SOUSLIN ALGEBRA EMBEDDINGS

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ABSTRACT. A Souslin algebra is a complete Boolean algebra whose main features are ruled by a tight combination of an antichain condition with an infinite distributive law.

The present article divides into two parts. In the first part a representation theory for the complete and atomless subalgebras of Souslin algebras is established (building on ideas of Jech and Jensen). With this we obtain some basic results on the possible types of subalgebras and their interrelation.

The second part begins with a review of some generalizations of results from descriptive set theory concerning Baire category which are then used in non-trivial Souslin tree constructions that yield Souslin algebras with a remarkable subalgebra structure. In particular, we use this method to prove that under the diamond principle there is a bi-embeddable though not isomorphic pair of homogeneous Souslin algebras.

Introduction

Souslin trees are well-known to most set theorists, Souslin algebras not so well any more, and subalgebras of Souslin algebras in general are suspected of being a messy business. I would like to put in a good word for them here.

We consider κ -Souslin algebras (for definitions see the following section) as well as their representations by normal κ -Souslin trees and address the problem how to describe and classify complete embeddings between κ -Souslin algebras. After introducing the representation for Souslin subalgebras we give a rough classification of the possible types of embeddings and find implications between existence statements involving them as well as counter examples proving non-implications.

The representation theory of Part 1 of this text was primarily developed exclusively for the case $\kappa = \aleph_1$ in my PhD thesis ([21]¹) in order to establish the consistency of

Date: March 4, 2022.

²⁰⁰⁰ Mathematics Subject Classification. 03E05, 06E10, 54H05.

Key words and phrases. Souslin algebra, Souslin tree, diamond principle, Baire category.

¹Readers familiar with [21] should note 1.) that the notion of tree equivalence relation (t.e.r.) as defined in this article corresponds to what is called a *decent* t.e.r. in [21], and 2.) that only Part 1 and Sections 5.2 and 5.3 of Part 2 consist of generalizations of results given in [21] while the remainder of Part 2 brings new material.

the existence of a chain homogeneous \aleph_1 -Souslin algebra (cf. the subsequent paper [22]). Here we take κ to be any regular cardinal. To describe Souslin subalgebras we define the notion of a tree equivalence relation on a κ -normal tree. The properties considered for the classification of embeddings are mainly niceness (introduced by Jensen, cf. [5]) and largeness and the global negations thereof. For example, we show that large subalgebras are always nice (Theorem 3.5) and that the existence of a nice and nowhere large subalgebra always implies that there also is a nowhere nice subalgebra (Theorem 4.9). We also study connections between the symmetric structure of κ -Souslin algebras (i.e. its automorphisms) and its subalgebra structure. Apart from the applications as given in this text, the representation theory might also be useful in the study of intermediate models of generic extensions built using κ -Souslin trees or κ -Souslin algebras or in other related areas of set theoretic research.

The topological notions developed at the beginning of Part 2 in Section 5.1 serve to facilitate the choice of the relevant limit levels in the κ -Souslin tree constructions and provide a nice tool for involved diagonalization procedures. The argument is a refinement of the *Diagonal Principle* as formulated by Jech in [10, p.63]:

If T is a countable normal tree of limit length, then there exists a branch through T which satisfies a countable number of prescribed conditions.

(E.g., given a countable set B of branches of T, there exists a branch which is not in B.)

Here we observe that the set of relevant branches is (a generalization of) a Polish space and "conditions" are comeagre subsets. The idea to use Baire category for Souslin tree constructions is not at all new. Taking into account the correspondence between Baire category and Cohen reals it was implicitly used by Jensen in his countable models constructions of Souslin trees (cf. [5, Chapters IV and V]) whose generic branches are close to being Cohen reals. Or, for a more recent example, it applies along with the parametrized diamond principles for Baire category as considered in [6].

Nevertheless, this tool has, as far as I know, not yet been used to perform advanced Souslin tree constructions. We use it to construct a rigid μ^+ -Souslin algebra with non-rigid Souslin subalgebras (Section 5.3), a rigid μ^+ -Souslin algebra with an essentially unique Souslin subalgebra (Section 6) and a pair of μ^+ -Souslin algebras that forms a counter example to the Schröder-Bernstein-Theorem for μ^+ -Souslin algebras (Section 7.2). In a subsequent paper ([22]) I will present constructions of chain homogeneous \aleph_1 -Souslin algebras in which the same method is applied. Of course, non of these constructions can be carried out in ZFC alone. We do not intend to give an exhaustive picture of what can be done under varying hypothesis and simply assume variants of the well-known diamond principle \diamondsuit .

The paper is fairly self-contained, though of course some acquaintance with Souslin tree constructions is an advantage for the reader.

1. Preliminaries

Our notation and terminology follow mainly [16] and [5] (Boolean) and [13] (set theoretic, exception: we use φ " M to denote the image of the set M under the mapping φ).

All Boolean algebras considered in this text are complete and all subalgebras are tacitly assumed to be regular, i.e., if A is a subalgebra of B and for some $M \subset A$ the infimum $\sum^A M$ with respect to A exists (and it does as A is assumed to be complete), then it coincides with the sum $\sum^B M$ taken in B. If M is a subset of the Boolean algebra B, then $\langle M \rangle^{cm}$ denotes the subalgebra of B completely generated by M, i.e., the least complete subalgebra of B which contains M as a subset.

A frequently used item is the *canonical (upper) projection* of a (complete) Boolean B algebra onto its subalgebra A:

$$h = h_{B,A} : B \to A, \quad b \mapsto \sum \{a \in A \mid ab = a\}.$$

As usual we omit subscripts if there is no danger of confusion. Note that h is not a homomorphism as it only respects sums but neither products nor complements in general.

Whenever we talk of the natural ordering of a Boolean algebra B, we mean the relation defined by $a \leq_B b : \iff ab = a$. We denote the relative algebra of B with respect to b by $B \upharpoonright b = \{a \in B \mid a \leq b\}$. When a (complete) subalgebra A of B and an element $b \in B$ are given, we might also consider the algebra of products $bA = b \cdot A = \{ba \mid a \in A\}$ which is a (complete) subalgebra of $B \upharpoonright b$. In this situation the projection $h = h_{B,A}$ gives rise to an isomorphism between bA and $A \upharpoonright h(b)$, the inverse map being multiplication with b.

A (complete) Boolean algebra B is called

simple, if it has no atomless (complete) subalgebras, rigid, if it admits no automorphisms except for the identical map, homogeneous, if for every $b \in B^+$ the relative algebra $B \upharpoonright b$ is isomorphic to B.

1.1. κ -Souslin algebras. Let κ be an regular uncountable cardinal. An antichain of a Boolean algebra is a subset consisting of pairwise disjoint elements, and the κ -chain condition states that every antichain is of cardinality less than κ . A κ -Souslin algebra is a complete Boolean algebra that satisfies both the κ -chain condition and the (κ, ∞) -distributive law, i.e., for index sets I, J where $|I| < \kappa$ and J is arbitrary

and each family $(a_{ij} \mid i \in I, j \in J)$ of elements the following equation holds:

$$\sum_{i \in I} \prod_{j \in J} a_{ij} = \prod \left\{ \sum_{i \in I} a_{if(i)} \mid f \in {}^{I}J \right\}.$$

This distributive law also has valuable characterizations in terms of common refinements of paritions of unity, i.e. maximal antichains (cf. [16, Propositions 14.8/9]): A Boolean algebra is (κ, ∞) -distributive if and only if every family $(X_i)_{i \in I}$ of less than κ maximal antichains has a common refinement, i.e., there is a maximal antichain X such that for every $i \in I$ and each member $a \in X_i$ there is an element $b \in X$ with ab = b, i.e., b lies below a in the natural partial ordering of B. As a consequence, if B is (κ, ∞) -distributive and A is a subalgebra of B which is completely generated by fewer than κ elements, then A is atomic. These two results are heavily used when a κ -Souslin algebra is represented as the regular open algebra of a κ -Souslin tree.

