $\mathcal{R}/\mathcal{E}$ is *cyclic* if it admits a sequence of the form $t \to^+_{\mathcal{R}/\mathcal{E}} t$. We say that $\mathcal{R}/\mathcal{E}$ is *looping* if there exist a term t, context C, and substitution σ such that $t \to^+_{\mathcal{R}/\mathcal{E}} C[t\sigma]$. Terminating ETRSs are non-looping and non-looping ETRSs are acyclic, but the reverse statements do not hold. A recent study of non-loopingness for TRSs is performed in Zantema and Geser [23].

By a straightforward modification of the proof of Lemma 3.8 we obtain the following result.

Lemma 5.1 *Let $\mathcal{R}/\mathcal{E}$ be an acyclic $\mathcal{S}$-sorted ETRS with $\mathcal{E}$ variable preserving and non-collapsing. If $\Theta(\mathcal{R}/\mathcal{E})$ is cyclic then there exists a cycle in which all terms have the same rank and which contains an outer duplicating and an inner collapsing $\Theta(\mathcal{R})$-step.* $\Box$

Corollary 5.2 *Acyclicity is persistent for many-sorted ETRSs $\mathcal{R}/\mathcal{E}$ such that $\mathcal{R}$ is non-collapsing or non-duplicating and $\mathcal{E}$ is non-collapsing and variable preserving.* $\Box$

The proof of the analogous result for non-loopingness is quite a bit more involved. The reason is that because the involved substitution may substitute a term of sort β for a variable of sort α we do not obtain a contradiction by considering a loop of minimal rank.

Definition 5.3 A substitution σ is called *consistent* if $\mathsf{sort}(x) = \mathsf{sort}(x\sigma)$ for all $x \in \mathcal{V}$.

We show that every looping $\Theta(\mathcal{R}/\mathcal{E})$ admits a loop with consistent substitution. Most of the work is done in the following lemma.

Lemma 5.4 *For every substitution σ and finite set of variables V with $\mathsf{dom}(\sigma) \subseteq V$ there exist a consistent substitution σ' and a variable substitution τ such that $\sigma\tau = \tau\sigma'$ $[V]$.*

Proof The desired substitutions σ' and τ are computed by the following algorithm:

W `:=` V`;`
σ' `:=` σ`;`
τ `:=` ε`;`
`while` $W \neq \varnothing$ `do`
    `if` $\exists x \in W$ with $x\sigma' \notin W$ `then`
        τ' `:=` $\{x \mapsto x'\}$ with x' a fresh variable of sort $\mathsf{sort}(x\sigma')$`;`
        σ' `:=` $\sigma'\tau'\restriction_{\mathsf{dom}(\sigma')\setminus\{x\}} \cup \{x' \mapsto x\sigma'\tau'\}$
    `else`
        τ' `:=` $\{x \mapsto \xi \mid x \in W\}$ with ξ a fresh variable (of arbitrary sort)`;`
        σ' `:=` $\sigma'\tau'\restriction_{\mathsf{dom}(\sigma')\setminus W}$
    `fi;`
    τ `:=` $\tau\tau'$`;`
    W`:=` $W \setminus \mathsf{dom}(\tau')$
`od`

Below we prove that the statements

1. τ is a variable substitution, i.e., a mapping from $\mathcal{V}$ to $\mathcal{V}$,
2. $\mathsf{var}(\tau) \cap W = \varnothing$,
3. $\sigma\tau = \tau\sigma'$ $[V]$,
4. $\mathsf{ran}(\sigma'{\restriction}_{\mathcal{V}\setminus W}) \cap W = \varnothing$, and
5. $\sigma'{\restriction}_{\mathcal{V}\setminus W}$ is consistent

are invariants of the while-loop which hold after the first three assignments. Here $\mathsf{var}(\tau)$ denotes the union of $\mathsf{dom}(\tau)$ and $\bigcup_{x\in\mathsf{dom}(\tau)} \mathsf{var}(x\tau)$. (Statements 2 and 4 are needed to show 3 and 5.) Note that $\mathcal{V}$ is the set of variables, not to be confused with V. Termination of the while-loop is obvious since in each iteration at least one element of W is removed and initially $W = V$ is finite by assumption. Upon termination we have $W = \varnothing$ and thus $\sigma'{\restriction}_{\mathcal{V}} = \sigma'$ is consistent.

Let W_i, σ'_i and τ_i denote the values of W, σ' and τ after the i-th iteration of the while-loop. After the first three assignments we have $W_0 = V$, $\sigma'_0 = \sigma$ and $\tau_0 = \varepsilon$. Statements 1, 2, and 3 are trivially true. For statements 4 and 5 we note that $\sigma'_0{\restriction}_{\mathcal{V}\setminus W_0} = \sigma{\restriction}_{\mathcal{V}\setminus V} = \varepsilon$ because $\mathsf{dom}(\sigma) \subseteq V$. Consider the $i+1$-th iteration. We distinguish two cases according to the condition of the if-statement.

a. Suppose there exists an $x \in W_i$ such that $x\sigma'_i \notin W_i$. By construction, $\tau' = \{x \mapsto x'\}$ with x' a fresh variable of sort $\mathsf{sort}(x\sigma'_i)$ and $\sigma'_{i+1} = \sigma'_i\tau'{\restriction}_{\mathsf{dom}(\sigma'_i)\setminus\{x\}} \cup \{x' \mapsto x\sigma'_i\tau'\}$. We have $W_{i+1} = W_i \setminus \{x\}$. Since τ' and τ_i (by induction hypothesis) are variable substitutions, so is their composition $\tau_{i+1} = \tau_i\tau'$. From the assumption $x \in W_i$ and the second part of the induction hypothesis we infer that $\mathsf{var}(\tau_{i+1}) = \mathsf{var}(\tau_i) \cup \{x, x'\}$. Since $x, x' \notin W_{i+1}$ and $x \notin \mathsf{var}(\tau_i)$ we obtain $\mathsf{var}(\tau_{i+1}) \cap W_{i+1} = \mathsf{var}(\tau_i) \cap W_{i+1} = \mathsf{var}(\tau_i) \cap W_i = \varnothing$.

