A complexity dichotomy for hypergraph partition functions^{*}

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Abstract

We consider the complexity of counting homomorphisms from an r-uniform hypergraph G to a symmetric r-ary relation H. We give a dichotomy theorem for r > 2, showing for which H this problem is in FP and for which H it is #P-complete. This generalises a theorem of Dyer and Greenhill (2000) for the case r = 2, which corresponds to counting graph homomorphisms. Our dichotomy theorem extends to the case in which the relation H is weighted, and the goal is to compute the *partition function*, which is the sum of weights of the homomorphisms. This problem is motivated by statistical physics, where it arises as computing the partition function for particle models in which certain combinations of r sites interact symmetrically. In the weighted case, our dichotomy theorem generalises a result of Bulatov and Grohe (2005) for graphs, where r = 2. When r = 2, the polynomial time cases of the dichotomy correspond simply to rank-1 weights. Surprisingly, for all r > 2 the polynomial time cases of the dichotomy have rather more structure. It turns out that the weights must be superimposed on a combinatorial structure defined by solutions of an equation over an Abelian group. Our result also gives a dichotomy for a closely related constraint satisfaction problem.

1 Introduction

We consider the complexity of counting homomorphisms from an r-uniform hypergraph G to a symmetric r-ary relation H. We will give a *dichotomy* theorem for r > 2, showing that counting is in polynomial time for certain H and is #P-complete for the remainder. Moreover our dichotomy is *effective*, meaning that there is an algorithm that takes H as input and determines whether the counting problem is polynomial time solvable or whether it is #P-complete. This generalises a theorem of Dyer and Greenhill [10] for the case r = 2, which corresponds to counting graph homomorphisms or H-colourings.

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Our dichotomy extends to the case in which the relation H is weighted, and we wish to compute the *partition function*, which is the sum of weights of all homomorphisms. Here our dichotomy theorem extends a result of Bulatov and Grohe [4] for the case of graphs, r = 2. In the graph dichotomy, the polynomial time cases correspond simply to weights which form rank-1 matrices. Surprisingly, for all r > 2, the polynomial time solvable cases are more structured. It turns out that the weights must be superimposed on a combinatorial structure defined by solutions of an equation over an Abelian group. We note that this already appears in a disguised form in the case r = 2. The bipartite case, which has no obvious analogue for r > 2, corresponds to the equation $\alpha_1 + \alpha_2 = 1$ over the group \mathbb{Z}_2 .

A motivation for considering this question comes from statistical physics. Identifying V(G) with a set of sites and D with a set of q spins, the quantity that we wish to compute, $Z^{g}(G)$, can be viewed as the partition function of a statistical physics model in which certain sets of r sites interact symmetrically, and their interaction contributes to the Hamiltonian of the system. The partition function then gives the normalising constant for the Gibbs distribution of the system. The sets of r interacting sites are the edges of G. (Sometimes, an edge of size greater than 2 is referred to as a "hyperedge", but we do not use that terminology here.) Clearly, the sites in an edge should be distinct, although their spins need not be. In this application, the edges would usually represent sets of sites which are in close physical proximity.

1.1 Notation and definitions

An r-uniform hypergraph G was defined by Berge [1] to be a system of subsets of a set V(G), where n = |V(G)|, in which each subset has cardinality r. The elements of V(G) are the vertices of the hypergraph, and the subsets are its edges. Then E(G) denotes the edge set of G. Let M = |E(G)|. Note that the edges of G are distinct sets, otherwise the set system is a multihypergraph. Note also that the edges are sets, not multisets, otherwise the multiset system has been called a hypergraph with multiplicities [12]. Note that "r-uniform hypergraph with multiplicities" is synonymous with "symmetric r-ary relation". A loop is then a (multiset) edge in which all r vertices are the same [12]. Therefore a simple graph G = (V, E) (having no loops or parallel edges) is a 2-uniform hypergraph, a graph with parallel edges is a 2-uniform multihypergraph, and a graph with loops is a 2-uniform hypergraph.

Let D be a finite set with q = |D|. We will assume $q \ge 2$, since the cases $q \le 1$ are trivial. For some $r \ge 3$, we consider a symmetric r-ary function g with domain D and codomain a set of real numbers. The codomain we will choose is the set of nonnegative algebraic numbers, $\overline{\mathbb{Q}}^{\ge 0}$. Thus $\overline{\mathbb{Q}}$ denotes the field of all algebraic numbers, and we let $\overline{\mathbb{Q}}^{>0}$ denote the positive numbers in $\overline{\mathbb{Q}}$. Our principal reason for this choice is that arithmetic operations and comparisons on such numbers can be carried out exactly on a Turing machine. See, for example, [6]. Moreover, since our analysis is entirely concerned with polynomial equations, it is natural to work in $\overline{\mathbb{Q}}$, which is the algebraic closure of the rational field \mathbb{Q} .

Given a symmetric function $g: D^r \to \overline{\mathbb{Q}}^{\geq 0}$ and an *r*-uniform hypergraph *G* as input, the partition function associated with *g* is

$$Z^{g}(G) = \sum_{\sigma: V(G) \to D} \prod_{(u_1, \dots, u_r) \in E(G)} g(\sigma(u_1), \dots, \sigma(u_r)).$$

$$\tag{1}$$

 $\mathsf{Eval}(g)$ is the problem of computing $Z^g(G)$, given the input G. Each choice for the function g leads to a computational problem which we will call $\mathsf{Eval}(g)$, and we may ask how the computational

complexity of $\mathsf{Eval}(g)$ varies with g.

We may view (1) as the evaluation of a multivariate polynomial function of the weights q(x) $(x \in D^r)$. If there are N different irrational weights $\xi_1, \xi_2, \ldots, \xi_N$, we can perform the necessary computations in the field $\mathbb{Q}(\xi_1,\xi_2,\ldots,\xi_N)$. It is known that this field is equivalent to $\mathbb{Q}(\theta)$ for a single algebraic number θ , the primitive element, and an algorithm to determine θ exists. We do not need to consider the efficiency of this algorithm, since N is a constant. The standard representation of a number in $\mathbb{Q}(\theta)$ is a constant degree polynomial in θ with rational coefficients. Arithmetic operations in $\mathbb{Q}(\theta)$ can be carried out in this representation. For details, see [6]. We assume that q is pre-processed so that all weights are given in this standard representation. Some of our intermediate reductions seemingly require computing in larger algebraic number fields. This is true even if all original weights are rational, and justifies our choice of $\overline{\mathbb{Q}}$ as the codomain of g. We will suppose, without further comment, that the necessary algebraic numbers are adjoined to $\mathbb{Q}(\theta)$ as required. In any case, we compute only in numbers fields which have constant degree over \mathbb{Q} . Despite this increase in field size during our reductions, we will show that the resulting algorithm for the polynomial time solvable cases can perform its computations entirely within $\mathbb{Q}(\theta)$. Note that the exact representation in $\mathbb{Q}(\theta)$ can also be used to compute in FP any polynomial number of bits of the binary expansion of $Z^{g}(G)$, if this is required.

It is easy to bound the number of different monomials which occur in (1). Suppose there are K nonzero weights, for some $0 \le K \le {q \choose r}$. Then the polynomial (1) has at most

$$\binom{M+K-1}{K-1} = O(M^{K-1})$$

monomial terms, which is polynomial in the size of the input. Each monomial can be computed exactly in FP, working in the field $\mathbb{Q}(\theta)$. The coefficient of each monomial is an integer, which is easily seen to be computable in #P. The nondeterministic Turing machine guesses $\sigma : V(G) \to D$, computes the term in (1) as a monomial in the weights and accepts if it is the chosen monomial. Therefore $Z^g(G)$ can be computed exactly in $\mathsf{FP}^{\#P}$ as an element of $\mathbb{Q}(\theta)$. Consequently, showing that $Z^g(G)$ is $\#\mathsf{P}$ -hard implies that it is complete for $\mathsf{FP}^{\#P}$. We make use of this observation below.

It will be helpful to describe a constraint satisfaction problem which is closely related to $\mathsf{Eval}(g)$. An instance I of $\#\mathsf{CSP}(g)$ consists of a set $V(I) = \{v_1, \ldots, v_n\}$ of variables and a multiset E(I) of constraints. Each constraint has a scope, (u_1, \ldots, u_r) , which is a tuple of r variables. The partition function $Z^g(I)$ is given by

$$Z^{g}(I) = \sum_{\sigma: V(I) \to D} \prod_{(u_1, \dots, u_r) \in E(I)} g(\sigma(u_1), \dots, \sigma(u_r)).$$

$$\tag{2}$$

Thus, every instance G of Eval(g) can be viewed as an instance of #CSP(g) by taking the vertices as variables and the edges as constraint scopes. The value of the partition function that gets output is the same in both cases. Thus, we have a trivial polynomial time reduction from Eval(g)to #CSP(g). The opposite is not necessarily true, because a constraint scope (u_1, \ldots, u_r) of an instance I of #CSP(g) might not be an edge – the same variable might appear more than once amongst u_1, \ldots, u_r . Also, the same scope might appear more than once in E(I). So an instance I of #CSP(g) might not be a properly-formed instance of Eval(g). In fact, I is a multihypergraph with multiplicities in general, rather than a hypergraph. Nevertheless, our main result applies also to the problem #CSP(g) — see Corollary 3. We note that both the Eval(g) and the #CSP(g)problems have been studied extensively. The problem $\#\mathsf{CSP}(g)$ may be generalised to the case in which the parameter g is replaced by a set of functions Γ . If Γ is a set of functions (of various arities) from D to $\overline{\mathbb{Q}}^{\geq 0}$, then $\#\mathsf{CSP}(\Gamma)$ is the problem of computing the partition function of an instance I in which each constraint with r-ary scope specifies a particular r-ary function from Γ which should be applied to the scope in the partition function. See [4] or [9] for further details. If the functions in Γ are not required to have any additional properties, like symmetry or given arity, $\#\mathsf{CSP}(\Gamma)$ is actually no more general than $\#\mathsf{CSP}(g)$, at least from the viewpoint of computational complexity. It can be shown that the two problems have the same complexity under polynomial time reductions [5]. Note, however, that the reduction from $\#\mathsf{CSP}(\Gamma)$ to $\#\mathsf{CSP}(g)$ given in [5] does not preserve symmetry. So this equivalence does not permit us to replace a family Γ of symmetric functions by a single symmetric function g. This holds even in the simplest possible case in which Γ has two unary functions. Hence, restricted to symmetric functions, $\#\mathsf{CSP}(\Gamma)$ may be a more general problem than $\#\mathsf{CSP}(g)$, but we do not consider it further here.

1.2 Previous work

The computational complexity of problems of the type we consider here was first investigated by Dyer and Greenhill [10], who examined the complexity of $\mathsf{Eval}(g)$ in the special case in which r = 2and $g : D^2 \to \{0, 1\}$, so g is equivalent to a symmetric relation on D. This is the problem of counting homomorphisms from an input simple graph G to a fixed (undirected) graph H, possibly with loops, where the function g represents the adjacency matrix of H. They showed that there is a polynomial time algorithm when each connected component of H is either a complete unlooped bipartite graph or a complete looped graph. In all other cases the counting problem $\mathsf{Eval}(g)$ is $\#\mathsf{P}$ -complete.