Note that every atomless (and complete) subalgebra of a κ -Souslin algebra is κ -Souslin itself. We therefore call these subalgebras *Souslin subalgebras* (omitting the parameter κ as it is determined by the context).

A result concerning κ -Souslin algebras and well-known only in the case where $\kappa = \aleph_1$ is Solovay's barrier for the cardinality of κ -Souslin algebras (cf. [13, Theorem 30.20]): A κ -Souslin algebra can have at most 2^{κ} elements. We will not use this result here as we concentrate on κ -Souslin algebras that can be represented by κ -Souslin trees and therefore always are of cardinality $2^{<\kappa}$.

1.2. **Trees.** A tree is a partial order $(T, <_T)$ with the additional property, that for every element $t \in T$, the set of its predecessors, $\{s <_T t\} := \{s \in T \mid s <_T t\}$, is well-ordered by the ordering $<_T$. Whenever possible, we omit the subscript $_T$ and denote the tree ordering just by <.

The elements of a tree are called its *nodes*, the minimal elements are *roots*. The *height* of a node, ht t, is the order type of the well-order ($\{s < t\}, <$). Nodes of limit height are also called *limit nodes*. If ht $t = \gamma + 1$ is a successor ordinal, then we denote by $t^- := t \upharpoonright \gamma$ the immediate predecessor of t.

For every node t we define the set of its immediate successors,

$$\operatorname{succ} t := \{ s \in T \mid t < s \text{ and } \operatorname{ht} s = \operatorname{ht} t + 1 \}.$$

For a cardinal μ we say that T is μ -splitting if every node has exactly μ immediate successors. For every ordinal α we define the α th level of T and denote it by $T_{\alpha} := \{t \in T \mid \text{ht } t = \alpha\}$. The height of T is the minimal ordinal α such that T_{α} is empty. For a subset C of ht T we consider the tree

$$T \upharpoonright C = \bigcup_{\alpha \in C} T_{\alpha}$$

with the ordering < inherited from T and call this tree the restriction of T to (the levels from) C. If $t \in T_{\alpha}$ and $\gamma < \alpha$ then $t \upharpoonright \gamma$ denotes the unique predecessor of t on level γ .

A subset b of a tree T is a branch if it is closed downwards and linearly ordered by <. The $length\ \ell(b)$ of a branch b is just its order type with respect to <. We sometimes take branches to be maps $\ell(b) \to b$ enumerating the nodes in a monotone way. A branch $x \subset T$ of limit length λ is extended if there is a node $t \in T_{\lambda}$ that dominates all members of x: t > s for all $s \in x$. A branch is cofinal if its length coincides with ht T. An antichain of T is a subset that consists of pairwise incomparable nodes. We call branches or antichains maximal if they cannot be extended. Note, that every (non-empty) level of T is a maximal antichain.

A tree T is μ -closed if all branches in T, whose length has cofinality less than μ , are extended. In particular, a μ -closed tree has no maximal branches with cofinality less than μ .

A tree T is *normal* if the following hold:

- T has a unique root,
- every node t has at least two successors on every level T_{α} with ht $t < \alpha <$ ht T
- branches of limit length λ have at most one extension to level $T\lambda$ (the *unique limits* condition)

A tree T is μ -normal if it is normal and every level of T has less than μ nodes.

For every node $t \in T$ we let $T(t) := \{s \in T \mid t \leq s\}$ and call it the tree T relativized to t. A homogeneous tree is a tree T, that admits tree isomorphisms between T(s) and T(t) for all pairs s, t of nodes from the same level T_{α} of T. A rigid tree has no tree automorphism but the identical map. Operations on trees sometimes used in the text are the tree product and the tree sum

$$S \otimes T := \bigcup_{\gamma < \alpha} S_{\gamma} \times T_{\gamma}$$
 and $S \oplus T := \{ \mathbf{root} \} \cup S \dot{\cup} T$,

where $\alpha = \operatorname{ht} T = \operatorname{ht} S$ and **root** is a new node, equipped with the obvious orderings. The apparatus used in Part 2 of the article rests entirely on the following definition, which is albeit useful also in Part 1: For a normal tree T of limit height let [T] be the set of cofinal branches of T. We topologise [T] with the basis that consists of the sets $\hat{s} := \{x \in [T] \mid s \in x\}$ for all $s \in T$. With this topology [T] is a regular Hausdorff (i.e. T_3) space of weight |T|. Moreover, if T is an \aleph_1 -normal tree of countable limit height, then [T] is a Polish space, i.e., it is completely metrizable and second countable.

1.3. κ -Souslin trees. Now let κ be an uncountable, regular cardinal. A κ -Souslin tree is a tree of height κ that has neither antichains nor branches of size κ . Note, that a κ -normal tree of height κ is κ -Souslin if and only if it has no cofinal branches.

A subtree is a subset which is a union of branches, i.e., it is closed downwards. (For example, $\{s < t\} \cup T(t)$ is always a subtree of T.) Every κ -Souslin tree has a normal subtree which is κ -Souslin. In this text we only consider normal Souslin trees.

The following Subtree Lemma is well-known for the case $\kappa = \omega_1$ but its proof (as given, e.g., in [19]) literally translates to the general, regular case. It captures the content of the notion of a κ -Souslin tree without recourse to related structures such as Souslin lines or κ -Souslin algebras.

Lemma 1.1 (Subtree Lemma). Let κ be an uncountable, regular cardinal and T a normal κ -Souslin tree. If S is a subtree of T with $|S| = \kappa$ then S contains a subtree $\{s < t\} \cup T(t)$ for some $t \in T$.

In order to turn a tree into a Boolean algebra we provide it with the (reversed) partial order topology: The basic open sets are T(s) for $s \in T$. Then we simply take the regular open algebra RO T of the space T with this topology. The basic representation lemma for Souslin algebras is given by (the proof of) [16, Theorem 14.20]:

- 1) For every (normal) κ -Souslin tree T its regular open algebra RO T is κ -Souslin, and
- 2) for every κ -Souslin algebra \mathbb{B} , if \mathbb{B} is completely generated by κ many of its elements, then there is a (normal) κ -Souslin tree T which can be (with reversed order) regularly embedded onto a dense subset of \mathbb{B} .

We stress once more that in the present paper only κ -Souslin algebras are considered which are completely generated by trees as in (2) above. Following [5] we call a subset T of the κ -Souslin algebra $\mathbb B$ a Souslinization of $\mathbb B$ if $(T,>_{\mathbb B})$ is a normal κ -Souslin tree and the limit nodes in T are obtained as products over their predecessors: $s = \prod\{t \in T \mid t>_{\mathbb B} s\}$. A minor inconvenience of this terminology is that we regard trees as growing upwards while Souslinizations grow downwards with respect to the natural Boolean order $\leq_{\mathbb B}$ of $\mathbb B$. If possible we prefer the tree order view, i.e., the common phrase "t is above s" is tantamount to "s is closer to the root than t" or in Boolean notation to $t \leq_{\mathbb B} s$.

Two Souslinizations of the same κ -Souslin algebra can look quite different, e.g. 2-splitting vs. infinitarily splitting. However, by the following Restriction Lemma they always coincide on a club set of levels. We will use this fact in a considerable portion of proofs.

Lemma 1.2 (Restriction Lemma). If the κ -Souslin algebras \mathbb{A} and \mathbb{B} are souslinized by S and T respectively and if $\varphi : \mathbb{A} \to \mathbb{B}$ is an isomorphism, then there is a club set $C \subset \kappa$ such that the restriction of φ to $S \upharpoonright C$ is an isomorphism onto $T \upharpoonright C$.

For a proof take the one of [12, Lemma 25.6] (or a solution to [13, Exercise 30.15]) and translate "countable" to "less than κ ".

1.4. \diamondsuit -principles. For a cardinal κ and a stationary subset $E \subset \kappa$ we denote the following statement by $\diamondsuit_{\kappa}(E)$:

There is a sequence $(R_{\alpha})_{\alpha < E}$ (the \lozenge -sequence) such that for every subset X of κ the set

$$\{\alpha \in E \mid X \cap \alpha = R_{\alpha}\}$$

is stationary in κ .