   For statement 3 we reason as follows. Let y be an arbitrary variable in V. We have to show that $y\sigma\tau_{i+1} = y\tau_{i+1}\sigma'_{i+1}$. The induction hypothesis yields $y\sigma\tau_{i+1} = y\sigma\tau_i\tau' = y\tau_i\sigma'_i\tau'$. If $y = x$ then $y\tau_i\sigma'_i\tau' = x\sigma'_i\tau' = x'\sigma'_{i+1} = x\tau'\sigma'_{i+1} = x\tau_i\tau'\sigma'_{i+1} = y\tau_{i+1}\sigma'_{i+1}$. In the first and fourth equality we use the fact that $x \notin \mathsf{dom}(\tau_i)$, which follows from the second part of the induction hypothesis. If $y \neq x$ then we obtain $y\tau_i \neq x$ from the second part of the induction hypothesis and hence $y\tau_i\sigma'_i\tau' = y\tau_i\sigma'_{i+1} = y\tau_i\tau'\sigma'_{i+1} = y\tau_{i+1}\sigma'_{i+1}$. For the first equality note that $y\tau_i$ is a variable different from x and x' and furthermore $y\tau_i\sigma'_i\tau' = y\tau_i = y\tau_i\sigma'_{i+1}$ whenever $y\tau_i \notin \mathsf{dom}(\sigma'_i)$.

   Next we show statement 4. From $x \in W_i$ and $x\sigma'_i \notin W_i$ we infer that $x \in \mathsf{dom}(\sigma'_i)$. Together with the freshness of x' this yields $\mathsf{dom}(\sigma'_i) = \mathsf{dom}(\sigma'_i\tau')$

and $x\sigma_i'\tau' \neq x, x'$. Hence $\mathsf{dom}(\sigma_{i+1}') = (\mathsf{dom}(\sigma_i'\tau') \cap (\mathsf{dom}(\sigma_i') \setminus \{x\})) \cup \{x'\} = (\mathsf{dom}(\sigma_i') \cap (\mathsf{dom}(\sigma_i') \setminus \{x\})) \cup \{x'\} = (\mathsf{dom}(\sigma_i') \setminus \{x\}) \cup \{x'\}$ and therefore

$$\begin{aligned}
\mathsf{dom}(\sigma_{i+1}' \restriction_{\mathcal{V} \setminus W_{i+1}}) &= ((\mathsf{dom}(\sigma_i') \setminus \{x\}) \cup \{x'\}) \cap ((\mathcal{V} \setminus W_i) \cup \{x\}) \\
&= ((\mathsf{dom}(\sigma_i') \setminus \{x\}) \cap ((\mathcal{V} \setminus W_i) \cup \{x\})) \cup \{x'\} \\
&= ((\mathsf{dom}(\sigma_i') \cap (\mathcal{V} \setminus W_i)) \setminus \{x\}) \cup \{x'\} \\
&= (\mathsf{dom}(\sigma_i') \cap (\mathcal{V} \setminus W_i)) \cup \{x'\} \\
&= \mathsf{dom}(\sigma_i' \restriction_{\mathcal{V} \setminus W_i}) \cup \{x'\}.
\end{aligned}$$

The fourth equality follows from the assumption that $x \in W_i$. Consequently,

$$\begin{aligned}
\mathsf{ran}(\sigma_{i+1}' \restriction_{\mathcal{V} \setminus W_{i+1}}) &= \{x'\sigma_{i+1}'\} \cup \bigcup_{y \in \mathsf{dom}(\sigma_i' \restriction_{\mathcal{V} \setminus W_i})} y\sigma_{i+1}' \\
&= \{x\sigma_i'\tau'\} \cup \bigcup_{y \in \mathsf{dom}(\sigma_i' \restriction_{\mathcal{V} \setminus W_i})} y\sigma_i'\tau' \restriction_{\mathcal{V} \setminus W_i} \\
&= \{x\sigma_i'\tau'\} \cup \mathsf{ran}(\sigma_i'\tau' \restriction_{\mathcal{V} \setminus W_i})
\end{aligned}$$

The last equality follows from $\mathsf{dom}(\sigma_i') = \mathsf{dom}(\sigma_i'\tau')$. From the fourth part of the induction hypothesis we learn that $x \notin \mathsf{ran}(\sigma_i' \restriction_{\mathcal{V} \setminus W_i})$ and thus

$$\begin{aligned}
\mathsf{ran}(\sigma_{i+1}' \restriction_{\mathcal{V} \setminus W_{i+1}}) \cap W_{i+1} &= (\mathsf{ran}(\sigma_i'\tau' \restriction_{\mathcal{V} \setminus W_i}) \cup \{x\sigma_i'\tau'\}) \cap (W_i \setminus \{x\}) \\
&= (\mathsf{ran}(\sigma_i'\tau' \restriction_{\mathcal{V} \setminus W_i}) \cup \{x\sigma_i'\tau'\}) \cap W_i \\
&= (\mathsf{ran}(\sigma_i'\tau' \restriction_{\mathcal{V} \setminus W_i}) \cap W_i) \cup (\{x\sigma_i'\tau'\} \cap W_i) \\
&= \mathsf{ran}(\sigma_i'\tau' \restriction_{\mathcal{V} \setminus W_i}) \cap W_i.
\end{aligned}$$

The last equality follows from $x\sigma_i'\tau' = x\sigma_i' \notin W_i$. Now suppose on the contrary that there exists a variable $y \in \mathcal{V} \setminus W_i$ such that $y\sigma_i'\tau' \in W_i$. If $y\sigma_i'\tau' = y\sigma_i'$ we obtain a contradiction with the fourth part of the induction hypothesis. Hence $y\sigma_i'\tau' \neq y\sigma_i'$, which implies that $x \in \mathsf{var}(y\sigma_i')$ and thus $x' \in \mathsf{var}(y\sigma_i'\tau')$. This is impossible since W_i is a set of variables that does not contain x'. This concludes the proof of statement 4.

Finally we show that $\sigma_{i+1}' \restriction_{\mathcal{V} \setminus W_{i+1}}$ is consistent. Let $y \in \mathcal{V}$. We have to show that $\mathsf{sort}(y) = \mathsf{sort}(y\sigma_{i+1}' \restriction_{\mathcal{V} \setminus W_{i+1}})$. If $y \notin \mathsf{dom}(\sigma_{i+1}' \restriction_{\mathcal{V} \setminus W_{i+1}})$ then the result is trivial. Let $y \in \mathsf{dom}(\sigma_{i+1}' \restriction_{\mathcal{V} \setminus W_{i+1}})$. In the proof of statement 4 above we observed that $\mathsf{dom}(\sigma_{i+1}' \restriction_{\mathcal{V} \setminus W_{i+1}}) = \mathsf{dom}(\sigma_i' \restriction_{\mathcal{V} \setminus W_i}) \cup \{x'\}$. We distinguish two cases. If $y = x'$ then $\mathsf{sort}(y) = \mathsf{sort}(x\sigma_i') = \mathsf{sort}(x\sigma_i'\tau') = \mathsf{sort}(y\sigma_{i+1}')$. The second equality follows from the inequality $x\sigma' \neq x$. In the other case we have $y \in \mathsf{dom}(\sigma'_i) \setminus W_i$ and thus $y\sigma'_{i+1} = y\sigma'_i\tau'$. From the fourth part of the induction hypothesis we infer that $y\sigma'_i \neq x$. So if $y\sigma'_i$ is a variable then $y\sigma'_i\tau' = y\sigma'_i$ and thus $\mathsf{sort}(y\sigma'_i\tau') = \mathsf{sort}(y\sigma'_i)$. If $y\sigma'_i$ is a non-variable term then we clearly also have $\mathsf{sort}(y\sigma'_i\tau') = \mathsf{sort}(y\sigma')$. The fifth part of the induction hypothesis yields $\mathsf{sort}(y) = \mathsf{sort}(y\sigma'_i)$. Hence $\mathsf{sort}(y) = \mathsf{sort}(y\sigma'_{i+1})$ as desired.