More generally, Bulatov and Grohe [4] considered the complexity of $\#\mathsf{CSP}(g)$ when g is a symmetric binary function on D. If the input is a simple graph G, we can think of this as counting weighted homomorphisms from G to an undirected graph H with nonnegative edge weights. The function g is equivalent to the weighted adjacency matrix \mathbf{A} of H. If H is connected, then we say that the matrix \mathbf{A} is "connected", otherwise the "connected components" of \mathbf{A} correspond to the connected components of the graph H. Similarly, we say that \mathbf{A} is bipartite if and only if H is bipartite. In this setting, Bulatov and Grohe [4] established the following important theorem, which is central to our analysis.

Theorem 1 (Bulatov and Grohe). Let A be a symmetric matrix with non-negative real entries.

- If A is connected and not bipartite, then Eval(A) is in polynomial time if the row rank of A is at most 1; otherwise Eval(A) is #P-hard.
- (2) If A is connected and bipartite, then Eval(A) is in polynomial time if the row rank of A is at most 2; otherwise Eval(A) is #P-hard.
- (3) If **A** is not connected, then $\mathsf{Eval}(\mathbf{A})$ is in polynomial time if each of its connected components satisfies the corresponding condition stated in (1) or (2); otherwise $\mathsf{Eval}(\mathbf{A})$ is $\#\mathsf{P}$ -hard.

Although Theorem 1 is stated for real numbers, we will make use of it only in the case of the algebraic numbers, since it is not clear to us how it extends to the models of real computation discussed in [4]. We prefer to work entirely in the standard Turing machine model of computation, though there may well be models of real computation in which Theorem 1 is valid. For algebraic

numbers, which include the rationals, all the arithmetic operations and comparisons required in our reductions, and those of [4], can be carried out exactly in the Turing machine model.

In the unweighted case of $\#\mathsf{CSP}(\Gamma)$, where all functions in Γ have codomain $\{0, 1\}$, Bulatov [2] has recently shown that there is a dichotomy between those Γ for which $\#\mathsf{CSP}(\Gamma)$ is polynomial time solvable, and those for which it is $\#\mathsf{P}$ -complete. The dichotomy can be extended to the case in which all functions in Γ have codomain $\mathbb{Q}^{\geq 0}$, the nonnegative rational numbers, using polynomial time reductions [5]. However, the reductions involved do not seem to extend to functions with codomain $\overline{\mathbb{Q}}^{\geq 0}$.

Establishing the existence of a dichotomy for $\#\mathsf{CSP}(\Gamma)$ is a major breakthrough. Nevertheless, the techniques of [2] shed very little light on which Γ render $\#\mathsf{CSP}(\Gamma)$ polynomial time solvable, and which Γ render it $\#\mathsf{P}$ -hard. In the current state of knowledge, Bulatov's dichotomy [2] is not effective, and its decidability is an open question.

1.3 The new results

Our main theorem, Theorem 2, gives a dichotomy for the case in which Γ contains a single symmetric function g. For this problem, we identify a set of functions g for which $\mathsf{Eval}(g)$ is computable in FP, and we show that, for every other function g, $\mathsf{Eval}(g)$ is complete for $\mathsf{FP}^{\#\mathsf{P}}$.

We examine both $\mathsf{Eval}(g)$ and $\#\mathsf{CSP}(g)$ in this setting, and give an explicit dichotomy theorem in both cases, extending the theorems of Dyer and Greenhill [10] and Bulatov and Grohe [4] to r > 2. In the r > 2 case, the problem $\mathsf{Eval}(g)$ can be understood as evaluating sums of weighted homomorphisms from an input hypergraph G to a fixed weighted hypergraph with multiplicities H. The weights of edges in H are represented by the function g.

As in the r = 2 case, there is a dichotomy, but this time some nontrivial algebraic structure is involved in the classification. The polynomial time solvable cases have rank-1 weights as before, but this time, these weights are superimposed on a combinatorial structure defined by solutions to an equation over an Abelian group. In particular, $\mathsf{Eval}(g)$ is polynomial time solvable if and only if each connected piece of the domain factors as the cartesian product of two sets A and [s]. Then, for any $\alpha_1, \ldots, \alpha_r \in A$ and $i_1, \ldots, i_r \in [s]$, the value of $g((\alpha_1, i_1), \ldots, (\alpha_r, i_r))$ is equal to 0 unless $(\alpha_1, \ldots, \alpha_r)$ is a solution to an equation in an Abelian group with domain A. In that case, the value $g((\alpha_1, i_1), \ldots, (\alpha_r, i_r))$ is just the product of some positive weights $\lambda_{i_1}, \ldots, \lambda_{i_r}$. A "connected piece" of the domain is defined as follows: two elements z and z' are linked if there are some z_2, \ldots, z_{r-1} such that $g(z, z_2, \ldots, z_{r-1}, z') > 0$. In general, two elements z and z' are connected if there is a sequence of c elements z_1, \ldots, z_c with $z = z_1$ and $z' = z_c$ such that each pair (z_i, z_{i+1}) is linked. See Theorem 2 for details.

In fact, it turns out that there is only one way to factor the connected component of the domain into A and [s] (see Theorem 4). Thus, there is a straightforward algorithm that takes g and determines whether Eval(g) is in FP or is #P-hard. See Sections 7 and 8.

Our result is in a similar spirit to the result of Klíma, Larose and Tesson [11] which gives a dichotomy for the problem of counting the number of solutions to a system of equations over a fixed semigroup. Although our application is rather different, parts of our proof draw inspiration from the proof of their theorem.

2 The main theorem

For $1 \le k \le r$, we will define

$$f^{[k]}(z_1, \dots, z_k) = \sum_{z_{k+1}, \dots, z_r \in D} g(z_1, \dots, z_r).$$

Note that $f^{[k]}$ is symmetric and that $f^{[r]}(z_1,\ldots,z_r) = g(z_1,\ldots,z_r)$. Let

$$R^{[k]} = \{(z_1, \dots, z_k) : f^{[k]}(z_1, \dots, z_k) > 0\}$$

be the relation underlying $f^{[k]}$. We will view relations either as subsets of D^k or as functions $D^k \to \{0,1\}$ according to convenience. To avoid trivialities, we assume that $R^{[1]}$ is the complete relation, i.e., that all elements of D participate in the relation; if not, an equivalent problem can be formed by simply removing the non-participating elements from D. For any k < r we have $f^{[k]}(z_1,\ldots,z_k) = \sum_{z_{k+1}\in D} f^{[k+1]}(z_1,\ldots,z_{k+1})$ so if $k \geq 2$ then $(z_1,z_2) \in R^{[2]}$ is equivalent to "there exist z_3,\ldots,z_k such that $(z_1,\ldots,z_k) \in R^{[k]}$ ". Let \equiv be the equivalence relation which is the transitive, reflexive closure of $R^{[2]}$. The domain D is partitioned into equivalence classes ("connected components") $D = D_1 \cup \cdots \cup D_m$ by \equiv .

We will use the following notation: We will let ℓ range over [m], and use it to refer to a particular connected component D_{ℓ} . When applied to any function as a subscript, it denotes the restriction of that function to the relevant connected component. For example, $f_{\ell}^{[k]} : (D_{\ell})^k \to \overline{\mathbb{Q}}^{\geq 0}$ denotes the restriction of $f^{[k]}$ to the ℓ th connected component D_{ℓ} . Likewise, g_{ℓ} is the restriction of g to D_{ℓ} . Given the definition of \equiv , it is clear that $f^{[k]} = f_1^{[k]} \oplus \cdots \oplus f_m^{[k]}$ (meaning that $f^{[k]}(z_1, \ldots, z_k) = 0$ unless z_1, \ldots, z_k are all in the same connected component). We can now state the main theorem.

Theorem 2. Let $g: D^r \to \overline{\mathbb{Q}}^{\geq 0}$ be a symmetric function with arity $r \geq 3$ and connected components D_1, \ldots, D_m as above. If g satisfies the following conditions, for all $\ell \in [m]$, then $\mathsf{Eval}(g)$ is in FP. Otherwise, $\mathsf{Eval}(g)$ is complete for $\mathsf{FP}^{\#\mathsf{P}}$. Moreover, the dichotomy is effective.

- There is a set A_{ℓ} and a positive integer s_{ℓ} , such that D_{ℓ} is the Cartesian product of A_{ℓ} and $[s_{\ell}]$ (which we write as $D_{\ell} \cong A_{\ell} \times [s_{\ell}]$).
- There are positive constants $\{\lambda_{\ell,i} : i \in [s_\ell]\}$ and a relation $S_\ell \subseteq A_\ell$ such that, for $\alpha_1, \ldots, \alpha_r \in A_\ell$ and $i_1, \ldots, i_r \in [s_\ell]$,

$$g_{\ell}((\alpha_1, i_1), \dots, (\alpha_r, i_r)) = \lambda_{\ell, i_1} \cdots \lambda_{\ell, i_r} S_{\ell}(\alpha_1, \dots, \alpha_r).$$

• There is an Abelian group $(A_{\ell}, +)$ and an equation $\alpha_1 + \cdots + \alpha_r = a$ (for some element $a \in A_{\ell}$) which defines S_{ℓ} in the sense that $(\alpha_1, \ldots, \alpha_r) \in S_{\ell}$ if and only if $\alpha_1 + \cdots + \alpha_r = a$.

The algorithm used in the polynomial time solvable cases of Theorem 2 still works if the instance is a CSP instance rather than a hypergraph. Thus, we have the following corollary.

Corollary 3. Let $g: D^r \to \overline{\mathbb{Q}}^{\geq 0}$ be a symmetric function with arity $r \geq 3$ and connected components D_1, \ldots, D_m as above. If g satisfies the conditions in Theorem 2 for all $\ell \in [m]$, then $\#\mathsf{CSP}(g)$ is in FP. Otherwise, $\#\mathsf{CSP}(g)$ is complete for $\mathsf{FP}^{\#\mathsf{P}}$. Moreover, the dichotomy is effective.

Some of the #P-hardness proofs in the proof of Theorem 2 could be simplified if we allowed ourselves a general CSP instance rather than a hypergraph, but we refrain from using this simplification in order to obtain the strongest-possible result (that is, to obtain Theorem 2 rather than just Corollary 3).

3 A restatement of the main theorem

We introduce some further notation and restate the main theorem more compactly. Along the way we gather more information, e.g., about the factorization $D_{\ell} \cong A_{\ell} \times [s_{\ell}]$.

We define the equivalence relation \sim_k on D as follows: $z_1 \sim_k z'_1$ iff there is a λ in $\overline{\mathbb{Q}}^{>0}$ such that, for all $z_2, \ldots, z_k \in D$, $f^{[k]}(z_1, z_2, \ldots, z_k) = \lambda f^{[k]}(z'_1, z_2, \ldots, z_k)$. Note that \sim_k refines \sim_{k-1} . Also, \sim_2 refines \equiv since, for any $z_1, z'_1 \in D$, $z_1 \sim_2 z'_1$ implies that there exists z_2 satisfying $R^{[2]}(z_1, z_2)$ and $R^{[2]}(z'_1, z_2)$, which in turn implies $z_1 \equiv z'_1$.

Let $[x]^{[k]} = \{y : y \sim_k x\}$ be the equivalence class of x under \sim_k . Choose a unique representative $\bar{x}^{[k]} \in [x]^{[k]}$. Thus $\bar{x}^{[k]} = \bar{y}^{[k]}$ if and only if $x \sim_k y$. Let $A^{[k]} = \{\bar{x}^{[k]} : x \in D\}$. Let $A^{[k]}_{\ell}$ denote the restriction of $A^{[k]}$ to D_{ℓ} so $A^{[k]}_{\ell} = \{\bar{x}^{[k]} : x \in D_{\ell}\}$.