The principle $\diamondsuit_{\kappa}(E)$ implies that $2^{\kappa} = \kappa^{+}$ and is therefore not a theorem of ZFC. But for many stationary sets E it follows from Gödel's axiom of constructibility and can be made true by forcing. We will use this principle in situations where κ is a successor cardinal $\kappa = \mu^{+}$ with $\mu = \mu^{<\mu}$ and $E = \operatorname{CF}_{\mu} = \{\alpha < \kappa \mid \operatorname{cf}(\alpha) = \mu\}$.

Part 1. Elementary representation and classification of Souslin subalgebras

Throughout all of Part 1 let κ denote a regular uncountable cardinal.

2. Tree equivalence relations

Subalgebras of Souslin algebras have been considered before, e.g. in [11] or [3, §5], [17] and more implicitly in [5] or [19, §8]. To represent a subalgebra \mathbb{A} of the Souslin algebra \mathbb{B} with respect to some Souslinization T of \mathbb{B} , the first three sources define a good equivalence relation on the Souslinization T, while the last two use maps between trees $T \upharpoonright C$ (for some club set $C \subseteq \omega_1$) and a Souslinization S of \mathbb{A} .

We combine the two approaches in so far as we will consider equivalence relations, which are designed in a way such that they directly induce the relevant mappings between the Souslinizations.

Definition 2.1. a) Let T be a κ -normal tree of height $\mu \leq \kappa$. An equivalence relation \equiv on T is a tree equivalence relation (t.e.r.) if

- i) \equiv respects levels, i.e., $s \equiv t$ only if $\operatorname{ht}_T s = \operatorname{ht}_T t$;
- ii) \equiv is compatible with $<_T$, i.e., for $s <_T s'$ and $t <_T t'$ with s and t of the same height, $s' \equiv t'$ implies $s \equiv t$;
- iii) the induced partial order on the set T/\equiv of \equiv -cosets given by

$$a <_{T/\equiv} b \iff (\exists s \in a, t \in b)s <_T t$$

for $a, b \in T/\equiv$ is a κ -normal tree order;

iv) \equiv is *honest*, by which we mean that for all triples (s, s', t) of nodes $s \equiv t$ in some level γ of T and $s' >_T s$ the following holds: If there is no successor of t that is equivalent to s', then the same holds already for $s' \upharpoonright (\gamma + 1)$, i.e., there is no $t' \in T_{\gamma+1}$ above t equivalent to $s' \upharpoonright (\gamma + 1)$.

b) If T souslinizes \mathbb{B} and \mathbb{A} is a Souslin subalgebra of \mathbb{B} , we say that the t.e.r. \equiv on T represents \mathbb{A} on T if the sums over the \equiv -classes form a dense subset of \mathbb{A} :

$$\left\langle \sum s/\!\equiv\!\mid s\in T\right\rangle^{\mathrm{cm}}=\mathbb{A}.$$

- Remark 2.2. (1) Note that in point (iii) the tree T/\equiv has unique limits. This implies that on a limit level T_{α} the t.e.r. \equiv is completely determined by its behavior on $T \upharpoonright \alpha$ below.
 - (2) Furthermore, as the tree order on T/\equiv splits in every node, we get that every t.e.r. represents an *atomless*, i.e. a Souslin subalgebra.
 - (3) Call a triple (s, s', t) of nodes a dispute $(on \equiv)$ if $s \equiv t$ and s < s' yet there is no successor of t equivalent to s', i.e., (s, s', t) is as in the definition of honesty above. Then \equiv is honest if and only if for every dispute (s, s', t) on \equiv already $(s, s' \upharpoonright (\operatorname{ht}(s) + 1), t)$ is a dispute. This is illustrated in figure 1.
 - (4) Honesty prevents a t.e.r. from associating two nodes of level T_{γ} that can be distinguished by the subalgebra that the t.e.r. represents. In partucular, if a κ -Souslin tree carries two different t.e.r.s, then the subalgebras represented by these t.e.r.s differ as well.

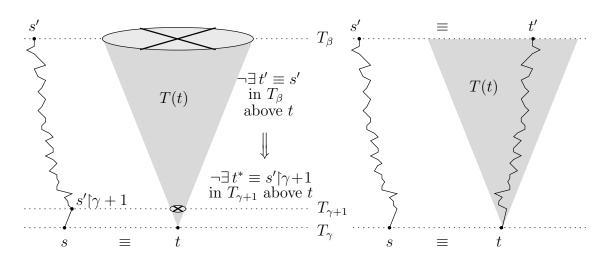


FIGURE 1. Honesty of a tree equivalence relation — the dispute case on the left hand side versus the nice case on the right

For the moment, let us denote by *pre-t.e.r.* an equivalence relation on a tree which satisfies conditions (i-iii) above but not necessarily honesty.

Part b) of the following proposition gives us a necessary criterion for testing whether a pre-t.e.r. is honest with respect to a limit level T_{α} . With its aid we can destroy unwanted t.e.r.s/subalgebras in recursive Souslin algebra constructions

during which we have to choose appropriate limit levels of a tree (cf. Example 2.8 and Theorem 6.1).

Proposition 2.3. Let T be a κ -normal tree of height $\beta \leq \kappa$ carrying a t.e.r. \equiv . Let $\alpha < \beta$ be a limit ordinal. Consider the equivalence relation \simeq on $[T | \alpha]$ induced by \equiv through

$$x \simeq y : \iff (\forall \gamma < \alpha) x | \gamma = y | \gamma.$$

- a) The \simeq -classes are closed subsets of [T].
- b) For $s \in T_{\alpha}$ denote by x_s the branch $\{r \in T \mid \alpha \mid r < s\} \in [T \mid \alpha]$. For each branch $x \in [T]$ consider its class x/\simeq as a subspace of [T]. Then for every $s \in T_{\alpha}$ the α -branches associated to the members of the \equiv -class of s, i.e. the set

$$\{x_r \mid r \in T_\alpha \text{ and } r \equiv s\},\$$

lies densely in the corresponding class x_s/\simeq . Stated in more elementary terms, for every node $s \in T_{\alpha}$, branch $y \simeq x_s$ and ordinal $\gamma < \alpha$ there is a node t in level T_{α} such that $t \equiv s$ and $t > y \upharpoonright \gamma$.

Proof. Part a) follows easily from the fact that for each $x \in [T]$ the set

$$S^x = \bigcup x/{\simeq} = \{s \in T \mid (\exists y \simeq x) \, s \in y\}$$

is a subtree of T and $x/\simeq = [S^x]$, which is always closed.

To prove b) by contradiction, assume that for s, y and γ as above there is no $t > y \upharpoonright \gamma$, $t \equiv s$. Then the triple $(s \upharpoonright \gamma, s, y \upharpoonright \gamma)$ would constitute a dispute on \equiv , but $s \upharpoonright (\gamma + 1) \equiv y \upharpoonright (\gamma + 1)$. This contradicts point (iv) of the last definition.

- Remark 2.4. (1) Note that, while in Proposition 2.3 we used different symbols for the t.e.r. \equiv and the induced equivalence relation \simeq on the space of branches of length α (because here this difference was crucial) we will further on denote the induced relation with the same symbol as the t.e.r. (in most cases: \equiv).
 - (2) In some of the later arguments we will identify the branches of the form x_s with the corresponding nodes s.
 - (3) Given an equivalence relation \equiv on some topological space \mathcal{X} , call a subset $M \subset \mathcal{X}$ suitable for \equiv if for every member $x \in M$ the intersection $M \cap (x/\equiv)$ is a dense subset of the space x/\equiv . With this notion at hand, the conclusion of Proposition 2.3.b) reads as:

The set of branches $\{x_s \mid s \in T_\alpha\}$ corresponding to the nodes of level T_α is suitable for the equivalence relation induced by \equiv on the α -branches of T.