b. Suppose $x\sigma'_i \in W_i$ for all $x \in W_i$. By construction, $\tau' = \{x \mapsto \xi \mid x \in W_i\}$ with ξ a fresh variable of arbitrary sort and $\sigma'_{i+1} = \sigma'_i\tau'\restriction_{\mathsf{dom}(\sigma'_i)\setminus W_i}$. Statement 1 follows as in the previous case. Since $\mathsf{dom}(\tau') = W_i$, $W_{i+1} = \varnothing$. Hence statements 2 and 4 are trivially true.

   For statement 3 we show that $y\sigma\tau_{i+1} = y\tau_{i+1}\sigma'_{i+1}$ for all $y \in V$. The third part of the induction hypothesis yields $y\sigma\tau_{i+1} = y\sigma\tau_i\tau' = y\tau_i\sigma'_i\tau'$. First consider the case that $y \in W_i$. From the second part of the induction hypothesis we infer that $y\tau_i = y$ and thus $y\tau_i\sigma'_i\tau' = y\sigma'_i\tau' = \xi$ by the assumption that $y\sigma'_i \in W_i$. Also, $y\tau_{i+1}\sigma'_{i+1} = y\tau'\sigma'_{i+1} = \xi\sigma'_{i+1} = \xi$. Next consider the case that $y \notin W_i$. From the second part of the induction hypothesis we infer that $y\tau_i \notin W_i$ and thus $y\tau_{i+1}\sigma'_{i+1} = y\tau_i\sigma'_{i+1}$. If $y\tau_i \in \mathsf{dom}(\sigma'_i)$ then $y\tau_i\sigma'_{i+1} = y\tau_i\sigma'_i\tau'$. If $y\tau_i \notin \mathsf{dom}(\sigma'_i)$ then $y\tau_i\sigma'_{i+1} = y\tau_i = y\tau_i\sigma'_i = y\tau_i\sigma'_i\tau'$. The last equality follows from $y\tau_i \notin W_i$.

   Finally we show that σ'_{i+1} is consistent. Let $y \in \mathsf{dom}(\sigma'_{i+1}) = \mathsf{dom}(\sigma'_i) \setminus W_i$. We have to show that $\mathsf{sort}(y) = \mathsf{sort}(y\sigma'_{i+1})$. We have $y\sigma'_{i+1} = y\sigma'_i\tau' = y\sigma'_i$. Here the last equality follows from the fourth part of the induction hypothesis. Now the fifth part of the induction hypothesis yields $\mathsf{sort}(y) = \mathsf{sort}(y\sigma'_i)$ and thus $\mathsf{sort}(y) = \mathsf{sort}(y\sigma'_{i+1})$.

$\square$

Lemma 5.5 *Let $\mathcal{R}/\mathcal{E}$ be an $\mathcal{S}$-sorted ETRS. If $\Theta(\mathcal{R}/\mathcal{E})$ is looping then there exists a loop $t \to^+ C[t\sigma]$ with consistent σ.*

Proof Let $t \to^+ C[t\sigma]$ be a loop in $\Theta(\mathcal{R}/\mathcal{E})$. Without loss of generality we assume that $\mathsf{dom}(\sigma) \subseteq \mathsf{var}(t)$. Let $V = \mathsf{var}(t)$. According to the previous lemma there exist a consistent substitution σ' and a variable substitution τ such that $\sigma\tau = \tau\sigma'$ $[V]$. Let $t' = t\tau$ and $C' = C\tau$. Since (equational) rewriting is closed under substitutions we obtain $t' = t\tau \to^+ C\tau[t\sigma\tau] = C'[t\tau\sigma'] = C'[t'\sigma']$, which shows that $\Theta(\mathcal{R}/\mathcal{E})$ admits a loop with consistent substitution. $\square$

Lemma 5.6 *Let $\mathcal{R}/\mathcal{E}$ be a non-looping $\mathcal{S}$-sorted ETRS with $\mathcal{E}$ variable preserving and non-collapsing. If $\Theta(\mathcal{R}/\mathcal{E})$ is looping then there exists a loop in which all* *terms have the same rank and which contains an outer duplicating and an inner collapsing* $\Theta(\mathcal{R})$*-step.*

Proof Let $\mathcal{A}$: $t \to^+_{\Theta(\mathcal{R}/\mathcal{E})} C[t\sigma]$ be a loop with consistent σ of minimal rank, the existence of which is guaranteed by Lemma 5.5. (The rank of $\mathcal{A}$ may be greater that the minimal rank of a loop because the construction in the proof of Lemma 5.4 may increase the rank by one.) Because σ is consistent, $\mathsf{rank}(t) = \mathsf{rank}(t\sigma)$ and thus $\mathsf{rank}(t) \leqslant \mathsf{rank}(C[t\sigma])$. From Lemmata 3.3 and 3.6 we obtain $\mathsf{rank}(t) \geqslant \mathsf{rank}(C[t\sigma])$ and therefore $\mathsf{rank}(t) = \mathsf{rank}(C[t\sigma])$. Hence, by Lemma 3.3, $\mathcal{A}$ contains no outer collapsing $\Theta(\mathcal{R})$-steps.

