Testing List H-Homomorphisms

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Abstract

Let H be an undirected graph. In the List H-Homomorphism Problem, given an undirected graph G with a list constraint $L(v) \subseteq V(H)$ for each variable $v \in V(G)$, the objective is to find a list H-homomorphism $f: V(G) \to V(H)$, that is, $f(v) \in L(v)$ for every $v \in V(G)$ and $(f(u), f(v)) \in E(H)$ whenever $(u, v) \in E(G)$.

We consider the following problem: given a map $f:V(G)\to V(H)$ as an oracle access, the objective is to decide with high probability whether f is a list H-homomorphism or far from any list H-homomorphisms. The efficiency of an algorithm is measured by the number of accesses to f.

In this paper, we classify graphs H with respect to the query complexity for testing list H-homomorphisms and show the following trichotomy holds: (i) List H-homomorphisms are testable with a constant number of queries if and only if H is a reflexive complete graph or an irreflexive complete bipartite graph. (ii) List H-homomorphisms are testable with a sublinear number of queries if and only if H is a bi-arc graph. (iii) Testing list H-homomorphisms requires a linear number of queries if H is not a bi-arc graph.

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1 Introduction

For two graphs G = (V(G), E(G)) and H = (V(H), E(H)), a map $f : V(G) \to V(H)$ is called a homomorphism from G to H if $(f(u), f(v)) \in E(H)$ whenever $(u, v) \in E(G)$. In the H-Homomorphism Problem (HOM(H)) for short), given an undirected graph G, the objective is to decide whether there exists a homomorphism from G to H. It is well known that HOM(H) is in \mathbf{P} if H is a bipartite graph and in \mathbf{NP} -Complete if H is not a bipartite graph [3, 8, 21].

List H-Homomorphism Problem (LHOM(H) for short) is a variant of H-Homomorphism Problem, in which we are also given a list $L(v) \subseteq V(H)$ for each vertex v in G. A map $f: V(G) \to V(H)$ is called a *list-homomorphism* from G to H if f is a homomorphism from G to H and $f(v) \in L(v)$ for every $v \in V(G)$. The objective is to decide whether there exists a list-homomorphism f from G to H. There are many results on the relationship between the graph H and the computational complexity of LHOM(H) [11, 12, 13, 14]. In particular, LHOM(H) is in \mathbf{P} iff H is a bi-arc graph [14].

In this paper, we consider testing list-homomorphisms. See [17, 27] for surveys on property testing. In our setting, a map f is given as an oracle access, i.e., the oracle returns f(v) if we specify a vertex $v \in V(G)$. A map f is called ϵ -far from list-homomorphisms if we must modify at least an ϵ -fraction of f to make f a list-homomorphism. An algorithm is called a tester for LHOM(H) if it accepts with probability at least 2/3 if f is a list-homomorphism from G to H and rejects with probability at least 2/3 if f is ϵ -far from list-homomorphisms. The efficiency of an algorithm is measured by the number of accesses to the oracle f. When we say that a query complexity is constant/sublinear/linear, it always means constant/sublinear/linear in |V(G)|, i.e., the domain size of f. We can assume that there exists a list-homomorphism from G to H. If otherwise, we can reject immediately without any query.

In this paper, we completely classify graphs H with respect to the query complexity for testing $\mathsf{LHOM}(H)$. Our result consists of the following two theorems.

Theorem 1.1. LHOM(H) is testable with a constant number of queries iff H is an irreflexive complete bipartite graph or a reflexive complete graph.

Theorem 1.2. LHOM(H) is testable with a sublinear number of queries iff H is a bi-arc graph.

The central question in the area of property testing is to classify properties into the following three categories: properties testable with a constant/sublinear/linear number of queries. Our result first establishes such a classification for a natural and general combinatorial problem.

We note that, from Theorem 1.2 and results given by [14], $\mathsf{LHOM}(H)$ is testable with a sublinear number of queries iff $\mathsf{LHOM}(H)$ is in **P**. However, it is not clear whether there is a computational class corresponding to properties testable with a constant number of queries.

To obtain our results, we exploit universal algebra, which is now a common tool to study computational complexity of constraint satisfaction problems (see, e.g., [24]). Another contribution of this paper is showing that universal algebraic approach is quite useful in the setting of property testing.

Related works: It is rare that we succeed to obtain characterizations of properties testable with a constant/sublinear number of queries. The only such a characterization we are aware of is one for graph properties in the *dense model* [18]. In this model, it is revealed that Szemerédi's regularity lemma [28] plays a crucial role [1]. Roughly speaking, the regularity lemma gives the constant-size sketch of a graph. It turns out that a property is testable in the dense model with a

constant number of queries iff the property is well-approximated by the union of constant number of sketches [1]. However, no characterization is known for properties testable with a sublinear number of queries. Similarly, for properties on Boolean functions, several partial classifications on constant-time testability are known [5, 25].

Let \mathcal{B} be a relational structure (see Section 2 for the definition). In Constraint Satisfaction Problem over \mathcal{B} (CSP(\mathcal{B}) for short), given another relational structure \mathcal{A} , the objective is to find a homomorphism from \mathcal{A} to \mathcal{B} . There have been a lot of research on classifying \mathcal{B} with respect to the computational complexity of CSP(\mathcal{B}) (see, e.g., [10, 22]). We can see that HOM(\mathcal{H}) and LHOM(\mathcal{H}) are special cases of CSP, and there are also many results classifying \mathcal{H} with respect to the computational complexity of HOM(\mathcal{H}) and LHOM(\mathcal{H}) (see, e.g., [20, 22]).

Testing homomorphisms on k-SAT is already studied [4, 16]. In k-SAT, given a CNF formula for which each clause consists of k literals, the objective is to find an assignment to variables so as to satisfy all the clauses. The problem k-SAT coincides with CSP(\mathcal{B}) for some appropriate relational structure \mathcal{B} . Testing homomorphisms for k-SAT can be restated as follows: Given an assignment, test whether the assignment is a satisfying assignment or far from any satisfying assignments. It is known that 2SAT is testable with $O(\sqrt{n})$ queries [16] and testing 3SAT requires $\Omega(n)$ queries [4], where n is the number of variables in an input CNF formula. We can see that our result extends those results to a large family of CSPs.

Given a relational structure \mathcal{B} , it is also natural to ask whether a relational structure \mathcal{A} has a homomorphism to \mathcal{B} or far from having homomorphisms. Here, \mathcal{A} is given as an oracle access. There are two major models in this setting, i.e., the *dense model* and the *bounded-degree model*. In the dense model, it is known that any CSP is testable in constant time [2]. In the bounded-degree model, it is known that Horn-SAT is testable in constant time [30, 29] and the number of queries needed to test 2SAT is $\widetilde{\Theta}(\sqrt{n})$ [19] where n is the number of variables in an input structure \mathcal{A} .

Organizations: In Section 2, we introduce definitions used throughout this paper. Sections 3 and 4 shows "if" and "only if" part of Theorem 1.1, respectively. Similarly, Sections 5 and 6 shows "if" and "only if" part of Theorem 1.2, respectively.

2 Preliminaries

Let $\boldsymbol{w}:A\to\mathbb{R}$ be a weight function such that $\sum_{a\in V}\boldsymbol{w}(a)=1$. Then, by $a\sim\boldsymbol{w}$, we mean that we pick an element $a\in A$ with probability $\boldsymbol{w}(a)$. For a function $f:A\to B$ and $A'\subseteq A$, we define $f|_{A'}:A'\to A$ as the function whose domain is restricted to A'. We also define $\boldsymbol{w}|_{A'}:A'\to\mathbb{R}$ as $\boldsymbol{w}|_{A'}(a)=\boldsymbol{w}(a)/\sum_{a'\in A'}\boldsymbol{w}(a')$. Note that $\boldsymbol{w}|_{A'}$ satisfies $\sum_{a'\in A'}\boldsymbol{w}|_{A'}(a')=1$.

Relational structures, polymorphisms and algebras: A vocabulary τ is a finite set of relational symbols; each symbol has an associated arity. A (finite) relational structure \mathcal{A} with vocabulary τ consists of a finite set A, its universe, and for every relational symbol $R \in \tau$ of arity n, an n-ary relation $R^{\mathcal{A}}$ on A, the interpretation of R by \mathcal{A} . A homomorphism of a structure \mathcal{A} to a structure \mathcal{B} with the same vocabulary τ is a mapping $\varphi : A \to B$ from the universe of \mathcal{A} to the universe of \mathcal{B} such that for each (n-ary) relational symbol $R \in \tau$ and any tuple $(a_1, \ldots, a_n) \in R^{\mathcal{A}}$ the tuple $(\varphi(a_1), \ldots, \varphi(a_n)) \in R^{\mathcal{B}}$. For relational structure \mathcal{B} , we define $\mathsf{HOM}(\mathcal{B})$ as the problem in which, given another relational structure \mathcal{A} , the objective is to find a homomorphism from \mathcal{A} to \mathcal{B} . We denote by $|\mathcal{A}|$ the size of universe of \mathcal{A} and denote by $|\mathcal{A}|$ the number of tuples in its

relations. For brevity, we use a capital letter to denote the universe of the corresponding relational structure (e.g., A denotes the universe of A).