Jensen defined a subalgebra \mathbb{A} of a κ -Souslin algebra \mathbb{B} to be a *nice subalgebra* if there is some Souslinization T of \mathbb{B} such that the image of T under the canonical

projection $h: \mathbb{B} \to \mathbb{A}$, $b \mapsto \prod \{a \in \mathbb{A} \mid b \leq a\}$ is a Souslinization of \mathbb{A} . We now define the corresponding notion for t.e.r.s.

Definition 2.5. a) A t.e.r. \equiv on T is called *nice*, if for all s, s', t in T with $s <_T s'$ and $s \equiv t$ there is some $t' >_T t$ with $s' \equiv t'$.

- b) A t.e.r. \equiv on T is called *almost nice*, if for all s, s', t in T with $s <_T s'$ and $s \equiv t$ and $\operatorname{ht}(s) = \alpha + 1$ for some α there is some $t' >_T t$ with $s' \equiv t'$.
- Remark 2.6. (1) Obviously niceness is the complete absence of disputes and almost niceness means that no dipute may have its lower nodes in a successor level of the tree. So both properties imply the honesty of the t.e.r. (from now on we can forget about pre-t.e.r.s).
 - (2) Honesty and niceness are handed down to any restriction to a club set of levels while almost niceness is not, because such a restriction can turn a limit level into a successor level. On the other hand it is easy to see, that every t.e.r. can be obtained as a restriction of an almost nice t.e.r. to some club set of levels.
 - (3) It is easy to see that the nice subalgebras (with respect to Jensen's definition) are exactly those that can be represented by nice t.e.r.s. Given a t.e.r. \equiv on a Souslinization T of \mathbb{B} let us denote the associated projection by

$$\pi_{\equiv}: T \to \mathbb{B}, \quad t \mapsto \sum t/\equiv .$$

The t.e.r. \equiv is nice if and only if $\pi_{\equiv} = h \upharpoonright T$, and it is almost nice if and only if π_{\equiv} and h coincide on all successor levels of T.

The next lemma will be called the Representation Lemma for Souslin subalgebras.

Lemma 2.7. Let \mathbb{A} be a Souslin subalgebra of the κ -Souslin algebra \mathbb{B} , and let S be any Souslinization of \mathbb{B} .

- a) There is a Souslinization T of \mathbb{B} that admits an almost nice t.e.r. \equiv representing \mathbb{A} .
- b) There are a club $C \subseteq \kappa$ and a t.e.r. \equiv on $S \upharpoonright C$ such that \equiv represents A.
- c) If \mathbb{A} is furthermore nice and represented by \equiv on S then there is a club $C \subseteq \kappa$ such that \equiv is nice on $S \upharpoonright C$.

Proof. We only prove part a) since parts b) and c) follow directly from part a) by the Restriction Lemma. Before constructing T and \equiv by recursion, we describe a method of refining a given partition P of unity in \mathbb{B} to a partition R in \mathbb{B} with the property, that h''R is a partition in \mathbb{A} . Let Q be the set of atoms of $\langle h''P\rangle^{\rm cm} \subset \mathbb{A}$ and define

$$R = \{ pq \mid p \in P, \, q \in Q \} \setminus \{0\}.$$

Then R refines P, and for $pq \in R$ we have h(pq) = qh(p) = q since q is an atom. So h''R = Q.

Now fix a dense subset $\{x_{\alpha+1} \mid \alpha \in \kappa\}$ of \mathbb{B} indexed by successor ordinals. Starting with the root level $T_0 = \{1_{\mathbb{B}}\}$ let P_{α} be any partition in \mathbb{B} refining T_{α} in such a way that every $s \in T_{\alpha}$ is divided in at least two parts, for all $s \in T_{\alpha}$ the image h(s) is not equal to the h-images of the parts of s, and $x_{\alpha} \in \langle P_{\alpha} \rangle^{\mathrm{cm}}$. Then let $T_{\alpha+1}$ be the refinement of P_{α} with respect to h as described above. So $h''T_{\alpha+1}$ is a partition in \mathbb{A} . The limit levels of T are canonically defined as

$$T_{\alpha} := \left\{ \prod b \mid b \in [S \upharpoonright \alpha] \right\} \setminus \{0\}.$$

Thus T is a Souslinization of \mathbb{B} . The t.e.r. is then given on successor levels by

$$s \equiv t : \iff h(s) = h(t).$$

This also determines \equiv on the limit levels and defines an almost nice t.e.r. on T. \square

As an illustration of the notion of t.e.r. and a first application of the Representation Lemma we reformulate Jech's construction of a simple κ -Souslin algebra, i.e., one having no non-trivial Souslin subalgebra. (cf. [11]).

Example 2.8 (a simple κ -Souslin algebra). We construct a Souslinization T of a simple κ -Souslin algebra \mathbb{B} . We assume that $\kappa = \mu^+$ is a successor cardinal and $\mu^{<\mu} = \mu$ and $\diamondsuit_{\kappa}(\mathrm{CF}_{\mu})$ hold². Let $(R_{\nu})_{\mathrm{cf}(\nu)=\mu}$ be a \diamondsuit -sequence.

We will define a κ -normal and μ -closed κ -Souslin tree order on the set κ . We let 0 be the root and provide every node of T with μ direct successors such that level T_{α} consists of the ordinal interval between³ $\mu \cdot \alpha$ and $\mu \cdot (\alpha + 1) = \mu \cdot \alpha + \mu$.

We take full limits on limit levels α of cofinality $< \mu$, i.e. we extend all branches of length α . Thanks to our our hypothesis on cardinal arithmetics there are only μ branches to extend, so our tree remains κ -normal.

On limit stage α of cofinality μ we consider the space $[T \upharpoonright \alpha]$ of cofinal branches through $T \upharpoonright \alpha$ and have to choose a dense subset of cardinality μ subject to some further restrictions imposed by our $\diamondsuit_{\kappa}(\mathrm{CF}_{\mu})$ -sequence $(R_{\nu})_{\mathrm{cf}\,\nu=\mu}$. If $\alpha < \mu\alpha$ then we can extend $T \upharpoonright \alpha$ by choosing any dense subset Q of $[T \upharpoonright \alpha]$ of size μ and extending the branches in Q to $T_{\alpha+1}$.

In the case where $\alpha = \mu \alpha$ we ask the \diamond -sequence for some information about $T \upharpoonright \alpha$. We let the first bit of R_{α} decide whether we care about antichains or about t.e.r.s. If $0 \in R_{\alpha}$ and $A = R_{\alpha} \setminus \{0\}$ is a maximal antichain of $T \upharpoonright \alpha$ then we choose our dense

²A similar construction (which also applies to an inaccessible cardinal κ that is not weakly compact) under an appropriate ($\Box + \diamondsuit$)-assumption is of course possible but more cumbersome, cf. [4, Theorem VII.1.3] for that framework.

³To be correct, $T_1 = \mu \setminus \{0\}$, $T_n = \mu \cdot n \setminus \mu(n-1)$ for $n \in \omega \setminus \{0,1\}$ and $T_\alpha = \mu(\alpha+1) \setminus \mu\alpha$ for all $\alpha \in \kappa \setminus \omega$.

subset Q from the dense open set $P = \{x \in [T \upharpoonright \alpha] \mid (\exists t \in A)t \in x\}$ to guarantee that A is still a maximal antichain when considered as a subset of $T_{\alpha+1}$.