We show that $\mathcal{A}$ contains an inner collapsing $\Theta(\mathcal{R})$-step. Because $\mathsf{rank}(t) = \mathsf{rank}(C[t\sigma])$ the displayed occurrence of $t\sigma$ is not a subterm of an alien subterm of $C[t\sigma]$ and thus $\mathsf{top}(C[t\sigma]) = C'[\Box_{\alpha_1}, \ldots, \mathsf{top}(t\sigma), \ldots, \Box_{\alpha_m}]$ for some context C' with $m > 0$. Let $\sigma' = \mathsf{top} \circ \sigma$. An easy induction on the structure of t yields $\mathsf{top}(t\sigma) = \mathsf{top}(t)\sigma'$. Here we make use of the consistency of σ. Now suppose for a proof by contradiction that $\mathcal{A}$ contains no inner collapsing $\Theta(\mathcal{R})$-steps. According to Lemma 3.4 we have $\mathsf{top}(s) \to_{\mathcal{R}} \mathsf{top}(s')$ for every outer step $s \to_{\Theta(\mathcal{R})} s'$ in $\mathcal{A}$ and $\mathsf{top}(s) = \mathsf{top}(s')$ for every inner step $s \to_{\Theta(\mathcal{R})} s'$ in $\mathcal{A}$. Furthermore, $\mathsf{top}(s) \sim_{\mathcal{E}} \mathsf{top}(s')$ for every $s \sim_{\Theta(\mathcal{E})} s'$ in $\mathcal{A}$ by Lemma 3.7. Hence if $\mathcal{A}$ contains an outer $\Theta(\mathcal{R})$-step then

$$\mathsf{top}(t) \to^+_{\mathcal{R}/\mathcal{E}} \mathsf{top}(C[t\sigma]) = C'[\Box_{\alpha_1}, \ldots, \mathsf{top}(t)\sigma', \ldots, \Box_{\alpha_m}],$$

contradicting the non-loopingness of $\mathcal{R}/\mathcal{E}$. Consequently, $\mathsf{top}(t) \sim_{\mathcal{E}} \mathsf{top}(C[t\sigma])$. Because $\mathcal{E}$ is variable preserving and non-collapsing this implies that $\mathsf{top}(t)$ and $\mathsf{top}(C[t\sigma])$ have the same number of holes and thus the context C must be well-typed. Let $\mathsf{alien}(t) = \{t_1, \ldots, t_n\}$. The consistency of σ yields $\mathsf{alien}(C[t\sigma])) = \mathsf{alien}(t\sigma) = \{t_1\sigma, \ldots, t_n\sigma\}$. With the help of Lemmata 3.5 and 3.7 we obtain a permutation π such that for all $1 \leqslant i \leqslant n$ either $t_i \sim_{\Theta(\mathcal{E})} t_{\pi(i)}\sigma$ or $t_i \to^+_{\Theta(\mathcal{R}/\mathcal{E})} t_{\pi(i)}\sigma$. Since there are inner $\Theta(\mathcal{R})$-steps in $\mathcal{A}$, the latter alternative must occur for some j. Let $k > 0$ satisfy $\pi^k(j) = j$. We obtain $t_j \to^+_{\Theta(\mathcal{R}/\mathcal{E})} t_j\sigma^k$ where σ^k denotes the k-fold composition of σ. Since $\mathsf{rank}(t_j) < \mathsf{rank}(t)$ and σ^k inherits consistency from σ, this contradicts the minimality of $\mathcal{A}$. We conclude that $\mathcal{A}$ contains an inner collapsing $\Theta(\mathcal{R})$-step.

It remains to show that $\mathcal{A}$ contains an outer duplicating $\Theta(\mathcal{R})$-step. Suppose on the contrary that there are no outer duplicating $\Theta(\mathcal{R})$-steps in $\mathcal{A}$. Consider the mapping $\sharp$ defined in the proof of Lemma 3.8. There we noticed that $\sharp(u) = \sharp(v)$ for every $\Theta(\mathcal{E})$-step $u \sim v$ in $\mathcal{A}$ and if $u \to v$ is a $\Theta(\mathcal{R})$-step in $\mathcal{A}$ then

1. $\sharp(u) \geqslant_{\mathrm{mul}} \sharp(v)$ if $u \to v$ is outer,
2. $\sharp(u) >_{\mathrm{mul}} \sharp(v)$ if $u \to v$ is inner collapsing, and

3. $\sharp(u) \geqslant_{\text{mul}} \sharp(v)$ if $u \to v$ is inner non-collapsing.

Since we know that the second alternative occurs at least once, we obtain $\sharp(t) >_{\text{mul}} \sharp(C[t\sigma])$. However, using the consistency of σ, one easily verifies that $\sharp(C[t\sigma]) \geqslant_{\text{mul}} \sharp(t\sigma) \geqslant_{\text{mul}} \sharp(t)$, yielding the desired contradiction. We conclude that $\mathcal{A}$ contains an outer duplicating $\Theta(\mathcal{R})$-step. $\square$

Corollary 5.7 *Non-loopingness is persistent for many-sorted ETRSs $\mathcal{R}/\mathcal{E}$ such that $\mathcal{R}$ is non-collapsing or non-duplicating and $\mathcal{E}$ is non-collapsing and variable preserving.* $\square$

We illustrate the usefulness of the above theorem by giving a simple proof of non-loopingness for the following TRS from [10], depending on arbitrary instance P of Post's Correspondence Problem over the alphabet Γ:

$$\mathcal{R} = \left\{ \begin{array}{rcll} h(F(c,c,a(z))) & \to & g(F(a(z),a(z),a(z))) & \forall\ a \in \Gamma \\ F(w_1(x),w_2(y),z) & \to & F(x,y,z) & \forall\ (w_1,w_2) \in P \\ h(g(x)) & \to & g(h(x)) & \\ f(g(x)) & \to & f(h(h(x))) & \end{array} \right\}$$

Here symbols of Γ are unary function symbols of $\mathcal{R}$ and if $w = a_1 \cdots a_n \in \Gamma^*$ then $w(x)$ denotes the term $a_1(\cdots(a_n(x))\cdots)$. In [10] this TRS is used to show that termination is an undecidable property of non-looping TRSs. Note that $\mathcal{R}$ is non-collapsing. Hence we can use type introduction to prove its non-loopingness. Consider $\mathcal{S} = \{\alpha,\beta,\gamma\}$ with sort declarations $F\colon \alpha \times \alpha \times \alpha \to \beta$, $c\colon \alpha$, $a\colon \alpha \to \alpha$ for all $a \in \Gamma$, $g,h\colon \beta \to \beta$, and $f\colon \beta \to \gamma$. Terms of sort α are in normal form, hence trivially non-looping. For terms of sort β we note that the rule $f(g(x)) \to f(h(h(x)))$ can never be applied, but since $\mathcal{R}$ minus this rule is terminating (by lexicographic path order) it follows that those terms are non-looping. So if $\mathcal{R}$ admits a loop $t \to^+ C[t\sigma]$ then $\mathsf{sort}(t) = \gamma$ and the rule $f(g(x)) \to f(h(h(x)))$ must be used. From $\mathsf{sort}(t) = \gamma$ we immediately infer that the root symbol of t is f and that C is empty. Hence $t \to^+ C[t\sigma]$ must be of the form

$$\begin{array}{rcl} t = f(C_1[F(s_1,s_2,s_3)]) & \to^* & f(g(C_2[F(t_1,t_2,t_3)])) \\ & \to & f(h(h(C_2[F(t_1,t_2,t_3)]))) \\ & \to^* & f(C_1[F(s_1\sigma,s_2\sigma,s_3\sigma)]) = t\sigma \end{array}$$

with C_1 and C_2 only containing g and h symbols. From the form of the rewrite rules of $\mathcal{R}$ we get the contradictory $|C_1| \leqslant |g(C_2)| = |C_2| + 1$ and $|C_2| + 2 = |h(h(C_2))| \leqslant |C_1|$. Hence also all terms of sort γ are non-looping. In [10] non-loopingness of $\mathcal{R}$ is shown by a more complicated ad-hoc argument.