Let R be a relation on a set A, An (n-ary) operation f on the same set is said to be a polymorphism of R if for any tuples $\mathbf{a}_1, \ldots, \mathbf{a}_n \in R$, the tuple $f(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ obtained by applying f component-wise also belongs to R. The relation R is called an *invariant* of f. An operation f is a polymorphism of a relational structure A if it is a polymorphism of each relation of the structure. The set of all polymorphism of A is denoted by $\mathsf{Pol}(A)$. From a collection C of operations $\mathsf{Inv}(C)$ denotes the set of invariants of all operations from C.

An algebra is a pair $\mathbb{A} = (A; F)$ consisting of a set A, the *universe* of \mathbb{A} , and a set F of operations, the *basic operations* of \mathbb{A} . Operations that can be obtained from the basic operations of \mathbb{A} and the projection operations on \mathbb{A} , that is operations of the form $f(x_1, \ldots, x_n) = x_i$, by means of compositions are called *term operations* of \mathbb{A} . Term(A) denotes the set of all term operations of \mathbb{A} . Any relational structure A can be associated with an algebra A = (A; F) corresponds to a class of relational structures A that includes all the structures A with universe A and A and A and A are A with universe A and A are A and A are A of A and A of A or A and A or A or A or A and A or A or

An operation f is called *idempotent* if $f(x_1, \ldots, x_k) \in \{x_1, \ldots, x_k\}$ for any x_1, \ldots, x_k where k is the arity of f. An algebra \mathbb{A} is called *idempotent* if any basic operation (thus, term operation) is idempotent.

Graph homomorphisms: We often identify a graph H = (V(H), E(H)) with a relational structure \mathcal{H} consisting of a universe V(H) and a binary relation E(H). In particular, $|\mathcal{H}| = |V(H)|$ and $||\mathcal{H}|| = |E(H)|$. Note that problem $\mathsf{HOM}(H)$ defined with graph terminologies and $\mathsf{HOM}(\mathcal{H})$ defined with relational structures coincide. Similarly, $\mathsf{LHOM}(H)$ can be restated with relational structures. Let \mathcal{H} be the relational structure associated with H and define \mathcal{H}^L as the relational structure obtained from \mathcal{H} by adding all unary relations $S^{\mathcal{H}} \subseteq V(H)$. Then, $\mathsf{LHOM}(\mathcal{H})$ coincides with $\mathsf{HOM}(\mathcal{H}^L)$. Let G be a graph with list constraints $\{L(v)\}_{v \in V(G)}$. Let G be the relational structure corresponding to G. Then, we can define another relational structure G^L obtained from G by adding unary relations $S^{\mathcal{G}} \subseteq V(G)$. Here, for each (unary) tuple $(v) \in S^{\mathcal{G}}$, there is a corresponding constraint $L(v) = S^{\mathcal{H}}$. Finally, we define $\mathbb{H}^L = \mathsf{Alg}(\mathcal{H}^L)$ for a graph H. Note that the algebra \mathbb{H}^L is idempotent since \mathcal{H}^L contains unary relations $S_v = \{(v)\}$ for every $v \in V(H)$, and an operation that has $\{S_v\}_{v \in V(H)}$ as invariants must be idempotent.

Testing homomorphisms: Let \mathcal{A} and \mathcal{B} be relational structures and $\mathbf{w}: A \to \mathbb{R}$ be a weight function with $\sum_{a \in A} \mathbf{w}(a) = 1$. The distance between two functions $f, f': A \to B$ is defined as $\operatorname{dist}(f, f') = \Pr_{a \sim \mathbf{w}}[f(a) \neq f'(a)]$. For a function $f: A \to B$, we define $\operatorname{dist}_{\mathcal{B}}(f) = \min_{f'} \operatorname{dist}(f, f')$ where f' is a homomorphism from \mathcal{A} to \mathcal{B} . We call a map f ϵ -far if $\operatorname{dist}_{\mathcal{B}}(f) \geq \epsilon$. For subset $A' \subseteq A$, we define $\operatorname{dist}_{\mathcal{B}}(f|_{A'})$ similarly using the weight function $\mathbf{w}|_{A'}$.

We consider testing homomorphisms to a relational structure \mathcal{B} . An input is $(\mathcal{A}, \boldsymbol{w}, f)$ where \mathcal{A} is a relational structure, $\boldsymbol{w}: A \to \mathbb{R}$ is a weight function with $\sum_{a \in A} \boldsymbol{w}(a) = 1$, and $f: A \to B$ is a map. The map f is given as an oracle access. Thus, by specifying $a \in A$, the oracle returns the value of f(a).

Definition 2.1. An algorithm is called a tester for $HOM(\mathcal{B})$ if, given an input $(\mathcal{A}, \mathbf{w}, f)$, it accepts with probability at least 2/3 when f is a homomorphism from \mathcal{A} to \mathcal{B} , and rejects with probability at least 2/3 when f is ϵ -far from homomorphisms.

A tester is called a *one-sided error tester* if it always accepts an input $(\mathcal{A}, \boldsymbol{w}, f)$ if f is a homomorphism. An algorithm is called having query complexity $q(n, m, \epsilon)$ if, given an input $(\mathcal{A}, \boldsymbol{w}, f)$, it queries at most $q(|\mathcal{A}|, ||\mathcal{A}||, \epsilon)$ times. We always assume that $q(n, m, \epsilon)$ is an increasing function on n, m and a decreasing function on ϵ .

We can similarly define testers and testability for $\mathsf{HOM}(H)$ and $\mathsf{LHOM}(H)$ for a graph H since they have equivalent formalizations using relational structures. For convenience, we write an input for $\mathsf{HOM}(H)$ and $\mathsf{LHOM}(H)$ as (G, \boldsymbol{w}, f) and $(G, L, \boldsymbol{w}, f)$, respectively, where G is a graph, $\{L(v)\}_{v\in V(G)}$ is a set of list constraints, \boldsymbol{w} is a weight function, and f is a map given as an oracle access.

We finally define bi-arc graphs for reference though we do not use the definition in our proof. Let C be a circle with two specified points p and q. A bi-arc is a pair of arcs (N, S) such that N contains p but not q and S contains q but not p. A graph H = (V, E) is a bi-arc graph if there is a family of bi-arcs $\{(N_x, S_x) \mid x \in V\}$ such that, for every $x, y \in V$, the following holds: (i) if x and y are adjacent, then neither N_x intersects S_y nor N_y intersects S_x , and (ii) if x is not adjacent to y then both N_x intersects S_y and N_y intersects S_x . An undirected graph is called reflexive if every vertex has a loop and irreflexive if no vertex has loop. It is known that a reflexive graph is bi-arc iff it is an interval graph, and that an irreflexive graph is bi-arc iff it is bipartite and its complement is an circular-arc graph [14].

3 Graphs Testable with a Constant Number of Queries

In this section, we show the following lemma, which is the "if" part of Theorem 1.1.

Lemma 3.1. When H is an irreflexive complete bipartite graph or a reflexive complete graph, then there exists a one-sided error tester for LHOM(H) with query complexity $O(1/\epsilon^2)$.

Let $(G, L, \boldsymbol{w}, f)$ be an input for LHOM(H). For a vertex $v \in V(G)$, let C(v) be the connected component containing v.

Proposition 3.2. Let
$$(G, L, \boldsymbol{w}, f)$$
 be an input for $LHOM(H)$. Suppose that $\operatorname{dist}_H(f) \geq \epsilon$. Then, $\underset{v \sim \boldsymbol{w}}{\mathbf{E}} [\operatorname{dist}_H(f|_{C(v)})] \geq \epsilon$.

Proof. Let
$$\overline{f}$$
 be a list-homomorphism closest to f and $\epsilon_v = \operatorname{dist}(f|_{C(v)}, \overline{f}|_{C(v)})$. It is clear that $\epsilon_v = \operatorname{dist}_H(f|_{C(v)})$. Since $\operatorname{dist}_H(f) \geq \epsilon$, we have $\mathbf{E}_{v \sim w}[\operatorname{dist}_H(f|_{C(v)})] = \mathbf{E}_{v \sim w}[\epsilon_v] \geq \epsilon$.

Corollary 3.3. Suppose that there exists a one-sided error tester A for LHOM(H) with query complexity $q(\epsilon)$ if an input graph is restricted to be connected. Then, there exists a one-sided error tester A' for LHOM(H) with query complexity $q(\epsilon)/\epsilon$ for any input graph.