Note that $R^{[k]}$ is consistent with \sim_k in the sense that $R^{[k]}(z_1, \ldots, z_k) = R^{[k]}(\bar{z}_1^{[k]}, \ldots, \bar{z}_k^{[k]})$, so we can quotient $R^{[k]}$ by \sim_k to get a relation $S^{[k]} = R^{[k]}/\sim_k$ on $A^{[k]}$. Note that $S^{[k]}$ is just the restriction of $R^{[k]}$ to $A^{[k]}$. Also, $S_{\ell}^{[k]}$ is the restriction of $R_{\ell}^{[k]}$ to $A_{\ell}^{[k]}$.

Suppose k is in the range $2 \le k \le r$. We say that g is k-factoring if the following conditions hold for every $\ell \in [m]$.

- 1. There is a positive integer $s_{\ell}^{[k]}$ such that D_{ℓ} is the Cartesian product of $A_{\ell}^{[k]}$ and $[s_{\ell}^{[k]}]$ (which we write as $D_{\ell} \cong A_{\ell}^{[k]} \times [s_{\ell}^{[k]}]$).
- 2. There are positive constants $\{\lambda_{\ell,i}^{[k]}: i \in [s_{\ell}^{[k]}]\}$ such that, for $\alpha_1, \ldots, \alpha_k \in A_{\ell}^{[k]}$ and $i_1, \ldots, i_k \in [s_{\ell}^{[k]}]$,

$$f_{\ell}^{[k]}((\alpha_1, i_1), \dots, (\alpha_k, i_k)) = \lambda_{\ell, i_1}^{[k]} \cdots \lambda_{\ell, i_k}^{[k]} S_{\ell}^{[k]}(\alpha_1, \dots, \alpha_k).$$

If g is k-factoring then we say that g is k-equational if, for every $\ell \in [m]$, there is an Abelian group $(A_{\ell}^{[k]}, +)$ and an equation $\alpha_1 + \cdots + \alpha_k = a$ (for some element $a \in A_{\ell}^{[k]}$) which defines $S_{\ell}^{[k]}$ in the sense that $(\alpha_1, \ldots, \alpha_k) \in S_{\ell}^{[k]}$ if and only if $\alpha_1 + \cdots + \alpha_k = a$.

Our main theorem (Theorem 2) can be restated as follows:

Theorem 4. Let $g: D^r \to \overline{\mathbb{Q}}^{\geq 0}$ be a symmetric function with arity $r \geq 3$. If g is r-factoring and r-equational then Eval(g) is in FP. Otherwise, Eval(g) is complete for $\text{FP}^{\#P}$. Moreover, the dichotomy is effective.

Before proving Theorem 4, we prove that it is equivalent to Theorem 2. First, it is easy to see that if g satisfies the conditions of Theorem 4 (that is, it is r-factoring and r-equational) then it also satisfies the conditions of Theorem 2 (taking A_{ℓ} to be $A_{\ell}^{[r]}$, s_{ℓ} to be $s_{\ell}^{[r]}$, and $\lambda_{\ell,i}$ to be $\lambda_{\ell,i}^{[r]}$). The other direction is a little less obvious. Suppose that g satisfies the conditions of Theorem 2. Fix any $\ell \in [m]$. From the first condition of Theorem 2, we have $D_{\ell} \cong A_{\ell} \times [s_{\ell}]$. Consider any $\alpha, \alpha' \in A_{\ell}$ and any $i, i' \in [s_{\ell}]$. We will argue that $(\alpha, i) \sim_r (\alpha', i')$ if and only if $\alpha = \alpha'$. First, suppose $\alpha = \alpha'$. Then, for any $\alpha_2, \ldots, \alpha_r \in A_{\ell}$ and $i_2, \ldots, i_r \in [s_{\ell}]$, the second condition of Theorem 2 gives

$$g_{\ell}((\alpha, i), (\alpha_2, i_2), \dots, (\alpha_r, i_r)) = \lambda_{\ell, i} \lambda_{\ell, i_2} \cdots \lambda_{\ell, i_r} S_{\ell}(\alpha, \alpha_2, \dots, \alpha_r)$$

and

$$g_{\ell}((\alpha',i'),(\alpha_2,i_2),\ldots,(\alpha_r,i_r)) = \lambda_{\ell,i'}\lambda_{\ell,i_2}\cdots\lambda_{\ell,i_r}S_{\ell}(\alpha,\alpha_2,\ldots,\alpha_r),$$

so, by the definition of \sim_r , $(\alpha, i) \sim_r (\alpha', i')$. Next, suppose $(\alpha, i) \sim_r (\alpha', i')$. Then there is a positive constant λ such that, for any $\alpha_2, \ldots, \alpha_r \in A_\ell$ and $i_2, \ldots, i_r \in [s_\ell]$,

$$\lambda_{\ell,i}\lambda_{\ell,i_2}\cdots\lambda_{\ell,i_r}S_\ell(\alpha,\alpha_2,\ldots,\alpha_r)=\lambda\lambda_{\ell,i'}\lambda_{\ell,i_2}\cdots\lambda_{\ell,i_r}S_\ell(\alpha',\alpha_2,\ldots,\alpha_r).$$

We conclude that, for any $\alpha_2, \ldots, \alpha_r \in A_{\ell}$, $S_{\ell}(\alpha, \alpha_2, \ldots, \alpha_r) = S_{\ell}(\alpha', \alpha_2, \ldots, \alpha_r)$. By the third condition in Theorem 2, we conclude that $\alpha = \alpha'$. We have now shown that $(\alpha, i) \sim_r (\alpha', i')$ if and only if $\alpha = \alpha'$. This implies that we can take the set $A_{\ell}^{[r]}$ of unique representatives to be A_{ℓ} and we can take $s_{\ell}^{[r]}$ to be s_{ℓ} . Then, taking $\lambda_{\ell,i}^{[r]}$ to be $\lambda_{\ell,i}$, g is r-factoring and r-equational (so it satisfies the conditions of Theorem 4). So we conclude that the two theorems are equivalent.

Now that we have shown that Theorem 4 is equivalent to Theorem 2, the rest of the paper will focus on proving Theorem 4. The case r = 2 is that of weighted graph homomorphism, which was analysed by Bulatov and Grohe [4]. Theorem 4 is true also when r = 2. In this situation, it could be viewed as a restatement of their result. Note, however, that "2-equational" is a restricted notion that places severe constraints on the groups $(A_{\ell}^{[2]}, +)$ that can arise. Indeed the only possibilities that are consistent with the connectivity relation \equiv are the 2-element group C_2 ("bipartite component") and the trivial group ("non-bipartite component").

It will follow from the proof of Theorem 4 (assuming that $\#P \not\subseteq FP$) that a symmetric function g of arity $r \geq 3$ that is r-factoring and r-equational is k-factoring and k-equational for all $2 \leq k < r$. In fact, the Abelian groups $(A_{\ell}^{[k]}, +)$ will all be trivial for k < r: non-trivial group structure is only possible at the top level. As a first step in the proof of Theorem 4, we verify that non-trivial group structure is only possible at the top level.

Lemma 5. Let $g: D^r \to \overline{\mathbb{Q}}^{\geq 0}$ be a symmetric function with arity $r \geq 3$. If g is k-factoring and k-equational for some k < r then for every $\ell \in [m]$ there are positive constants $\{\lambda_{\ell,i}^{[k]}: i \in D_\ell\}$ such that, for $i_1, \ldots, i_k \in D_\ell$,

$$f_{\ell}^{[k]}(i_1,\ldots,i_k) = \lambda_{\ell,i_1}^{[k]} \cdots \lambda_{\ell,i_k}^{[k]}$$

Proof. g is k-factoring so $D_{\ell} \cong A_{\ell}^{[k]} \times [s_{\ell}^{[k]}]$ and, for $\alpha_1, \ldots, \alpha_k \in A_{\ell}^{[k]}$ and $i_1, \ldots, i_k \in [s_{\ell}^{[k]}]$,

$$f_{\ell}^{[k]}((\alpha_1, i_1), \dots, (\alpha_k, i_k)) = \lambda_{\ell, i_1}^{[k]} \cdots \lambda_{\ell, i_k}^{[k]} S_{\ell}^{[k]}(\alpha_1, \dots, \alpha_k).$$

Now consider $\alpha_1, \ldots, \alpha_{k+1} \in A_{\ell}^{[k]}$ and $i_1, \ldots, i_{k+1} \in [s_{\ell}^{[k]}]$. If

$$f_{\ell}^{[k+1]}((\alpha_1, i_1), \dots, (\alpha_{k+1}, i_{k+1})) > 0$$

then

$$f_{\ell}^{[k]}((\alpha_1, i_1), \dots, (\alpha_{k-1}, i_{k-1}), (\alpha_k, i_k)) > 0$$

and

$$f_{\ell}^{[k]}((\alpha_1, i_1), \dots, (\alpha_{k-1}, i_{k-1}), (\alpha_{k+1}, i_{k+1})) > 0.$$

So since g is k-equational,

$$\alpha_1 + \dots + \alpha_{k-1} + \alpha_k = \alpha_1 + \dots + \alpha_{k-1} + \alpha_{k+1} = a$$

so $\alpha_k = \alpha_{k+1}$. By symmetry, $\alpha_1 = \cdots = \alpha_{k+1}$.

Now if $((\alpha, i), (\beta, j)) \in \mathbb{R}^{[2]}$ then there exist $\alpha_2, \ldots, \alpha_k$ and i_2, \ldots, i_k such that

$$f^{[k+1]}((\alpha, i), (\beta, j), (\alpha_2, i_2), \dots, (\alpha_k, i_k)) > 0.$$

Thus, $\alpha = \beta$. Taking the transitive closure, we note that if (α, i) and (β, j) are both in D_{ℓ} then $\alpha = \beta$. Hence $|A_{\ell}^{[k]}| = 1$ so $D_{\ell} = [s_{\ell}^{[k]}]$.

Our strategy for proving Theorem 4 is now as follows. Suppose Eval(g) is not #P-hard. We prove, for k = 2, 3, ..., r in turn, that g is k-factoring and k-equational. For k = 2 this follows straightforwardly from Theorem 1. The inductive step from k to k + 1 is where the work lies, but Lemma 5 plays a role. Ultimately, we deduce that g is r-factoring and r-equational. Conversely, if g is r-factoring and r-equational, the partition function Z^g may be computed in polynomial time using existing algorithms for counting solutions to systems over Abelian groups, and hence Eval(g) is polynomial time solvable.

4 Preliminaries

An easy observation that will be frequently used in the rest of this paper is the following.

Lemma 6. If $\text{Eval}(f^{[k]})$ is #P-hard, for some $2 \le k < r$, then so is Eval(g).

Proof. An instance of $\mathsf{Eval}(f^{[k]})$ is a k-uniform hypergraph. Simply pad each edge $e = (u_1, \ldots, u_k)$ to size r by adding r - k fresh vertices as follows: $(u_1, \ldots, u_k, z_{k+1}^e, \ldots, z_r^e)$. It is easy to verify that this is a polynomial time reduction from $\mathsf{Eval}(f^{[k]})$ to $\mathsf{Eval}(g)$.

Another easy observation is that the partition function $Z^{g}(G)$ factorises if G is not connected. So we may assume henceforth that the instance hypergraph G is connected.

