# Inapproximability of the Tutte polynomial of a planar graph* 

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#### Abstract

The Tutte polynomial of a graph $G$ is a two-variable polynomial $T(G ; x, y)$ that encodes many interesting properties of the graph. We study the complexity of the following problem, for rationals $x$ and $y$ : given as input a planar graph $G$, determine $T(G ; x, y)$. Vertigan completely mapped the complexity of exactly computing the Tutte polynomial of a planar graph. He showed that the problem can be solved in polynomial time if $(x, y)$ is on the hyperbola $H_{q}$ given by $(x-1)(y-1)=q$ for $q=1$ or $q=2$ or if $(x, y)$ is one of the two special points $(x, y)=(-1,-1)$ or $(x, y)=(1,1)$. Otherwise, the problem is \#P-hard. In this paper, we consider the problem of approximating $T(G ; x, y)$, in the usual sense of "fully polynomial randomised approximation scheme" or FPRAS. Roughly speaking, an FPRAS is required to produce, in polynomial time and with high probability, an answer that has small relative error. Assuming that NP is different from RP, we show that there is no FPRAS for the Tutte polynomial in a large portion of the $(x, y)$ plane. In particular, there is no FPRAS if $x>1, y<-1$ or if $y>1, x<-1$ or if $x<0, y<0$ and $q>5$. Also, there is no FPRAS if $x<1, y<1$ and $q=3$. For $q>5$, our result is intriguing because it shows that there is no FPRAS at $(x, y)=(1-q /(1+\varepsilon),-\varepsilon)$ for any positive $\varepsilon$ but it leaves open the limit point $\varepsilon=0$, which corresponds to approximately counting $q$-colourings of a planar graph.


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## 1 Introduction

### 1.1 The Tutte Polynomial

The Tutte polynomial of a graph $G=(V, E)$ (see [11, 13]) is the two-variable polynomial

$$
\begin{equation*}
T(G ; x, y)=\sum_{A \subseteq E}(x-1)^{\kappa(V, A)-\kappa(V, E)}(y-1)^{|A|-n+\kappa(V, A)}, \tag{1}
\end{equation*}
$$

where $\kappa(V, A)$ denotes the number of connected components of the graph $(V, A)$ and $n=$ $|V|$. Following the usual convention for the Tutte polynomial [10] a graph is allowed to have loops and/or multiple edges.

Many interesting properties of a graph correspond to evaluations of the Tutte polynomial at different points $(x, y)$. For example, the number of spanning trees of a connected graph $G$ is $T(G ; 1,1)$, the number of acyclic orientations is $T(G ; 2,0)$, and the reliability probability $R(G ; p)$ of the graph is an easily-computed multiple of $T(G ; 1,1 /(1-p))$. For a positive integer $q$, the Tutte polynomial along the hyperbola $H_{q}$ given by $(x-1)(y-1)=q$ corresponds to the Partition function of the $q$-state Potts model. See Welsh's book [13] for details.

Two particularly interesting Tutte invariants correspond to evaluations along the $x$ axis and the $y$ axis. In particular,

- The chromatic polynomial $P(G ; \lambda)$ of a graph $G$ with $n$ vertices, $m$ edges and $k$ connected components is given by

$$
P(G ; \lambda)=(-1)^{n-k} \lambda^{k} T(G ; 1-\lambda, 0) .
$$

When $\lambda$ is a positive integer, $P(G ; \lambda)$ counts the proper $\lambda$-colourings of $G$.

- The flow polynomial $F(G ; \lambda)$ is given by

$$
F(G ; \lambda)=(-1)^{m-n+k} T(G ; 0,1-\lambda) .
$$

When $\lambda$ is a positive integer, $F(G ; \lambda)$ counts the nowhere-zero $\lambda$-flows of $G$.

### 1.2 Evaluating the Tutte Polynomial

For fixed rational numbers $x$ and $y$, consider the following computational problem.
Name. Tutte $(x, y)$.
Instance. A graph $G=(V, E)$.
Output. $T(G ; x, y)$.
The parameters $x$ and $y$ are fixed in advance and are not considered part of the problem instance. Each choice for $x$ and $y$ defines a distinct computational problem.

Jaeger, Vertigan and Welsh [6] have completely mapped the complexity of $\operatorname{Tutte}(x, y)$. They have shown that $\operatorname{Tutte}(x, y)$ is in FP for any point $(x, y)$ on the hyperbola $H_{1}$ and when $(x, y)$ is one of the special points $(1,1),(0,-1),(-1,0)$, and $(-1,-1)$. They showed that $\operatorname{Tutte}(x, y)$ is \#P-hard for every other pair of rationals $(x, y)$. See [9] for definitions of FP and \#P; informally, FP is the extension of the class P from predicates to more general functions, and \#P is the counting analogue of NP. Jaeger et al. also investigated the complexity of evaluating the Tutte polynomial when $x$ and $y$ are real or complex numbers, but that is beyond the scope of this paper.

Vertigan [12] considered the restriction of $\operatorname{TuTte}(x, y)$ in which the input is restricted to be a planar graph.

Name. PlanarTutte $(x, y)$.
Instance. A planar graph $G=(V, E)$.
Output. $T(G ; x, y)$.
He showed that $\operatorname{PlanarTutte}(x, y)$ is in FP for any point $(x, y)$ on the hyperbolas $H_{1}$ or $H_{2}$, and when $(x, y)$ is one of the special points $(1,1)$ and $(-1,-1)$. He showed that $\operatorname{PlanarTutte}(x, y)$ is \#P-hard for every other pair of rationals $(x, y)$. The hyperbola $H_{2}$ is of particular interest, as the Tutte polynomial here corresponds to the partition function of the celebrated Ising model in statistical physics.

### 1.3 Approximating the Tutte polynomial

A fully polynomial randomised approximation scheme (FPRAS) for $\operatorname{TUTTE}(x, y)$ is a randomised algorithm that takes as input a graph $G$ and a constant $\varepsilon \in(0,1)$ and outputs a value $Y$ such that, with probability at least $3 / 4, e^{-\varepsilon} T(G ; x, y) \leq Y \leq e^{\varepsilon} T(G ; x, y)$. The running time of the algorithm is bounded from above by a polynomial in $n$ (the number of vertices of $G$ ) and $\varepsilon^{-1}$. An $\operatorname{FPRAS}$ for $\operatorname{Planartutte}(x, y)$ is defined similarly. See [7] for further details on fully polynomial randomised approximation schemes.

In earlier work [4, we considered the problem of determining for which points $(x, y)$ there is an $\operatorname{FPRAS}$ for $\operatorname{Tutte}(x, y)$. Our results are summarized in Figure 1. In particular, under the assumption $R P \neq N P$, we showed the following.
(1) If $x<-1$ and $(x, y)$ is not on $H_{0}$ or $H_{1}$, then there is no FPRAS at $(x, y)$.
(2) If $y<-1$ and $(x, y)$ is not on $H_{1}$ or $H_{2}$, then there is no FPRAS at $(x, y)$.
(3) There is no FPRAS at points $(x, y)$ lying in certain regions in the vicinity of the origin, contained in the square $-1<x, y<1$.
(4) If $(x, y)$ is on $H_{2}$ and $y<-1$ then approximating $T(G ; x, y)$ is equivalent in difficulty to approximately counting perfect matchings (resolving the complexity of this is a well-known and interesting open problem).


Figure 1: The result from [4]. Green points are FPRASable, red points are equivalent to counting perfect matchings and gray points are not FPRASable unless RP = NP. We don't know about white points. The black line, which is the portion of the hyperbola $q=4$ lying in $y \in(-1,0)$, is \#P-hard. The black points are harder than gray in a complexity-theoretic sense. The black region is presumably more extensive than shown.

An interesting consequence of these results is that, under the assumption $R P \neq N P$, there is no FPRAS at the point $(x, y)=(0,1-\lambda)$ when $\lambda>2$ is a positive integer. Thus, there is no FPRAS for counting nowhere-zero $\lambda$ flows for $\lambda>2$. This is interesting since the corresponding decision problem is in P , for example, for $\lambda=6$. See [4] for details.

### 1.4 Approximating the Tutte polynomial of a planar graph

In this paper we consider the problem of determining for which points $(x, y)$ there is an $\operatorname{FPRAS}$ for $\operatorname{PlanarTutte}(x, y)$. The results of 4 do not help us here because all of the constructions are badly non-planar. Our results are summarised in Figure 2,

In particular, under the assumption RP $\neq \mathrm{NP}$, Corollary 9 and Lemma 13 show that there is no FPRAS for PlanarTutte $(x, y)$ in the following cases:

1. $x<0, y<0$ and $q>5$;
2. $x<1, y<1$ and $q=3$;
3. $x>1, y<-1$;


Figure 2: The complexity of the planar case. The shaded gray regions are shown to be intractable in Corollary 9 . The lower branch of the $q=3$ hyperbola (also depicted in gray) is shown to be intractable in Lemma 13. As Vertigan has shown [12], it is easy to compute the polynomial exactly on the hyperbolas $q=1$ and $q=2$ and at the two special points $(1,1)$ and $(-1,-1)$. (These are shown in green.)
4. $y>1, x<-1$.

For integer $q \geq 4$, the point $x=1-q, y=0$ is of particular interest. As noted earlier $T(G ; x, y)$ gives the number of proper $q$-colourings of $G$. By the 4 -colour theorem, there is at least one $q$-colouring, so the corresponding decision problem is trivial, but it is not clear whether there is an FPRAS. For $q \geq 5$, our result shows that there is no FPRAS for any nearby point $x=1-q /(1+\varepsilon), y=-\varepsilon$ on the hyperbola $H_{q}$ (for any $\left.\varepsilon>0\right)$. However, the case of colourings itself (corresonding to the limit point $\varepsilon=0$ ) remains open. The same intriguing situation occurs with the flow polynomial points $x=0, y=1-q$.

In a recent posting on ArXiv, Kuperberg [8] independently offers a proof sketch, based on the complexity theory of quantum computation, of a result closely related to ours. If the details in the proof sketch can be filled in, then it will strengthen our result in the negative quadrant by (i) relaxing the condition $q \geq 5$ to $q \geq 4$, and (ii) strengthening the conclusion to \#P-hardness.

### 1.5 The multivariate formulation of the Tutte polynomial

As in [4, we need the multivariate formulation of the Tutte polynomial in order to prove our results. The multivariate formulation is also known as the random cluster model [13, 10]. For $q \in \mathbb{Q}$ and a graph $G=(V, E)$ with edge weights $w: E \rightarrow \mathbb{Q}$, the multivariate Tutte polynomial of $G$ is defined by $Z(G ; q, w)=\sum_{A \subseteq E} w(A) q^{\kappa(V, A)}$, where $w(A)=\prod_{e \in A} w(e)$.

Suppose $(x, y) \in \mathbb{Q}^{2}$ and $q=(x-1)(y-1)$. For a graph $G=(V, E)$, let $w: E \rightarrow \mathbb{Q}$ be the constant function which maps every edge to the value $y-1$. Then (see, for example [10, (2.26)])

$$
\begin{equation*}
T(G ; x, y)=(y-1)^{-n}(x-1)^{-\kappa(E)} Z(G ; q, w) . \tag{2}
\end{equation*}
$$

So approximating $T(G ; x, y)$ is equivalent in difficulty to approximating $Z(G ; q, w)$ for the constant function $w(e)=y-1$. However, the multivariate formulation is more general, because we can assign different weights to different edges of $G$.

Consider the following computational problem, which is a planar version of one that we considered in [4].

Name. $\operatorname{MultiTutte}\left(q ; \alpha_{1}, \alpha_{2}, \alpha_{3}\right)$.
Instance. A planar graph $G=(V, E)$ with edge labelling $w: E \rightarrow\left\{\alpha_{1}, \alpha_{2}, \alpha_{3}\right\}$.
Output. $Z(G ; q, w)$.
Our main tool in proving inapproximability (Lemma 7 below) is showing that

$$
\operatorname{MultiTutte}\left(q ; \alpha_{1}, \alpha_{2}, \alpha_{3}\right)
$$

is difficult to approximate if $\alpha_{1} \notin[-2,0], \alpha_{2} \in(-2,0)$ and $\alpha_{3}<-1$. (Note that $\alpha_{3}$ might be equal to $\alpha_{1}$ or $\alpha_{2}$.)