Proof. Let $(G, L, \boldsymbol{w}, f)$ be an input. Our algorithm \mathcal{A}' is as follows: Let S be a set of $\Theta(1/\epsilon)$ vertices chosen from G according to \boldsymbol{w} . For each vertex $v \in S$, we run \mathcal{A} with an error parameter ϵ on the input whose domain is restricted to C(v). We reject if \mathcal{A} rejects for some $v \in S$. We accept otherwise.

It is easy to see that the above algorithm always accepts when f is a list-homomorphism, and the query complexity is at most $q(\epsilon)/\epsilon$.

Suppose that $\operatorname{dist}_H(f) \geq \epsilon$. From Proposition 3.2, we have $\underset{v \sim \boldsymbol{w}}{\mathbf{E}}[\operatorname{dist}_H(f|_{C(v)})] \geq \epsilon$. Thus, $\Pr_{v \sim \boldsymbol{w}}[\operatorname{dist}_H(f|_{C(v)}) \geq \epsilon] \geq \epsilon$ holds. For a vertex v with $\operatorname{dist}_H(f|_{C(v)}) \geq \epsilon$, the algorithm \mathcal{A} rejects with probability at least 2/3. Thus, \mathcal{A}' rejects with probability at least $1 - (1 - 2/3 \cdot \epsilon)^{\Theta(1/\epsilon)} \geq 2/3$. \square

Lemma 3.4. Suppose that there exists a one-sided error tester \mathcal{A} for LHOM(H) with query complexity $q(\epsilon)$ if an input map is restricted to satisfy list constraints. Then, there also exists a one-sided error tester \mathcal{A}' for LHOM(H) with query complexity $O(1/\epsilon) + q(\epsilon/2)$ for any input.

Proof. Let (G, L, \mathbf{w}, f) be an input. We define $f': G \to H$ as

$$f'(v) = \begin{cases} f(v) & \text{if } f(v) \in L(v) \\ \text{any element in } L(v) & \text{otherwise} \end{cases}$$

Note that we can compute f'(v) by querying f once, i.e., f(v).

Our algorithm \mathcal{A}' is as follows: Let S be a set of $\Theta(1/\epsilon)$ vertices chosen from G according to \boldsymbol{w} . We reject if $f(v) \notin L(v)$ for some $v \in S$. If otherwise, we simply return the output by \mathcal{A} running on f' with an error parameter $\epsilon/2$.

It is easy to see that the above test always accepts when f is a list-homomorphism, and the query complexity is at most $O(1/\epsilon) + q(\epsilon/2)$.

Suppose that $\operatorname{dist}_H(f) \geq \epsilon$. If $\operatorname{dist}(f, f') \geq \epsilon/2$, then \mathcal{A}' rejects with probability at least $1 - (1 - \epsilon/2)^{\Theta(1/\epsilon)} \geq 2/3$ when checking $f(v) \in L(v)$ for $v \in S$. If $\operatorname{dist}(f, f') < \epsilon/2$, from the triangle inequality, we have $\operatorname{dist}_H(f') \geq \epsilon/2$. Thus, \mathcal{A}' rejects with probability at least 2/3.

Lemma 3.5. Let H be a reflexive complete graph. Then, there exists a one-sided error tester for LHOM(H) with query complexity $O(1/\epsilon)$.

Proof. Let (G, L, \mathbf{w}, f) be an input and assume that f satisfies list constraints. Then, we can always accept since any map is a list-homomorphism to H. The lemma follows from Lemma 3.4.

Lemma 3.6. Let K_2 be an irreflexive complete graph with two vertices. There exists a one-sided error tester \mathcal{A} for LHOM (K_2) with query complexity $O(1/\epsilon^2)$.

Proof. Let (G, L, \mathbf{w}, f) be an input and assume that G is connected and f satisfies list constraints. Note that G must be bipartite in order to have a homomorphism to H, and let $V_1 \cup V_2$ be the bipartition of G. Let a, b be two vertices in K_2 . Then, we have two homomorphisms, i.e., f^1 and f^2 where $f^1|_{V_1} \equiv a, f^1|_{V_2} \equiv b$, and $f^2|_{V_1} \equiv b, f^2|_{V_2} \equiv a$.

Our algorithm \mathcal{A} is as follows: Let S_1 (resp., S_2) be a set of $\Theta(1/\epsilon)$ vertices chosen from V_1 (resp., V_2) according to $\boldsymbol{w}|_{V_1}$ (resp., $\boldsymbol{w}|_{V_2}$). Then, we check f(u) = f(v) for every $u, v \in S_1$, f(u) = f(v) for every $u, v \in S_2$, and $f(u) \neq f(v)$ for every $u \in S_1$, $v \in S_2$. We reject if any of them do not hold. We accept otherwise.

It is easy to see that the above test always accepts when f is a list-homomorphism, and the query complexity is $O(1/\epsilon)$.

Suppose that $\operatorname{dist}_H(f) \geq \epsilon$. Note that f^1 or f^2 must satisfy list constraints. We assume below that both f^1 and f^2 satisfy list constraints. The analysis is similar when either of them does not satisfy list constraints.

Since f is ϵ -far from f^1 , we have $\sum_{v \in V_1, f(v) = b} \boldsymbol{w}(v) \ge \epsilon/2$ or $\sum_{v \in V_2, f(v) = a} \boldsymbol{w}(v) \ge \epsilon/2$. Similarly, we have $\sum_{v \in V_1, f(v) = a} \boldsymbol{w}(v) \ge \epsilon/2$ or $\sum_{v \in V_2, f(v) = b} \boldsymbol{w}(v) \ge \epsilon/2$. In any case, the probability that we reject is at least $1 - 2(1 - \epsilon/2)^{\Theta(1/\epsilon)} \ge 2/3$.

From Corollary 3.3 and Lemma 3.4, LHOM (K_2) is testable with $O(1/\epsilon^2)$ queries.

Now, we show that any complete bipartite graph is testable with a constant number of queries. For two graphs G and H, we call a map $f:V(G)\to V(H)$ a full-homomorphism if $(u,v)\in E(G)$ iff $(f(u),f(v))\in E(H)$. The difference from a homomorphism is that we must have $(f(u),f(v))\not\in E(H)$ when $(u,v)\not\in E(G)$.

Proposition 3.7. Let h be a full-homomorphism from H to H'. For a graph G and a homomorphism f' from G to H', let $f: V(G) \to V(H)$ be a map such that f(v) is any element in $h^{-1}(f'(v))$. Then, f is also a homomorphism from G to H.

Proof. Suppose that f is not a homomorphism. Then, there exist $u, v \in V(G)$ such that $(u, v) \in E(G)$ while $(f(u), f(v)) \notin E(H)$. Then, we have $(f'(u), f'(v)) = (h(f(u)), h(f(v))) \notin E(H')$ since h is a full-homomorphism, which is contradicting the fact that f' is a homomorphism.

Lemma 3.8. Let H be a graph and suppose that there exists a full-homomorphism h from H to H'. If there exists a one-sided error tester for $\mathsf{LHOM}(H')$ with query complexity $q(\epsilon)$, then there exists a one-sided error tester for $\mathsf{LHOM}(H)$ with query complexity $O(1/\epsilon) + q(\epsilon/2)$.

Proof. Let $(G, L, \boldsymbol{w}, f)$ be an input and assume that f satisfies list constraints. We define $L' = h \circ L$ and $f' = h \circ f$. Then, we run the tester for $\mathsf{LHOM}(H')$ on an input $(G, L', \boldsymbol{w}, f')$ with an error parameter ϵ .

If f is a list-homomorphism, then f' is also a list-homomorphism, and the tester always accepts. Suppose that $\operatorname{dist}_H(f) \geq \epsilon$ and let $\overline{f'}$ be the list-homomorphism closest to f'. We define $\widetilde{f}: V(G) \to V(H)$ as follows.

$$\widetilde{f}(v) = \begin{cases} f(v) & \text{if } \overline{f'}(v) = f'(v), \\ \text{any element in } L(v) \cap h^{-1}(\overline{f'}(v)) & \text{otherwise.} \end{cases}$$

Note that, in the latter case, $L(v) \cap h^{-1}(\overline{f'}(v))$ is not empty since $\overline{f'}(v) \in L'(v) = h(L(v))$. Thus, \widetilde{f} is well-defined. We can easily see that \widetilde{f} satisfies list constraints from the construction. When $\overline{f'}(v) = f'(v)$, we have $f(v) \in h^{-1}(\overline{f'}(v))$. Thus, \widetilde{f} is a list-homomorphism from Proposition 3.7. It means that $\operatorname{dist}(f', \overline{f'}) \geq \operatorname{dist}(f, \widetilde{f}) \geq \operatorname{d$

We have shown that $\mathsf{LHOM}(H)$ is testable with $q(\epsilon)$ queries when an input is restricted to satisfy list constraints. The lemma follows from Lemma 3.4.

Lemma 3.9. Let H be an irreflexive complete bipartite graph. Then, there exists a one-sided error tester for LHOM(H) with query complexity $O(1/\epsilon^2)$.