If $0 \notin R_{\alpha}$ and if R_{α} codes a t.e.r. \equiv on $T \upharpoonright C$ for some club set $C \subset \alpha$ then we want to choose the new level T_{α} in a way that destroys \equiv , i.e., the unique extension of \equiv to $T_{\alpha+1}$ violates the honesty criterion of Proposition 2.3. For this consider the equivalence relation induced by \equiv on the space $[T \upharpoonright \alpha]$ of cofinal branches via

$$x \equiv y : \iff (\forall \gamma \in C) x \upharpoonright \gamma \equiv y \upharpoonright \gamma.$$

The \equiv -classes of $[T \upharpoonright \alpha]$ are closed and nowhere dense subsets of $[T \upharpoonright \alpha]$. If \equiv is a non-trivial t.e.r. then there is certainly a \equiv -class x/\equiv of size > 1. Fix a representative x of such a class. In order to define T_{α} we choose a dense subset Q of $[T \upharpoonright \alpha] \setminus (x/\equiv)$ of elements not equivalent to x. Then extend every branch in $Q \cup \{x\}$. The node extending x violates the conclusion of Proposition 2.3, so the extension of \equiv to $T \upharpoonright C \cup \{\alpha\}$ is no longer honest and therefore no t.e.r.

If $\operatorname{cf}(\alpha) = \mu$ yet R_{α} neither is an antichain nor does it code some t.e.r. on T, then we simply choose any dense μ -subset Q of $[T \upharpoonright \alpha]$ and extend the branches in Q to level T_{α} . This finishes the recursive construction of T.

By standard \diamondsuit_{κ} -arguments, the result T of this construction is a κ -Souslin tree that admits no t.e.r. So by the Representation Lemma $\mathbb{B} = \operatorname{RO} T$ has no proper and atomless complete subalgebra.

Note that, while in the above construction we explicitly talk about a *non-trivial* t.e.r., we will from now on tacitly assume the t.e.r.s proposed by a \diamond -sequence not to be trivial, i.e., not to be the identity.

We close this section with a proposition on the local nature of niceness. For this and also for later purposes, we say that a Souslin subalgebra \mathbb{A} is nowhere nice in the κ -Souslin algebra \mathbb{B} if for every $b \in \mathbb{B}^+$ the relative subalgebra $b\mathbb{A} = \{ba \mid a \in \mathbb{A}\}$ is not nice in the relative algebra \mathbb{B} [b.

Proposition 2.9. Let \mathbb{B} be a κ -Souslin algebra and \mathbb{A} a Souslin subalgebra of \mathbb{B} . Let $b := \sum \{x \in \mathbb{B} \mid x \mathbb{A} \text{ is nice in } \mathbb{B} \upharpoonright x\}$. Then $b \mathbb{A}$ is nice in $\mathbb{B} \upharpoonright b$ and $(-b) \mathbb{A}$ is nowhere nice in $\mathbb{B} \upharpoonright (-b)$.

Proof. It follows directly from the definitions that $(-b)\mathbb{A}$ is nowhere nice.

Clearly, the property " $x\mathbb{A}$ is nice in $\mathbb{B} \upharpoonright x$ " descends from x to $y \leq_{\mathbb{B}} x$. We prove that this property is also preserved under taking arbitrary sums. So let M be a subset of \mathbb{B} , such that all elements of M have this property. We want to show that for $x := \sum M$ the subalgebra $x\mathbb{A}$ is nice in $\mathbb{B} \upharpoonright x$. We can without loss of generality assume that M is an antichain. Then M is of cardinality $< \kappa$. Furthermore we can assume that also h''M is an antichain by the argument used at the beginning of the proof of the Representation Lemma 2.7. We finally assume that there is a

Souslinization T of B such that M is a subset of T_1 , the first nontrivial level of T, and T carries a t.e.r. \equiv which represents \mathbb{A} .

Now for every element r of M there is by part c) of the Representation Lemma 2.7 a club C_r of κ , such that \equiv is nice on $T(r) \upharpoonright C_r$. Let C be the club intersection of all sets C_r for $r \in M$. We claim that \equiv is nice on the subtree

$$S = \bigoplus_{r \in M} T(r) \restriction C$$

of $T \upharpoonright C$. So let $s \equiv t$ in S and s' > s. If there is a unique member r of M below both nodes s and t, then we can directly apply the hypothesis on r. Otherwise we would still have $r_s := s \upharpoonright 1 \equiv t \upharpoonright 1 =: r_t$ and $h(r_s) = h(r_t)$ by our assumption that h''M is an antichain. But then we have that $r_th(s') > 0$. So there is a node $t^* > r_t$ equivalent to s'. Finally, by niceness above r_t , there also is a node t' above t such that $t' \equiv t^* \equiv s'$.

3. Large subalgebras

Large subalgebras can be regarded as the simplest type of subalgebras⁴. They are closely related to symmetries of the Souslin algebra and admit a detailed yet clear representation.

Definition 3.1. Let B be a complete Boolean algebra. We say that C is a large subalgebra of B, if there is an antichain M of B, such that $\langle A \cup M \rangle^{cm} = B$. We say that a large subalgebra A of B is μ -large for some cardinal μ if there is an antichain M of size μ such that $\langle A \cup M \rangle^{cm} = B$.

Note that large subalgebras of κ -Souslin algebras are always atomless and therefore Souslin subalgebras, since for every atom a of \mathbb{A} , the set $M \cup \{a\}$ of size $< \kappa$ would have to generate the relative algebra $\mathbb{B} \upharpoonright a$. But this is impossible, because $\langle M \rangle^{\mathrm{cm}}$ is itself atomic.

As a first example we consider a κ -Souslin algebra that has exactly one non-trivial subalgebra, and this subalgebra is large.

Example 3.2. Let \mathbb{B} be a simple κ -Souslin algebra, i.e., that \mathbb{B} has no proper atomless and complete subalgebra, cf. Example 2.8.

We claim that the κ -Souslin algebra $\mathbb{C} := \mathbb{B} \times \mathbb{B}$ has exactly one proper atomless and complete subalgebra, which is furthermore 1-large in \mathbb{C} .

Clearly, \mathbb{C} has the large subalgebra

$$\mathbb{A} := \{(b,b) \mid b \in \mathbb{B}\},\$$

⁴In [12, pp.266] such subalgebras are called "locally equal" and studies in the general context of forcing with complete Boolean algebras.

and \mathbb{A} is 1-large in \mathbb{C} , because $\langle \mathbb{A} \cup \{(1,0)\}\rangle^{\mathrm{cm}} = \mathbb{C}$. As we have $\mathbb{A} \cong \mathbb{B}$, there are no (atomless and complete) subalgebras of \mathbb{C} below \mathbb{A} .

On the other hand we have

$$\mathbb{C}[(1,0) \cong \mathbb{C}[(0,1) \cong \mathbb{B}.$$

So if there was any other atomless and complete subalgebra \mathbb{A}' of \mathbb{C} , then $(0,1) \cdot \mathbb{A}'$ or $(1,0) \cdot \mathbb{A}'$ would be a nontrivial subalgebra of the respective relative algebra of \mathbb{C} . But the latter are simple. So the existence of such a subalgebra \mathbb{A}' is impossible.

In general, $(2^{\aleph_0}$ -)large subalgebras always occur whenever a κ -Souslin algebra has non-trivial symmetries.

Theorem 3.3. Let \mathbb{B} be a κ -Souslin algebra and $\varphi \in \operatorname{Aut} \mathbb{B}$. Then the set of fixed points of φ is a large subalgebra \mathbb{A} of \mathbb{B} . In particular, if $\kappa > 2^{\aleph_0}$ then \mathbb{A} is 2^{\aleph_0} -large.

Proof. We use Frolik's Theorem, a deep result from the theory of complete Boolean algebras (cf. [16, Theorem 13.23]): For every automorphism f of a complete Boolean algebra A, there is a partition of unity $\{a_0, a_1, a_2, a_3\}$ in A such that $f \upharpoonright (A \upharpoonright a_0)$ is the identity and for i > 0 we have $f(a_i) \cdot a_i = 0$.