For $z \in D$ let $\lambda'_{z}^{[k]}$ be defined so that, for all $z_{2}, \ldots, z_{k} \in D$,

$$f^{[k]}(z, z_2, \dots, z_k) = \lambda'^{[k]}_z f^{[k]}(\bar{z}^{[k]}, z_2, \dots, z_k)$$

(Recall from the definition of \sim_k that $\lambda'_z^{[k]}$ does not depend on $z_2, ..., z_k$.) Then, by symmetry, we have

$$f^{[k]}(z_1, \dots, z_k) = \lambda'^{[k]}_{z_1} \cdots \lambda'^{[k]}_{z_k} f^{[k]}(\bar{z}_1^{[k]}, \dots, \bar{z}_k^{[k]}).$$
(3)

Define

$$\tilde{f}^{[k]}(z_1, z_1') = \sum_{z_2, \dots, z_k \in D} f^{[k]}(z_1, z_2, \dots, z_k) f^{[k]}(z_1', z_2, \dots, z_k).$$

Let $\widetilde{R}^{[k]}$ be the (symmetric) binary relation underlying $\widetilde{f}^{[k]}$. It will turn out that $\widetilde{R}^{[k]}$ and \sim_k coincide when g is not #P-hard.

For the purposes of this paper, a symmetric relation $R \subset \mathcal{A}^k$ is said to be a *Latin hypercube* if, for all $\alpha_1, \ldots, \alpha_{k-1} \in \mathcal{A}$, there exists a unique $\alpha_k \in \mathcal{A}$ such that $(\alpha_1, \ldots, \alpha_k) \in R$. Note that symmetry implies similar statements with the α_i s permuted. This definition specialises to the familiar notion of Latin square if we take k = 3 and think of α_1, α_2 and α_3 as ranging over rows, columns and symbols, respectively. For k > 3 it is consistent with the existing, if less familiar, notion of Latin (k-1)-hypercube.

We use the following interpolation result, which is [10, Lemma 3.2]

Lemma 7. Let η_1, \ldots, η_m be known distinct nonzero constants Suppose that we know values Z_1, \ldots, Z_m such that $Z_p = \sum_{\ell=1}^m \gamma_\ell \eta_\ell^p$ for $1 \leq p \leq m$. The coefficients $\gamma_1, \ldots, \gamma_m$ can be evaluated in polynomial time.

Lemma 7 has the following consequence, since if we have $\eta_i = \eta_j$ below we can combine γ_i and γ_j into $\gamma_i + \gamma_j$.

Corollary 8. Let η_1, \ldots, η_m be known nonzero constants Suppose that we know values Z_1, \ldots, Z_m such that $Z_p = \sum_{\ell=1}^m \gamma_\ell \eta_\ell^p$ for $1 \le p \le m$. The value $Z_0 = \sum_{\ell=1}^m \gamma_\ell$ can be computed in polynomial time.

As mentioned earlier, the base case (k = 2) in the proof of Theorem 4 will follow from the result of Bulatov and Grohe [4]. They examined the complexity of $\#\mathsf{CSP}(g)$ and there is no immediate polynomial time reduction from $\#\mathsf{CSP}(g)$ to $\mathsf{Eval}(g)$. The next lemma provides such a reduction for the case that we require.

Lemma 9. Suppose $h: D^2 \to \overline{\mathbb{Q}}^{\geq 0}$ has connected components D_1, \ldots, D_ℓ , and underlying relation R_h . Suppose also that R_h has no bipartite components. If the restriction h_ℓ of h to any component D_ℓ is not rank 1, then $\mathsf{Eval}(h)$ is $\#\mathsf{P}$ -hard.

Proof. Let *I* be an instance of $\#\mathsf{CSP}(h)$. View *I* as a multigraph with possible loops and parallel edges. Form the graph *G* as the "2-stretch" of *I*; that is to say, subdivide each edge of *I* by introducing a new vertex. Note that *G* is a simple graph without loops. Define the symmetric function $h^{(2)}: D^2 \to \overline{\mathbb{Q}}^{\geq 0}$ by $h^{(2)}(x,y) = \sum_{z \in D} h(x,z)h(y,z)$. Note that $Z^h(G) = Z^{h^{(2)}}(I)$, and hence $\#\mathsf{CSP}(h^{(2)})$ reduces to $\mathsf{Eval}(h)$.

Suppose Eval(h) is not #P-hard. Then $\#\text{CSP}(h^{(2)})$ is not #P-hard. By [4, Thm 1(1)], $h^{(2)}$, viewed as a matrix, is a direct sum of rank-1 matrices; i.e., each $h_{\ell}^{(2)}$ has rank 1. But each $h_{\ell}^{(2)}$ is the "Gram matrix" of h_{ℓ} (the product of h_{ℓ} , viewed as a matrix, and its transpose), and it is a elementary fact that the rank of a matrix and its corresponding Gram matrix are equal [13]. Thus, for all ℓ , the restrictions h_{ℓ} of h to D_{ℓ} are rank 1.

5 Factoring

Lemma 10. Let $g: D^r \to \overline{\mathbb{Q}}^{\geq 0}$ be a symmetric function with arity $r \geq 3$. Either $\mathsf{Eval}(f^{[2]})$ is $\#\mathsf{P}$ -hard (which implies that $\mathsf{Eval}(g)$ is $\#\mathsf{P}$ -hard) or g is 2-factoring and 2-equational.

Proof. First, note that $R^{[2]}$ has no bipartite components: If $(z_1, z_2) \in R^{[2]}$ then there is a z_3 such that $(z_1, z_2, z_3) \in R^{[3]}$. By the symmetry of $f^{[3]}$, we find that (z_1, z_3) and (z_2, z_3) are also in $R^{[2]}$, so the component containing z_1 and z_2 is not bipartite.

Now, by [4] (using Lemma 9), $f_{\ell}^{[2]}$ has rank 1. Thus, there are positive constants $\{\mu_z : z \in D\}$ such that, for every $\ell \in [m]$ and every z_1, z_2 in D_{ℓ} , the following holds.

$$f_{\ell}^{[2]}(z_1, z_2) = \mu_{z_1} \mu_{z_2}.$$
(4)

We conclude that all elements in D_{ℓ} are related by \sim_2 , so $|A_{\ell}^{[2]}| = 1$. Thus, we can take $s_{\ell}^{[2]} = |D_{\ell}|$ and $\lambda_{\ell,z}^{[2]} = \mu_z$ and the trivial equation (since $|A_{\ell}^{[2]}| = 1$).

The parenthetical claim in the statement of this lemma and subsequent ones comes from Lemma 6. \Box

Lemma 11. Let $g: D^r \to \overline{\mathbb{Q}}^{\geq 0}$ be a symmetric function with arity $r \geq 3$. Let k be an integer in $\{3, \ldots, r\}$. Suppose that g is (k-1)-factoring and (k-1)-equational. Either $\mathsf{Eval}(f^{[k]})$ is $\#\mathsf{P}$ -hard (which implies that $\mathsf{Eval}(g)$ is $\#\mathsf{P}$ -hard), or all the following hold: (i) there are positive constants $\{\lambda_z^{[k]}: z \in D\}$ such that $f^{[k]}(z_1, \ldots, z_k) = \lambda_{z_1}^{[k]} \cdots \lambda_{z_k}^{[k]} R^{[k]}(z_1, \ldots, z_k)$, (ii) for every connected component $\ell \in [m]$, the relation $S_{\ell}^{[k]}$ is a Latin hypercube, and (iii) for every $\ell \in [m]$, the sum $\sum_{z \in [\alpha]^{[k]}} \lambda_z^{[k]}$ is independent of $\alpha \in A_{\ell}^{[k]}$.

Proof. Assume $\mathsf{Eval}(f^{[k]})$ is not $\#\mathsf{P}$ -hard. Fix $\ell \in [m]$ and $z_1, z'_1 \in D_\ell$. By the Cauchy-Schwarz inequality,

$$\left(\sum_{z_2,\dots,z_k\in D_\ell} f_\ell^{[k]}(z_1,z_2,\dots,z_k) f_\ell^{[k]}(z_1',z_2,\dots,z_k)\right)^2 \le \sum_{z_2,\dots,z_k\in D_\ell} f_\ell^{[k]}(z_1,z_2,\dots,z_k)^2 \sum_{z_2,\dots,z_k\in D_\ell} f_\ell^{[k]}(z_1',z_2,\dots,z_k)^2,$$

i.e.,

 $\tilde{f}_{\ell}^{[k]}(z_1, z_1')^2 \le \tilde{f}_{\ell}^{[k]}(z_1, z_1) \tilde{f}_{\ell}^{[k]}(z_1', z_1'),$ (5)

with equality precisely when $z_1 \sim_k z'_1$. Note that the difference between the right-hand-side and the left-hand-side in Equation (5) can be seen as a 2 by 2 determinant.

Now $\mathsf{Eval}(\tilde{f}^{[k]}) \leq \mathsf{Eval}(f^{[k]})$ since $\tilde{f}^{[k]}(u, v)$ can be simulated by a pair of constraints $f^{[k]}(u, w_2, \dots, w_k) f^{[k]}(v, w_2, \dots, w_k)$

using new variables w_2, \ldots, w_k , so $\mathsf{Eval}(\tilde{f}^{[k]})$ is not $\#\mathsf{P}$ -hard. $\tilde{R}^{[k]}$ has no bipartite components since it is reflexive, so by [4] and Lemma 9, $\tilde{f}^{[k]}$ decomposes into a sum of rank-1 blocks.

When $z_1 \not\sim_k z'_1$ we have strict inequality in (5), which implies

$$\tilde{f}_{\ell}^{[k]}(z_1, z_1') = \sum_{z_2, \dots, z_k \in D} f^{[k]}(z_1, z_2, \dots, z_k) f^{[k]}(z_1', z_2, \dots, z_k) = 0,$$
(6)

since otherwise $\tilde{f}^{[k]}$ would not decompose into rank 1 blocks.

So for each choice of canonical representatives $\alpha_2, \ldots, \alpha_k$ in $A_{\ell}^{[k]}$ there is at most one representative $\alpha_1 \in A_{\ell}^{[k]}$ such that $f_{\ell}^{[k]}(\alpha_1, \ldots, \alpha_k) > 0$. There is at least one such representative α_1 since, by Lemma 5,

$$f_{\ell}^{[k-1]}(\alpha_2,\ldots,\alpha_k) = \lambda_{\ell,\alpha_2}^{[k-1]}\cdots\lambda_{\ell,\alpha_k}^{[k-1]},$$

and the $\lambda_{\ell,\alpha_j}^{[k-1]}$ values are positive. This is part (ii) of the lemma.