Proof. Since there exists a full-homomorphism from H to K_2 , LHOM(H) is testable with $O(1/\epsilon^2)$ queries from Lemmas 3.6 and 3.8.

Proof of Lemma 3.1. The claim immediately follows from Lemmas 3.5 and 3.9. \Box

4 Graphs Not Testable with a Constant Number of Queries

In this section, we show the following lemma, which is the "only if" part of Theorem 1.1.

Lemma 4.1. If H is neither an irreflexive complete bipartite graph nor a reflexive complete graph, testing LHOM(H) requires $\Omega(\frac{\log n}{\log\log n})$ queries.

The following is immediate since we can freely restrict the range of a map by list constraints.

Proposition 4.2. Let H be a graph and H' be an induced subgraph of H. If testing LHOM(H') requires q queries, then testing LHOM(H) also requires q queries.

The following lemma is implicitly stated in [16] when showing lower bounds for testing 2SAT.

Lemma 4.3 ([16]). Let H be a graph and G be a graph with list constraints $\{L(v)\}_{v \in V(G)}$. For vertices $u, v \in V(G)$ with $u \neq v$, let $R = \{(f(u), f(v)) \mid f \text{ is a list-homomorphism from } G \text{ to } H\}$. If $R = \{(a, c), (b, c), (b, d)\}$ for vertices $a, b, c, d \in H, a \neq b, c \neq d$, then testing LHOM(H) requires $\Omega(\frac{\log n}{\log \log n})$ queries.

We also use the following lemma, which is a special case of Lemma 6.1 (see Section 6).

Lemma 4.4. Let K_3 be an irreflexive complete graph with three vertices. Then, testing LHOM (K_3) requires $\Omega(n)$ queries.

Note that K_3 is not a bi-arc graph.

The following two lemmas deal with reflexive graphs and irreflexive graphs, respectively.

Lemma 4.5. Let H be a reflexive graph. If H is not a complete graph, testing LHOM(H) requires $\Omega(\frac{\log n}{\log \log n})$ queries.

Proof. Let $P = (\{a,b,c\}; \{(a,a),(b,b),(c,c),(a,b),(b,c)\})$ be a reflexive path of length 2 and $G = (\{u,v\}; \{(u,v)\})$ be an irreflexive edge with list constraints $L(u) = \{a,b\}$ and $L(v) = \{b,c\}$. It is easy to see that the relation R in Lemma 4.3 becomes $R = \{(a,b),(b,b),(b,c)\}$. It follows that testing LHOM(P) requires $\Omega(\frac{\log n}{\log \log n})$ queries. From Proposition 4.2, if LHOM(H) is testable with a constant number of queries, H must have a diameter 1, implying that H is a reflexive complete graph.

Lemma 4.6. Let H be an irreflexive graph. If H is not a complete bipartite graph, testing LHOM(H) requires $\Omega(\frac{\log n}{\log\log n})$ queries.

Proof. Let $P = (\{a, b, c, d\}; \{(a, b), (b, c), (c, d)\})$ be an irreflexive path of length 3 and $G = (\{u, v\}; \{(u, v)\})$ be an irreflexive edge with list constraints $L(u) = \{a, c\}$ and $L(v) = \{b, d\}$. It is easy to see that the relation R in Lemma 4.3 becomes $R = \{(a, b), (c, b), (c, d)\}$. It follows that testing LHOM(P) requires $\Omega(\frac{\log n}{\log \log n})$ queries. Also, for an irreflexive triangle T, testing LHOM(T) requires $\Omega(n)$ queries from Lemma 4.4.

Thus, if $\mathsf{LHOM}(H)$ is testable with a constant number of queries, H must not have a path of length 3 or a triangle as induced subgraphs from Proposition 4.2. Thus, H must be a complete bipartite graph.

Proof of Lemma 4.1. Let $P = (\{a,b\}; \{(a,b),(b,b)\})$ be an edge with a loop and $G = (\{u,v\}; \{(u,v)\})$ be an irreflexive edge with list constraints $L(u) = L(v) = \{a,b\}$. Then, the relation R in Lemma 4.3 becomes $\{(a,b),(b,b),(b,a)\}$, and it follows that testing LHOM(P) requires $\Omega(\frac{\log n}{\log \log n})$ queries. Thus, if LHOM(H) is testable with a constant number of queries, H must be reflexive or irreflexive from Proposition 4.2. Then, the lemma follows from Lemmas 4.5 and 4.6.

5 Graphs Testable with a Sublinear Number of Queries

In this section, we show the following lemma, which is the "if" part of Theorem 1.2.

Lemma 5.1. Let H be a bi-arc graph. Then, there exists a one-sided error tester for LHOM(H) with query complexity $O(\sqrt{n/\epsilon})$.

We first describe a propagation algorithm to solve CSPs. Let \mathcal{A} and \mathcal{B} be two relational structures. To check whether there exists a homomorphism f from \mathcal{A} to \mathcal{B} , we can use the following algorithm. Let k, ℓ be integers with $k \leq \ell$. For each subset $U \subseteq A, |U| \leq \ell$, we keep track of a set \mathcal{S}_U of tuples corresponding to maps from $A|_U$ to B. First, we initialize \mathcal{S}_U to the set of solutions to the partial instance $\mathcal{A}|_U$. Then, for each subsets $U, U' \subseteq A$ with $|U \cap U'| \leq k$, we eliminate tuples \mathbf{a} in \mathcal{S}_U if $\mathbf{a}|_{U'}$ is not contained in $\mathcal{S}_{U'}|_U$. We continue this process until no update occurs. This propagation algorithm is called (k,ℓ) -Minimality [9]. If \mathcal{S}_U becomes empty for some $U \subseteq A$, we can conclude that \mathcal{A} has no homomorphism to \mathcal{B} . Even if no \mathcal{S}_U is empty when propagation stops, \mathcal{A} may not have a homomorphism to \mathcal{B} . If \mathcal{A} has a homomorphism to \mathcal{B} in such a case, then \mathcal{B} is called having width (k,ℓ) .

A ternary operation $f: B^3 \to B$ is called a majority if f(x, x, y) = f(x, y, x) = f(y, x, x) = x for $x, y \in B$. It is known that a relational structure \mathcal{B} such that the associated algebra $\mathsf{Alg}(\mathcal{B})$ admits a majority operation has width (2,3) [24]. We say a map $f: A \to B \cup \{\bot\}$ is extendable to a homomorphism if there exists a homomorphism $f': A \to B$ such that f'(v) = f(v) whenever $f(v) \in B$. We call a vertex v a violating vertex if $f(v) \notin \mathcal{S}_{\{v\}}$ and a pair of vertices (v, u) a violating pair if $(f(v), f(u)) \notin \mathcal{S}_{\{v,u\}}$. It is known that the majority operation implies the following property.

Lemma 5.2 (2-Helly property, [15]). Let \mathcal{B} be a relational structure such that $Alg(\mathcal{B})$ admits a majority operation. For a relational structure \mathcal{A} and $U \subseteq A, |U| \leq 3$, let \mathcal{S}_U be the set of tuples obtained by (2,3)-Minimality running on \mathcal{A} . If a map $f: A \to B \cup \{\bot\}$ is not extendable to a homomorphism, then there is a violating vertex v or a violating pair (v,u) for some $u,v \in f^{-1}(B)$.

Lemma 5.3 ([11]). Let H be a bi-arc graph. Then, \mathbb{H}^L admits a majority operation.

Proof of Lemma 5.1. Let $(G, L, \boldsymbol{w}, f)$ be an input for LHOM(H). From Lemmas 5.2 and 5.3, we can assume the 2-Helly property. Our algorithm is described below.

Algorithm 1 LHOM(H) tester for a bi-arc graph H

- 1: Run (2,3)-Minimality and let S_U be the set of tuples obtained for $U \subseteq V(G), |U| \leq 3$.
- 2: Let X be a set of $\Theta(1/\epsilon)$ vertices chosen according to \boldsymbol{w}
- 3: if $\exists v \in X$ such that $f(v) \notin \mathcal{S}_{\{v\}}$ then
- 4: Reject the input.
- 5: Let Y_1, Y_2 be sets of $\Theta(\sqrt{n/\epsilon})$ vertices chosen according to \boldsymbol{w} .
- 6: if $\exists v \in Y_1, u \in Y_2$ such that $(f(v), f(u)) \notin \mathcal{S}_{\{v,u\}}$ then
- 7: Reject the input.
- 8: Accept the input.

Note that (2,3)-Minimality updates S_U using not only the graph G but also the list constraint L. The query complexity is clearly $O(\sqrt{n/\epsilon})$. It is clear that the tester always accepts when f is a list-homomorphism.