We consider the at most countable family $(\varphi^n \mid n \in \mathbb{Z})$ of automorphisms of \mathbb{B} and let $(a_{n0}, a_{n1}, a_{n2}, a_{n3})$ be a partition of unity given by Frolík's Theorem for φ^n , $n \in \mathbb{Z}$. Let M be the set of atoms of the complete subalgebra of \mathbb{B} that is (completely) generated by the elements $\varphi^k(a_{ni})$ for $k, n \in \mathbb{Z}$ and i < 4. Then M has by distributivity of \mathbb{B} at most 2^{\aleph_0} elements. Note that $\varphi \upharpoonright M$ is a permutation of M and if for some $x \in M$ and $n \in \omega$ we have $\varphi^n(x) = x$, then the restriction of φ^n to $\mathbb{B} \upharpoonright x$ is the identity map.

We claim that $\langle \mathbb{A} \cup M \rangle^{\mathrm{cm}} = \mathbb{B}$. Since M is an antichain, it suffices to show that for all $x \in M$ and $b \in \mathbb{B} \upharpoonright x$ there is a member $a \in \mathbb{A}$, i.e., a fixed point of φ , with ax = b. For all integers n we know that either $\varphi^n(b) = b$ or $\varphi^n(b)$ is disjoint from x. Let $a = \sum \{\varphi^n(b) \mid n \in \mathbb{Z}\}$ and it is easy to check that the proof is finished. \square

Note that the algebra \mathbb{C} from Example 3.2 has exactly two automorphisms: the identical mapping and flipping of coordinates.

The following technical lemma states the existence of *optimal witnesses of large-ness*. With these witnesses at hand we can easily deduce the main structural properties of large subalgebras.

Lemma 3.4. Let \mathbb{A} be a μ -large subalgebra of the κ -Souslin algebra \mathbb{B} . Define $X := \{x \in \mathbb{B} \mid \mathbb{B} | x = x\mathbb{A}\}.$

- a) The set X is dense in \mathbb{B} , and $x <_{\mathbb{B}} y \in X$ imply $x \in X$.
- b) For every $x \in X$ the restriction of the canonical projection $h : \mathbb{B} \to \mathbb{A}$ to $\mathbb{B} \upharpoonright x$, i.e. the map

$$\varphi: \mathbb{B} \upharpoonright x \to \mathbb{A} \upharpoonright h(x), \quad b \mapsto h(b)$$

is an isomorphism between $\mathbb{B} \upharpoonright x$ and $\mathbb{A} \upharpoonright h(x)$. The inverse map of $h \upharpoonright (\mathbb{B} \upharpoonright x)$ is given by multiplication with x:

$$\varphi^{-1}: \mathbb{A} \upharpoonright h(x) \to \mathbb{B} \upharpoonright x, \quad a \mapsto ax.$$

- c) Every subset $M \subseteq X$ with $\sum M = 1$ (or even $1 \sum M \in X$) witnesses that \mathbb{A} is large.
- d) For every $x \in X$ there is a maximal element y of X above x.

If additionaly $\mathbb{A} \upharpoonright a \neq \mathbb{B} \upharpoonright a$ for all $a \in \mathbb{A}^+$ and Y denotes the set of maximal elements of X, then the following hold as well.

- e) The image of Y under h is a maximal antichain of \mathbb{A} .
- f) Every set of pairwise disjoint elements of Y is extendible to a maximal antichain $\subset Y$ of \mathbb{B} .
- g) For every maximal antichain $M \subseteq Y$ we have h''M = h''Y.

The announced optimal witnesses of largeness are simply the partitions of unity that are subsets of the set Y defined in the lemma.

Proof. We only give proofs of points c-e). The rest is then trivial or follows by standard arguments.

For the proof of c) pick a subset $M \subset X$ with $\sum M = 1$. We want to show that every $b \in \mathbb{B}^+$ is of the form

$$b' = \sum \{xh(bx) \mid x \in M, xb >_{\mathbb{B}} 0\}.$$

It is clear that $b' \geq_{\mathbb{B}} b$, because $h(b), \sum M \geq b$. On the other hand we conclude from part b) that xh(bx) = bx for $x \in X$, so $b' \leq_{\mathbb{B}} b$ as well. So we have $\langle \mathbb{A} \cup M \rangle^{\mathrm{cm}} = \mathbb{B}$.

To prove the existence of maximal elements of X, it is enough to verify that X is closed under taking sums over increasing sequences of length $< \kappa$. So let $x_{\alpha} \in X$ and $x_{\alpha+1} >_{\mathbb{B}} x_{\alpha}$ for all $\alpha < \delta$ ($< \kappa$). Set $x = \sum x_{\alpha}$. We prove that every x_{α} is in x A as follows. Fix α . For every $\beta > \alpha$ pick an element $a_{\beta} \in A$ that satisfies $x_{\beta}a_{\beta} = x_{\alpha}$. Defining $a = \prod_{\beta > \alpha} a_{\beta}$ we get $x_{\beta}a = x_{\alpha}$ for all $\beta > \alpha$ and therefore (using the infinite distributive law available in \mathbb{B})

$$xa = \sum x_{\beta}a = x_{\alpha}.$$

But then we already have $x\mathbb{A} = \mathbb{B} \upharpoonright x$, because every element $y \in \mathbb{B} \upharpoonright x$ can be decomposed into a sum

$$y = \sum_{\alpha \leq y} y_{\alpha}$$
 with $y_{\alpha} := y(x_{\alpha+1} - x_{\alpha})$.

By the same argument as above we have $y_{\alpha} \in x\mathbb{A}$ for all $\alpha < \delta$.

Concerning the proof of part e) of the lemma, we know by a) and d) that $1 = \sum Y$ and therefore $1 = \sum h''Y$. It remains to show that for all pairs $x, y \in Y$ with

 $h(x)h(y) >_{\mathbb{B}} 0$ we have h(x) = h(y). To reach a contradiction we assume the existence of a pair $x, y \in Y$ with a non-empty intersection of the h-images, $h(x)h(y) >_{\mathbb{B}} 0$, yet $h(x) - h(y) >_{\mathbb{B}} 0$. This implies $x - h(y) >_{\mathbb{B}} 0$, for otherwise h(x)h(y) = 0. We set

$$z := y + (x - h(y))$$

and get that $z >_{\mathbb{B}} y$ and zh(y) = y. This shows that $y, z - y \in z\mathbb{A}^+$ and implies thus that $z \in X$ (because $z - y <_{\mathbb{B}} x$ so $z - y \in X$), contradicting the maximality of y in X.

We are prepared to state and prove the key properties of large subalgebras. Figure 2 below corresponds to part c) of the theorem in terms of Souslinizations and illustrates the strong resemblance between superalgebra and large subalgebra.

Theorem 3.5. Let \mathbb{A} be a large subalgebra of the κ -Souslin algebra \mathbb{B} . Then the following hold.

- a) \mathbb{A} is a nice subalgebra of \mathbb{B} .
- b) There is a group G of size less than κ of automorphisms of \mathbb{B} such that

$$\mathbb{A} = \{ x \in \mathbb{B} \mid (\forall \varphi \in G) \, \varphi(x) = x \}.$$

If furthermore μ is the minimal cardinal such that \mathbb{A} is μ -large in \mathbb{B} , then G can be chosen of size $\aleph_0 \cdot \mu$.

c) There are a maximal antichain N in \mathbb{A} and a map f associating a cardinal f(a) to each member a of N such that we have the following representation of \mathbb{B} over \mathbb{A} :

$$\mathbb{B} \cong \prod_{a \in N} (\mathbb{A} \upharpoonright a)^{f(a)}.$$

d) If \mathbb{B} is homogeneous, then \mathbb{A} and \mathbb{B} are isomorphic.

Proof. Let $h: \mathbb{B} \to \mathbb{A}$ be the canonical projection. To prove a) we construct a Souslinization of T such that h''T souslinizes \mathbb{A} . Let T_1 , the first non-trivial level of T, be a maximal antichain of \mathbb{B} consisting of maximal elements x with $x\mathbb{A} = \mathbb{B} \upharpoonright x$, i.e., $T_1 \subset Y$ in the notation used above. Then $N := h''T_1$ is an antichain in \mathbb{A} . Now fix a pre-image $b_a \in h^{-1}(a) \cap T_1$ for each $a \in N$. To construct the higher successor levels, we first refine the nodes above b_a for each $a \in N$ and then copy these refinements by virtue of the isomorphisms

$$\psi_b: \mathbb{B} \upharpoonright b_a \to \mathbb{B} \upharpoonright b, \ x \mapsto bh(x)$$

for all $b \in h^{-1}(a) \cap T_1$. This automatically transfers to limit levels and guarantees that also for limit α the set $h''T_{\alpha}$ consists of products over cofinal branches in $T \upharpoonright \alpha$.