## 2 Technical Preparation

In this section we introduce a gadget (weighted graph) with certain useful properties. Although the graph is very simple, the edge weights must be carefully tuned to achieve the desired properties. We need to be able to "implement" these particular edge weights in terms of the actual weights that are available to us.

### 2.1 Implementing new edge weights

Let $W$ be a set of edge weights (for example, $W$ might contain the edge weights $\alpha_{1}, \alpha_{2}$ and $\alpha_{3}$ from above) and fix a value $q$. Let $w^{*}$ be a weight (which may not be in $W$ ) which we want to "implement". Suppose that there is a planar graph $\Upsilon$, with distinguished vertices $s$ and $t$ on the outer face, and a weight function $\widehat{w}: E(\Upsilon) \rightarrow W$ such that

$$
\begin{equation*}
w^{*}=q Z_{s t}(\Upsilon) / Z_{s \mid t}(\Upsilon), \tag{3}
\end{equation*}
$$

where $Z_{s t}(\Upsilon)$ denotes the contribution to $Z(\Upsilon ; q, \widehat{w})$ arising from edge-sets $A$ in which $s$ and $t$ are in the same component. That is, $Z_{s t}(\Upsilon)=\sum_{A} \widehat{w}(A) q^{\kappa(V, A)}$, where the sum is over subsets $A \subseteq E(\Upsilon)$ in which $s$ and $t$ are in the same component. Similarly, $Z_{s \mid t}$ denotes the contribution to $Z(\Upsilon ; q, \widehat{w})$ arising from edge-sets $A$ in which $s$ and $t$ are in different components. In this case, we say that $\Upsilon$ and $\widehat{w}$ implement $w^{*}$ (or even that $W$ implements $w^{*}$ ).

The purpose of "implementing" edge weights is this. Let $G$ be a graph with edgeweight function $w$. Let $f$ be some edge of $G$ with edge weight $w(f)=w^{*}$. Suppose that $W$ implements $w^{*}$. Let $\Upsilon$ be a planar graph with distinguished vertices $s$ and $t$ with a weight function $\widehat{w}$ satisfying (3). Construct the weighted graph $G^{\prime}$ by replacing edge $f$ with a copy of $\Upsilon$ (identify $s$ with either endpoint of $f$ (it doesn't matter which one) and identify $t$ with the other endpoint of $f$ and remove edge $f$ ). Let the weight function $w^{\prime}$ of $G^{\prime}$ inherit weights from $w$ and $\widehat{w}$ (so $w^{\prime}(e)=\widehat{w}(e)$ if $e \in E(\Upsilon)$ and $w^{\prime}(e)=w(e)$ otherwise). Then the definition of the multivariate Tutte polynomial gives

$$
\begin{equation*}
Z\left(G^{\prime} ; q, w^{\prime}\right)=\frac{Z_{s \mid t}(\Upsilon)}{q^{2}} Z(G ; q, w) \tag{4}
\end{equation*}
$$

So, as long as $q \neq 0$ and $Z_{s \mid t}(\Upsilon)$ is easy to evaluate, evaluating the multivariate Tutte polynomial of $G^{\prime}$ with weight function $w^{\prime}$ is essentially the same as evaluating the multivariate Tutte polynomial of $G$ with weight function $w$.

Two especially useful implementations are series and parallel compositions. These are explained in detail in [5, Section 2.3]. So we will be brief here. Parallel composition is the case in which $\Upsilon$ consists of two parallel edges $e_{1}$ and $e_{2}$ with endpoints $s$ and $t$ and $\widehat{w}\left(e_{1}\right)=$ $w_{1}$ and $\widehat{w}\left(e_{2}\right)=w_{2}$. It is easily checked from Equation (3) that $w^{*}=\left(1+w_{1}\right)\left(1+w_{2}\right)-1$. Also, the extra factor in Equation (4) cancels, so in this case $Z\left(G^{\prime} ; q, w^{\prime}\right)=Z(G ; q, w)$.

Series composition is the case in which $\Upsilon$ is a length-2 path from $s$ to $t$ consisting of edges $e_{1}$ and $e_{2}$ with $\widehat{w}\left(e_{1}\right)=w_{1}$ and $\widehat{w}\left(e_{2}\right)=w_{2}$. It is easily checked from Equation (3) that $w^{*}=w_{1} w_{2} /\left(q+w_{1}+w_{2}\right)$. Also, the extra factor in Equation (4) is $q+w_{1}+w_{2}$, so in this case $Z\left(G^{\prime} ; q, w^{\prime}\right)=\left(q+w_{1}+w_{2}\right) Z(G ; q, w)$. It is helpful to note that $w^{*}$ satisfies

$$
\left(1+\frac{q}{w^{*}}\right)=\left(1+\frac{q}{w_{1}}\right)\left(1+\frac{q}{w_{2}}\right) .
$$

We say that there is a "shift" from $(q, \alpha)$ to $\left(q, \alpha^{\prime}\right)$ if there is an implementation of $\alpha^{\prime}$ consisting of some $\Upsilon$ and $\widehat{w}: E(\Upsilon) \rightarrow W$ where $W$ is the singleton set $W=\{\alpha\}$. This is the same notion of "shift" that we used in [4]. Taking $y=\alpha+1$ and $y^{\prime}=\alpha^{\prime}+1$ and defining $x$ and $x^{\prime}$ by $q=(x-1)(y-1)=\left(x^{\prime}-1\right)\left(y^{\prime}-1\right)$ we equivalently refer to this as a shift from $(x, y)$ to $\left(x^{\prime}, y^{\prime}\right)$.

Thus, the $k$-thickening of Jaeger, Vertigan and Welsh [6] is the parallel composition of $k$ edges of weight $\alpha$. It implements $\alpha^{\prime}=(1+\alpha)^{k}-1$ and is a shift from $(x, y)$ to $\left(x^{\prime}, y^{\prime}\right)$ where $y^{\prime}=y^{k}$ (and $x^{\prime}$ is given by $\left.\left(x^{\prime}-1\right)\left(y^{\prime}-1\right)=q\right)$. Similarly, the $k$-stretch is the series composition of $k$ edges of weight $\alpha$. It implements an $\alpha^{\prime}$ satisfying

$$
1+\frac{q}{\alpha^{\prime}}=\left(1+\frac{q}{\alpha}\right)^{k}
$$

It is a shift from $(x, y)$ to $\left(x^{\prime}, y^{\prime}\right)$ where $x^{\prime}=x^{k}$. (In the classical bivariate $(x, y)$ parameterisation, there is effectively one edge weight, so the stretching or thickening is applied uniformly to every edge of the graph.)

Since it is useful to switch freely between ( $q, \alpha$ ) coordinates and $(x, y)$ coordinates we also refer to the implementation in Equation (3) as an implementation of the point $(x, y)=\left(q / w^{*}+1, w^{*}+1\right)$ using the points

$$
\{(x, y)=(q / w+1, w+1) \mid w \in W\} .
$$

### 2.2 Global Constants

Our proofs will use several global constants which depend upon $q$ but do not depend upon the problem instances in our reductions. The definitions of these constants are provided here for easy reference. The purpose of all of these constants will become clear later, but as a rough guide, the constants $A^{-}, A^{+}, B^{-}$and $B^{+}$will be lower and upper bounds on the (absolute values of the) edge weights, $a$ and $b$, that we use in our gadgets. The edge weights themselves will depend on the problem instance, but it is important for the proof that these lower and upper bounds do not depend upon the problem instance - they only depend upon $q$.

Let $f(x)$ be the function $f(x)=x^{3}+3 x^{2}$. We start by defining several quantities for which the definitions differ depending on whether $q<0$ or $q>5$.
Case 1: $q>5: \quad \chi=\min (1,(q-5) / 6), \eta=3 / 4, A^{-}=1 / 2, A^{+}=q, B^{-}=q$ and $B^{+}=10 q^{3}$.
Case 2: $q<0: \quad \chi=\min (1,|q|)$. To define the other constants, it helps to make a few observations. Let $g(y)=f(-3-y)$ and note that $g(0)=0$ and that $g^{\prime}(y)<0$ for $y>0$ so $g(y)$ decreases as $y$ increases from 0 . Now let $\eta>0$ be the real solution of $g(\eta)=q / 2$. Let $A^{-}=3+\eta$. Then let $y^{*}>0$ be the real solution of $g\left(y^{*}\right)=q$. Let $A^{+}=3+y^{*}$. Let $B^{-}=|q| / 3$ and let $B^{+}=4|q| / 3+2$.

Note that, in both cases, $0<A^{-}<A^{+}$and $0<B^{-}<B^{+}$and $\eta>0$. Finally, define

- $A^{*}=1+3\left(A^{+}\right)^{4}+9\left(A^{+}\right)^{3}+3\left(A^{+}\right)^{2}+3 A^{+}(1+|q|)$,
- $Q=\max \left(|q|,|q|^{-1}\right)$,
- $\mu=q^{2} A^{+}\left(B^{+}\right)^{2}$,
- $\tau=|q|\left(A^{+}\right)^{2}\left(A^{+}+3\right)\left(B^{+}\right)^{3}$, and
- $M=\max (1, \mu, \tau)$.


### 2.3 Implementing useful edge weights

In much of the technical part of the paper, we will have at our disposal three edge weights, $\alpha_{1}, \alpha_{2}$ and $\alpha_{3}$ such that $\alpha_{1} \notin[-2,0], \alpha_{2} \in(-2,0)$ and $\alpha_{3}<-1$. $\left(\alpha_{3}\right.$ might be equal to $\alpha_{1}$ or $\alpha_{2}$.) Now that we have defined the global constants in Section 2.2, we state some lemmas showing that we can use $\alpha_{1}, \alpha_{2}$ and $\alpha_{3}$ to implement certain edge weights, $a, b$ and $\beta$, which we will later use in our gadgets. As will become apparent below, the precise definitions of $a, b$ and $\beta$ will depend upon two accuracy parameters $\varrho$ and $\varrho$. When we use the lemmas, we will take these to be very small (depending on the input sizes in our reductions). We defer the proofs of the lemmas until Section 3.6 because are they mainly technical, and are not necessary for understanding our main argument.

Lemma 1. Suppose $q \notin[0,5]$ and that $\alpha_{1} \notin[-2,0], \alpha_{2} \in(-2,0)$ and $\alpha_{3}<-1$. Given a positive constant $\varrho$ which is sufficiently small with respect to $q, \alpha_{1}, \alpha_{2}$, and $\alpha_{3}$, there is a planar graph $\Upsilon$ (depending on $\varrho$ ) and a weight function $\widehat{w}: E(\Upsilon) \rightarrow\left\{\alpha_{1}, \alpha_{2}, \alpha_{3}\right\}$ that implements a weight $a$, such that

$$
\begin{gather*}
A^{-} \leq|a| \leq A^{+},  \tag{5}\\
q+\varrho<f(a) \leq q+2 \varrho, \text { and }  \tag{6}\\
|f(a)| \geq \eta . \tag{7}
\end{gather*}
$$

The size of $\Upsilon$ is at most a polynomial in $\log \left(\varrho^{-1}\right)$.
Lemma 2. Suppose $q \notin[0,5]$ and that $\alpha_{1} \notin[-2,0], \alpha_{2} \in(-2,0)$ and $\alpha_{3}<-1$. Suppose, for a positive value $\varrho$, which is sufficiently small with respect to $q, \alpha_{1}, \alpha_{2}$, and $\alpha_{3}$, the value a satisfies inequalities (5), (6) and (7). Let

$$
\begin{equation*}
c=a^{2}+3 a+q \tag{8}
\end{equation*}
$$

Given a positive constant $\hat{\varrho}$ which is sufficiently small with respect to $q, \alpha_{1}, \alpha_{2}$, and $\alpha_{3}$, there is a planar graph $\Upsilon$ (depending on $\hat{\varrho}$ ) and a weight function $\widehat{w}: E(\Upsilon) \rightarrow\left\{\alpha_{1}, \alpha_{2}, \alpha_{3}\right\}$ that implements a weight b, such that

$$
\begin{align*}
& B^{-} \leq|b| \leq B^{+}, \text {and }  \tag{9}\\
& -\hat{\varrho} \leq b+c \leq \hat{\varrho} . \tag{10}
\end{align*}
$$

The size of $\Upsilon$ is at most a polynomial in $\log \left(\hat{\varrho}^{-1}\right)$.
Lemma 3. Suppose $q \notin[0,5]$ and that $\alpha_{2} \in(-2,0)$. Given a positive constant $\varrho$ which is sufficiently small with respect to $q$ and $\alpha_{2}$, there is a planar graph $\Upsilon$ (depending on @) and a weight function $\widehat{w}: E(\Upsilon) \rightarrow\left\{\alpha_{2}\right\}$ that implements a weight $\beta$, such that $|1+\beta| \leq \varrho$. The size of $\Upsilon$ is at most a polynomial in $\log \left(\varrho^{-1}\right)$.