Suppose that $\operatorname{dist}_H(f) \geq \epsilon$. Let U be a subset of V(G) with maximum weight such that the partial homomorphism $f|_U$ is extendable to a list-homomorphism. Clearly, we have $\boldsymbol{w}(U) + \epsilon \leq 1$. Let v be a vertex in $V(G) \setminus U$. From the maximality of U, we cannot extend $f|_{U \cup \{v\}}$ to a list-homomorphism. Thus, from the 2-Helly property, v is a violating vertex or (v,u) is a violating pair for some $u \in U$. Let A be the set of violating vertices in $V(G) \setminus U$, and $B = V(G) \setminus (U \cup A)$. Note that for any $v \in B$, a pair (v,u) is a violating pair for some $u \in U$. Since $A \cup B = V(G) \setminus U$, we have $\boldsymbol{w}(A) + \boldsymbol{w}(B) \geq 1 - \boldsymbol{w}(U) \geq \epsilon$.

When $w(A) \ge \epsilon/2$, we reject at Line 4 with probability at least $1 - (1 - \epsilon/2)^{\Theta(1/\epsilon)} \ge 2/3$.

Suppose that $w(B) \geq \epsilon/2$. For a subset $B' \subseteq B$, we define $N(B') = \{u \in U \mid \exists v \in B', (v, u) \text{ is a violating pair}\}$. Note that $f|_{U'}$ is extendable to a list-homomorphism where $U' = (U \setminus N(B')) \cup B'$. Thus, we must have $w(B') \leq w(N(B'))$ from the maximality of U.

Let $q=|Y_1|=|Y_2|$ and c>0 be a parameter. Let $B'=Y_1\cap B$ and F_1 be the event that $\boldsymbol{w}(B')\leq \frac{\epsilon q}{cn}$. By choosing c large enough, we have $\Pr[F_1]\leq \frac{1}{10}$ from Chernoff's bound. Note that $\boldsymbol{w}(N(B'))\geq \boldsymbol{w}(B')\geq \frac{\epsilon q}{cn}$. Let F_2 be the event that no violated edge is detected. Then, $\Pr[F_2]\leq \Pr[F_1]+\Pr[F_2\mid \overline{F_1}]\leq \frac{1}{10}+(1-\frac{\epsilon q}{cn})^q\leq \frac{1}{3}$ by choosing the hidden constant in $q=\Theta(\sqrt{n/\epsilon})$ large enough.

Note that we only use the fact that \mathbb{H}^L admits a majority operation. Thus, our algorithm can be also used to testing homomorphisms to other CSPs admitting majority operations, e.g., 2SAT.

6 Graphs Not Testable with a Sublinear Number of Queries

In this section, we prove the following, which is "only if" part of Theorem 1.2.

Lemma 6.1. If a graph H is not a bi-arc graph, then testing LHOM(H) requires $\Omega(n)$ queries.

To prove Lemma 6.1, we make use of a sequence of reductions. First, we define reductions used in this section.

Definition 6.2. Let \mathcal{A} and \mathcal{B} be relational structures. We say that there is a (randomized) gappreserving local reduction from \mathcal{B} to \mathcal{A} if there exist functions $t_1(n,m), t_2(n,m)$ and constants c_1, c_2 satisfying the following: there exists a (randomized) construction such that given an input $(\mathcal{J}, \mathbf{w}_J, f_J)$ for $HOM(\mathcal{B})$, it generates an input $(\mathcal{I}, \mathbf{w}_J, f_J)$ for $HOM(\mathcal{A})$ such that

- 1. $|\mathcal{I}| \leq t_1(|\mathcal{J}|, ||\mathcal{J}||)$,
- 2. $\|\mathcal{I}\| \leq t_2(|\mathcal{J}|, \|\mathcal{J}\|),$
- 3. if f_J is a homomorphism, then f_I is also a homomorphism,
- 4. if $\operatorname{dist}_{\mathcal{B}}(f_J) \geq \epsilon$, then $\Pr[\operatorname{dist}_{\mathcal{A}}(f_I) \geq c_1 \epsilon] \geq 9/10$, and
- 5. we can compute $f_I(v)$ for any $v \in I$ by querying f_J at most c_2 times.

Lemma 6.3. Let \mathcal{A} be a relational structure such that there exists a tester for $HOM(\mathcal{A})$ with query complexity $q(n, m, \epsilon)$. If there exists a gap-preserving local reduction from a relational structure \mathcal{B} to \mathcal{A} , there exists a tester for $HOM(\mathcal{B})$ with query complexity $O(q(t_1(n, m), t_2(n, m), O(\epsilon)))$.

Proof. Let $(\mathcal{J}, \boldsymbol{w}_J, f_J)$ be an input for $\mathsf{HOM}(\mathcal{B})$. Let $(\mathcal{I}, \boldsymbol{w}_I, f_I)$ be the (random) input for $\mathsf{HOM}(\mathcal{A})$ given by the reduction.

We run \mathcal{A} with an error parameter $c_1\epsilon$. If f_J is a homomorphism, then \mathcal{A} accepts with probability at least 2/3. If f_J is ϵ -far from homomorphisms, then \mathcal{A} rejects with probability at least $9/10 \cdot 2/3 = 3/5$. In both cases, we can increase the probability by running \mathcal{A} a constant number of times and take the majority of outputs.

Since we can compute the value of $f_I(v)$ by querying f_J at most c_2 times, the number of queries to f_J is at most $O(c_2q(t_1(n,m),t_2(n,m),c_1\epsilon))$.

For an algebra \mathbb{A} , we say that $\mathsf{HOM}(\mathbb{A})$ is testable with $q(n, m, \epsilon)$ queries if, for any relational structure $\mathcal{A} \in \mathsf{Str}(\mathbb{A})$, $\mathsf{HOM}(\mathcal{A})$ is testable with $q(n, m, \epsilon)$ queries.

Lemma 6.4. Let A be a relational structure such that HOM(A) is testable with $q(n, m, \epsilon)$ queries. Then, HOM(A) is also testable with $O(1/\epsilon + q(O(n+m), O(m), O(\epsilon)))$ queries.

Proof. Let \mathcal{B} be a relational structure in $\mathsf{Str}(\mathbb{A})$. Then, each relation of \mathcal{B} is obtained from relations of \mathcal{A} in finitely many steps by using the following constructions [6, 7]:

- 1. removing a relation,
- 2. adding a relation obtained by permuting the variables of a relation,
- 3. adding the intersection of two relations of the same arity,
- 4. adding the product of two relations,
- 5. adding the equality relation, and
- 6. adding a relation obtained by projecting an n-ary relation to its first n-1 variables.

It thus suffices to prove that \mathcal{B} is testable if \mathcal{B} is obtained by any of those constructions from \mathcal{A} . To this end, we will give gap-preserving local reductions from \mathcal{B} to \mathcal{A} with $t_1(n,m) \leq n + m, t_2(n,m) \leq 2m, c_1 = c_2 = O(1)$. (For Case 5, we need reprocessing that costs $O(1/\epsilon)$ queries.) Let (\mathcal{J}, w_J, f_J) be an input of $\mathsf{HOM}(\mathcal{B})$. Then, we construct another input (\mathcal{I}, w_I, f_I) of $\mathsf{HOM}(\mathcal{A})$ so that the construction satisfies conditions of gap-preserving local reductions. Since checking conditions (1),(2),(3) and (5) are straightforward, we will only check the condition (4). For any case below, we define $\overline{f}_I:I\to A$ as the homomorphism closest to f_I . Then, we will construct a homomorphism $\widetilde{f}_J:J\to B$ using \overline{f}_I and show that $\mathrm{dist}(f_I,\overline{f}_I)$ must be large using the fact that $\mathrm{dist}(f_J,\widetilde{f}_J)\geq \epsilon$. Then, the lemma follows by iteratively applying Lemma 6.3.

Case 1: Let us suppose first that \mathcal{B} is obtained from \mathcal{A} by removing a relation of \mathcal{A} . Let \mathcal{I} be the relational structure obtained from \mathcal{J} by supplementing the relations of \mathcal{J} by an empty relation corresponding to the relation removed from \mathcal{A} . We set $w_I = w_J$ and $f_I = f_J$. Then, we define $\widetilde{f}_J = \overline{f}_I$. It is clear that \widetilde{f}_J is a homomorphism from \mathcal{J} to \mathcal{B} . Thus, $\operatorname{dist}(f_I, \overline{f}_I) = \operatorname{dist}(f_J, \widetilde{f}_J) \geq \epsilon$ holds.

Case 2: Let us suppose that \mathcal{B} is obtained from \mathcal{A} by adding a relation S obtained from a relation R of \mathcal{A} by permuting the variables according to a permutation π . Let \mathcal{I} be the relational structure obtained from \mathcal{J} by deleting $S^{\mathcal{J}}$ and replacing $R^{\mathcal{J}}$ by $R^{\mathcal{I}} = R^{\mathcal{J}} \cup S_{\pi}^{\mathcal{J}}$ where $S_{\pi}^{\mathcal{J}}$ is obtained by permuting the variables of $S^{\mathcal{J}}$ according to π^{-1} . We set $\mathbf{w}_I = \mathbf{w}_J$ and $f_I = f_J$. Then, we define $f_J = \overline{f}_I$. It is clear that f_J is a homomorphism from \mathcal{J} to \mathcal{B} . Thus, $\operatorname{dist}(f_I, \overline{f}_I) = \operatorname{dist}(f_J, \widetilde{f}_J) \geq \epsilon$ holds.