We conclude this section by remarking that the proofs of non-loopingness of several of the examples in [23] can be simplified by an appeal to Corollary 5.7.

6 Modularity

Persistence is closely related to the notion of *modularity*. A property of ETRSs is said to be modular if the union of two ETRSs with the property and disjoint signatures has the property. Modularity has been extensively studied in the literature, see Ohlebusch [17] for a recent overview. The following result for TRSs is from Zantema [22]. The easy proof in [22] applies to ETRSs as well. Here a property P of ETRSs is called *component closed* if it can be defined in terms of the induced rewrite relation (so an ETRS $\mathcal{R}/\mathcal{E}$ has the property P if and only if the relation $\rightarrow_{\mathcal{R}/\mathcal{E}}$ has the property P) and the following statements are equivalent for every ETRS $\mathcal{R}/\mathcal{E}$:

1. $\mathcal{R}/\mathcal{E}$ has the property P,
2. for every equivalence class (with respect to $\leftrightarrow^*_{\mathcal{R}/\mathcal{E}}$) C of terms, the restriction of $\rightarrow_{\mathcal{R}/\mathcal{E}}$ to C has the property P.

Lemma 6.1 *Let P be a component closed property of ETRSs. If P is persistent then P is modular.* $\square$

Most properties of ETRSs, including the ones we study in this paper (viz. termination, acyclicity, and non-loopingness), are component closed. An example of a persistent property that is neither component closed nor modular is “non-collapsing or non-duplicating”. It is an open problem whether the converse of Lemma 6.1 holds, even for TRSs. Van de Pol [18] showed that a component closed property is persistent if and only if it is modular for $\mathcal{S}$-*sorted* TRSs.

Combining Lemma 6.1 with Corollaries 3.9 and 5.7 and Theorem 5.2 yields the following result.

Corollary 6.2 *The union of disjoint terminating (acyclic, non-looping) ETRSs $\mathcal{R}_1/\mathcal{E}_1$ and $\mathcal{R}_2/\mathcal{E}_2$ is terminating (acyclic, non-looping) provided $\mathcal{E}_1 \cup \mathcal{E}_2$ is both non-collapsing and variable preserving and $\mathcal{R}_1 \cup \mathcal{R}_2$ is either non-collapsing or non-duplicating.* $\square$

For disjoint terminating TRSs $\mathcal{R}_1$ and $\mathcal{R}_2$ it is well-known that their union is also terminating if one of $\mathcal{R}_1$, $\mathcal{R}_2$ is both non-collapsing and non-duplicating (Middeldorp [14]). This result, which also holds for acyclicity and non-loopingness, extends to ETRSs.

Theorem 6.3 *The union of disjoint terminating (acyclic, non-looping) ETRSs $\mathcal{R}_1/\mathcal{E}_1$ and $\mathcal{R}_2/\mathcal{E}_2$ is terminating (acyclic, non-looping) provided $\mathcal{E}_1 \cup \mathcal{E}_2$ is both* *non-collapsing and variable preserving and one of $\mathcal{R}_1$, $\mathcal{R}_2$ is both non-collapsing and non-duplicating.*

Proof Suppose on the contrary that $(\mathcal{R}_1 \cup \mathcal{R}_2)/(\mathcal{E}_1 \cup \mathcal{E}_2)$ is not terminating (acyclic, non-looping). Let $\mathcal{S} = \{\alpha, \beta\}$ and consider the sort declarations $f\colon \alpha \times \cdots \times \alpha \to \alpha$ for all function symbols f occurring in $\mathcal{R}_1/\mathcal{E}_1$ and $g\colon \beta \times \cdots \times \beta \to \beta$ for all function symbols g occurring in $\mathcal{R}_2/\mathcal{E}_2$. Note that $(\mathcal{R}_1 \cup \mathcal{R}_2)/(\mathcal{E}_1 \cup \mathcal{E}_2)$ is trivially $\mathcal{S}$-sorted. Lemma 3.8 (Lemma 5.1, Lemma 5.6) yields a rewrite sequence (cycle, loop) $\mathcal{A}$ in which all terms have the same rank and which contains an outer duplicating and inner collapsing $(\mathcal{R}_1 \cup \mathcal{R}_2)$-step. Because all terms in $\mathcal{A}$ have the same rank we may assume without loss of generality that $\mathsf{sort}(t) = \alpha$ for every term t in $\mathcal{A}$. This implies that outer duplicating $(\mathcal{R}_1 \cup \mathcal{R}_2)$-steps are $\mathcal{R}_1$-steps and inner collapsing $(\mathcal{R}_1 \cup \mathcal{R}_2)$-steps are $\mathcal{R}_2$-steps. Hence $\mathcal{R}_1$ is duplicating and $\mathcal{R}_2$ is collapsing, yielding the desired contradiction. $\square$

Note that the above proof also implies Corollary 6.2, eliminating the need for Lemma 6.1 for obtaining our modularity results.

Modularity results are rather restrictive because of the disjointness requirement. Next we show how persistence gives rise to preservation results for non-disjoint combinations of ETRSs, generalizing and simplifying one of the main results of Aoto and Toyama [1].

Definition 6.4 An $\mathcal{S}$-sorted signature $\mathcal{F}$ is called *decomposable* if $\mathcal{S} = \{0, 1, 2\}$ and every sort declaration $\alpha_1 \times \cdots \times \alpha_n \to \alpha$ of a function symbol in $\mathcal{F}$ satisfies $\alpha_1, \ldots, \alpha_n \in \{0, \alpha\}$. Let $\mathcal{R}/\mathcal{E}$ be an $\mathcal{S}$-sorted ETRS over a decomposable signature $\mathcal{F}$. Let $\mathcal{S}_1 = \{0, 1\}$ and $\mathcal{S}_2 = \{0, 2\}$. We define the $\mathcal{S}_i$-sorted ETRS $\mathcal{R}_i/\mathcal{E}_i$ for $i \in \{1, 2\}$ as follows: $\mathcal{R}_i = \{l \to r \in \mathcal{R} \mid \mathsf{sort}(l) \in \{0, i\}\}$ and $\mathcal{E}_i = \{l \approx r \in \mathcal{E} \mid \mathsf{sort}(l) \in \{0, i\}\}$.