Figure 3: The gadget $Y$.

### 2.4 A useful gadget

Suppose $a$ and $b$ are edge weights. Let $Y$ be a weighted graph with weight function $w$ defined as follows. $Y$ will have vertex set $V(Y)=\{0,1,2, \overline{0}, \overline{1}, \overline{2}\}$. The edge set $E(Y)$ of $Y$ consists of three edges $(0, \overline{0}),(1, \overline{1})$ and $(2, \overline{2})$ of weight $b$ and three edges $(\overline{0}, \overline{1}),(\overline{1}, \overline{2})$ and $(\overline{2}, \overline{0})$ of weight $a$. See Figure 3 .

For a fixed $q$, let $Z_{0|1| 2}$ denote the contribution to $Z(Y ; q, w)$ arising from edge sets $A$ in which the vertices 0,1 and 2 are in distinct components. Thus, $Z_{0|1| 2}=\sum_{A} \prod_{e \in A} w(e) q^{\kappa(V(Y), A)}$, where the sum is over all subsets $A \subseteq E(Y)$ such that 0,1 and 2 are all in distinct components. Similarly, let $Z_{0 \mid 12}$ denote the contribution to $Z(Y ; q, w)$ arising from edge sets $A$ in which the vertex 0 is in one component and the vertices 1 and 2 are in another, distinct, component. Finally, let $Z_{012}$ denote the contribution to $Z(Y ; q, w)$ denote the contribution to $Z(Y ; q, w)$ arising from edge sets $A$ in which the vertices 0,1 and 2 are all in the same component. Define $c$ via Equation (8). From the definition of $Z_{0 \mid 12}$ we see that

$$
\begin{equation*}
Z_{0 \mid 12}=q^{2} a b^{2}(c+b) \tag{11}
\end{equation*}
$$

Similarly,

$$
Z_{0|1| 2}=q^{3}\left(b^{3}+3 b^{2}(2 a+q)+(3 b+q)\left(a^{3}+3 a^{2}+3 a q+q^{2}\right)\right) .
$$

Let

$$
\begin{equation*}
d=a^{2}+3 a+q+b \tag{12}
\end{equation*}
$$

and

$$
\begin{equation*}
e=a^{3}+3 a^{2}-q . \tag{13}
\end{equation*}
$$

Then

$$
\begin{align*}
Z_{0| | \mid 2} & =-q^{3} a^{2}(a+3)\left(a^{3}+3 a^{2}-q\right)+d^{3} q^{3}+d^{2}\left(-3 a-3 a^{2}\right) q^{3}+d q^{3}\left(9 a^{3}+3 a^{4}-3 a q\right) \\
& =-q^{3} a^{2}(a+3) e+d^{3} q^{3}-d^{2}\left(3 a+3 a^{2}\right) q^{3}+d q^{3}\left(9 a^{3}+3 a^{4}-3 a q\right) \\
& =-q^{3} a^{2}(a+3)\left(e-\frac{d^{3}}{a^{2}(a+3)}+\frac{d^{2}\left(3 a+3 a^{2}\right)}{a^{2}(a+3)}-\frac{d\left(9 a^{3}+3 a^{4}-3 a q\right)}{a^{2}(a+3)}\right) \tag{14}
\end{align*}
$$

Also

$$
\begin{equation*}
Z_{012}=q a^{2}(a+3) b^{3} . \tag{15}
\end{equation*}
$$

### 2.5 A lower bound on the Tutte polynomial

We conclude our technical preparations by presenting a lemma which gives a positive lower bound on the (multivariate) Tutte polynomial of a planar graph for $q>5$. The lemma is essentially due to Woodall [14, Theorem 1]. (The method can be traced back to [1].) However, we need two slight generalisations. First, Woodall's proof was actually about the chromatic polynomial, which corresponds to the specialisation of the Tutte polynomial in which $w(e)=-1$ for every edge $e$. In our lemma, we will ensure that $w(e)$ is always close to -1 but it will not be exactly equal to -1 . Second, Woodall's objective was to show that the polynomial is positive. We will need something slightly stronger - namely, a strictly positive lower bound. Woodall's proof technique suffices to provide this.

Lemma 4. Suppose $q>5$. Suppose $\varrho \in(0,1)$ and $\zeta \in(0,1)$ satisfy $q \geq 5(1+\varrho)+\zeta$. For any simple planar graph $G=(V, E)$ and any edge-weight function $w$ satisfying $|1+w(e)| \leq$ $\varrho$ for all $e \in E, Z(G ; q, w) \geq \zeta^{|V|}$.

Proof. We follow the proof of [14, Theorem 1] due to Woodall. We can assume without loss of generality that $G$ is connected (otherwise consider the components separately). The proof is by induction on $n$, the number of vertices of $G$. The base case, in which $n=1$, is straightforward since $G$ has no loops. Suppose $n>1$. Since $G$ is planar, it has a vertex $v$ whose degree, $\ell$, is between 1 and 5 . Let $e_{1}=\left(v, v_{1}\right), \ldots, e_{\ell}=\left(v, v_{\ell}\right)$ be the edges incident at $v$. Let $\mathcal{A}_{0}=\left\{A \subseteq E \mid A \cap\left\{e_{1}, \ldots, e_{\ell}\right\}=\emptyset\right\}$ and $\mathcal{A}_{i}=\left\{A \subseteq E \mid A \cap\left\{e_{1}, \ldots, e_{i}\right\}=\right.$ $\left.\left\{e_{i}\right\}\right\}$. Let $Z_{i}(G ; q, w)=\sum_{A \in \mathcal{A}_{i}} w(A) q^{\kappa(V, A)}$, so $Z(G ; q, w)=\sum_{i=0}^{\ell} Z_{i}(G ; q, w)$. It is easy to see (using the definition of the multivariate Tutte polynomial) that $Z_{0}(G ; q, w)=$ $q Z(G-v ; q, w)$ where, in the expression $Z(G-v, q, w)$, we view $w$ as a weight function $w: E \backslash\left\{e_{1}, \ldots, e_{\ell}\right\} \rightarrow \mathbb{Q}$. Also, for $i \in[\ell], Z_{i}(G ; q, w)=w\left(e_{i}\right) Z\left(G_{i}^{\prime} ; q, w\right)$, where $G_{i}^{\prime}$ is the multigraph formed from $G$ by deleting $e_{1}, \ldots, e_{i-1}$ and contracting $e_{i}$ (i.e., identifying its endpoints and then deleting it). Note that $G_{i}^{\prime}$ may have parallel edges (though it has no loops). However, if we consider two parallel edges $e$ and $f$ with weights $w(e)$ and $w(f)$, we know from Section 2.1 that the parallel composition of these two edges implements the single edge weight $w^{*}=(1+w(e))(1+w(f))-1$. Thus, we can replace these two parallel edges with a single edge $e^{\prime}$ with weight $w^{*}$ without changing the value of the Tutte polynomial. Also, note that since $|1+w(e)| \leq \varrho,|1+w(f)| \leq \varrho$ and $\varrho \in(0,1)$, we also have $\left|1+w^{*}\right| \leq \varrho$. We conclude that $Z\left(G_{i}^{\prime} ; q, w\right)=Z\left(G_{i} ; q, w_{i}\right)$, where $G_{i}$ is the simple graph underlying $G_{i}^{\prime}$ and $w_{i}$ is the induced weight function, which is "good" in the sense that $\left|1+w_{i}(e)\right| \leq \varrho$ for every edge $e$ of $G_{i}$. The graph $G_{i}$ has vertex set $V-v$ and edge set $E_{i}=E \backslash\left\{e_{1}, \ldots, e_{\ell}\right\} \cup\left\{f_{1}, \ldots, f_{\ell_{i}}\right\}$, where $\ell_{i}$ is the number of vertices in $v_{i+1}, \ldots, v_{\ell}$ which are not neighbours of $v_{i}$ in $G$ and $f_{1}, \ldots, f_{\ell_{i}}$ are new edges connecting $v_{i}$ to these vertices.

Let $\mathcal{B}_{0}=\left\{A \subseteq E_{i} \mid A \cap\left\{f_{1}, \ldots, f_{\ell_{i}}\right\}=\emptyset\right\}$ and $\mathcal{B}_{j}=\left\{A \subseteq E_{i} \mid A \cap\left\{f_{1}, \ldots, f_{j}\right\}=\left\{f_{j}\right\}\right\}$. Let $Z_{j}^{\prime}\left(G_{i} ; q, w_{i}\right)=\sum_{A \in \mathcal{B}_{i}} w_{i}(A) q^{\kappa(V-v, A)}$, so $Z\left(G_{i} ; q, w_{i}\right)=\sum_{j=0}^{\ell_{i}} Z_{j}^{\prime}\left(G_{i} ; q, w_{i}\right)$. Once again, $Z_{0}^{\prime}\left(G_{i} ; q, w\right)=Z\left(G-v ; q, w_{i}\right)$. Also, for $j \in\left[\ell_{i}\right], Z_{j}^{\prime}\left(G_{i} ; q, w_{i}\right)=w_{i}\left(f_{j}\right) Z\left(G_{i, j}^{\prime} ; q, w_{i}\right)$, where $G_{i, j}^{\prime}$ is the multigraph formed from $G_{i}$ by deleting $f_{1}, \ldots, f_{j-1}$, and contracting $f_{j}$. As before, there is a simple graph $G_{i, j}$ and a "good" weight function $w_{i, j}$ such that
$Z\left(G_{i, j}^{\prime} ; q, w_{i}\right)=Z\left(G_{i, j} ; q, w_{i, j}\right) . G_{i, j}$ has vertex set $V-v$, so we know by induction that $Z\left(G_{i, j} ; q, w_{i, j}\right) \geq 0$.

Putting all of the above together,

$$
\begin{aligned}
Z(G ; q, w) & =q Z(G-v ; q, w)+\sum_{i=1}^{\ell} w\left(e_{i}\right) Z\left(G_{i} ; q, w_{i}\right) \\
& =q Z(G-v ; q, w)+\sum_{i=1}^{\ell} w\left(e_{i}\right)\left(Z\left(G-v ; q, w_{i}\right)+\sum_{j=1}^{\ell_{i}} w_{i}\left(f_{j}\right) Z\left(G_{i, j} ; q, w_{i, j}\right)\right) .
\end{aligned}
$$

Since $w$ and $w_{i}$ are "good" weight functions, both $w\left(e_{i}\right)$ and $w_{i}\left(f_{j}\right)$ are at most 0 . Thus, $w\left(e_{i}\right) w_{i}\left(f_{j}\right) \geq 0$ so we get

$$
\begin{aligned}
Z(G ; q, w) & \geq q Z(G-v ; q, w)+\sum_{i=1}^{\ell} w\left(e_{i}\right) Z\left(G-v ; q, w_{i}\right) . \\
& =Z(G-v ; q, w)\left(q+\sum_{i=1}^{\ell} w\left(e_{i}\right)\right) . \\
& \geq Z(G-v ; q, w) \zeta
\end{aligned}
$$

where the final inequality follows from $q>5(1+\varrho)+\zeta$ and from the fact that the weight function $w$ is good. The result follows by induction.