Recall the definition of $\lambda'_{z}^{[k]}$ from Equation (3). For $\alpha \in A_{\ell}^{[k]}$, let $\bar{\lambda}_{\alpha}$ denote the sum $\bar{\lambda}_{\alpha} = \sum_{z \in [\alpha]^{[k]}} \lambda'_{z}^{[k]}$. Similarly, let $\bar{\mu}_{\alpha} = \sum_{z \in [\alpha]^{[k]}} \lambda_{\ell,z}^{[k-1]}$. Fix $z_{2}, \ldots, z_{k} \in D_{\ell}$. By Lemma 5,

$$\lambda_{\ell,z_2}^{[k-1]} \cdots \lambda_{\ell,z_k}^{[k-1]} = f_{\ell}^{[k-1]}(z_2, \dots, z_k) = \sum_{z_1 \in D_{\ell}} f_{\ell}^{[k]}(z_1, \dots, z_k)$$
$$= \sum_{z_1 \in D_{\ell}} \lambda_{z_1}^{'[k]} \cdots \lambda_{z_k}^{'[k]} f_{\ell}^{[k]}(\bar{z}_1^{[k]}, \dots, \bar{z}_k^{[k]})$$
$$= \bar{\lambda}_{\alpha_1} \lambda_{z_2}^{'[k]} \cdots \lambda_{z_k}^{'[k]} f_{\ell}^{[k]}(\alpha_1, \bar{z}_2^{[k]}, \dots, \bar{z}_k^{[k]}),$$

where α_1 is the unique representative in $A_{\ell}^{[k]}$ such that $f^{[k]}(\alpha_1, \bar{z}_2^{[k]}, \dots, \bar{z}_k^{[k]}) > 0$. So for fixed $\alpha_2, \dots, \alpha_k \in A_{\ell}^{[k]}$, there is a representative $\alpha_1 \in A_{\ell}^{[k]}$ such that

$$\bar{\mu}_{\alpha_2}\cdots\bar{\mu}_{\alpha_k} = \sum_{z_2\in[\alpha_2]^{[k]}}\cdots\sum_{z_k\in[\alpha_k]^{[k]}}\lambda_{\ell,z_2}^{[k-1]}\cdots\lambda_{\ell,z_k}^{[k-1]}$$
$$= \sum_{z_2\in[\alpha_2]^{[k]}}\cdots\sum_{z_k\in[\alpha_k]^{[k]}}\bar{\lambda}_{\alpha_1}\lambda_{z_2}^{\prime [k]}\cdots\lambda_{z_k}^{\prime [k]}f_{\ell}^{[k]}(\alpha_1,\ldots,\alpha_k)$$
$$= \bar{\lambda}_{\alpha_1}\cdots\bar{\lambda}_{\alpha_k}f_{\ell}^{[k]}(\alpha_1,\ldots,\alpha_k).$$

Since we have a Latin hypercube (Part (ii) of the lemma), any of $\alpha_1, \ldots, \alpha_k$ is determined by the other k-1 of them. Thus, we can derive a similar equality omitting any other $\bar{\mu}_{\alpha_i}$ on the left-hand-side. Now the right-hand-side of the above equality is symmetric in the α_j 's, and the lefthand-side has exactly one α_j missing, so by symmetry we conclude $\bar{\mu}_{\alpha_1} = \cdots = \bar{\mu}_{\alpha_k}$ and, further, $\bar{\mu}_{\alpha_j}$ is constant for $\alpha_j \in A_{\ell}^{[k]}$. Moreover, $\bar{\lambda}_{\alpha_1} \cdots \bar{\lambda}_{\alpha_k} f_{\ell}^{[k]}(\alpha_1, \ldots, \alpha_k)$ is constant on representatives $\alpha_1, \ldots, \alpha_k \in A_{\ell}^{[k]}$ with $f_{\ell}^{[k]}(\alpha_1, \ldots, \alpha_k) > 0$. That is, for any set of k representatives $\alpha'_1, \alpha'_2, \ldots, \alpha'_k \in A_{\ell}^{[k]}$ with $f_{\ell}^{[k]}(\alpha'_1, \ldots, \alpha'_k) > 0$, the value of that expression $\bar{\lambda}_{\alpha'_1} \cdots \bar{\lambda}_{\alpha'_k} f_{\ell}^{[k]}(\alpha'_1, \ldots, \alpha'_k)$ is the same. Now define $\lambda_x^{[k]} = c_\ell \lambda'_x^{[k]} / \bar{\lambda}_{[x]^{[k]}}$, where c_ℓ is a constant, depending only on ℓ , to be determined

below. Then, whenever $f_{\ell}^{[k]}(z_1,\ldots,z_k) > 0$,

$$f_{\ell}^{[k]}(z_1, \dots, z_k) = \lambda_{z_1}^{\prime [k]} \cdots \lambda_{z_k}^{\prime [k]} f_{\ell}^{[k]}(\bar{z}_1^{[k]}, \dots, \bar{z}_k^{[k]}) = c_{\ell}^{-k} \lambda_{z_1}^{[k]} \cdots \lambda_{z_k}^{[k]} \bar{\lambda}_{[z_1]^{[k]}} \cdots \bar{\lambda}_{[z_k]^{[k]}} f_{\ell}^{[k]}(\bar{z}_1^{[k]}, \dots, \bar{z}_k^{[k]})$$

But

$$c_{\ell}^{-k} \bar{\lambda}_{[z_1]^{[k]}} \cdots \bar{\lambda}_{[z_k]^{[k]}} f_{\ell}^{[k]}(\bar{z}_1^{[k]}, \dots, \bar{z}_k^{[k]})$$

is independent of z_1, \ldots, z_k (assuming, as we are, that $f_{\ell}^{[k]}(z_1, \ldots, z_k) > 0$), so, by appropriate choice of c_{ℓ} ,

$$f_{\ell}^{[k]}(z_1,\ldots,z_k) = \lambda_{z_1}^{[k]}\cdots\lambda_{z_k}^{[k]}R_{\ell}^{[k]}(z_1,\ldots,z_k).$$

The choice of component D_{ℓ} was arbitrary, so a similar statement holds for $f^{[k]}$ over its whole range, as required by part (i) of the lemma.

Finally,

$$\sum_{z \in [\alpha]^{[k]}} \lambda_z^{[k]} = c_\ell \sum_{z \in [\alpha]^{[k]}} \lambda_z'^{[k]} / \bar{\lambda}_\alpha = c_\ell,$$

establishing part (iii).

Lemma 12. Let $g: D^r \to \overline{\mathbb{Q}}^{\geq 0}$ be a symmetric function with arity $r \geq 3$. Let k be an integer in $\{3, \ldots, r\}$. Suppose that g is (k-1)-factoring and (k-1)-equational. Suppose there are positive constants $\{\lambda_z^{[k]}: z \in D\}$ such that $f^{[k]}(z_1, \ldots, z_k) = \lambda_{z_1}^{[k]} \cdots \lambda_{z_k}^{[k]} R^{[k]}(z_1, \ldots, z_k)$. Either $\mathsf{Eval}(f^{[k]})$ is $\#\mathsf{P}$ -hard (which implies that $\mathsf{Eval}(g)$ is $\#\mathsf{P}$ -hard), or, for every $\ell \in [m]$, the multiset $\{\lambda_z^{[k]}: z \in [\alpha]^{[k]}\}$ is independent of the choice of $\alpha \in A_{\ell}^{[k]}$.

Proof. In preparation for the proof, consider the unary constraint U(x) applied to a variable x and defined as follows: Take k-1 new variables x_2, \ldots, x_k then add the constraint $f^{[k]}(x, x_2, \ldots, x_k)$.

The resulting unary relation U(x) will be used in the reduction that follows. For any $\ell \in [m]$ and $\alpha \in A_{\ell}^{[k]}$, let $n_{\ell} = |A_{\ell}^{[k]}|$ and $c_{\ell} = \sum_{z \in [\alpha]^{[k]}} \lambda_z^{[k]}$ (which, by Lemma 11, is independent of the choice of $\alpha \in A_{\ell}^{[k]}$). For any $z_1 \in D_{\ell}$,

$$U(z_{1}) = \sum_{z_{2},...,z_{k}\in D_{\ell}} f_{\ell}^{[k]}(z_{1},...,z_{k}) = \sum_{z_{2},...,z_{k}\in D_{\ell}} \lambda_{z_{1}}^{[k]} \cdots \lambda_{z_{k}}^{[k]} R_{\ell}^{[k]}(z_{1},...,z_{k})$$

$$= \sum_{\alpha_{2},...,\alpha_{k}\in A_{\ell}^{[k]}:(\bar{z}_{1}^{[k]},\alpha_{2},...,\alpha_{k})\in R_{\ell}^{[k]}} \sum_{z_{2}\in[\alpha_{2}]^{[k]},...,z_{k}\in[\alpha_{k}]^{[k]}} \lambda_{z_{1}}^{[k]} \cdots \lambda_{z_{k}}^{[k]}$$

$$= \lambda_{z_{1}}^{[k]} \sum_{\alpha_{2},...,\alpha_{k}\in A_{\ell}^{[k]}:(\bar{z}_{1}^{[k]},\alpha_{2},...,\alpha_{k})\in R_{\ell}^{[k]}} \left(\sum_{z_{2}\in[\alpha_{2}]^{[k]}} \lambda_{z_{2}}^{[k]}\right) \cdots \left(\sum_{z_{k}\in[\alpha_{k}]^{[k]}} \lambda_{z_{k}}^{[k]}\right)$$

$$= \lambda_{z_{1}}^{[k]} n_{\ell}^{k-2} c_{\ell}^{k-1},$$

where the final equality uses part (ii) of Lemma 11.

The idea of the proof is to use U to "power up" vertex weights $\lambda_z^{[k]}$. In this way we discover that not only is $\sum_{z \in [\alpha]^{[k]}} \lambda_z^{[k]}$ independent of $\alpha \in A_\ell^{[k]}$, but so also is $\sum_{z \in [\alpha]^{[k]}} (\lambda_z^{[k]})^j$ for any positive integer j. This implies that the multiset of weights on an equivalence class $[\alpha]^{[k]}$ is independent of $\alpha \in A_\ell^{[k]}$.

For $z_1, \ldots, z_k \in D_\ell$ and $j \ge 1$, define

$$\psi_{z_1} = (\lambda_{z_1}^{[k]} n_{\ell}^{k-2} c_{\ell}^{k-1})^{j-1} \lambda_{z_1}^{[k]}$$

and

$$h_{\ell}^{[j]}(z_1,\ldots,z_k) = \psi_{z_1}\cdots\psi_{z_k}R_{\ell}^{[k]}(z_1,\ldots,z_k).$$

Let $h^{[j]} = h_1^{[j]} \oplus \cdots \oplus h_m^{[j]}$. We will give a reduction from $\mathsf{Eval}(h^{[j]})$ to $\mathsf{Eval}(f^{[k]})$. Suppose G = (V, E) is a k-uniform hypergraph (an input to $\mathsf{Eval}(h^{[j]})$). For $j \ge 1$, the hypergraph $G^{[j]}$ is obtained from G as follows: for each vertex v in G of degree d_v , add $(k-1)(j-1)d_v$ new vertices and $(j-1)d_v$ new edges, each one incident at v and at k-1 of the new vertices. Then

$$\begin{split} Z^{h_{\ell}^{[j]}}(G) &= \sum_{\sigma: V \to D_{\ell}} \prod_{(u_1, \dots, u_k) \in E} h_{\ell}^{[j]}(\sigma(u_1), \dots, \sigma(u_k)) \\ &= \sum_{\sigma: V \to D_{\ell}} \prod_{(u_1, \dots, u_k) \in E} \psi_{\sigma(u_1)} \cdots \psi_{\sigma(u_k)} R_{\ell}^{[k]}(\sigma(u_1), \dots, \sigma(u_k)) \\ &= \sum_{\sigma: V \to D_{\ell}} \prod_{v \in V} (\lambda_{\sigma(v)}^{[k]} n_{\ell}^{k-2} c_{\ell}^{k-1})^{(j-1)d_v} \prod_{(u_1, \dots, u_k) \in E} \lambda_{\sigma(u_1)}^{[k]} \cdots \lambda_{\sigma(u_k)}^{[k]} R_{\ell}^{[k]}(\sigma(u_1), \dots, \sigma(u_k)) \\ &= \sum_{\sigma: V \to D_{\ell}} \prod_{v \in V} (\lambda_{\sigma(v)}^{[k]} n_{\ell}^{k-2} c_{\ell}^{k-1})^{(j-1)d_v} \prod_{(u_1, \dots, u_k) \in E} f_{\ell}^{[k]}(\sigma(u_1), \dots, \sigma(u_k)) \\ &= Z^{f_{\ell}^{[k]}}(G^{[j]}). \end{split}$$

Thus (for connected G)

$$Z^{h^{[j]}}(G) = \sum_{\ell \in [m]} Z^{h_{\ell}^{[j]}}(G) = \sum_{\ell \in [m]} Z^{f_{\ell}^{[k]}}(G^{[j]}) = Z^{f^{[k]}}(G^{[j]}),$$

so $\operatorname{Eval}(h^{[j]}) \leq \operatorname{Eval}(f^{[k]}).$

Assume $\mathsf{Eval}(f^{[k]})$ is not $\#\mathsf{P}$ -hard. Then $\mathsf{Eval}(h^{[j]})$ is not $\#\mathsf{P}$ -hard for any $j \ge 1$. Recall from the statement of the lemma that g is (k-1)-factoring and (k-1)-equational. Then from Lemma 11 part (iii),

$$\sum_{z \in [\alpha]^{[k]}} \psi_z = (n_\ell^{k-2} c_\ell^{k-1})^{j-1} \sum_{z \in [\alpha]^{[k]}} (\lambda_z^{[k]})^j$$

is independent of $\alpha \in A_{\ell}^{[k]}$ for all $j \ge 1$. This can only occur if the multiset $\{\lambda_z^{[k]} : z \in [\alpha]^{[k]}\}$ is independent of $\alpha \in A_{\ell}^{[k]}$.