Note that $\mathcal{R}/\mathcal{E} = (\mathcal{R}_1 \cup \mathcal{R}_2)/(\mathcal{E}_1 \cup \mathcal{E}_2)$ but the intersection of $\mathcal{R}_1$ and $\mathcal{R}_2$ ($\mathcal{E}_1$ and $\mathcal{E}_2$) need not be empty. The partitioning of $\mathcal{R}/\mathcal{E}$ (with empty $\mathcal{E}$) into $\mathcal{R}_1/\mathcal{E}_1$ and $\mathcal{R}_2/\mathcal{E}_2$ is called decomposition with naive sort attachment in [1].

Lemma 6.5 *Let $\mathcal{R}/\mathcal{E}$ be a terminating $\mathcal{S}$-sorted ETRS over a decomposable signature with $\mathcal{E}$ variable preserving and non-collapsing. If $\Theta(\mathcal{R}/\mathcal{E})$ is not terminating and $\mathcal{R}_i$ is non-collapsing and non-duplicating for some $i \in \{1, 2\}$ then there exists an infinite rewrite sequence which contains infinitely many outer duplicating and infinitely many inner collapsing $\Theta(\mathcal{R}_j)$-steps with $j \neq i$.*

Proof According to Lemma 3.8 there exists an infinite $\Theta(\mathcal{R}/\mathcal{E})$-rewrite sequence $\mathcal{A}$ in which all terms have the same rank and which contains an outer duplicating and an inner collapsing $\Theta(\mathcal{R})$-step. Actually, we may assume that $\mathcal{A}$ contains infinitely many outer duplicating and infinitely many inner collapsing $\Theta(\mathcal{R})$-steps, cf. the remark following Lemma 3.8. Let $\alpha \in \{0,1,2\}$ be the sort of the terms in $\mathcal{A}$. Because the only rewrite rules that apply to well-typed terms of sort 0 come from $\mathcal{R}_1 \cap \mathcal{R}_2 \subseteq \mathcal{R}_i$, which by assumption is non-duplicating, $\alpha \neq 0$. Since $\mathcal{R}_i$ is non-duplicating, it follows that $\alpha = j$ with $j \neq i$ and thus $\mathcal{A}$ contains infinitely many outer duplicating $\Theta(\mathcal{R}_j)$-steps. Aliens in $\mathcal{A}$ have sort 0, i, or j. Since well-typed terms of sort 0 and i do not admit collapsing rewrite rules, it follows that every inner collapsing step in $\mathcal{A}$ uses a rule from $\Theta(\mathcal{R}_j)$. $\square$

The following result appears in [1] for the special case $\mathcal{E} = \varnothing$. Our proof is much simpler.

Theorem 6.6 *Let $\mathcal{R}/\mathcal{E}$ be an $\mathcal{S}$-sorted ETRS over a decomposable signature with $\mathcal{E}$ variable preserving and non-collapsing. The ETRS $\Theta(\mathcal{R}/\mathcal{E})$ is terminating if both $\Theta(\mathcal{R}_1/\mathcal{E}_1)$ and $\Theta(\mathcal{R}_2/\mathcal{E}_2)$ are terminating and one of the following conditions holds:*

1. *$\mathcal{R}$ is non-collapsing,*
2. *$\mathcal{R}$ is non-duplicating,*
3. *$\mathcal{R}_1$ or $\mathcal{R}_2$ is both non-collapsing and non-duplicating.*

Proof First note that by definition of decomposability, any $\mathcal{R}/\mathcal{E}$-rewrite sequence is an $\mathcal{R}_1/\mathcal{E}_1$-rewrite sequence or an $\mathcal{R}_2/\mathcal{E}_2$-rewrite sequence. Since both $\mathcal{R}_1/\mathcal{E}_1$ and $\mathcal{R}_2/\mathcal{E}_2$ are terminating, we conclude that $\mathcal{R}/\mathcal{E}$ is terminating. Hence parts 1 and 2 are just a special case of Corollary 3.9. For part 3 we reason as follows. We assume without loss of generality that $\mathcal{R}_1$ is non-collapsing and non-duplicating. Suppose on the contrary that $\Theta(\mathcal{R}/\mathcal{E})$ is not terminating. According to the previous lemma $\Theta(\mathcal{R}/\mathcal{E})$ admits an infinite rewrite sequence $\mathcal{A}$ that contains infinitely many outer duplicating and infinitely many inner collapsing $\Theta(\mathcal{R}_2)$-steps. Hence, by replacing all maximal subterms of sort 1 in $\mathcal{A}$ by an arbitrary but fixed variable we obtain an infinite $\Theta(\mathcal{R}_2/\mathcal{E}_2)$-rewrite sequence, contradicting the termination of $\Theta(\mathcal{R}_2/\mathcal{E}_2)$. We conclude that $\Theta(\mathcal{R}/\mathcal{E})$ is terminating. $\square$

Termination of $\Theta(\mathcal{R}_1/\mathcal{E}_1)$ and $\Theta(\mathcal{R}_2/\mathcal{E}_2)$ cannot be weakened to termination of $\mathcal{R}_1/\mathcal{E}_1$ and $\mathcal{R}_2/\mathcal{E}_2$, as shown by the following example. Consider

$$\mathcal{R} = \left\{ \begin{array}{rcl} f(a,b,x) & \to & f(x,x,x) \\ g(x,y) & \to & x \\ g(x,y) & \to & y \end{array} \right\}$$

(with $\mathcal{E} = \varnothing$) and let $\mathcal{S} = \{0, 1, 2\}$ with sort declarations $a, b\colon 0$, $f\colon 0 \times 0 \times 0 \to 1$, and $g\colon 1 \times 1 \to 1$. These sort declarations clearly satisfy the conditions of Definition 6.4. We have $\mathcal{R}_1 = \mathcal{R}$ and $\mathcal{R}_2 = \varnothing$. The many-sorted TRS $\mathcal{R}_1$ is terminating and $\mathcal{R}_2$ is trivially terminating, in addition to being non-collapsing and non-duplicating. However, $\Theta(\mathcal{R})$ lacks termination.