## 3 Proving inapproximability

### 3.1 The starting point

Our starting point is the following problem.
Name. Planar cubic Maximum Independent Set.
Instance. A cubic planar graph $G$ and a positive integer $K$.
Question. Does $G$ contain an independent set of size at least $K$ ?
Lemma 5. Planar cubic Maximum Independent Set is NP-complete.
Proof. This problem is essentially the same as "Node cover in planar graphs with maximum degree 3", which was shown to be NP-complete by Garey and Johnson [2, Lemma 1]. First, the complement of a minimum node (or vertex) cover in a graph is a maximum independent set. Thus Garey and Johnson's problem is the same as "Maximum independent set in a planar graph with maximum degree 3 ". So we just need to show that we can transform a planar graph with maximum degree 3 into a cubic graph in such a way that the size of a maximum independent set changes in a controlled way.


Figure 4: The graph $T$.

It is easily checked that there is a (unique) simple planar graph $T$ with degree sequence $(1,3,3,3,3,3)$. See Figure 4. Denote by $r$ the unique vertex of degree 1. Given a planar graph with maximum degree 3, we can form a planar cubic graph by attaching (via vertex $r$ ) the appropriate number of copies of $T$ to the deficient vertices. It is easily checked that each copy of $T$ increases the size of a maximum independent set by 2 .

We will use the following variant of Planar cubic Maximum Independent Set. This variant will help us to maintain planarity in our constructions.

## Name. Planar stretched cubic Maximum Independent Set.

Instance. A graph $G$ which is the 3 -stretch of a cubic planar graph $H$ and a positive integer $K$.
Question. Does $G$ contain an independent set of size at least $K$ ?
Lemma 6. Planar stretched cubic Maximum Independent Set is NP-complete.
Proof. Let $m^{\prime}$ be the number of edges of $H$. We claim that the size of a maximum independent set of $G$ is equal to $m^{\prime}$ plus the size of a maximum independent set of $H$.

First, suppose that $H$ has an independent set of size $k$. We use this independent set to construct an independent set of size $m^{\prime}+k$ in $G$ : For every IN-OUT edge of $H$ (that is, for every edge $(u, v)$ of $H$ such that $u$ is in the independent set, and $v$ is out), the corresponding configuration in $G$ can be IN-OUT-IN-OUT. For every OUT-OUT edge of $H$ the corresponding configuration of $G$ can be OUT-IN-OUT-OUT.

Next, suppose that $G$ has an independent set of size $m^{\prime}+k^{\prime}$ for some $k^{\prime} \geq 0$. We construct an independent set of size $k^{\prime}$ in $H$. Consider an independent set in $G$ of size $m^{\prime}+k^{\prime}$ which contains as many degree-2 vertices as possible. Consider the configuration corresponding to an edge of $H$. It cannot be IN-OUT-OUT-IN, because one of the IN vertices could be moved to a degree-2 vertex without changing the size of the independent set. Thus, this independent set induces an independent set of $H$. Since at most $m^{\prime}$ degree-2 vertices are contained in the independent set, the induced independent set in $H$ is size at least $k^{\prime}$.

### 3.2 Some global variables

In our proofs, we will work with an instance $G$ and $K$ of Planar stretched cubic Maximum Independent Set where $G$ has $n$ vertices and $m$ edges. For now, in order to do the preliminary work, let's view $n, m$ and $K$ as parameters corresponding to the size of the instance that we'll work with. Using the global constants from Section 2.2, we define the following quantities.

$$
\begin{aligned}
\nu & =3 n-m-2 K . \\
\varepsilon & =\frac{\left(B^{-}\right)^{3}}{3} \chi^{\nu} 2^{-(n+2 m+4)} Q^{-3 n+m} . \\
L & =\left|q^{3}\right| \eta \varepsilon / 2 . \\
R & =\left(B^{-}\right)^{3} / 3 \varepsilon . \\
\delta & =\frac{L^{n} \chi^{\nu}}{16 A^{*} 5^{n} M^{n} 2^{2 m} Q^{9 n}}
\end{aligned}
$$

Since $G$ is the 3 -stretch of a cubic planar graph, we will have $m=\frac{9}{8} n$. We will also assume that $K \leq \frac{5}{8} n$, since this is an easy upper bound on the size of any independent set in $G$. We will rely on the following inequalities, which follow from these considerations as long as $n$ is sufficiently large.

$$
\begin{align*}
& L \leq 1  \tag{16}\\
& R \geq 1  \tag{17}\\
& \nu \geq \frac{5}{8} n \geq 1  \tag{18}\\
& 0<\delta<\varepsilon<\chi \leq 1  \tag{19}\\
& \delta \leq \varepsilon \eta /\left(6 A^{*}\right) \tag{20}
\end{align*}
$$

### 3.3 The gadget revisited

Suppose that quantity $a$ satisfies (5), (6) and (7) with $\varrho=\varepsilon$ and that $b$ satisfies (9) and (10) with $\hat{\varrho}=\delta$. Define $c, d$ and $e$ via equations (8), (12) and (13) respectively. Note that $d \in[-\delta, \delta]$ and $e \in[\varepsilon, 2 \varepsilon]$. Note also that (11) implies

$$
\begin{equation*}
\left|Z_{0 \mid 12}\right| \leq \delta \mu \tag{21}
\end{equation*}
$$

Now, $1 \leq A^{*}$, and the constraints on $a$ imply $\left|3 a+3 a^{2}\right| \leq A^{*}$ and $\left|9 a^{3}+3 a^{4}-3 a q\right| \leq A^{*}$. Thus, using (7), the absolute value of each of the right-most three terms in (14) is at most $\delta A^{*} / \eta$ and by (20), this is at most $\varepsilon / 6$. Thus,

$$
\begin{equation*}
L \leq\left|q^{3} a^{2}(a+3)\right| \frac{\varepsilon}{2} \leq\left|Z_{0|1| 2}\right| \leq\left|q^{3} a^{2}(a+3)\right| 3 \varepsilon . \tag{22}
\end{equation*}
$$

Also, from (15),

$$
\begin{equation*}
\left|q a^{2}(a+3)\right|\left(B^{-}\right)^{3} \leq\left|Z_{012}\right| \leq \tau \tag{23}
\end{equation*}
$$

Finally, we combine these to see

$$
\begin{equation*}
\frac{q^{2}\left|Z_{012}\right|}{\left|Z_{0|1| 2}\right|} \geq \frac{q^{2}\left|q a^{2}(a+3)\right|\left(B^{-}\right)^{3}}{\left|q^{3} a^{2}(a+3)\right| 3 \varepsilon}=R . \tag{24}
\end{equation*}
$$

We will also use the following quantity, defined in terms of the $Y$-gadget.

$$
\Psi=\left|\frac{q^{2} Z_{012}}{Z_{0 \mid 12}}\right|^{K-1} R\left|Z_{0|1| 2}\right|^{n}|q|^{-3 n} \chi^{\nu} .
$$

### 3.4 The Main lemma

We can now state, and prove, our main lemma.
Lemma 7. Suppose $q \notin[0,5]$ and that $\alpha_{1} \notin[-2,0], \alpha_{2} \in(-2,0)$ and $\alpha_{3}<-1$. Then there is no FPRAS for $\operatorname{MultiTutte}\left(q ; \alpha_{1}, \alpha_{2}, \alpha_{3}\right)$ unless $\mathrm{RP}=\mathrm{NP}$.

Proof. Suppose $H$ is a cubic planar graph and $G$ is the 3 -stretch of $H$. Let $n=|V(G)|$ and $m=|E(G)|$ and note that Suppose $G$ and $K$ are inputs to Planar stretched cubic Maximum Independent Set. Recall the definitions of the global variables from Section 3.2. Our ultimate goal is to construct a planar instance $\left(G^{\prime}, w^{\prime}\right)$ of $\operatorname{MultiTutte}\left(q ; \alpha_{1}\right.$, $\left.\alpha_{2}, \alpha_{3}\right)$ such that a close approximation to $Z\left(G^{\prime} ; q, w^{\prime}\right)$ enables us to determine whether $G$ has an independent set of size $K$. To do this, we'll construct a weighted planar graph $(\widehat{G}=(\widehat{V}, \widehat{E}), w)$ such that a close approximation to $Z(\widehat{G} ; q, w)$ enables us to determine whether $G$ has an independent set of size $K$, where $w: \widehat{E} \rightarrow\{\beta, a, b\}$ and the edge-weight $a$ satisfies (5), (6) and (7) with $\varrho=\varepsilon$, the edge-weight $b$ satisfies (9) and (10) with $\hat{\varrho}=\delta$ and the edge-weight $\beta$ satisfies $|1+\beta| \leq \delta$. Lemmas 1, 2 and 3 show that such values can be implemented using weights $\alpha_{1}, \alpha_{2}$ and $\alpha_{3}$. Thus, applying these implementations to the weight graph $\widehat{G}$ will give us $G^{\prime}$ and $w^{\prime}$ so that $Z(\widehat{G} ; q, w)$ is an easily computable multiple of $Z\left(G^{\prime}, q, w^{\prime}\right)$, completing the proof.

Here is the construction of $\widehat{G}=(\widehat{V}, \widehat{E})$. (See Figure 5.) First, fix any ordering on the vertices of $H$. Next, let's set up some useful notation for the graph $Y$, which we will use as a gadget. A particular copy $Y^{x}$ of this gadget will have vertex set $V^{x}=\{\langle x, 0\rangle,\langle x, 1\rangle,\langle x, 2\rangle$, $\langle x, \overline{0}\rangle,\langle x, \overline{1}\rangle,\langle x, \overline{2}\rangle\}$ with vertices $\langle x, 0\rangle,\langle x, 1\rangle$ and $\langle x, 2\rangle$ arranged in clockwise order around the outer face. The edge set $E^{x}$ consists of three edges $(\langle x, 0\rangle,\langle x, \overline{0}\rangle),(\langle x, 1\rangle,\langle x, \overline{1}\rangle)$, and $(\langle x, 2\rangle,\langle x, \overline{2}\rangle)$ of weight $b$ and three edges $(\langle x, \overline{0}\rangle,\langle x, \overline{1}\rangle),(\langle x, \overline{1}\rangle,\langle x, \overline{2}\rangle)$ and $(\langle x, \overline{2}\rangle,\langle x, \overline{0}\rangle)$ of weight $a$. We will construct a copy $Y^{u}$ of the $Y$-gadget for every vertex $u \in V(H)$.

Now, associate every edge $(u, v)$ of $H$ with two indices $i_{u, v}$ and $i_{v, u}$ in $\{0,1,2\}$ in such a way that (a) for every $u$, the three neighbors $v$ of $u$ get distinct indices $i_{u, v}$, and (b) the graph with vertex set $\bigcup_{u \in V(H)} V^{u}$ and edge set

$$
\bigcup_{u \in V(H)} E^{u} \cup \bigcup_{(u, v) \in E(H)}\left(\left\langle u, i_{u, v}\right\rangle,\left\langle v, i_{v, u}\right\rangle\right)
$$



Figure 5: The portion of $\widehat{G}$ corresponding to edge $(u, v)$ of $H$. In the picture, we assume $i_{u, v}=0$ and $i_{v, u}=1$. Edges of $E$ are depicted as solid black lines and edges of $E^{\prime}$ are depicted as dashed red lines. Where two vertices have been identified, both the original labels have been displayed.
is planar. We will construct two copies, $Y^{u v}$ and $Y^{v u}$, of the $Y$-gadget for every edge ( $u, v$ ) of $H$. These correspond to the vertices of $G$ along the three-stretched edge $(u, v)$ of $H$. Thus, we have one $Y$ gadget for every vertex of $G$.