We will use the following corollary of Lemmas 10, 11 and 12.

Corollary 13. Let $g: D^r \to \overline{\mathbb{Q}}^{\geq 0}$ be a symmetric function with arity $r \geq 3$. Let k be an integer in $\{3, \ldots, r\}$. Suppose that g is (k-1)-factoring and (k-1)-equational. Either $\mathsf{Eval}(f^{[k]})$ is $\#\mathsf{P}$ -hard (which implies that $\mathsf{Eval}(g)$ is $\#\mathsf{P}$ -hard), or g is k-factoring.

Proof. By Lemma 11 part (i) there are positive constants $\{\lambda_z^{[k]} : z \in D\}$ such that

$$f^{[k]}(z_1,\ldots,z_k) = \lambda_{z_1}^{[k]}\cdots\lambda_{z_k}^{[k]}R^{[k]}(z_1,\ldots,z_k).$$

Fix any $\ell \in [m]$. By Lemma 12, the multiset $\{\lambda_z^{[k]} : z \in [\alpha]^{[k]}\}$ is independent of the choice of $\alpha \in A_{\ell}^{[k]}$ Let $s_{\ell}^{[k]}$ be the size of this multiset. Then $D_{\ell} \cong A_{\ell}^{[k]} \times [s_{\ell}^{[k]}]$ giving condition (1) in the definition of k-factoring. Also, if the element $z \in D_{\ell}$ corresponds to the *i*'th element of the \sim_k class $[z]^{[k]}$ then the value $\lambda_z^{[k]}$ just depends upon *i* (and on ℓ) — it is independent of the equivalence class $[z]^{[k]}$. We denote this value as $\lambda_{\ell,i}^{[k]}$. Thus, for $\alpha_1, \ldots, \alpha_k \in A_{\ell}^{[k]}$ and $i_1, \ldots, i_k \in [s_{\ell}^{[k]}]$,

$$f_{\ell}^{[k]}((\alpha_1, i_1), \dots, (\alpha_k, i_k)) = \lambda_{\ell, i_1}^{[k]} \cdots \lambda_{\ell, i_k}^{[k]} R_{\ell}^{[k]}(\alpha_1, \dots, \alpha_k)$$

giving condition (2) in the definition of k-factoring.

Lemma 14. Let $g: D^r \to \overline{\mathbb{Q}}^{\geq 0}$ be a symmetric function with arity $r \geq 3$. Let k be an integer in $\{3, \ldots, r\}$. Suppose that g is k-factoring. Then, for every $\ell \in [m]$,

$$Z^{f_{\ell}^{[k]}}(G) = \Lambda_{\ell}^{[k]}(G) \ Z^{S_{\ell}^{[k]}}(G),$$

where

$$\Lambda_{\ell}^{[k]}(G) = \prod_{v \in V(G)} \sum_{i \in [s_{\ell}^{[k]}]} (\lambda_{\ell,i}^{[k]})^{d_{v}}.$$
(7)

Proof. For G = (V, E),

$$\begin{split} Z^{f_{\ell}^{[k]}}(G) &= \sum_{\sigma: V \to A_{\ell}^{[k]}, \tau: V \to [s_{\ell}^{[k]}]} \prod_{(u_1, \dots, u_k) \in E} f_{\ell}^{[k]}((\sigma(u_1), \tau(u_1)), \dots, (\sigma(u_k), \tau(u_k))) \\ &= \sum_{\sigma: V \to A_{\ell}^{[k]}, \tau: V \to [s_{\ell}^{[k]}]} \prod_{(u_1, \dots, u_k) \in E} \lambda_{\ell, \tau(u_1)}^{[k]} \cdots \lambda_{\ell, \tau(u_k)}^{[k]} S_{\ell}^{[k]}(\sigma(u_1), \dots, \sigma(u_k)) \\ &= \sum_{\sigma: V \to A_{\ell}^{[k]}} \left(\prod_{(u_1, \dots, u_k) \in E} S_{\ell}^{[k]}(\sigma(u_1), \dots, \sigma(u_k)) \right) \left(\sum_{\tau: V \to [s_{\ell}^{[k]}]} \prod_{v \in V} (\lambda_{\ell, \tau(v)}^{[k]})^{d_v} \right) \\ &= Z^{S_{\ell}^{[k]}}(G) \ \Lambda_{\ell}^{[k]}(G). \end{split}$$

Lemma 15. Let $g: D^r \to \overline{\mathbb{Q}}^{\geq 0}$ be a symmetric function with arity $r \geq 3$. Let k be an integer in $\{3, \ldots, r\}$. Suppose that g is (k-1)-factoring and (k-1)-equational. Either $\mathsf{Eval}(f^{[k]})$ is $\#\mathsf{P}$ -hard (which implies that $\mathsf{Eval}(g)$ is $\#\mathsf{P}$ -hard), or $\mathsf{Eval}(S^{[k]}) \leq \mathsf{Eval}(f^{[k]})$.

Proof. Suppose that G is a connected k-uniform hypergraph. For any positive integer, p, let G^1, \ldots, G^p be copies of G. Let $\{v_1^j, \ldots, v_n^j\}$ be the vertices of G^j . Construct $G^{[p]}$ by taking the union of G^1, \ldots, G^p along with n(k-1)p new vertices and 2np new edges: For each $i \in [n], t \in [k-1]$ and $j \in [p]$ we add a vertex $u_{i,t}^j$. Then we add edges $(u_{i,1}^j, \ldots, u_{i,k-1}^j, v_i^j)$ and $(u_{i,1}^j, \ldots, u_{i,k-1}^j, v_i^{(j \mod n)+1})$.

Now by Corollary 13, g is $k\text{-factoring, so }D_\ell\cong A_\ell^{[k]}\times [s_\ell^{[k]}].$ By Lemma 14,

$$Z^{f^{[k]}}(G^{[p]}) = \sum_{\ell \in [m]} \Lambda_{\ell}^{[k]}(G^{[p]}) \ Z^{S_{\ell}^{[k]}}(G^{[p]}).$$
(8)

We now look at the constituent parts of the right-hand-side of Equation (8). First,

$$Z^{S_{\ell}^{[k]}}(G^{[p]}) = \sum_{\sigma: V(G^{[p]}) \to A_{\ell}^{[k]}} \prod_{(w_1, \dots, w_k) \in E(G^{[p]})} S_{\ell}^{[k]}(\sigma(w_1), \dots, \sigma(w_k))$$

By Part (ii) of Lemma 11, $S_{\ell}^{[k]}$ is a Latin hypercube. So, given the values $\sigma(v_1^j), \ldots, \sigma(v_n^j)$, the values $\sigma(u_{i,1}^j), \ldots, \sigma(u_{i,k-2}^j)$ (for $i \in [n]$) can be chosen arbitrarily from $A_{\ell}^{[k]}$. Then there is exactly one choice for each $\sigma(u_{i,k-1}^j)$ so that

$$(\sigma(u_{i,1}^j),\ldots,\sigma(u_{i,k-1}^j),\sigma(v_i^j)) \in S_{\ell}^{[k]}.$$

Then for j < n to have

$$(\sigma(u_{i,1}^j),\ldots,\sigma(u_{i,k-1}^j),\sigma(v_i^{(j \bmod n)+1})) \in S_\ell^{[k]}$$

we must have $\sigma(v_i^{j+1}) = \sigma(v_i^j)$. (If j = n then

$$(\sigma(u_{i,1}^j), \dots, \sigma(u_{i,k-1}^j), \sigma(v_i^{(j \mod n)+1})) \in S_{\ell}^{[k]}$$

just ensures $v_i^1 = v_i^n$ so it adds no new constraint.) Thus,

$$Z^{S_{\ell}^{[k]}}(G^{[p]}) = \sum_{\sigma: V(G^{1}) \to A_{\ell}^{[k]}} \prod_{(w_{1}, \dots, w_{k}) \in E(G^{1})} S_{\ell}^{[k]}(\sigma(w_{1}), \dots, \sigma(w_{k})) \left| A_{\ell}^{[k]} \right|^{n(k-2)p} = \left| A_{\ell}^{[k]} \right|^{n(k-2)p} Z^{S_{\ell}^{[k]}}(G).$$

Also, using $d_{\Gamma}(w)$ to denote the degree of vertex w in hypergraph Γ ,

$$\begin{split} \Lambda_{\ell}^{[k]}(G^{[p]}) &= \prod_{w \in V(G^{[p]})} \sum_{h \in [s_{\ell}^{[k]}]} \left(\lambda_{\ell,h}^{[k]}\right)^{d_{G^{[p]}}(w)} \\ &= \left(\prod_{i \in [n]} \sum_{h \in [s_{\ell}^{[k]}]} \left(\lambda_{\ell,h}^{[k]}\right)^{d_{G}(v_{i})+2}\right)^{p} \left(\prod_{i \in [n]} \prod_{t \in [k-1]} \sum_{h \in [s_{\ell}^{[k]}]} \left(\lambda_{\ell,h}^{[k]}\right)^{2}\right)^{p}, \end{split}$$

where the first factor on the right-hand-side is the product over vertices v_i^j and the second factor is the product over vertices $u_{i,t}^j$.