The preceding result applies to the ETRS $\mathcal{R}/\mathcal{E}$ with $\mathcal{R}$ consisting of the rewrite rules

$$\begin{array}{rcl@{\qquad}rcl}
\mathit{list}(0) & \to & [\,] & [\,] \mathbin{+\!\!+} x & \to & x \\
\mathit{list}(s(x)) & \to & \mathit{list}(x) \mathbin{+\!\!+} [s(x)] & [x \mid y] \mathbin{+\!\!+} z & \to & [x \mid y \mathbin{+\!\!+} z] \\
x \geqslant 0 & \to & \mathit{true} & x > y & \to & x \geqslant s(y) \\
s(x) \geqslant s(y) & \to & x \geqslant y & \mathit{true} \wedge \mathit{true} & \to & \mathit{true} \\
0 \geqslant s(x) & \to & \mathit{false} & \mathit{false} \wedge x & \to & \mathit{false}
\end{array}$$

and $\mathcal{E} = \mathrm{A}(\mathbin{+\!\!+}) \cup \mathrm{AC}(\wedge)$. (The expression $[s(x)]$ denotes the term $[s(x) \mid [\,]]$.) Let $\mathcal{S} = \{0, 1, 2\}$ with sort declarations $0\colon 0$, $s\colon 0 \to 0$, $[\,]\colon 1$, $[\,\mid\,]\colon 0 \times 1 \to 1$, $\mathit{list}\colon 0 \to 1$, $\mathbin{+\!\!+}\colon 1 \times 1 \to 1$, $\mathit{false}, \mathit{true}\colon 2$, $\geqslant, >\colon 0 \times 0 \to 2$, and $\wedge\colon 2 \times 2 \to 2$. The conditions of Definition 6.4 are satisfied and $\mathcal{R}_2$ is non-collapsing as well as non-duplicating. Hence Theorem 6.6 yields that termination of $\Theta(\mathcal{R}/\mathcal{E})$ follows from termination of $\Theta(\mathcal{R}_1/\mathrm{A}(\mathbin{+\!\!+}))$ and $\Theta(\mathcal{R}_2/\mathrm{AC}(\wedge))$. Termination of $\Theta(\mathcal{R}_1/\mathrm{A}(\mathbin{+\!\!+}))$ is e.g. established by the AC-RPO ordering of [20] with precedence $\mathit{list} \succ \mathbin{+\!\!+} \succ [\,\mid\,] \succ [\,]$ (so $\Theta(\mathcal{R}_1/\mathrm{AC}(\mathbin{+\!\!+}))$ is terminating which implies that $\Theta(\mathcal{R}_1/\mathrm{A}(\mathbin{+\!\!+}))$ is terminating) and for $\Theta(\mathcal{R}_2/\mathrm{AC}(\wedge))$ we can use the polynomial interpretations $[\![\, 0 \,]\!] = [\![\, \mathit{true} \,]\!] = [\![\, \mathit{false} \,]\!] = 0$, $[\![\, s \,]\!](x) = x + 1$, $[\![\, \geqslant \,]\!](x, y) = [\![\, \wedge \,]\!](x, y) = x + y + 1$, $[\![\, > \,]\!](x, y) = x + y + 3$.

Theorem 6.6 extends to acyclicity and non-loopingness.

Theorem 6.7 *Let $\mathcal{R}/\mathcal{E}$ be an $\mathcal{S}$-sorted ETRS over a decomposable signature with $\mathcal{E}$ variable preserving and non-collapsing. The ETRS $\Theta(\mathcal{R}/\mathcal{E})$ is acyclic (non-looping) if both $\Theta(\mathcal{R}_1/\mathcal{E}_1)$ and $\Theta(\mathcal{R}_2/\mathcal{E}_2)$ are acyclic (non-looping) and one of the following conditions holds:*

1. *$\mathcal{R}$ is non-collapsing,*

2. *$\mathcal{R}$ is non-duplicating,*

3. *$\mathcal{R}_1$ or $\mathcal{R}_2$ is both non-collapsing and non-duplicating.*

Proof We only consider non-loopingness here. The proof for acyclicity is simpler. We obtain the non-loopingness of $\mathcal{R}/\mathcal{E}$ as in the proof of Theorem 6.6. Hence parts 1 and 2 are a special case of Corollary 5.7. For part 3 assume without loss of generality that $\mathcal{R}_1$ is non-collapsing and non-duplicating. Suppose on the contrary that $\Theta(\mathcal{R}/\mathcal{E})$ is looping. We conclude from Lemma 5.6 that $\Theta(\mathcal{R}/\mathcal{E})$ admits a loop $\mathcal{A}\colon t \to^{+}_{\Theta(\mathcal{R}/\mathcal{E})} C[t\sigma]$ in which all terms have the same rank and which contains an outer duplicating and an inner collapsing $\Theta(\mathcal{R}_2)$-step. In particular, $\mathsf{sort}(t) = 2$.

Let ϕ be the mapping that replaces all maximal subterms of sort 1 by an arbitrary but fixed variable. From $\mathcal{A}$ we obtain the rewrite sequence $\phi(t) \to^{+}_{\Theta(\mathcal{R}_2/\mathcal{E}_2)} \phi(C[t\sigma])$. Note that all symbols above the hole in C have sort 2 for otherwise $\mathsf{rank}(C[t\sigma]) > \mathsf{rank}(t)$. Therefore $\phi(C[t\sigma]) = C'[\phi(t)\sigma']$ with $C' = \phi(C)$ and $\sigma' = \phi \circ \sigma$. Hence $\Theta(\mathcal{R}_2/\mathcal{E}_2)$ admits a loop, which contradicts the assumption. $\square$

7 Future Work

An obvious question for future work is whether persistence can be proved for other properties of ETRSs. A more important question is whether persistence results still hold if we allow order-sorted signatures. This would result in more useful decomposability results by relaxing the typing conditions imposed on the respective subsystems.

Nevertheless, persistence in a many-sorted setting is already very useful. Below we illustrate this by giving a simple proof of the completeness of the recent powerful dependency pair approach of Arts and Giesl [2] for proving termination. More precisely, we prove that termination of a TRS $\mathcal{R}$ implies termination of the TRS $\mathcal{R} \cup \mathsf{DP}(\mathcal{R})$. Here $\mathsf{DP}(\mathcal{R})$ denotes the set of dependency pairs of $\mathcal{R}$. These are defined as follows. Let $\mathcal{F}$ be the signature of $\mathcal{R}$. A function symbol $f \in \mathcal{F}$ is said to be *defined* if $f = \mathsf{root}(l)$ for some rewrite rule $l \to r \in \mathcal{R}$. With every defined function symbol f we associate a so-called *tuple symbol* F of the same arity. Now if $f(l_1, \ldots, l_n) \to r \in \mathcal{R}$ and $g(t_1, \ldots, t_m)$ is a subterm of r with g a defined symbol then $F(l_1, \ldots, l_n) \to G(t_1, \ldots, t_m)$ is a *dependency pair* of $\mathcal{R}$.