The vertex set $\widehat{V}$ is constructed from $\bigcup_{u \in V(H)} V^{u} \cup \bigcup_{(u, v) \in E(H)}\left(V^{u v} \cup V^{v u}\right)$ by identifying some vertices, In particular, for every edge $(u, v)$ of $E(H)$ with $u<v$, identify $\left\langle u, i_{u, v}\right\rangle$ with $\langle u v, 0\rangle$. Also, identify $\langle u v, 2\rangle$ with $\langle v u, 1\rangle$. Finally, identify $\langle v u, 0\rangle$ with $\left\langle v, i_{v, u}\right\rangle$. Note that $G$ has one vertex for each vertex of $H$ and two vertices for each edge of $H$ so $n=$ $|V(H)|+2|E(H)|$. Also $m=3|E(H)|$. So $|\widehat{V}|=6|V(H)|+12|E(H)|-3|E(H)|=6 n-m$.

Let $E=\bigcup_{u \in V(H)} E^{u} \cup \bigcup_{(u, v) \in E(H)}\left(E^{u v} \cup E^{v u}\right) . E$ is all of the internal edges in the $Y$ gadgets. So $|E|=6|V(H)|+12|E(H)|=6 n$. Let $E^{\prime}$ be the set of $m$ edges with weight $\beta$ constructed as follows. For each edge $(u, v)$ of $E(H)$ with $u<v$, let $i=i_{u, v}-1 \bmod 3$ and let $j=i_{v, u}-1 \bmod 3$. Add edges $(\langle u, i\rangle,\langle u v, 1\rangle),(\langle u v, 1\rangle,\langle v u, 2\rangle)$ and $(\langle v u, 1\rangle,\langle v, j\rangle)$ to $E^{\prime}$. Let $\widehat{E}=E \cup E^{\prime}$. Note that $\widehat{G}$ is planar.

The Tutte polynomial of $\widehat{G}$ is given by

$$
Z(\widehat{G} ; q, w)=\sum_{A \subseteq E} \sum_{B \subseteq E^{\prime}} w(A) w(B) q^{k(\widehat{V}, A \cup B)}
$$

where we have used the obvious fact $w(A \cup B)=w(A) w(B)$. We would like to go further and factor $\kappa(\widehat{V}, A \cup B)$, in a similar way, but we cannot do this directly because of the complex way that components in $(\widehat{V}, A)$ and $(\widehat{V}, B)$ may interact. To control this interaction, we partition sets $A \subseteq E$ according to the patterns of connectivities they induce within the various gadgets. Specifically, let $\Pi=\left(S, D_{0}, D_{1}, D_{2}, T\right)$ be a labelled partition of $V(G)$
into five sets $S, D_{0}, D_{1}, D_{2}$ and $T$, some of which could be empty. By "labelled" here, we mean that the five parts of the partition are distinguished by $\Pi$. In the following, it will help to think of $S$ as "singleton", $D$ as "doubleton" and $T$ as "triple". Let $\mathcal{A}_{\Pi}$ denote the set of subsets $A \subseteq E$ such that the following statements are true.

- For every $x \in S$, the vertices $\langle x, 0\rangle,\langle x, 1\rangle$ and $\langle x, 2\rangle$ are in a single component of ( $V^{x}, A \cap E^{x}$ ). Informally, all three vertices are connected within the gadget $Y^{x}$.
- For every $x \in D_{i}$, the vertex $\langle x, i\rangle$ is in one component of ( $V^{x}, A \cap E^{x}$ ) and the other two vertices $\{\langle x, 0\rangle,\langle x, 1\rangle,\langle x, 2\rangle\}-\langle x, i\rangle$ are in another.
- For every $x \in T$, the vertices $\langle x, 0\rangle,\langle x, 1\rangle$ and $\langle x, 2\rangle$ are in three distinct components of $\left(V^{x}, A \cap E^{x}\right)$.

For a labelled partition $\Pi$ of $V(G)$ as above, let $Z_{\Pi}$ be the contribution to $Z(\widehat{G} ; q, w)$ from edge sets $A \in \mathcal{A}_{\Pi}$, specifically

$$
\begin{equation*}
Z_{\Pi}=\sum_{A \in \mathcal{A}_{\Pi}} \sum_{B \subseteq E^{\prime}} w(A) w(B) q^{\kappa(\widehat{V}, A \cup B)} \tag{25}
\end{equation*}
$$

It is clear that $Z(\widehat{G} ; q, w)=\sum_{\Pi} Z_{\Pi}$, where $\Pi$ ranges over all labelled partitions $\Pi=$ ( $S, D_{0}, D_{1}, D_{2}, T$ ) of $V(G)$ into five parts. By constraining $A$ to come from a particular collection $\mathcal{A}_{\Pi I}$ it now becomes possible to factor $\kappa(\widehat{V}, A \cup B)$. To formalise this claim, let $V^{\prime}=\widehat{V} \backslash \bigcup_{x \in V(G)}\{\langle x, \overline{0}\rangle,\langle x, \overline{1}\rangle,\langle x, \overline{2}\rangle\}$, so that $\left|V^{\prime}\right|=|\widehat{V}|-3 n=3 n-m$, and let $\Gamma$ denote the graph $\Gamma=\left(V^{\prime}, E^{\prime}\right)$. Suppose $\Pi=\left\{S, D_{0}, D_{1}, D_{2}, T\right)$ is some labelled partition, and denote by $\Gamma_{\Pi}$ the graph obtained from $\Gamma$ by identifying certain vertices. Specifically, $\langle x, 0\rangle,\langle x, 1\rangle$ and $\langle x, 2\rangle$ are identified if $x \in S,\langle x, 1\rangle$ and $\langle x, 2\rangle$ are identified if $x \in D_{0}$ (and symmetrically for $D_{1}$ and $D_{2}$ ), and none of the vertices are identified if $x \in T$.

With a view to factorising $\kappa(\widehat{V}, A \cup B)$ into an $A$ - and a $B$-part, divide the connected components of $(\widehat{V}, A \cup B)$ into two kinds: those that contain no vertices in $V^{\prime}$ (and therefore are contained entirely within a single $Y^{x}$ ), and the others. For convenience, let $D=$ $D_{0} \cup D_{1} \cup D_{2}$. The the number of connected components of the first kind is just

$$
\begin{align*}
& \sum_{x \in S}\left(\kappa\left(V^{x}, A \cap E^{x}\right)-1\right)+\sum_{x \in D}\left(\kappa\left(V^{x}, A \cap E^{x}\right)-2\right)+\sum_{x \in T}\left(\kappa\left(V^{x}, A \cap E^{x}\right)-3\right) \\
& \quad=\sum_{x \in V} \kappa\left(V^{x}, A \cap E^{x}\right)-|S|-2|D|-3|T| . \tag{26}
\end{align*}
$$

We argue that the connected components of the second kind are in 1-1 correspondence with the connected components of $\left(V\left(\Gamma_{\Pi}\right), B\right)$. Suppose two vertices $\langle x, i\rangle$ and $\langle y, j\rangle$ are connected by a path in $(\widehat{V}, A \cup B)$; then that same path can be traced out in $\left(V\left(\Gamma_{\Pi}\right), B\right)$ just by omitting the $A$-edges. (Any pair of vertices joined by a sequence of $A$-edges will have been identified in the construction of $\Gamma_{\Pi}$.) Conversely, given a path in $\left(V\left(\Gamma_{\Pi}\right), B\right)$, we can recover a path in $(\widehat{V}, A \cup B)$ by interpolating $A$-edges. (We identify vertices in the
construction of $\Gamma_{\Pi}$ only if they are in the same $A$-component. Note that the "recovered" path may not be unique. We conclude that the number of connected components of the second type is $\kappa\left(V\left(\Gamma_{\Pi}\right), B\right)$. Combining this with the count (26) of connected components of the first type, we obtain

$$
\kappa(\widehat{V}, A \cup B)=\sum_{x \in V} \kappa\left(V^{x}, A \cap E^{x}\right)-|S|-2|D|-3|T|+\kappa\left(V\left(\Gamma_{\Pi}\right), B\right) .
$$

Substituting for $\kappa(\widehat{V}, A \cup B)$ in 25

$$
\begin{aligned}
Z_{\Pi} & =\sum_{A \in \mathcal{A}_{\Pi}} \sum_{B \subseteq E^{\prime}} w(A) w(B)\left(\prod_{x \in V} q^{k\left(V^{x}, A \cap E^{x}\right)}\right) q^{-|S|-2|D|-3|T|} q^{k\left(V\left(\Gamma_{\Pi}\right), B\right)} \\
& =q^{-|S|-2|D|-3|T|}\left(\sum_{A \in \mathcal{A}_{\Pi}} \prod_{x \in V} w\left(A \cap E^{x}\right) q^{k\left(V^{x}, A \cap E^{x}\right)}\right)\left(\sum_{B \subseteq E^{\prime}} w(B) q^{\kappa\left(V\left(\Gamma_{\Pi}\right), B\right)}\right) .
\end{aligned}
$$

This immediately leads to the key identity

$$
\begin{equation*}
Z_{\Pi}=q^{-|S|-2|D|-3|T|} Z_{012}^{|S|} Z_{0 \mid 12}^{|D|} Z_{0|1| 2}^{|T|} Z\left(\Gamma_{\Pi} ; q, \beta\right), \tag{27}
\end{equation*}
$$

where we recall that $\Pi=\left(S, D_{0}, D_{1}, D_{2}, T\right)$ and $D=D_{0} \cup D_{1} \cup D_{2}$. Note that this identity captures the sought-for factorisation of $Z_{\Pi}$ into a part that is internal to the gadgets, and an part that is external, namely $Z\left(\Gamma_{\Pi} ; q, \beta\right)$.

With an eye on (27), it is possible to give a short overview of the rest of the proof. Recall that $Z(\widehat{G} ; q, w)$ is the sum over partitions $\Pi=\left(S, D_{0}, D_{1}, D_{2}, T\right)$ of $Z_{\Pi}$. If $D=$ $D_{0} \cup D_{1} \cup D_{2} \neq \emptyset$, then $Z_{\Pi}$ is negligible because $\left|Z_{0 \mid 12}\right|$ is tiny. If $S$ is not an independent set in $G$, then $Z_{\Pi}$ is negligible because $\Gamma_{\Pi}$ has a loop, and hence $Z\left(\Gamma_{\Pi} ; q, \beta\right)$ is tiny. Finally, if $S$ is a maximum independent set, then $Z_{\Pi}$ dominates because $\left|Z_{012}\right|$ is much larger than $\left|Z_{0|1| 2}\right|$. So $Z(\widehat{G} ; q, w)$ is dominated by the contribution from maximum independent sets.