So $Z^{f^{[k]}}(G^{[p]})$ is equal to

$$\sum_{\ell \in [m]} \left(\prod_{i \in [n]} \sum_{h \in [s_{\ell}^{[k]}]} \left(\lambda_{\ell,h}^{[k]} \right)^{d_{G}(v_{i})+2} \right)^{p} \left(\prod_{i \in [n]} \prod_{t \in [k-1]} \sum_{h \in [s_{\ell}^{[k]}]} \left(\lambda_{\ell,h}^{[k]} \right)^{2} \right)^{p} |A_{\ell}^{[k]}|^{n(k-2)p} Z^{S_{\ell}^{[k]}}(G).$$

We can now use Corollary 8 with $Z_p = Z^{f^{[k]}}(G^{[p]}), \, \gamma_\ell = Z^{S_\ell^{[k]}}(G)$ and

$$\eta_{\ell} = \left(\prod_{i \in [n]} \sum_{h \in [s_{\ell}^{[k]}]} (\lambda_{\ell,h}^{[k]})^{d_{G}(v_{i})+2} \right) \left(\prod_{i \in [n]} \prod_{t \in [k-1]} \sum_{h \in [s_{\ell}^{[k]}]} (\lambda_{\ell,h}^{[k]})^{2} \right) |A_{\ell}^{[k]}|^{n(k-2)}.$$

Let us take stock. Suppose g is not #P-hard and that g is (k-1)-factoring and (k-1)-equational. We know by Corollary 13 that g is k-factoring, and by Part (ii) of Lemma 11 that the various relations $S_{\ell}^{[k]}$ are Latin hypercubes. The final step, the subject of the following section, is to show that the latter have additional structure, namely that they are defined by equations over an Abelian groups. It will follow that g is k-equational.

6 Constraint satisfaction and Abelian group equations

Let S be an arity-k relation on a ground set A. Recall our earlier discussion, in Section 1, on the relation between $\mathsf{Eval}(S)$ and $\#\mathsf{CSP}(S)$. Every instance G of $\mathsf{Eval}(S)$ can be viewed as an instance of $\#\mathsf{CSP}(S)$ by taking the vertices as variables and the edges as constraint scopes. However, we noted that the converse is not true, since an instance I of $\#\mathsf{CSP}(S)$ might not be a properly-formed instance of $\mathsf{Eval}(S)$. Nevertheless, by copying variables, we can view an instance I of $\#\mathsf{CSP}(S)$ as being a k-uniform hypergraph G, together with some binary equality constraints on variables. For variables U and W, the constraint = (U, W) is satisfied if and only if $\sigma(U) = \sigma(W)$. The following lemma shows that, in our setting, these equality constraints do not add any real power - they can be implemented by interpolation.

Lemma 16. Let $S = S_1 \oplus \cdots \oplus S_m$ be a symmetric k-ary relation on a ground set A, such that each S_ℓ is a Latin hypercube. Then $\#CSP(S) \leq Eval(S)$.

Proof. For $\ell \in [m]$, let A_{ℓ} be the ground set of S_{ℓ} .

Let I be an instance of $\#\mathsf{CSP}(S)$ comprising a connected hypergraph G with vertices $\{v_1, \ldots, v_n\}$ and ν equality constraints. Note that this is without loss of generality – an instance I may be represented as a hypergraph G together with equality constraints in which equality is only applied to variables in the same connected component of G.

For a positive integer p, construct a hypergraph $G^{[p]}$ by combining G with $\nu p(k-1)$ new vertices and $2\nu p$ new edges: For $j \in [p]$ and $i \in [\nu]$ add vertices $u_{i,1}^j, \ldots, u_{i,k-1}^j$. If the *i*'th equality constraint is $= (v_s, v_t)$ then add the 2p edges $(v_s, u_{i,1}^j, \ldots, u_{i,k-1}^j)$ and $(v_t, u_{i,1}^j, \ldots, u_{i,k-1}^j)$ for $j \in [p]$.

Now, suppose we are given the values $\sigma(v_1), \ldots, \sigma(v_n)$ in A_{ℓ} . By the Latin hypercube property, we can have $(\sigma(v_s), \sigma(u_{i,1}^j), \ldots, \sigma(u_{i,k-1}^j)) \in S$ and $(\sigma(v_t), \sigma(u_{i,1}^j), \ldots, \sigma(u_{i,k-1}^j)) \in S$ only if $\sigma(v_s) = C_{i,k-1}$

 $\sigma(v_t)$. In that case, there are $|A_\ell|^{k-2}$ choices for $\sigma(u_{i,1}^j), \ldots, \sigma(u_{i,k-1}^j)$. So

$$Z^{S}(G^{[p]}) = \sum_{\ell \in [m]} Z^{S_{\ell}}(I) |A_{\ell}|^{(k-2)p}.$$

We can now use Corollary 8.

The following lemma establishes the algebraic structure of the S_{ℓ} , using a result of Bulatov and Dalmau [3]. The proof itself has similarities to that of Pálfy's theorem [14] (see, for example, [7]).

Lemma 17. Suppose $k \geq 3$. Let $S = S_1 \oplus \cdots \oplus S_m$ be a symmetric k-ary relation on a ground set A such that, for each $\ell \in [m]$, S_ℓ is a Latin hypercube. Suppose $\mathsf{Eval}(S)$ is not $\#\mathsf{P}$ -hard. Then for each $\ell \in [m]$, the relation S_ℓ is defined by an equation over an Abelian group $\mathcal{G}_\ell = \langle A_\ell, + \rangle$ as follows: for some element $a \in A_\ell$, $(\alpha_1, \ldots, \alpha_k) \in S_\ell$ if and only if $\alpha_1 + \cdots + \alpha_k = a$.

Proof. Suppose Eval(S) is not #P-hard. Fix $\ell \in [m]$, and fix any element $a_{\ell} \in A_{\ell}$ and denote it by 0. If $(\alpha, \beta, \gamma, 0, \dots, 0) \in S_{\ell}$ we will write $\gamma = \alpha \cdot \beta$. Then we will call (α, β, γ) a triple and denote the set of triples by T_{ℓ} . We will call $(\alpha, \beta, \gamma, 0, \dots, 0) \in S_{\ell}$ the corresponding padded triple. For given α and β , the existence and uniqueness of γ in a padded triple follows directly from the fact that S_{ℓ} is a Latin hypercube. Thus we may regard $\alpha \cdot \beta$ as a binary operation on A_{ℓ} , and hence $\mathcal{A}_{\ell} = \langle A_{\ell}, \cdot \rangle$ is an algebra. By symmetry, the binary operation of \mathcal{A}_{ℓ} is commutative, and satisfies the identity $\alpha \cdot (\alpha \cdot \beta) = \beta$ for all $\alpha, \beta \in A_{\ell}$. However, the operation is not necessarily associative.

By Lemma 16, $\#\mathsf{CSP}(S) \leq \mathsf{Eval}(S)$, so $\#\mathsf{CSP}(S)$ is not $\#\mathsf{P}$ -hard. Thus, by [3], there is a Mal'tsev polymorphism $\varphi(\alpha, \beta, \gamma)$ on A which preserves S. Recall that a Mal'tsev operation $\varphi: A^3 \to A$ is any function which satisfies the identities $\varphi(\alpha, \beta, \beta) = \varphi(\beta, \beta, \alpha) = \alpha$ for all $\alpha, \beta \in A$. We may use φ to calculate, as follows. Each line of a table is a triple in T_{ℓ} , and the Mal'tsev polymorphism implies that the bottom line is also a triple in T_{ℓ} , using the fact that $\varphi(0, 0, 0) = 0$ in the padded triples (which follows from the Mal'tsev property). Thus

$$egin{array}{cccc} lpha & \gamma & lpha \cdot \gamma \ eta & \gamma & eta \cdot \gamma \ \gamma & eta & \gamma & eta \cdot \gamma \ \gamma & eta & eta \cdot \gamma \ \hline arphi(lpha,eta,\gamma) & eta & lpha \cdot \gamma \end{array}$$

and hence $\varphi(\alpha, \beta, \gamma) = \beta \cdot (\alpha \cdot \gamma)$ is a term of the algebra \mathcal{A}_{ℓ} . We have

$$\varphi(\alpha,\beta,\gamma) = \beta \cdot (\alpha \cdot \gamma) = \beta \cdot (\gamma \cdot \alpha) = \varphi(\gamma,\beta,\alpha),$$

so φ is a symmetric Mal'tsev operation (in the sense that it is symmetric in the first and third arguments).

Define a new binary operation + on A_{ℓ} by $\alpha + \beta = \varphi(\alpha, 0, \beta) = 0 \cdot (\alpha \cdot \beta)$. It follows immediately that + is commutative. Hence

$$0 + \alpha = \alpha + 0 = 0 \cdot (\alpha \cdot 0) = \varphi(\alpha, 0, 0) = \alpha,$$

so 0 is an identity for +. Denote $0 \cdot 0$ by 0^2 , and define $-\alpha$ by $\alpha \cdot 0^2$. Then

$$(-\alpha) + \alpha = \alpha + (-\alpha) = 0 \cdot (\alpha \cdot (\alpha \cdot 0^2)) = 0 \cdot (0^2) = 0 \cdot (0 \cdot 0) = 0,$$

so $-\alpha$ is an inverse for α . As usual, we write $\alpha - \beta$ for $\alpha + (-\beta)$.

We have

$$\begin{array}{cccc} \alpha & 0^2 & \alpha \cdot 0^2 \\ 0 & 0^2 & 0 \\ \beta & \beta \cdot 0 & 0 \\ \hline \alpha + \beta & \beta \cdot 0 & \alpha \cdot 0^2 \end{array}$$

so $\alpha + \beta = (\beta \cdot 0) \cdot (\alpha \cdot 0^2)$ and since + is commutative, $\alpha + \beta = \beta + \alpha = (\alpha \cdot 0) \cdot (\beta \cdot 0^2)$. Then

$$\begin{array}{cccc} \alpha \cdot 0 & \beta \cdot 0^2 & \alpha + \beta \\ 0^2 & 0 & 0 \\ \hline \gamma \cdot 0 & 0 & \gamma \\ \hline \varphi(\alpha \cdot 0, 0^2, \gamma \cdot 0) & \beta \cdot 0^2 & (\alpha + \beta) + \gamma \end{array}$$

Therefore, since φ is symmetric in its first and third arguments,

$$\begin{aligned} (\alpha + \beta) + \gamma &= \varphi(\alpha \cdot 0, 0^2, \gamma \cdot 0) \cdot (\beta \cdot 0^2) &= \varphi(\gamma \cdot 0, 0^2, \alpha \cdot 0) \cdot (\beta \cdot 0^2) \\ &= (\gamma + \beta) + \alpha &= \alpha + (\gamma + \beta) &= \alpha + (\beta + \gamma). \end{aligned}$$

The operation + is therefore associative, and hence the algebra $\mathcal{G}_{\ell} = \langle A_{\ell}, +, -, 0 \rangle$ is an Abelian group. Hence, since -X is defined to be $X \cdot 0^2$ and $\alpha - 0^2 = -(-\alpha + 0^2)$, we have, for any $\alpha, \beta \in A_{\ell}$,

where we used the fact that, by definition, $\varphi(x, 0, y) = x + y$. Thus $\alpha \cdot \beta = -\alpha - \beta + 0^2$., and it follows that

$$T_{\ell} = \{ (\alpha, \beta, -\alpha - \beta + 0^2) \in A_{\ell}^3 : \alpha, \beta \in \mathcal{A}_{\ell} \}$$

= $\{ (\alpha, \beta, \gamma) \in A_{\ell}^3 : \alpha + \beta + \gamma = 0^2 \text{ in } \mathcal{G}_{\ell} \}.$ (9)

In particular, $(\alpha, -\alpha, 0^2) \in T_{\ell}$ for all $\alpha \in A_{\ell}$, and hence $(0, 0, 0^2) \in T_{\ell}$. It follows further that

$$\varphi(\alpha,\beta,\gamma) = \beta \cdot (\alpha \cdot \gamma) = -\beta - (\alpha \cdot \gamma) + 0^2 = -\beta - (-\alpha - \gamma + 0^2) + 0^2 = \alpha - \beta + \gamma,$$

so the Mal'tsev operation is the term $\alpha - \beta + \gamma$ in the Abelian group \mathcal{G}_{ℓ} .