Case 3: Let R and S be two relations of the same arity of A. Let T denote the intersection of R and S. Let us suppose that \mathcal{B} is obtained from A by adding the relation T. Let \mathcal{I} be the relational structure obtained from \mathcal{J} by deleting $T^{\mathcal{J}}$ and replacing $R^{\mathcal{J}}$ by $R^{\mathcal{I}} = R^{\mathcal{J}} \cup T^{\mathcal{J}}$ and $S^{\mathcal{J}}$ by $S^{\mathcal{I}} = S^{\mathcal{J}} \cup T^{\mathcal{J}}$. We set $\mathbf{w}_I = \mathbf{w}_J$ and $f_I = f_J$. Then, we define $\widetilde{f}_J = \overline{f}_I$. It is clear that \widetilde{f}_J is a homomorphism from \mathcal{J} to \mathcal{B} . Thus, $\operatorname{dist}(f_I, \overline{f}_I) = \operatorname{dist}(f_J, \widetilde{f}_J) \geq \epsilon$ holds.

Case 4: Let R and S be two relations of A. Let T denote the product of R and S, and let B be obtained from A by adding the relation T. Let T be the relational structure obtained from T by deleting $T^{\mathcal{J}}$ and replacing $T^{\mathcal{J}}$ by $T^{\mathcal{J}} = T^{\mathcal{J}} = T^{\mathcal$

Case 5: Let us suppose that \mathcal{B} is obtained from \mathcal{A} by adding the equality relation. Let θ be the reflexive, symmetric, transitive closure of θ' where θ' is the relation of \mathcal{J} corresponding to equality in \mathcal{B} . Clearly, θ is an equivalent relation on J. For a variable $v \in J$, we define v/θ as the corresponding θ -block. For a θ -block u, we define $\mathbf{w}_J(u,b) = \sum_{v \in u: f_J(v) = b} \mathbf{w}_J(v)$, $\mathbf{w}_J(u) = \sum_{v \in u: f_J(v) = b} \mathbf{w}_J(v)$

$$\sum_{b \in B} \boldsymbol{w}_J(u,b), \ \boldsymbol{w}_J^{\mathrm{maj}}(u) = \max_{b \in B} \boldsymbol{w}_J(u,b), \ \mathrm{and} \ f_J^{\mathrm{maj}}(u) = \operatorname*{argmax}_{b \in B} \boldsymbol{w}_J(u,b).$$

Before we reduce the problem to $\mathsf{HOM}(\mathcal{A})$, we run the following algorithm first: Pick a set of $\Theta(1/\epsilon)$ variables according to \boldsymbol{w} and check whether those variables obey θ . Let $\widehat{f}_J: J \to B$ be the map such that $\widehat{f}_J(v) = f_J^{\mathrm{maj}}(v/\theta)$. Note that \widehat{f}_J is the map closest to f_J obeying θ . It is easy to see that the above algorithm always accepts when f_J is a homomorphism from \mathcal{J} to \mathcal{B} and the query complexity is $O(1/\epsilon)$.

Suppose that $\operatorname{dist}(f_J, \widehat{f}_J) \geq \epsilon/20$. This indicates $\sum_{u:\theta\text{-block}} (\boldsymbol{w}_J(u) - \boldsymbol{w}_J^{\text{maj}}(u)) \geq \epsilon/20$. It is not hard to show that, in such a case, the algorithm above rejects \widehat{f}_J with probability at least 2/3. Thus, we assume that $\operatorname{dist}(f_J, \widehat{f}_J) = \sum_{u:\theta\text{-block}} (\boldsymbol{w}_J(u) - \boldsymbol{w}_J^{\text{maj}}(u)) < \epsilon/20$ in what follows.

Now, we define the input $(\mathcal{I}, \boldsymbol{w}_I, f_I)$ for $\mathsf{HOM}(\mathcal{A})$. We define a structure \mathcal{I} so that its base set is the set of θ -blocks, and its relations are defined as follows: For each relation $R^{\mathcal{I}}$ of \mathcal{J} distinct from θ' , we define a relation $R^{\mathcal{I}}$ on the θ -blocks by stipulating that

$$(u_1, \ldots, u_r) \in R^{\mathcal{I}}$$
 iff $\exists v_1 \in u_1, \ldots, \exists v_r \in u_r$ such that $(v_1, \ldots, v_r) \in R^{\mathcal{I}}$.

Also, we set $\mathbf{w}_I(u) = \mathbf{w}_J(u)$. We define $f_I(u) = f_J(v)$ where v is a variable in u randomly chosen according to $\mathbf{w}_J|_u$, i.e., v is chosen with probability $\mathbf{w}_J(v)/\mathbf{w}_J(u)$.

We define $\operatorname{dist}(f_J, f_I) = \sum_{u:\theta\text{-block }v\in u: f_I(u)\neq f_J(v)} \sum_{\boldsymbol{w}_J(v)} \boldsymbol{w}_J(v)$. For a θ -block u, we define $\operatorname{dist}_u(f_J, f_I) = \sum_{u:\theta\text{-block }v\in u: f_I(u)\neq f_J(v)} \sum_{\boldsymbol{w}_J(v)=1} \boldsymbol{w}_J(v)$.

$$\sum_{v \in u: f_I(u) \neq f_J(v)} \frac{\boldsymbol{w}_J(v)}{\boldsymbol{w}_J(u)}.$$
 It is clear that $\operatorname{dist}(f_J, f_I) = \sum_{u: \theta\text{-block}} \boldsymbol{w}_J(u) \operatorname{dist}_u(f_J, f_I).$

We have

$$\mathbf{E}[\operatorname{dist}_{u}(f_{J}, f_{I})] = \sum_{b \in B} \frac{\boldsymbol{w}_{J}(u, b)}{\boldsymbol{w}_{J}(u)} \left(1 - \frac{\boldsymbol{w}_{J}(u, b)}{\boldsymbol{w}_{J}(u)}\right)$$

$$\leq \frac{\boldsymbol{w}_{J}^{\operatorname{maj}}(u)}{\boldsymbol{w}_{J}(u)} \left(1 - \frac{\boldsymbol{w}_{J}^{\operatorname{maj}}(u)}{\boldsymbol{w}_{J}(u)}\right) + \left(1 - \frac{\boldsymbol{w}_{J}^{\operatorname{maj}}(u)}{\boldsymbol{w}_{J}(u)}\right) \cdot 1$$

$$\leq 2 \left(1 - \frac{\boldsymbol{w}_{J}^{\operatorname{maj}}(u)}{\boldsymbol{w}_{J}(u)}\right).$$

Thus,

$$\mathbf{E}[\operatorname{dist}(f_J, f_I)] = \mathbf{E}[\sum_{u:\theta\text{-block}} \boldsymbol{w}_J(u)\operatorname{dist}_u(f_J, f_I)] \leq \sum_{u:\theta\text{-block}} 2(\boldsymbol{w}_J(u) - \boldsymbol{w}_J^{\text{maj}}(u)) < \frac{\epsilon}{10}.$$

Also, we have

$$\begin{aligned} & \mathbf{Var}[\operatorname{dist}_{u}(f_{J}, f_{I})] \\ &= \mathbf{E}[(\operatorname{dist}_{u}(f_{J}, f_{I}) - \mathbf{E}[\operatorname{dist}_{u}(f_{J}, f_{I})])^{2}] \\ &\leq \frac{\boldsymbol{w}_{J}^{\operatorname{maj}}(u)}{\boldsymbol{w}_{J}(u)} \left(1 - \frac{\boldsymbol{w}_{J}^{\operatorname{maj}}(u)}{\boldsymbol{w}_{J}(u)} - \mathbf{E}[\operatorname{dist}_{u}(f_{J}, f_{I})]\right)^{2} + \left(1 - \frac{\boldsymbol{w}_{J}^{\operatorname{maj}}(u)}{\boldsymbol{w}_{J}(u)}\right) \cdot 1 \\ &= \left(1 - \frac{\boldsymbol{w}_{J}^{\operatorname{maj}}(u)}{\boldsymbol{w}_{J}(u)}\right)^{2} + \left(1 - \frac{\boldsymbol{w}_{J}^{\operatorname{maj}}(u)}{\boldsymbol{w}_{J}(u)}\right) \leq 2\left(1 - \frac{\boldsymbol{w}_{J}^{\operatorname{maj}}(u)}{\boldsymbol{w}_{J}(u)}\right). \end{aligned}$$