The rest of the proof is concerned with providing the estimates required to make the above proof sketch rigorous. The number of vertices in $\Gamma_{\Pi}$ is at most $3 n-m$ and the number of edges is $m$. Since $\max \{|\beta|, 1\} \leq 1+\delta \leq 2$ and $|q| \leq Q$, we have the following general upper bound on $Z\left(\Gamma_{\Pi} ; q, \beta\right)=\sum_{B \subseteq E^{\prime}} \beta^{|B|} q^{\kappa\left(V\left(\Gamma_{\Pi}\right), B\right)}$ :

$$
\begin{equation*}
\left|Z\left(\Gamma_{\Pi} ; q, \beta\right)\right| \leq 2^{m}(1+\delta)^{m} Q^{3 n-m} \leq 2^{2 m} Q^{3 n-m} \tag{28}
\end{equation*}
$$

If $\Gamma_{\Pi}$ has a loop we have the tighter bound

$$
\begin{equation*}
\left|Z\left(\Gamma_{\Pi} ; q, \beta\right)\right| \leq|1+\beta| 2^{2 m} Q^{3 n-m} \leq \delta 2^{2 m} Q^{3 n-m} \tag{29}
\end{equation*}
$$

This comes about because the loop contributes $\beta$ when it is included and 1 when it is excluded, but the number of connected components is the same in both cases. Recall the following general bounds on the other factors in (27) which follow from (21), (22) and (23):

$$
\begin{aligned}
\left|Z_{012}\right|,\left|Z_{0 \mid 12}\right|,\left|Z_{0|1| 2}\right| & \leq \tau \\
\left|Z_{0 \mid 12}\right| & \leq \delta \mu<\tau \\
\left|q^{-|S|-2|D|-3|T|}\right| & \leq Q^{3 n} .
\end{aligned}
$$

Following the proof sketch, first fix a partition $\Pi$ in which $D$ is non-empty. Then, from (27), (28) and the bounds just noted,

$$
\left|Z_{\Pi}\right| \leq Q^{3 n} \cdot \delta \mu \cdot \tau^{n-1} \cdot 2^{2 m} Q^{3 n-m} \leq 2^{2 m} \delta \mu \tau^{n-1} Q^{6 n}
$$

So we get the following upper bound on contributions in which $D \neq \emptyset$ :

$$
\begin{equation*}
\sum_{\Pi: D \neq \emptyset}\left|Z_{\Pi}\right| \leq 5^{n} \cdot 2^{2 m} \delta \mu \tau^{n-1} Q^{6 n} \leq \Psi / 16 \tag{30}
\end{equation*}
$$

To see that the final inequality in (30) holds, first use 24) to obtain $\Psi \geq R^{K}\left|Z_{0|1| 2}\right|^{n}|q|^{-3 n} \chi^{\nu}$. Then use 17 ) $\left(R^{K} \geq 1\right)$ and 22$)\left(\left|Z_{0|1| 2}\right| \geq L\right)$ to see that this is at least $L^{n}|q|^{-3 n} \chi^{\nu}$. Now divide the centre term in (30) by this lower bound for $\Psi$. Plug in the definition of $\delta$ and cancel the $\mu$ and $\tau$ in the numerator with $M$ in the denominator. The remaining terms cancel, and the result is at most $1 / 16$.

Next, fix a partition $\Pi$ in which $D$ is empty and $S$ is not an independent set of $G$. In this case, $\Gamma_{\Pi}$ has a loop, which arises from two adjacent gadgets being contracted. So from (27), (29) and the usual upper bounds,

$$
\left|Z_{\Pi}\right| \leq Q^{3 n} \cdot \tau^{n} \cdot \delta 2^{2 m} Q^{3 n-m} \leq 2^{2 m} \delta \tau^{n} Q^{6 n} .
$$

So we get the following upper bound on contributions in which $S$ is not an independent set:

$$
\begin{equation*}
\sum_{\substack{\Pi: D=\emptyset, \\ \text { It independent }}}\left|Z_{\Pi}\right| \leq 2^{n} \cdot 2^{2 m} \delta \tau^{n} Q^{6 n} \leq \Psi / 16 . \tag{31}
\end{equation*}
$$

The derivation of the final inequality in (31) is essentially the same as the derivation of (30). The only difference is that a $5^{n}$ there has been replaced with an (even smaller) $2^{n}$ here. Also, a $\mu$ has been replaced with a $\tau$ - this still cancels against an $M$ as before.

Finally, fix a partition $\Pi$ in which $D$ is empty and $S$ is an independent set of $G$ of size $k$. From identity (27),

$$
\begin{equation*}
Z_{\Pi}=q^{-k-3(n-k)} Z_{012}^{k} Z_{0|1| 2}^{n-k} Z\left(\Gamma_{\Pi} ; q, \beta\right)=\left(\frac{q^{2} Z_{012}}{Z_{0|1| 2}}\right)^{k} Z_{0| | \mid 2}^{n} q^{-3 n} Z\left(\Gamma_{\Pi} ; q, \beta\right) \tag{32}
\end{equation*}
$$

Thus, by (28),

$$
\left|Z_{\Pi}\right| \leq\left|\frac{q^{2} Z_{012}}{Z_{0|1| 2}}\right|^{k}\left|Z_{0|1| 2}\right|^{n}|q|^{-3 n} 2^{2 m} Q^{3 n-m}
$$

and by (24) and (17),

$$
\begin{equation*}
\sum_{\substack{\Pi: D=\emptyset,|S|<K, S \text { is independent }}}\left|Z_{\Pi}\right| \leq 2^{n} \cdot\left|\frac{q^{2} Z_{012}}{Z_{0|1| 2}}\right|^{K-1}\left|Z_{0|1| 2}\right|^{n}|q|^{-3 n} 2^{2 m} Q^{3 n-m} \leq \Psi / 16 \tag{33}
\end{equation*}
$$

To see that the final inequality in (33) holds, plug in the definition of $\Psi$, and, inside that, plug in the definition of $R$, and inside that, plug in the definition of $\varepsilon$. Everything cancels exactly.

Finally, we consider the situation $D=\emptyset, S$ is an independent set in $G$, and $|S|=K$. Fix a partition $\Pi$ for which these conditions hold. We are interested in obtaining a lower bound on $\left|Z_{\Pi}\right|$. Note that $\left|V\left(\Gamma_{\Pi}\right)\right|=\nu=3 n-m-2 K$.
Case 1: $q>5$. For a lower bound on $\left|Z\left(\Gamma_{\Pi} ; q, \beta\right)\right|$ and information about its sign, we use Woodall's Lemma 4. Since $\chi=(q-5) / 6$ and $\delta<\chi$, we have $q>5(1+\delta)+\chi$. Furthermore, $|1+\beta| \leq \delta$. Thus, Lemma 4 ensures that the sign of $Z\left(\Gamma_{\Pi} ; q, \beta\right)$ is the same for all $\Pi$ (it is always positive). Also, we have shown

$$
\begin{equation*}
\left|Z\left(\Gamma_{\Pi} ; q, \beta\right)\right| \geq \chi^{\nu} . \tag{34}
\end{equation*}
$$

Case 2: $q<0$. To determine the same facts for $q<0$ we use [5, Theorem 4.1]. Note that $\Gamma_{\Pi}$ has no loops. Let $C_{1}, \ldots, C_{\nu}$ denote the coefficients of $Z\left(\Gamma_{\Pi} ; q, \beta\right)$, viewed as a polynomial in $q$, so $Z\left(\Gamma_{\Pi} ; q, \beta\right)=\sum_{j=1}^{\nu} C_{j} q^{j}$. Let $\pi_{j}=1$ if $C_{j}>0, \pi_{j}=0$ if $C_{j}=0$ and $\pi_{j}=-1$ if $C_{j}<0$. Then

$$
Z\left(\Gamma_{\Pi} ; q, \beta\right)=(-1)^{\nu} \sum_{j=1}^{\nu}(-1)^{\nu-j} \pi_{j}\left|C_{j}\right||q|^{j} .
$$

Jackson and Sokal [5, Theorem 4.1] showed (assuming $\delta \leq 1$ which holds by inequality (19)) that $(-1)^{\nu-j} \pi_{j} \geq 0$. So

$$
Z\left(\Gamma_{\Pi} ; q, \beta\right)=(-1)^{\nu} \sum_{j \in\{1, \ldots, \nu\}, \pi_{j} \neq 0}\left|C_{j}\right||q|^{j} .
$$

Note that for $j=\nu, C_{j}=1$ so Equation (34) holds and the sign of $Z\left(\Gamma_{\Pi} ; q, \beta\right)$ is the same for all partitions $\Pi$ in which $D=\emptyset$ and $S$ is an independent set of size $K$ (the sign depends on the parity of $\nu$ ). This concludes Case $2 \xrightarrow{\text { ग }}$

Now, for a partition $\Pi$ in which $D$ is empty and $S$ is an independent set of $G$ of size $K$, Equations (32), (24) and (34) give

$$
\begin{equation*}
\left|Z_{\Pi}\right| \geq \Psi \tag{35}
\end{equation*}
$$

Since the sign of $Z\left(\Gamma_{\Pi} ; q, \beta\right)$ is the same for all $\Pi$ under consideration, it is apparent from (32) that the sign of $Z_{\Pi}$ depends only on the sign of $q$, the sign of $Z_{012}$, the sign of $Z_{0|1| 2}$ and the parity of $K$ and $n$. It does not depend on the set $S$.

So if $G$ has $N>0$ independent sets of size $K$ then by Equations (30), (31), (33) and (35), $|Z(\widehat{G} ; q, w)| \geq N \Psi-3 \Psi / 16 \geq 3 \Psi / 4$. On the other hand, if $G$ has no independent sets of size $K$ then the same equations give $|Z(\widehat{G} ; q, w)| \leq 3 \Psi / 16<\Psi / 4$. So if we could approximate $Z(\widehat{G} ; q, w)$ within a factor of $\frac{3}{2}$ then we could determine whether or not $G$ has an independent set of size $K$.

[^1]
### 3.5 The main result

Theorem 8. Suppose $(x, y) \in \mathbb{Q}^{2}$ satisfies $q=(x-1)(y-1) \notin[0,5]$. Suppose also that it is possible to shift the point $(x, y)$ to a point $\left(x_{1}, y_{1}\right)$ with $y_{1} \notin[-1,1]$ and to a point $\left(x_{2}, y_{2}\right)$ with $y_{2} \in(-1,1)$ and to a point $\left(x_{3}, y_{3}\right)$ with $y_{3}<0$. Then there is no FPRAS for $\operatorname{Tutte}(x, y)$ unless $\mathrm{RP}=\mathrm{NP}$.

Proof. This follows easily from Lemma 7 and is similar to the proof of [4, Theorem 2]. For completeness, here is a proof.

Let $\alpha=y-1$ and $\alpha_{i}=y_{i}-1$. Let $\Upsilon_{i}$ be a planar graph with distinguished vertices $s_{i}$ and $t_{i}$ that shifts $(x, y)$ to $\left(x_{i}, y_{i}\right)$. Note that $\Upsilon_{i}$ shifts $(q, \alpha)$ to $\left(q, \alpha_{i}\right)$.
$\operatorname{Suppose}(G, w)$ is an instance of $\operatorname{MultiTutte}\left(q ; \alpha_{1}, \alpha_{2}, \alpha_{3}\right)$ and note that $\alpha_{1}, \alpha_{2}$ and $\alpha_{3}$ satisfy the conditions of Lemma 7. Suppose that $G$ has $m_{i}$ edges with weight $\alpha_{i}$. Denote by $\widehat{G}$ the graph derived from $G$ by applying the above shifts - replacing each edge with weight $\alpha_{i}$ with a copy of $\Upsilon_{i}$ (using the distinguished vertices $s_{i}$ and $t_{i}$ ). Let $\widehat{w}$ be the constant weight function which assigns weight $\alpha$ to every edge in $\widehat{G}$. Then by Equation (4),

$$
Z(\widehat{G} ; q, \widehat{w})=\left(\frac{Z_{s \mid t}\left(\Upsilon_{1}\right)}{q^{2}}\right)^{m_{1}}\left(\frac{Z_{s \mid t}\left(\Upsilon_{2}\right)}{q^{2}}\right)^{m_{2}}\left(\frac{Z_{s \mid t}\left(\Upsilon_{3}\right)}{q^{2}}\right)^{m_{3}} Z(G ; q, w),
$$

so by Equation (2),

$$
(y-1)^{n}(x-1)^{\kappa} T(\widehat{G} ; x, y)=\left(\frac{Z_{s \mid t}\left(\Upsilon_{1}\right)}{q^{2}}\right)^{m_{1}}\left(\frac{Z_{s \mid t}\left(\Upsilon_{2}\right)}{q^{2}}\right)^{m_{2}}\left(\frac{Z_{s \mid t}\left(\Upsilon_{3}\right)}{q^{2}}\right)^{m_{3}} Z(G ; q, w),
$$

where $n$ is the number of vertices in $\widehat{G}$, and $\kappa$ is the number of connected components in $\widehat{G}$. Note that $Z_{s \mid t}\left(\Upsilon_{i}\right) \neq 0$ since $q Z_{s t}\left(\Upsilon_{i}\right) / Z_{s \mid t}\left(\Upsilon_{i}\right)=\alpha_{i}$. Thus an FPRAS for TUTTE $(x, y)$ would yield an FPRAS for the problem MultiTutte $\left(q ; \alpha_{1}, \alpha_{2}, \alpha_{3}\right)$, contrary to Lemma 7 .