Lemma 7.1 *If $\mathcal{R}$ is a terminating TRS then so is $\mathcal{R} \cup \mathsf{DP}(\mathcal{R})$.*

Proof Let $\mathcal{F}$ be the signature of $\mathcal{R}$. Consider two sorts, α and β, with sort declarations $f\colon \alpha \times \cdots \times \alpha \to \alpha$ for every $f \in \mathcal{F}$ and $F\colon \alpha \times \cdots \times \alpha \to \beta$ for every tuple symbol F. Note that these sort declarations are compatible with the rewrite rules in $\mathcal{R} \cup \mathsf{DP}(\mathcal{R})$. Let $\mathcal{S} = \{\alpha, \beta\}$. First we show that the $\mathcal{S}$-sorted TRS $\mathcal{R} \cup \mathsf{DP}(\mathcal{R})$ is terminating. Suppose on the contrary that there exists a well-typed term t that admits an infinite $\mathcal{R} \cup \mathsf{DP}(\mathcal{R})$-rewrite sequence $\mathcal{A}$. If $\mathsf{sort}(t) = \alpha$ then $\mathcal{A}$ consists of $\mathcal{R}$-rewrite steps, contradicting the termination of $\mathcal{R}$. If $\mathsf{sort}(t) = \beta$ then $\mathcal{A}$ must have the form

$$t \to^{*}_{\mathcal{R}} s_1 \to_{\mathsf{DP}(\mathcal{R})} t_1 \to^{*}_{\mathcal{R}} s_2 \to_{\mathsf{DP}(\mathcal{R})} t_2 \to^{*}_{\mathcal{R}} s_3 \to_{\mathsf{DP}(\mathcal{R})} t_3 \to^{*}_{\mathcal{R}} \cdots$$

where all $\mathcal{R}$-steps take place below the root and all $\mathsf{DP}(\mathcal{R})$-steps at the root. By replacing every tuple symbol by its corresponding symbol in $\mathcal{F}$ and by putting appropriate contexts around the t_i terms, we easily obtain an infinite $\mathcal{R}$-rewrite sequence starting from t, again contradicting the termination of $\mathcal{R}$.

Now suppose that the unsorted TRS $\mathcal{R} \cup \mathsf{DP}(\mathcal{R})$ is not terminating. According to Lemma 3.8 it admits a rewrite sequence that contains an inner collapsing rewrite step. According to the sort declarations, aliens must have sort β. However, the only rewrite rules applicable to a subterm of sort β stem from $\mathsf{DP}(\mathcal{R})$ and these rules are non-collapsing. Hence inner collapsing rewrite steps do not exist. We conclude that $\mathcal{R} \cup \mathsf{DP}(\mathcal{R})$ is terminating. $\square$

The reader is invited to compare our proof with the one in [2, Theorem 7].

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References

[1] T. Aoto and Y. Toyama, *On Composable Properties of Term Rewriting Systems*, Proceedings of the 6th International Conference on Algebraic and Logic Programming, Southampton, LNCS 1298 (1997) 114–128.

[2] T. Arts and J. Giesl, *Termination of Term Rewriting Using Dependency Pairs*, Technical report IBN-97/46, Department of Computer Science, Darmstadt University of Technology, Germany, 1997. To appear in Theoretical Computer Science.

[3] F. Baader and T. Nipkow, *Term Rewriting and All That*, Cambridge University Press, 1998.

[4] A. Ben Cherifa and P. Lescanne, *Termination of Rewriting Systems by Polynomial Interpretations and its Implementation*, Science of Computer Programming 9(2) (1987) 137–159.

[5] N. Dershowitz, *Termination of Rewriting*, Journal of Symbolic Computation 3 (1987) 69–116.

[6] N. Dershowitz and J.-P. Jouannaud, *Rewrite Systems*, in: Handbook of Theoretical Computer Science, Vol. B (ed. J. van Leeuwen), North-Holland (1990) 243–320.

[7] N. Dershowitz and Z. Manna, *Proving Termination with Multiset Orderings*, Communications of the ACM 22 (1979) 465–476.

[8] M.C.F. Ferreira, *Dummy Elimination in Equational Rewriting*, Proceedings of the 7th International Conference on Rewriting Techniques and Applications, New Brunswick, LNCS 1103 (1996) 78–92.

[9] M.C.F. Ferreira and H. Zantema, *Dummy Elimination: Making Termination Easier*, Proceedings of the 10th International Conference on Fundamentals of Computation Theory, Dresden, LNCS 965 (1995) 243–252.

[10] A. Geser, A. Middeldorp, E. Ohlebusch, and H. Zantema, *Relative Undecidability in Term Rewriting*, Proceedings of the 10th Annual Conference of the European Association for Computer Science Logic, Utrecht, LNCS 1258 (1997) 150–166.

[11] J.-P. Jouannaud and M. Muñoz, *Termination of a Set of Rules Modulo a Set of Equations*, Proceedings of the 7th International Conference on Automated Deduction, Napa, LNCS 170 (1984) 175–193.

[12] D. Kapur and G. Sivakumar, *A Total, Ground Path Ordering for Proving Termination of AC-Rewrite Systems*, Proceeding of the 8th International Conference on Rewriting Techniques and Applications, Sitges, LNCS 1232 (1997) 142–155.

[13] J.W. Klop, *Term Rewriting Systems*, in: Handbook of Logic in Computer Science, Vol. 2 (eds. S. Abramsky, D. Gabbay and T. Maibaum), Oxford University Press (1992) 1–116.

[14] A. Middeldorp, *A Sufficient Condition for the Termination of the Direct Sum of Term Rewriting Systems*, Proceedings of the 4th IEEE Symposium on Logic in Computer Science, Pacific Grove (1989) 396–401.

[15] A. Middeldorp and B. Gramlich, *Simple Termination is Difficult*, Applicable Algebra in Engineering, Communication and Computing 6 (1995) 115–128.

[16] E. Ohlebusch, *A Simple Proof of Sufficient Conditions for the Termination of the Disjoint Union of Term Rewriting Systems*, Bulletin of the EATCS 49 (1993) 178–183.

[17] E. Ohlebusch, *Modular Properties of Composable Term Rewriting Systems*, Ph.D. thesis, Universität Bielefeld (1994).

[18] J. van de Pol, *Modularity in Many-Sorted Term Rewriting Systems*, Master's thesis, report INF/SCR-92-37, Utrecht University (1992).

[19] A. Rubio, *A fully syntactic AC-RPO*, Proceedings of the 10th International Conference on Rewriting Techniques and Applications, Trento, LNCS 1631 (1999) 133–147.

[20] A. Rubio and R. Nieuwenhuis, *A Total AC-Compatible Ordering Based on RPO*, Theoretical Computer Science 142 (1995) 209–227.

[21] J. Steinbach, *Termination of Rewriting: Extensions, Comparison and Automatic Generation of Simplification Orderings*, Ph.D. thesis, Universität Kaiserslautern (1994).

[22] H. Zantema, *Termination of Term Rewriting: Interpretation and Type Elimination*, Journal of Symbolic Computation 17 (1994) 23–50.

[23] H. Zantema and A. Geser, *Non-Looping Rewriting*, report UU-CS-1996-03, Utrecht University, Department of Computer Science (1996).