Thus,

$$\mathbf{Var}[\mathrm{dist}(f_J, f_I)] = \mathbf{Var}[\sum_{u:\theta\text{-block}} \boldsymbol{w}_J(u)\mathrm{dist}_u(f_J, f_I)] \leq \sum_{u:\theta\text{-block}} 2(\boldsymbol{w}_J(u) - \boldsymbol{w}_J^{\mathrm{maj}}(u)) \leq \frac{\epsilon}{10}.$$

Thus, from Chebyshev's inequality, $\Pr[\operatorname{dist}(f_J, f_I) \geq \epsilon/2] \leq 1/16$. We check the condition (4). We define \widetilde{f}_J as $\widetilde{f}_J(v) = \overline{f}_I(v/\theta)$. It is clear that \widetilde{f}_J is a homomorphism from \mathcal{J} to \mathcal{B} . Since we have $\operatorname{dist}(f_J, f_I) + \operatorname{dist}(f_I, \overline{f}_I) \geq \operatorname{dist}(f_J, \widetilde{f}_J) \geq \epsilon$, it follows that $\Pr[\operatorname{dist}(f_I, \overline{f}_I) \ge \epsilon/2] \ge 15/16.$

Case 6: Let us suppose that \mathcal{B} is obtained from \mathcal{A} by adding the projection S of an r-ary relation R of A to its first r-1 variables. Let \mathcal{I} be the relational structure with the base set J extended by a new element for each (r-1)-tuple in $S^{\mathcal{J}}$. The relations of \mathcal{I} are those of \mathcal{J} , except that $S^{\mathcal{J}}$ is removed and $R^{\mathcal{J}}$ is replaced by $R^{\mathcal{I}} = R^{\mathcal{J}} \cup S^{\mathcal{J}}_{r-1}$ where $S^{\mathcal{J}}_{r-1}$ is obtained from $S^{\mathcal{J}}$ by extending every (r-1)-tuple of $S^{\mathcal{I}}$ with the corresponding new element in the base set of I. Note that $|\mathcal{I}| \leq |\mathcal{I}| + ||\mathcal{I}||$ and $||\mathcal{I}|| = ||\mathcal{I}||$. We set $w_I(v) = w_J(v)$ for a variable $v \in J$ is and $w_I(v) = 0$ for a variable v corresponding to (r-1)-tuple in $S^{\mathcal{I}}$. We define f_I as follows: We set $f_I(v) = f_J(v)$ for $v \in J$ and arbitrary value for a variable v corresponding to an (r-1)-tuple in $S^{\mathcal{J}}$. (Indeed, we do not have to care about those values since there weights are zero.) Then, we define $\overline{f}_{J}(v) = \overline{f}_{I}(v)$ for $v \in J$. It is easy to check that f_J is a homomorphism, and $\operatorname{dist}(f_I, \overline{f}_I) \geq \operatorname{dist}(f, f_J) \geq \epsilon$.

We introduce some notions related to algebras. Let $\mathbb{A} = (A; F)$ be an algebra. A set $B \subseteq A$ is a subuniverse of \mathbb{A} if for every basic operation $f \in F$ restricted to B has all the results in B. For a nonempty subuniverse B of an algebra \mathbb{A} , $f|_B$ is the restriction of f to B. The algebra $\mathbb{B} = (B, F|_B)$ where $F|_B = \{f|_B \mid f \in F\}$ is a subalgebra of \mathbb{A} . Algebras \mathbb{A} , \mathbb{B} are of the same type if they have the same number of basic operations and corresponding operations have equal arities. Given algebras \mathbb{A} , \mathbb{B} of the same type, a product $\mathbb{A} \times \mathbb{B}$ is the algebra with the same type as \mathbb{A} and \mathbb{B} with universe $A \times B$ and basic operations computed coordinate-wise. An equivalence relation θ on A is called a congruence of an algebra \mathbb{A} if θ is a subalgebra of $\mathbb{A} \times \mathbb{A}$. Given a congruence θ on A, we can form the homomorphic image $\mathbb{A}/_{\theta}$, whose elements are the equivalence classes of \mathbb{A} and the basic operations are defined so that the natural projection mapping is a homomorphism $\mathbb{A} \to \mathbb{A}/_{\theta}$. If \mathbb{A} is idempotent, θ -classes are subuniverse of \mathbb{A} . It is known that a relational structure \mathbb{A} is in \mathbb{B} if each relation in \mathbb{A} is a subuniverse of a finite power of \mathbb{A} .

A variety is a class of algebras of the same type closed under formation of subalgebras, homomorphic images and finite products. For any algebra \mathbb{A} , there is a smallest variety containing \mathbb{A} , denoted by $\mathcal{V}(\mathbb{A})$ and called the variety generated by \mathbb{A} . It is well known that any variety is generated by an algebra and that any member of $\mathcal{V}(\mathbb{A})$ is a homomorphic image of a subalgebra of a power of \mathbb{A} .

Lemma 6.5. Let \mathbb{A} be an algebra such that $HOM(\mathbb{A})$ is testable with $q(n, m, \epsilon)$ queries. Then, for any finite algebra $\mathbb{B} \in \mathcal{V}(\mathbb{A})$, $HOM(\mathbb{B})$ is also testable with $O(q(O(n), O(m), O(\epsilon)))$ queries.

Proof. It suffices to show that every subalgebra, homomorphic image and finite power of \mathbb{A} is testable with $O(q(O(n), O(m), O(\epsilon)))$ queries. Let \mathbb{B} be a subalgebra, a homomorphic image, or a finite power of \mathbb{A} and let \mathcal{B} be a relational structure on B such that the relations of \mathcal{B} are subalgebras of finite powers of \mathbb{B} . From Lemma 6.3, it suffices to show a gap-preserving local reduction from \mathcal{B} to \mathcal{A} . We follow the approach similar to the proof of Lemma 6.4. Given an input $(\mathcal{J}, \boldsymbol{w}_J, f_J)$ for $\mathsf{HOM}(\mathcal{B})$, we define another structure $(\mathcal{I}, \boldsymbol{w}_I, f_I)$ for $\mathsf{HOM}(\mathcal{A})$. Then, we show that the construction satisfies the conditions of gap-preserving local reductions. Since checking conditions (1), (2), (3), and (5) are straightforward, we will only check the condition (4). For any case below, we define $\overline{f}_I: I \to A$ as a homomorphism closest to f_I . Then, we will construct a homomorphism $f_J: J \to B$ from \overline{f}_I and show that $\mathrm{dist}(f_I, \overline{f}_I)$ must be large by using the fact that $\mathrm{dist}(f_I, f_J) \geq \epsilon$.

Suppose first that \mathbb{B} is a subalgebra of \mathbb{A} . Let \mathcal{A} be the relational structure whose base set is A and whose relations are all the relations of \mathcal{B} and B as a unary relation. Notice that the relations of \mathcal{A} are subalgebras of finite powers of \mathbb{A} , and $\mathcal{A} \in \mathsf{Str}(\mathbb{A})$. We take \mathcal{I} to be \mathcal{J} with all of its relations adding B as a unary relation. In particular, I = J. Then, we define $\mathbf{w}_I = \mathbf{w}_J$ and $f_I = f_J$. Due to the unary relations, $\overline{f}_I(v) \in B$ must hold for every $v \in I$. Thus, \overline{f}_I is also a homomorphism from \mathcal{J} to \mathcal{B} . Thus, $\mathrm{dist}_{\mathcal{A}}(f) = \mathrm{dist}(f_I, \overline{f}_I) \geq \epsilon$.

Secondly, suppose that \mathbb{B} is a homomorphic image of \mathbb{A} under the homomorphism $h:A\to B$. This time, let \mathcal{A} be the relational structure whose base set is A and whose relations are the preimages under the homomorphism h of all the relations of \mathcal{B} . Notice that the relations of \mathcal{A} are subalgebras of finite powers of \mathbb{A} , and $\mathcal{A} \in \mathsf{Str}(\mathbb{A})$. We take \mathcal{I} to be the relational structure whose base set is J and whose relations are the preimages under the homomorphism h of all the relations of \mathcal{I} . We define $\mathbf{w}_I = \mathbf{w}_J$ and $f_I: I \to A$ so that $f_I(v)$ is any element in $h^{-1}(f_J(v))$. Note that $h \circ \overline{f}_I$ is a homomorphism from I to B. From the construction, we have $\mathrm{dist}(f_I, \overline{f}_I) \geq \mathrm{dist}(h \circ f_I, h \circ \overline{f}_I) = \mathrm{dist}(f_J, h \circ \overline{f}_I) \geq \epsilon$.