The following corollary identifies regions where approximating $\operatorname{Tutte}(x, y)$ is intractable. It is illustrated in Figure 2

Corollary 9. Suppose $\mathrm{RP} \neq \mathrm{NP}$. Then there is no FPRAS for $\operatorname{PlanarTutte}(x, y)$ when $(x, y)$ is a point in the following regions, where $q$ denotes $(x-1)(y-1)$ :

1. $x<0, y<0$ and $q>5$;
2. $x>1, y<-1$;
3. $y>1, x<-1$.

Proof. We will show that for each point $(x, y)$ in the following regions, we can shift to points $\left(x_{1}, y_{1}\right),\left(x_{2}, y_{2}\right)$ and $\left(x_{3}, y_{3}\right)$ satisfying the conditions of Theorem 8 .

For the remaining cases, we use the fact that, when $G$ is a planar graph and $G^{*}$ is any plane dual of $G, T(G ; x, y)=T\left(G^{*} ; y, x\right)[13, \S 3.3 .7]$ (so the fact that there is no FPRAS at $(x, y)$ implies that there is no FPRAS at $(y, x)$ and vice-versa). The regions that we consider are as follows.

1. $x<-1, y<-1$ and $q>5$ : We can take $\left(x_{3}, y_{3}\right)$ and $\left(x_{1}, y_{1}\right)$ to be $(x, y)$ since $y<-1$. We can realise $\left(x_{2}, y_{2}\right)$ using a large, odd, $k$-stretch so $y_{2}=q /\left(x^{k}-1\right)+1$ which is in the range $(-1,1)$.
2. $-1 \leq x<0, y \leq-3 / 2$ and $q>5$ : Note that the condition $y \leq-3 / 2$ is implied by the bounds on $x$ and $q$. As above, we can take $\left(x_{3}, y_{3}\right)$ and $\left(x_{1}, y_{1}\right)$ to be $(x, y)$ since $y<-1$. Next, realise ( $x^{\prime}, y^{\prime}$ ) using a 2 -thickening so

$$
x^{\prime}=\frac{q}{y^{2}-1}+1>1 .
$$

Choose $j$ so that $x^{\prime j}>q /|x|$. Realise $\left(x_{2}, y_{2}\right)$ by $j$-stretching $x^{\prime}$ and combining this in series with $x$ so $x_{2}=x^{\prime j} x<-q$. Since $\left|x_{2}-1\right| \geq q$, we have $\left|y_{2}-1\right| \leq 1$. Also $y_{2}-1<0$ Thus $y_{2} \in(0,1)$.
3. $x>1, y<-1$ : Note that $q<0$. Again, we can take both $\left(x_{1}, y_{1}\right)$ and $\left(x_{3}, y_{3}\right)$ to be the point $(x, y)$ since that gives $y_{1}=y_{3}<-1$. We get to $\left(x_{2}, y_{2}\right)$ by a $j$-stretch, for sufficiently large $j$. This gives

$$
y_{2}=\frac{q}{x^{j}-1}+1 \in(-1,1) .
$$

### 3.6 The deferred proofs from Section 2.3

In this section we provide the proofs of Lemmas 1, 2 and 3. We start with some technical lemmas which show that we can use $\alpha_{1}, \alpha_{2}$ and $\alpha_{3}$ to implement a very close approximation to any target weight $T^{*}$, provided $T^{*} \notin[-2,0]$.

Lemma 10. Suppose $q>5$ and that $\alpha_{1} \notin[-2,0], \alpha_{2} \in(-2,0)$ and $\alpha_{3}<-1$. Suppose that $T^{-}$and $T^{+}$satisfy $0<T^{-} \leq T^{+}$. Given a target edge-weight $T^{*} \in\left[T^{-}, T^{+}\right]$and a positive value $\pi$ which is sufficiently small with respect to $q, \alpha_{1}, \alpha_{2}, \alpha_{3}, T^{-}$, and $T^{+}$, there is a planar graph $\Upsilon$ (depending on $T^{*}$ and $\pi$ ) and a weight function $\widehat{w}: E(\Upsilon) \rightarrow\left\{\alpha_{1}, \alpha_{2}, \alpha_{3}\right\}$ that implements a weight $w^{*}$ with $T^{*}-\pi \leq w^{*} \leq T^{*}$. The size of $\Upsilon$ is at most a polynomial in $\log \left(\pi^{-1}\right)$. (This upper bound on the size of $\Upsilon$ does not depend on $T^{*}$, though it does depend on the fixed bounds $T^{-}$and $T^{+}$.)

Proof. The weights that we have available for our implementations are $\alpha_{1}, \alpha_{2}$ and $\alpha_{3}$ and the target edge weight is $T^{*}$. It will be useful to use $(x, y)$ coordinates as well as $(q, \alpha)$ coordinates since series compositions power $x$ and parallel compositions power $y$. Recall that the relationship between the two coordinate systems is given by $q=(x-1)(y-1)$ and $\alpha=y-1$. Thus, we define

- $y_{i}^{\prime}=1+\alpha_{i}$ for $i \in\{1,2,3\}$,
- $T=1+T^{*}$, and
- $x_{i}^{\prime}=q /\left(y_{i}^{\prime}-1\right)+1$ for $i \in\{1,2,3\}$.
(The primes are just there because we use the notation $y_{i}$ for something else below.)
We will show how to use the values $y_{1}^{\prime}, y_{2}^{\prime}$ and $y_{3}^{\prime}$ to implement an edge-weight whose $y$-coordinate is between $T-\pi$ and $\pi$. (In fact, we won't use $y_{2}^{\prime}$ in the proof of this lemma, but we will use it in the proof of some of the related lemmas.) To do this efficiently (keeping the size of $\Upsilon$ at most a polynomial in $\log \left(\pi^{-1}\right)$ ) we need to be somewhat careful about decomposing $T$. Let $\left(x_{1}, y_{1}\right)$ be the point on the hyperbola $\left(x_{1}-1\right)\left(y_{1}-1\right)=q$ given by $y_{1}=y_{1}^{\prime 2}$. Note that $y_{1}>1$ so $x_{1}>1$ and that we we can implement $\left(x_{1}, y_{1}\right)$ by 2 -thickening from $\left(x_{1}^{\prime}, y_{1}^{\prime}\right)$.

Let

$$
y_{j}=\frac{q}{x_{1}^{j}-1}+1 .
$$

Let $x_{j}$ be the corresponding value so that $\left(x_{j}-1\right)\left(y_{j}-1\right)=q$. Note that, for every integer $j \geq 1$, we can implement $\left(x_{j}, y_{j}\right)$ by $j$-stretching from $\left(x_{1}, y_{1}\right)$. Also, since $x_{1}>1$, we have $y_{j}>1$ and $y_{j}>y_{j+1}$.

Now, for every integer $j \geq 1$, we recursively define a quantity $d_{j}$ in terms of the values of $d_{1}, \ldots, d_{j-1}$. In particular, we first find the largest power of $y_{1}$ not exceeding $T$, and divide $T$ by this power to obtain $d_{1}$; then we divide $d_{1}$ by the largest power of $y_{2}$ to obtain $d_{2}$, and so on. Formally,

$$
d_{j}=\left\lfloor\frac{\log \left(T \prod_{\ell=1}^{j-1} y_{\ell}-d_{\ell}\right)}{\log \left(y_{j}\right)}\right\rfloor .
$$

Let $y_{m}^{\prime \prime}=\prod_{\ell=1}^{m} y_{\ell}^{d_{\ell}}$. Note that $T / y_{m} \leq y_{m}^{\prime \prime} \leq T$. Also, since $d_{j}$ is a non-negative integer, we can implement $y_{m}^{\prime \prime}$ with a graph $\Upsilon_{m}$ by $d_{\ell}$-thickening $y_{\ell}$ (for $\ell \in\{1, \ldots, m\}$ ) and then combining these in parallel.

Let

$$
m=\left\lceil\frac{\log (q T / \pi+1)}{\log \left(x_{1}\right)}\right\rceil .
$$

Note that $y_{m} \leq 1+\pi / T \leq 1 /(1-\pi / T)$, so $1 / y_{m} \geq 1-\pi / T$. Let $y=y_{m}^{\prime \prime}$ and let $\Upsilon=\Upsilon_{m}$. Note that $T-\pi \leq y \leq T$, as required.

To see that this implementation is feasible, note that $m$ is not too large. In particular, for fixed $q, y_{1}^{\prime}, T^{-}$and $T^{+}, m$ is bounded from above by a polynomial in the logarithm of $\pi^{-1}$. To finish, we must show that the same is true of $d_{1}, \ldots, d_{m}$. Here, the key observation is that $y_{j}^{d_{j}} \leq T / y_{j-1}^{\prime \prime} \leq y_{j-1}$, so $d_{j} \leq \log \left(y_{j-1}\right) / \log \left(y_{j}\right)$. Then for $y_{j} \leq 5 / 4$, say, we have $\frac{3}{4}\left(y_{j}-1\right) \leq \log \left(y_{j}\right) \leq y_{j}-1$ which suffices.

Lemma 11. Suppose $q<0$ and that $\alpha_{1} \notin[-2,0], \alpha_{2} \in(-2,0)$ and $\alpha_{3}<-1$. Suppose that $T^{-}$and $T^{+}$satisfy $0<T^{-} \leq T^{+}$. Given a target edge-weight $T^{*} \in\left[T^{-}, T^{+}\right]$and a positive value $\pi$ which is sufficiently small with respect to $q, \alpha_{1}, \alpha_{2}, \alpha_{3}, T^{-}$, and $T^{+}$, there is a planar graph $\Upsilon\left(\right.$ depending on $T^{*}$ and $\pi$ ) and a weight function $\widehat{w}: E(\Upsilon) \rightarrow\left\{\alpha_{1}, \alpha_{2}, \alpha_{3}\right\}$ that implements a weight $w^{*}$ with $T^{*}-\pi \leq w^{*} \leq T^{*}$. The size of $\Upsilon$ is at most a polynomial in $\log \left(\pi^{-1}\right)$.

Proof. The situation is the same as that of Lemma 10 except that $q<0$. The proof is very similar to the proof of Lemma 10, and we use the notation from that proof. Here, we start by implementing a point $\left(x_{1}, y_{1}\right)$ with $x_{1}<-1$. Then we just use odd values of $j$ and it suffices to take

$$
m=\left\lceil\frac{\log (|q| T / \pi)}{\log \left(\left|x_{1}\right|\right)}\right\rceil \text {, }
$$

and to follow the proof of Lemma 10 .
The point $\left(x_{1}, y_{1}\right)$ is reached as follows. If $y_{1}^{\prime 2}<1+|q| / 2$ then we can take $\left(x_{1}, y_{1}\right)=$ $\left(q /\left(y_{1}^{\prime 2}-1\right)+1, y_{1}^{\prime 2}\right)$ since $x_{1}<-1$. Otherwise, proceed as follows. Let

$$
\xi=\frac{|q|}{2} \frac{1}{1+|q| / 2} .
$$

Choose a positive integer $j$ so that

$$
-\xi<\frac{q}{\left(q /\left(y_{2}^{\prime}-1\right)+1\right)^{j}-1}<0
$$

There is such a $j$ since $y_{2}^{\prime} \in(-1,1)$. Now let $(\hat{x}, \hat{y})$ be the $j$-stretch of $\left(x_{2}^{\prime}, y_{2}^{\prime}\right)$ so $1-\xi<$ $\hat{y}<1$. Now let

$$
k=1+\left\lfloor\frac{\log \left((1+|q| / 2) / y_{1}^{\prime 2}\right)}{\log (\hat{y})}\right\rfloor .
$$

Note that $k$ is a positive integer since $y_{1}^{\prime 2} \geq 1+|q| / 2$. Let $\left(x_{1}, y_{1}\right)$ be the parallel composition of $\left(q /\left(y_{1}^{\prime 2}-1\right)+1, y_{1}^{\prime 2}\right)$ with the $k$-thickening of $(\hat{x}, \hat{y})$. Thus, $y_{1}=\hat{y}^{k} y_{1}^{\prime 2}$. Note that $1<\hat{y}(1+|q| / 2) \leq y_{1}<1+|q| / 2$ so $x_{1}<-1$.