Finally, in order to prove that the relation

$$s \equiv t : \iff \operatorname{ht} s = \operatorname{ht} t \text{ and } h(s) = h(t)$$

is a nice t.e.r., let $s \equiv t$ on level T_{α} and let s' be a T-successor of s. Then the node $t' = (t \upharpoonright 1) \cdot h(s')$ is the witness for this instance of niceness.

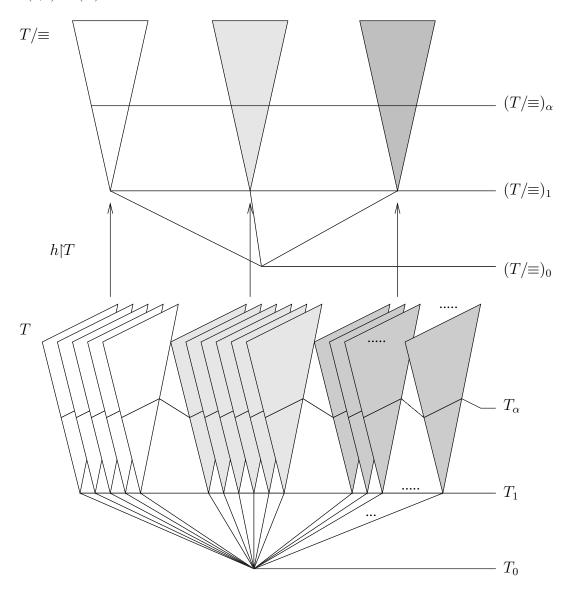


FIGURE 2. Nice representation of a large subalgebra $\mathbb{A}=\mathrm{RO}(T/\equiv)$ (above) of $\mathbb{B}=\mathrm{RO}\,T$ (below). The algebras and their Souslinizations can be decomposed in relative parts (here there are 3), such that the \mathbb{B} -part consists of a Cartesian product (resp. a tree sum) of the corresponding \mathbb{A} -part. The factors/tree summands of the \mathbb{B} -part are interconnected by the isomorphism coming from the projection h

For the proof of b) let $A \subset Y$ be a maximal antichain in \mathbb{B} such that $(\mathbb{A} \cup A)^{cm} = \mathbb{B}$. For $a, b \in A$ with h(a) = h(b) let

$$\varphi_{ab}: \mathbb{B} \to \mathbb{B}, \quad x \mapsto x - (a+b) + a \cdot h(bx) + b \cdot h(ax),$$

which is a self-inverse automorphism of \mathbb{B} interchanging a with b. The fixed points of φ_{ab} form the subalgebra

$$\mathbb{B}[(-(a+b)) \cup (a+b)\mathbb{A}]$$
.

Letting G be the group of automorphisms of \mathbb{B} generated by the set

$$\{\varphi_{ab} \mid a, b \in A, h(a) = h(b)\}\$$

we see that this is as stated in the theorem.

Part d) readily follows from part c) which we prove now. Let $A \subset Y$ be as above and set $N := h^{n}A = h^{n}Y$. Define $f: N \to \kappa$ by $f(a) := |h^{-1}(a) \cap A|$. Taking into account that for each $b \in a$ we have that

$$\mathbb{B} \upharpoonright b = b \mathbb{A} \cong \mathbb{A} \upharpoonright h(b)$$

we get as a Cartesian product

$$\mathbb{B} \cong \prod_{b \in A} \mathbb{B} \upharpoonright b \cong \prod_{b \in A} \mathbb{A} \upharpoonright h(b) \cong \prod_{a \in N} (\mathbb{A} \upharpoonright a)^{f(a)}.$$

This finishes the proof.

4. Nowhere large subalgebras

We now consider more general algebras with more involved representation features.

Definition 4.1. Let \mathbb{B} be a κ -Souslin algebra, T be a Souslinization of \mathbb{B} and \mathbb{A} a complete subalgebra of \mathbb{B} .

- a) A is nowhere large (in \mathbb{B}) if for all $b \in \mathbb{B}^+$ we have $b\mathbb{A} \neq \mathbb{B}|b$.
- b) A t.e.r. \equiv on T is μ -nice (for a cardinal $\mu < \kappa$) if it is nice and for all $\alpha < \beta < \operatorname{ht}(T)$ and

$$(\forall r \in (s \upharpoonright \alpha)/\equiv) \quad |\{t \in s/\equiv | t \upharpoonright \alpha = r\}| \ge \mu,$$

i.e., for all $s \in T_{\beta}$, the projections $t \mapsto t \upharpoonright \alpha$, when restricted to the \equiv -class of s, are $(\geq \mu)$ -to-one.

c) A is ∞ -nice in B if for one/any cardinal $1 < \mu < \kappa$ there is a club C of κ , such that $T \upharpoonright C$ carries an μ -nice t.e.r. \equiv that represents A.

Note that in point c) one cardinal μ suffices as an easy argument shows that a 2-nice t.e.r. on a κ -Souslin tree can be turned into a μ -nice t.e.r. for any $\mu < \kappa$ by concentrating the tree on a club set of levels.

Remark 4.2. If A is any atomless complete subalgebra of the κ -Souslin algebra \mathbb{B} , then obviously we have for

$$x := \sum \{b \in \mathbb{B} \mid b\mathbb{A} = \mathbb{B} \restriction b\}$$

that $x\mathbb{A}$ is large in $\mathbb{B} \upharpoonright x$ while $(-x)\mathbb{A}$ is nowhere large in $\mathbb{B} \upharpoonright (-x)$. This corresponds to the situation for niceness as stated in Proposition 2.9. Note that, if a nice t.e.r. \equiv for the large portion $x\mathbb{A}$ as in the last section is found, then the \equiv -classes on limit levels are discrete (above x).

Before we give a first example of an ∞ -nice subalgebra, we turn to clarify the interrelationship between the new notions.

Proposition 4.3. Let \mathbb{A} be a nice subalgebra of the κ -Souslin algebra \mathbb{B} . Then \mathbb{A} is ∞ -nice if and only if it is nowhere large.

Plainly: the ∞ -nice subalgebras are just the nice and nowhere large ones.

Proof. Let T souslinize \mathbb{B} , and let the nice t.e.r. \equiv on T represent \mathbb{A} . We start from left to right, so let \equiv be 2-nice. We show for every node s of T, that $\mathbb{A} \cdot s \neq \mathbb{B} \upharpoonright s$. Pick any node t above s in T. Since \equiv is 2-nice, there is a node r above s and equivalent to t, so $t \notin \mathbb{A} \cdot s$. So what we actually have shown, is $\mathbb{A} \cdot s = \{0, s\}$.

For the other implication let \mathbb{A} be nowhere large. We define a club set $C \subset \kappa$, such that the restriction of Ξ to $T \upharpoonright C$ is 2-nice. The inductive construction of C is straightforward once we have proven the following claim.

Claim. Given any $\alpha < \kappa$ there is a $\beta \in (\alpha, \kappa)$ such that for all nodes $t \in T_{\beta}$ there is a node $t' \in T_{\beta} \setminus \{t\}$ above $t \upharpoonright \alpha$ and equivalent to t.