Finally, suppose that $\mathbb{B} = \mathbb{A}^k$. Let \mathcal{A} be the relational structure with the following relations: If R is an s-ary relation of \mathcal{B} , define R_0 to be the sk-ary relation such that, if $(b_1, \ldots, b_s) \in R$

with $b_i = (a_{1,i}, \ldots, a_{k,i})$, we put the sk-tuple $(a_{1,1}, \ldots, a_{1,s}, \ldots, a_{k,1}, \ldots, a_{k,s})$ in R_0 . Note that the sk-ary relations obtained in this way are subalgebras of finite powers of \mathbb{A} , and $\mathbb{A} \in \mathsf{Str}(\mathbb{A})$. We take \mathcal{I} to be the union of k disjoint copies of J with one sk-ary relation for each s-ary relation of \mathcal{J} . An sk-tuple in the new relation on \mathcal{I} is formed by the k copies of an s-tuple in the old relation on \mathcal{J} , that is, if (x_1, \ldots, x_s) is in the old relation on \mathcal{J} and $x_{i,j}$ is the i-th copy of x_j in J, then $(x_{1,1}, \ldots, x_{1,s}, \ldots, x_{k,1}, \ldots, x_{k,s})$ is in the new relation. We define $\mathbf{w}_I(x_i) = \mathbf{w}_J(x)/k$ if x_i is a copy of x. For a map $f_J: J \to B$, we define $f_I: I \to A$ as follows: If $x_{i,j}$ is the i-th copy of x_j , we define $f_I(x_{i,j})$ as the i-th element of $f_J(x_j)$. We make $f_J: J \to B$ from f_I by $f_J(x_j) = (\overline{f}_I(x_{1,j}), \ldots, \overline{f}_I(x_{n,j}))$. Clearly, f_J is a homomorphism. Thus, $\mathrm{dist}(f_I, \overline{f}_I) \geq \mathrm{dist}(f_J, \widetilde{f}_J)/k \geq \epsilon/k$.

We define $3LIN = (\{0,1\}; \{(0,0,0), (0,1,1), (1,0,1), (1,1,0)\}, \{(0,0,1), (0,1,0), (1,0,0), (1,1,1)\})$ as the relational structure expressing a system of linear equations over \mathbb{F}_2 such that each equation has arity three.

Lemma 6.6 ([4]). Testing HOM(3LIN) requires $\Omega(n)$ queries even if m = O(n).

We introduce the notion of type set of an algebra and of a variety. Roughly speaking, the type set of a finite algebra is a subset of the set 1, 2, 3, 4, 5 whose elements are called types and correspond to certain classes of algebras: 1 to unary algebras, 2 to vector spaces over finite fields, 3 to Boolean algebras, 4 to distributive lattices and 5 to semilattices. See [23] for details. If \mathcal{V} is a variety, the type set of \mathcal{V} is the union of the type sets of the finite algebras in \mathcal{V} . We say that an algebra or variety admits (omits) type i when i is (is not) in its type set.

Lemma 6.7 ([26]). Let \mathbb{A} be an idempotent algebra such that $\mathcal{V}(\mathbb{A})$ admits type 1. There exists an algebra $\mathbb{B} \in \mathcal{V}(\mathbb{A})$ such that $\mathsf{3LIN} \in \mathit{Str}(\mathbb{B})$.

Lemma 6.8 ([14]). Let H be a non bi-arc graph. Then, $V(\mathbb{H}^L)$ admits type 1.

Proof of Lemma 6.1. Assume that $\mathsf{LHOM}(H)$ is testable with o(n) queries when m = O(n). Then, from Lemma 6.4, $\mathsf{HOM}(\mathbb{H}^L)$ is testable with o(n) queries. Then, from Lemma 6.5, $\mathsf{HOM}(\mathbb{H}^L)$ is testable with o(n) queries for any $\mathbb{H}' \in \mathcal{V}(\mathbb{H}^L)$. However, \mathbb{H}^L is idempotent and $\mathcal{V}(\mathbb{H}^L)$ admits type 1 from Lemma 6.8. Thus, $\mathsf{3LIN} \in \mathcal{V}(\mathbb{H}^L)$ from Lemmas 6.7, and testing $\mathsf{3LIN}$ requires $\Omega(n)$ queries from Lemma 6.6. Contradiction.

References

- [1] N. Alon, E. Fischer, I. Newman, and A. Shapira. A combinatorial characterization of the testable graph properties: it's all about regularity. In *Proc. 38th ACM symposium on Theory of computing*, pages 251–260, 2006.
- [2] N. Alon and A. Shapira. Testing satisfiability. Journal of Algorithms, 47:87–103, 2003.
- [3] L. Barto and M. Kozik. Cyclic terms in algebraic approach to CSP. In *Proc. 25th IEEE Symposium on Logic in Computer Science*, volume 1, 2010.
- [4] E. Ben-Sasson, P. Harsha, and S. Raskhodnikova. Some 3cnf properties are hard to test. *SIAM Journal on Computing*, 35(1):1–21, 2006.

- [5] A. Bhattacharyya, E. Grigorescu, and A. Shapira. A unified framework for testing linearinvariant properties. In *Proc. 51st IEEE Symposium on Foundations of Computer Science*, pages 478–487, 2010.
- [6] V. G. Bodnarchuk, L. A. Kaluzhnin, V. N. Kotov, and B. A. Romov. Galois theory for post algebras. i. Cybernetics and Systems Analysis, 5(3):243–252, 1969.
- [7] V. G. Bodnarchuk, L. A. Kaluzhnin, V. N. Kotov, and B. A. Romov. Galois theory for post algebras. ii. *Cybernetics and Systems Analysis*, 5(5):531–539, 1969.
- [8] A. Bulatov. H-coloring dichotomy revisited. Theoretical Computer Science, 349(1):31–39, 2005.
- [9] A. Bulatov. Combinatorial problems raised from 2-semilattices. *Journal of Algebra*, 298(2):321–339, 2006.
- [10] A. Bulatov and M. Valeriote. Recent results on the algebraic approach to the CSP. *Complexity of Constraints*, pages 68–92, 2008.
- [11] L. Egri, A. Krokhin, B. Larose, and P. Tesson. The complexity of the list homomorphism problem for graphs. In *Proc. 27th International Symposium on Theoretical Aspects of Computer Science*, volume 5, pages 335–346, 2010.
- [12] T. Feder and P. Hell. List homomorphisms to reflexive graphs. *Journal of Combinatorial Theory, Series B*, 72(2):236–250, 1998.
- [13] T. Feder, P. Hell, and J. Huang. List homomorphisms and circular arc graphs. *Combinatorica*, 19(4):487–505, 1999.
- [14] T. Feder, P. Hell, and J. Huang. Bi-arc graphs and the complexity of list homomorphisms. Journal of Graph Theory, 42(1):61–80, 2003.
- [15] T. Feder and M.Y. Vardi. The computational structure of monotone monadic snp and constraint satisfaction: A study through datalog and group theory. SIAM Journal of Computing, 28(1):57–104, 1998.
- [16] E. Fischer, E. Lehman, I. Newman, S. Raskhodnikova, R. Rubinfeld, and A. Samorodnitsky. Monotonicity testing over general poset domains. In *Proc. 34th ACM symposium on Theory of computing*, pages 474–483, 2002.
- [17] O. Goldreich. Introduction to testing graph properties, 2010.
- [18] O. Goldreich, S. Goldwasser, and D. Ron. Property testing and its connection to learning and approximation. *Journal of the ACM*, 45(4):653–750, 1998.
- [19] O. Goldreich and D. Ron. A sublinear bipartiteness tester for bounded degree graphs. *Combinatorica*, 19(3):335–373, 1999.
- [20] P. Hell. Algorithmic aspects of graph homomorphisms. Surveys in Combinatorics, 307:239–276, 2003.
- [21] P. Hell and J. Nešetřil. On the complexity of h-coloring. *Journal of Combinatorial Theory*, Series B, 48(1):92–110, 1990.

- [22] P. Hell and J. Nešetřil. Colouring, constraint satisfaction, and complexity. *Computer Science Review*, 2(3):143–163, 2008.
- [23] D.C. Hobby, R. McKenzie, and American Mathematical Society. *The structure of finite algebras*. American Mathematical Society, 1988.
- [24] P. Jeavons, D. Cohen, and M. Gyssens. Closure properties of constraints. *Journal of the ACM*, 44(4):527–548, 1997.
- [25] T. Kaufman and M. Sudan. Algebraic property testing: the role of invariance. In *Proc.* 40th ACM symposium on Theory of computing, pages 403–412, 2008.
- [26] A. Krokhin, A. Bulatov, and P. Jeavons. The complexity of constraint satisfaction: an algebraic approach. Structural theory of automata, semigroups, and universal algebra, pages 181–213, 2005.
- [27] D. Ron. Algorithmic and analysis techniques in property testing, 2009.
- [28] E. Szemerédi. Regular partitions of graphs. *Problèmes combinatoires et théorie des graphes*, 260:399–401, 1975.
- [29] Y. Yoshida. Optimal constant-time approximation algorithms and (unconditional) inapproximability results for every bounded-degree CSP. In *Proc. 43rd ACM symposium on Theory of computing*, pages 665–674, 2011.
- [30] Y. Yoshida and Y. Kobayashi. Testing (s,t)-disconnectivity of graphs and digraphs, 2011. manuscript.