Lemma 12. Suppose $q \notin[0,5]$ and that $\alpha_{1} \notin[-2,0], \alpha_{2} \in(-2,0)$ and $\alpha_{3}<-1$. Suppose that $T^{-}$and $T^{+}$satisfy $2<T^{-} \leq T^{+}$. Given a target edge-weight $T^{*}$ with $-T^{*} \in\left[T^{-}, T^{+}\right]$ and a positive value $\pi$ which is sufficiently small with respect to $q, \alpha_{1}, \alpha_{2}, \alpha_{3}, T^{-}$, and $T^{+}$, there is a planar graph $\Upsilon$ (depending on $T^{*}$ and $\pi$ ) and a weight function $\widehat{w}: E(\Upsilon) \rightarrow$ $\left\{\alpha_{1}, \alpha_{2}, \alpha_{3}\right\}$ that implements a weight $w^{*}$ with $T^{*} \leq w^{*} \leq T^{*}+\pi$. The size of $\Upsilon$ is at most a polynomial in $\log \left(\pi^{-1}\right)$.

Proof. Once again, we use the notation from the proof of Lemma 10. The situation is the same as that of Lemmas 10 and 11 except that the target edge weight is negative (in fact, it is less than -2 ).

Choose an even positive integer $j$ so that $y_{3}^{\prime} y_{2}^{\prime j} \in(-1,0)$ and let $\hat{y}=-y_{3}^{\prime} y_{2}^{j}$ (so $\hat{y} \in(0,1))$. Recall that $T=T^{*}+1$. Now let $U^{*}=\frac{-T}{\hat{y}}-1$. Note that

$$
U^{*} \in\left[\frac{T^{-}-1}{\hat{y}}-1, \frac{T^{+}-1}{\hat{y}}-1\right]
$$

and that the lower bound $\left(T^{-}-1\right) / \hat{y}-1$ is positive. Using Lemma 10 or 11 (whichever is appropriate, depending on the sign of $q$ ) with target edge weight $U^{*}$ and error value $\pi / \hat{y}$,
implement an edge-weight whose $y$-coordinate $y^{\prime}$ satisfies

$$
\frac{-T}{\hat{y}}-\frac{\pi}{\hat{y}} \leq y^{\prime} \leq \frac{-T}{\hat{y}}
$$

Then take $y^{\prime}$ in parallel with $y_{3}^{\prime}$ and $j$ copies of $y_{2}^{\prime}$ to get a value $y=-y^{\prime} \hat{y}$ satisfying $T \leq y \leq T+\pi$.

We can now provide the proof of Lemmas 1, 2 and 3. For convenience, we re-state these lemmas here.

Lemma 1. Suppose $q \notin[0,5]$ and that $\alpha_{1} \notin[-2,0], \alpha_{2} \in(-2,0)$ and $\alpha_{3}<-1$. Given a positive constant $\varrho$ which is sufficiently small with respect to $q, \alpha_{1}, \alpha_{2}$, and $\alpha_{3}$, there is a planar graph $\Upsilon($ depending on $\varrho)$ and a weight function $\widehat{w}: E(\Upsilon) \rightarrow\left\{\alpha_{1}, \alpha_{2}, \alpha_{3}\right\}$ that implements a weight a, such that

$$
\begin{gather*}
A^{-} \leq|a| \leq A^{+}  \tag{5}\\
q+\varrho<f(a) \leq q+2 \varrho,  \tag{6}\\
|f(a)| \geq \eta \tag{7}
\end{gather*}
$$

The size of $\Upsilon$ is at most a polynomial in $\log \left(\varrho^{-1}\right)$.
Proof. First, suppose $q>5$. We start by noting that any $a$ that satisfies (6) also has $\frac{1}{2} \leq a \leq q$, so $A^{-} \leq|a| \leq A^{+}$and $|f(a)|>\eta$. So we just need to see how to implement a value of $a$ that satisfies (6). We will use Lemma 10 with the target value $T^{*}$ being the solution to the equation $f\left(T^{*}\right)=q+2 \varrho$ and the error value $\pi=\varrho^{2}$. As noted above, $T^{*} \in\left[A^{-}, A^{+}\right]$. To see that $f\left(T^{*}-\varrho^{2}\right)>q+\varrho$, note that

$$
f\left(T^{*}\right)-f\left(T^{*}-\varrho^{2}\right)=\left(6 T^{*}+3\left(T^{*}\right)^{2}\right) \varrho^{2}-\left(3+3 T^{*}\right) \varrho^{4}+\varrho^{6}
$$

This is at most $\varrho$, as required, as long as $\varrho$ is sufficiently small with respect to $T^{*}$ (which is in between $A^{-}$and $A^{+}$, which depend only on $q$ ). Since $\varrho$ is assumed to be sufficiently small with respect to $q$, this is the desired result.

Now suppose $q<0$. Start by noting that if $y$ satisfies $f(-3-y) \leq q+2 \varrho \leq q / 2$ then $y \geq \eta$. Also, if $y$ satisfies $f(-3-y) \geq q+\varrho \geq q$ then $y \leq y^{*}$. Thus, if $a=-3-y$ satisfies (6) then $-a \in\left[A^{-}, A^{+}\right]$. We will now argue that these conditions also imply $|f(a)|>\eta$. To see this, check that $f(-3-x / 2)+x / 2$ is negative for $x>0$. Therefore, taking $x=-q$, we get $f(-3+q / 2)<q / 2$. By the definition of $\eta$, we find that $q / 2<-\eta$. Thus, for $-a \in\left[A^{-}, A^{+}\right]$, we have $f(a) \leq q / 2<-\eta$, as required. So we just need to see how to find a value of $a$ that satisfies (6). This is now essentially the same as the $q>5$ case except that we use Lemma 12 .

Lemma 2. Suppose $q \notin[0,5]$ and that $\alpha_{1} \notin[-2,0], \alpha_{2} \in(-2,0)$ and $\alpha_{3}<-1$. Suppose, for a positive value $\varrho$, which is sufficiently small with respect to $q, \alpha_{1}, \alpha_{2}$, and $\alpha_{3}$, the value a satisfies inequalities (5), (6) and (7). Let

$$
\begin{equation*}
c=a^{2}+3 a+q \tag{8}
\end{equation*}
$$

Given a positive constant $\hat{\varrho}$ which is sufficiently small with respect to $q, \alpha_{1}, \alpha_{2}$, and $\alpha_{3}$, there is a planar graph $\Upsilon$ (depending on $\hat{\varrho}$ ) and a weight function $\widehat{w}: E(\Upsilon) \rightarrow\left\{\alpha_{1}, \alpha_{2}, \alpha_{3}\right\}$ that implements a weight b, such that

$$
\begin{gather*}
B^{-} \leq|b| \leq B^{+}, \text {and }  \tag{9}\\
-\hat{\varrho} \leq b+c \leq \hat{\varrho} . \tag{10}
\end{gather*}
$$

The size of $\Upsilon$ is at most a polynomial in $\log \left(\widehat{\varrho}^{-1}\right)$.
Proof. First, suppose $q>5$. Note that $B^{-} \leq c-1 \leq c-\hat{\varrho}$ and $c+\hat{\varrho} \leq c+1 \leq B^{+}$so it suffices to implement a value $b$ with $-c-\hat{\varrho} \leq b \leq-c+\hat{\varrho}$. So we use Lemma 12 choosing $\operatorname{target} T^{*}=-c$ and $\pi=\hat{\varrho}$. From our observation above, $-T^{*} \in\left[B^{-}+1, B^{+}-1\right]$.

Next, suppose $q<0$. By equation (6) (using the fact that $a<0$ ),

$$
\frac{q+2 \varrho}{a} \leq a^{2}+3 a \leq \frac{q+\varrho}{a}
$$

so since $a \leq-3$,
$-\left(\frac{4}{3}|q|+1\right) \leq \frac{q+2}{a}+q<\frac{q+2 \varrho}{a}+q \leq a^{2}+3 a+q \leq \frac{q+\varrho}{a}+q \leq q\left(1+\frac{1}{a}\right) \leq-\frac{2}{3}|q|<0$.
Thus, $-c-\hat{\varrho} \geq B^{-}$and $-c+\hat{\varrho} \leq B^{+}$. So it suffices to implement a value $b$ with $-c-\hat{\varrho} \leq b \leq-c+\hat{\varrho}$. For this, we just use the argument in Lemma 11 choosing target $T^{*}=-c$ and $\pi=\hat{\varrho}$.

Lemma 3 Suppose $q \notin[0,5]$ and that $\alpha_{2} \in(-2,0)$. Given a positive constant $\varrho$ which is sufficiently small with respect to $q$ and $\alpha_{2}$, there is a planar graph $\Upsilon$ (depending on $\varrho$ ) and a weight function $\widehat{w}: E(\Upsilon) \rightarrow\left\{\alpha_{2}\right\}$ that implements a weight $\beta$, such that $|1+\beta| \leq \varrho$. The size of $\Upsilon$ is at most a polynomial in $\log \left(\varrho^{-1}\right)$.

Proof. This is already done in [4]. To implement $\beta$, choose a positive integer $k$ such that $\left|\left(\alpha_{2}+1\right)^{k}\right|<\varrho$ then implement $\beta$ by $k$-thickening $\alpha_{2}$.

## 4 The lower branch of $q=3$

The following is NP-hard [3].
Name. Planar 3-Colouring.
Instance. A planar graph $G$.
Question. Does $G$ have a proper 3-colouring?
The following lemma gives hardness for approximating the Tutte polynomial on the lower branch of the $q=3$ hyperbola. See Figure 2.

Lemma 13. Suppose $\mathrm{RP} \neq \mathrm{NP}$. Then there is no $\operatorname{FPRAS}$ for $\operatorname{Tutte}(x, y)$ when $(x, y)$ satisfies $(x-1)(y-1)=3$ and $x, y<1$.

Proof. We will consider a point $(x, y)$ with $-1<y<1$. The remaining cases follow by symmetry between $x$ and $y$ as in the proof of Corollary 9. Let $G=(V, E)$ be an input to Planar 3-Colouring with $n$ vertices. For an even positive integer $k$, let $G^{k}$ be the graph formed from $G$ by $k$-thickening every edge and let $E^{k}$ be its edge set. It is well-known (see, for example, [4, Section 5.1]) that, assuming $(x-1)(y-1)=3$,

$$
T\left(G^{k} ; x, y\right)=(y-1)^{-n}(x-1)^{-\kappa\left(V, E^{k}\right)} \sum_{\sigma: V \rightarrow\{1,2,3\}} y^{\operatorname{mono}(\sigma)},
$$

where $\operatorname{mono}(\sigma)$ is the number of edges in $E^{k}$ that are monochromatic under the map $\sigma$. Note that $\operatorname{mono}(\sigma)$ is an even number, since $k$ is. Thus, $\sum_{\sigma: V \rightarrow\{1,2,3\}} y^{\operatorname{mono}(\sigma)}$ is a positive number which is at least 1 if $G$ has a proper 3 colouring and is at most $3^{n} y^{k}$ otherwise. Choosing

$$
k=\left\lceil\frac{\log \left(4 \cdot 3^{n}\right)}{\log (1 / y)}\right\rceil
$$

we have $3^{n} y^{k} \leq 1 / 4$, so a 2 -approximation to $T\left(G^{k} ; x, y\right)$ would enable us to determine whether or not $G$ is 3 -colourable.

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[^1]:    ${ }^{1}$ Establishing (34) is the main barrier to extending our result to $q \in[0,5]$. The Tutte polynomial with $\beta$ close to -1 is similar to the chromatic polynomial. Essentially, we are using the fact that this polynomial is non-zero with $\operatorname{sign}(-1)^{\nu}$ when $q<0$ and is positive when $q>5$. There are known to be many zeroes of chromatic polynomials in between 0 and 5 .