Now assume by induction that the conclusion of the lemma is true for any S of arity less than k. It is true for arity 3 by (9), since then, for any $\ell \in [m]$, $S_{\ell} = T_{\ell}$. For larger k, suppose $(\alpha_1, \alpha_2, \ldots, \alpha_k) \in S_{\ell}$ is arbitrary. Then, using the Mal'tsev operation and padding the triples $(\alpha_1, -\alpha_1, 0^2), (0, 0, 0^2)$, we have

α_1	α_2	α_3	α_4	• • •	$lpha_k$
α_1	$-\alpha_1$	0^{2}	0		0
0	0	0^{2}	0	•••	0
0	$\alpha_1 + \alpha_2$	α_3	α_4	•••	$lpha_k$

Now the (k-1)-ary relation

$$S'_{\ell} = \{ (\alpha'_2, \alpha'_3, \dots, \alpha'_k) \in A^{k-1}_{\ell} : (0, \alpha'_2, \alpha'_3, \dots, \alpha'_k) \in S_{\ell} \}$$

is symmetric and has the same Mal'tsev operation as S_{ℓ} . Thus we can define the same Abelian group \mathcal{G}_{ℓ} , and by induction we will have

$$S'_{\ell} = \{ (\alpha'_2, \alpha'_3, \dots, \alpha'_k) \in A^{k-1}_{\ell} : \sum_{j=2}^k \alpha'_j = a' \text{ in } \mathcal{G}_{\ell} \},\$$

for some $a' \in A_{\ell}$. But we have shown that, for all $(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k) \in S_{\ell}$, we have $(\alpha_1 + \alpha_2, \alpha_3, \ldots, \alpha_k) \in S'_{\ell}$. Thus, since \mathcal{G}_{ℓ} is an Abelian group,

$$S_{\ell} = \{ (\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_k) \in A_{\ell}^k : \sum_{j=1}^k \alpha_j = a \text{ in } \mathcal{G}_{\ell} \},\$$

where a = a', completing the induction and the proof.

7 Proof of Theorem 4

Proof. Let $g: D^r \to \overline{\mathbb{Q}}^{\geq 0}$ be a symmetric function with arity $r \geq 3$. First, suppose that g is r-factoring and r-equational. Then applying Lemma 14 with k = r, we find that, for connected G,

$$Z^{g}(G) = \sum_{\ell \in [m]} \Lambda_{\ell}^{[r]}(G) \ Z^{S_{\ell}^{[r]}}(G).$$
(10)

Now since g is r-equational, $S_{\ell}^{[r]}$ is defined by an equation over an Abelian group $(A_{\ell}^{[r]}, +)$. Now, by [11, Lemma 13], $\mathsf{Eval}(S_{\ell}^{[r]})$ is polynomial time solvable: The Abelian group is a direct product of cyclic groups of prime power. For each of these cyclic groups, we just need to count the solutions to a system of linear equations over the field \mathbb{Z}_p and this can be done in polynomial time (see [11]). Thus, $\mathsf{Eval}(S_{\ell}^{[r]})$ is in FP. To show that $\mathsf{Eval}(g)$ is in FP, it remains to show that $\Lambda_{\ell}^{[r]}(G)$, as defined in (7), can be computed in FP. This is immediate over the number field $\mathbb{Q}(\theta, \lambda_{\ell,1}^{[r]}, \ldots, \lambda_{\ell,s_{\ell}}^{[r]})$. In Section 8, we show that it can even be computed in FP over the number field $\mathbb{Q}(\theta)$.

Suppose now that $\mathsf{Eval}(g)$ is not $\#\mathsf{P}$ -hard. Then by Lemma 10, g is both 2-factoring and 2-equational. Next suppose that, for some $k \in \{3, \ldots, r\}$, g is (k-1)-factoring and (k-1)-equational. Since $\mathsf{Eval}(g)$ is not $\#\mathsf{P}$ -hard, we know that $\mathsf{Eval}(f^{[k]})$ is not $\#\mathsf{P}$ -hard. By Corollary 13, g is k-factoring. Suppose, for contradiction, that g is not k-equational. By Part (ii) of Lemma 11, each $S_{\ell}^{[k]}$ is a Latin hypercube, so by Lemma 17, $\mathsf{Eval}(S^{[k]})$ is $\#\mathsf{P}$ -hard. By Lemma 15, $\mathsf{Eval}(f^{[k]})$ is $\#\mathsf{P}$ -hard, giving the contradiction. So g is k-equational. By induction, g is r-factoring and r-equational.

It remains to consider the effectiveness of the dichotomy. For this, we must show that there is an algorithm that determines whether g is r-factoring and r-equational. This is nearly identical to a proof that the dichotomy in Theorem 2 is effective, however the notation is simpler in the latter context, so we provide this proof next.

Lemma 18. The dichotomy in Theorem 2 is effective.

Proof. We must show that there is an algorithm that determines whether the conditions in Theorem 2 are satisfied. The connected components D_1, \ldots, D_m can easily be determined. Then, for each $\ell \in [m]$, there are a constant number of possibilities for the decompositions $D_{\ell} \cong A_{\ell} \times [s_{\ell}]$ $(\ell \in [m])$ which can all be checked, if necessary. Then, for the third condition, there are only a finite number of possibilities for the group structure, corresponding to the factorisations of $|A_{\ell}|$. Again, these can all be checked to see if any defines S_{ℓ} , for each $\ell \in [m]$.

For the second condition, for each $\ell \in [m]$, we need to decide the satisfiability of a system of the form

$$g((\alpha_1, i_1), \dots, (\alpha_r, i_r)) = \lambda_{\ell, i_1} \cdots \lambda_{\ell, i_r} \text{ for all } (\alpha_1, \dots, \alpha_r) \in S_\ell \text{ and } i_1, \dots, i_r \in [s_\ell].$$
(11)

Thus we have

$$\lambda_{\ell,i} = g((\alpha_1, i), \dots, (\alpha_r, i))^{1/r} \quad \text{for all} \quad (\alpha_1, \dots, \alpha_r) \in S_\ell \quad \text{and} \quad i \in [s_\ell], \tag{12}$$

and hence (11) is equivalent to the system

$$g((\alpha_1, i_1), \dots, (\alpha_r, i_r))^r = \prod_{j=1}^r g((\alpha_1, i_j), \dots, (\alpha_r, i_j))$$

for all $(\alpha_1, \ldots, \alpha_r) \in S_\ell$ and $i_1, \ldots, i_r \in [s_\ell]$, which can be decided in constant time by computation in the number field $\mathbb{Q}(\theta)$.

8 Computation of $Z^{g}(G)$ in $\mathbb{Q}(\theta)$

Observe that (7), (10) and (12) seem together to imply that, in the polynomial time computable cases, we must compute $Z^{g}(G)$ in the number field $\mathbb{Q}(\theta, \lambda_{1,1}, \ldots, \lambda_{1,s_1}, \ldots, \lambda_{m,1}, \ldots, \lambda_{m,s_m})$, where, for $\ell \in [m]$ and $i \in [s_{\ell}], \lambda_{\ell,i} = \lambda_{\ell,i}^{[r]}$ is an r^{th} root of one of the original weights. This seems anomalous, since $Z^{g}(G)$ is actually an element of $\mathbb{Q}(\theta)$. We conclude by showing that the computation of $Z^{g}(G)$ can be done entirely within $\mathbb{Q}(\theta)$, as might be hoped.

To do this, we must expand the expressions

$$\Lambda_{\ell}^{[r]}(G) = \prod_{v \in V(G)} \sum_{i=1}^{s_{\ell}} (\lambda_{\ell,i})^{d_v}.$$

To simplify the text, we drop the subscript ℓ in the rest of this section, writing s for s_{ℓ} and λ_i for $\lambda_{\ell,i}$ and $\Lambda^{[r]}$ for $\Lambda^{[r]}_{\ell}$. Thus, we wish to expand

$$\Lambda^{[r]}(G) = \prod_{v \in V(G)} \left(\sum_{i=1}^{s} \lambda_i^{d_v}\right).$$

The exponents of λ_i $(i \in [s])$ in the monomials of the expansion of $\Lambda^{[r]}(G)$ are given by

$$\sum_{v \in V(G)} \delta_{v,i} d_v, \text{ where } \sum_{i=1}^s \delta_{v,i} = 1 \text{ and } \delta_{v,i} \in \{0,1\} \quad (i \in [s], v \in V(G)).$$
(13)

Recall that M denotes the number of edges of G. Thus there are $O(M^s)$ possible monomials in the λ_i , and the integer coefficient of each monomial $\prod_{i=1}^s \lambda_i^{M_i}$ are given by computing the number of solutions to systems of equations of the form

$$\sum_{v \in V(G)} \delta_{v,i} d_v = M_i, \text{ where } \sum_{i=1}^{s} \delta_{v,i} = 1 \text{ and } \delta_{v,i} \in \{0,1\} \quad (i \in [s], v \in V(G)).$$
(14)

This can be done for all $0 \leq M_i \leq rM$ $(i \in [s])$ in $O(nM^s)$ time by dynamic programming. An easy counting argument shows that $\sum_{v \in V(G)} d_v = rM$, so this returns a nonzero coefficient for the monomial $\prod_{i=1}^{s} \lambda_i^{M_i}$ only if $\sum_{i=1}^{s} M_i = rM$. Thus, in fact, there are at most

$$\binom{rM+s-1}{s-1} = O(M^{s-1})$$

such monomials, which is clearly polynomial in the input size.

Thus we can compute in FP a representation of $\Lambda^{[r]}(G)$ as a multivariate polynomial with monomials $\prod_{i=1}^{s} \lambda_i^{M_i}$ such that $\sum_{i=1}^{s} M_i = rM$ and $M_i \ge 0$ $(i \in [s])$. We can express each such monomial in terms of the original weights, as follows. Let r_{ij} $(i \in [s], j \in [M])$ be nonnegative integers such that $\sum_{i=1}^{s} r_{ij} = r$ $(j \in [M])$ and $\sum_{j=1}^{M} r_{ij} = M_i$ $(i \in [s])$. Such numbers always exist, though they will usually be far from unique, and can be computed in O(M) time. They are the entries of a contingency table with row totals M_i $(i \in [s])$ and column totals r $(j \in [M])$. See, for example, [8]. Now each column r_{ij} $(j \in [M])$ can be interpreted as an r-multiset $\{i_{1j}, \ldots, i_{rj}\} \subseteq [s]$, where $i \in [s]$ appears with multiplicity r_{ij} . Thus, choosing any $(\alpha_1, \ldots, \alpha_r) \in S$, we have

$$\prod_{i=1}^{s} \lambda_{i}^{M_{i}} = \prod_{j=1}^{M} \prod_{i=1}^{s} \lambda_{i}^{r_{ij}} = \prod_{j=1}^{M} \left(\lambda_{i_{1j}} \cdots \lambda_{i_{rj}} \right) = \prod_{j=1}^{M} g((\alpha_{1}, i_{1j}), \dots, (\alpha_{r}, i_{rj})),$$

using (11). This can be computed in O(M) time in $\mathbb{Q}(\theta)$, so $Z^g(G)$ can be evaluated in $O(M^s)$ time. The most demanding part of the computation seems to be the $O(nM^s)$ time needed to determine the relevant monomials by dynamic programming. But clearly all computations can be done in FP, and by working entirely within $\mathbb{Q}(\theta)$.

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