To prove the claim by contradiction, assume that there is an ordinal α such that for all $\beta > \alpha$ there is a node t^{β} of level T_{β} such that above its predecessor $t^{\beta} | \alpha$ there is no other node equivalent to t^{β} . By the pigeon hole principle one of the $<\kappa$ many nodes of level T_{α} sits underneath κ many of these nodes t^{β} . So we can assume, that we have one node $s^* \in T_{\alpha}$ such that $s^* < t^{\beta}$ for all $\beta > \alpha$. But then these nodes t^{β} for $\beta > \alpha$ span a tree of height κ which by the Subtree Lemma 1.1 contains a canonical subtree $\{s < r\} \cup T(r)$ of T for some node r above s^* . But then in turn we have that $\mathbb{A} \cdot r = \mathbb{B} | r$, which contradicts the hypothesis on \mathbb{A} to be nowhere large in \mathbb{B} .

The basic example we consider now can easily be generalized to κ -Souslin algebras for regular $\kappa > \aleph_0$.

Example 4.4. Let S and T be \aleph_0 -splitting ω_1 -trees such that their tree product $S \otimes T$ is ω_1 -Souslin⁵. Set $\mathbb{B} := \mathrm{RO}(S \otimes T)$ and $(s,t) \sim (u,v)$ if and only if s = u.

⁵For example, the principle \diamondsuit implies, that for every given ω_1 -Souslin tree S there is an ω_1 -Souslin tree T, such that $S \otimes T$ is c.c.c.. For a proof of this fact see [19, Lemma 7.3].

Then \sim is an \aleph_0 -nice t.e.r.: If $s <_S s'$ and $\operatorname{ht}_S(s) = \operatorname{ht}_T(t) = \operatorname{ht}_T(r)$ and $t' >_T t$, then for any r' > r we have that $(s', t') \sim (s', r')$. So \sim is nice. The \aleph_0 -part follows from the splitting assumption on T. The quotient tree $(S \otimes T)/\sim$ is obviously isomorphic to S, and the subalgebra \mathbb{A} represented by \sim is ∞ -nice in \mathbb{B} .

Remark 4.5. Note that not not all ∞ -nice subalgebras of κ -Souslin algebras do have a complement as in the example above. For example, one of the subalgebras, that will be constructed in Section 7.2, call it \mathbb{C} , is ∞ -nice, yet isomorphic to the superalgebra \mathbb{B} . If there was a subalgebra \mathbb{C}' of \mathbb{B} independent of \mathbb{C} , then an isomorphic copy of \mathbb{C}' would exist inside of \mathbb{C} . This contradicts the chain condition satisfied by \mathbb{B} .

4.1. Homogeneity and ∞ -nice subalgebras. Recall that a Boolean algebra B is homogeneous, if for all pairs $b \in B^+$ there is a Boolean isomorphism between B and $B \upharpoonright b$, while homogeneity of the tree T means that for all pairs of nodes s,t of the same height in T the trees T(s) and T(t) of nodes above s and t respectively are isomorphic

Proposition 4.6. Let κ be an uncountable, regular cardinal. Then every homogeneous κ -Souslin algebra has a homogeneous Souslinization.

Proof. Let \mathbb{B} be homogeneous and T be any Souslinization of \mathbb{B} . Our task is to find a club $C \subset \kappa$ such that $T \upharpoonright C$ is a homogeneous κ -Souslin tree. By the homogeneity of \mathbb{B} we can choose a Boolean isomorphism $\psi_{st} : \mathbb{B} \upharpoonright s \to \mathbb{B} \upharpoonright t$ for every pair $s, t \in T$ of the same height $\alpha < \kappa$. By the Restriction Lemma 1.2 for Isomorphisms, there is also a club $C_{st} \subset \kappa$ containing α , such that $\psi_{st} \upharpoonright (T(s) \upharpoonright C_{st})$ is an isomorphism onto $T(t) \upharpoonright C_{st}$.

Finally, we define C to be the range of the normal sequence (γ_{ν}) which is given as follows: Set $\gamma_0 = 0$ and let for $\nu < \kappa$

$$\gamma_{\nu+1} := \min \bigcap_{s,t \in T_{\gamma_{\nu}}} C_{st} \setminus (\gamma_{\nu} + 1),$$

the limit values of the sequence are then determined by normality.

In Section 3 we have seen that the existence of large subalgebras is linked to the existence of automorphisms. Yet if there are enough automorphisms, which here means: if \mathbb{B} is homogeneous, then we even get subalgebras of different kinds (∞ -nice and nowhere nice, see also Theorem 4.9).

Theorem 4.7. Every homogeneous κ -Souslin algebra has an ∞ -nice subalgebra.

Proof. Let T be a homogeneous κ -Souslin tree, i.e., for every pair s, t of nodes on the same level of T there is a tree isomorphism between T(s) and T(t). We inductively show for $\alpha < \kappa$ that $T \upharpoonright \alpha$ carries an 2-nice t.e.r. \equiv using the homogeneity of T. After

construction stage $\alpha < \kappa$ we will have fixed the t.e.r. \equiv on $T \upharpoonright \alpha + 1$, sets $I_{\gamma} \subset T_{\gamma}$ of representatives of the \equiv -classes for $\gamma \leq \alpha$ and a family of isomorphisms

$$\{\varphi_{st}: T(s) \cong T(t) \mid s \equiv t, \, \operatorname{ht}(s) \leq \alpha\}.$$

These isomorphisms commute in the sense that

$$\varphi_{tt} = \mathrm{id}_{T(t)}$$
 and $\varphi_{st} = \varphi_{rt} \circ \varphi_{sr}$ (for all $r \equiv_{\gamma} s \equiv_{\gamma} t$).

Furthermore they have the following coherence property: for $s, t \in T_{\alpha}$ and $r = s \upharpoonright \gamma$, $u = t \upharpoonright \gamma$ where $\gamma < \alpha$ and $\varphi_{ru}(s) = t$ we have $\varphi_{st} = \varphi_{ru} \upharpoonright T(s)$. These isomorphisms φ_{st} will help to guarantee that \equiv always remains honest.

We will use the representatives from the set I_{α} for the constructions of both the t.e.r. and the tree isomorphisms. We will first define the relevant piece of structure above the representative nodes $r \in I_{\alpha}$ and then copy it over to the equivalent nodes by virtue of the tree isomorphisms that have already been fixed.

In the case of the successor ordinal $\alpha + 1$, we consider the equivalence relation \equiv on T_{α} , the set of representatives $I_{\alpha} \subset T_{\alpha}$ and the isomorphisms φ_{st} for $s \equiv_{\alpha} t$, all given by the inductive hypothesis. For $s \in T_{\alpha}$ denote by r_s the unique element of $s/\equiv \cap I_{\alpha}$. In order to define \equiv on $T_{\alpha+1}$, we first choose for each $r \in I_{\alpha}$ a partition of succ(r) into 2 sets P_0^r, P_1^r of equal cardinality.

Then for all $s, t \in T_{\alpha+1}$ we let s and t be equivalent if their (immediate) predecessors s^- and t^- are and if their images under the tree isomorphisms sending them above the representative node $r = r_{s^-} = r_{t^-}$ lie in the same member of the partition, both in P_0^r or both in P_1^r :

$$s \equiv t : \iff s^- \equiv t^- \text{ and } (\exists i \in \{0,1\}) \varphi_{s^-r}(u), \varphi_{t^-r}(v) \in P_i^r.$$

Afterwards, we pick a set of representatives $I_{\alpha+1} \subset \bigcup_{r \in I_{\alpha}} \operatorname{succ}(r)$.

Finally, we have to choose the tree isomorphisms φ_{st} for all equivalent pairs $s, t \in T_{\alpha+1}$ such that the coherence requirement as formulated above is satisfied. Fix a representative $r \in I_{\alpha+1}$ and choose for a pair of successors (s,t) of $r^- := r \upharpoonright \alpha$, both equivalent but unequal to r, isomorphisms φ_{st} and φ_{sr} respectively and let $\varphi_{st} = \varphi_{rt} \circ \varphi_{rs}^{-1}$. For s,t both equivalent to r, but not necessarily successors of r^- , define

$$\varphi_{st} := (\varphi_{r^-t^-} \upharpoonright T(v)) \circ \varphi_{uv} \circ (\varphi_{s^-r^-} \upharpoonright T(s)),$$

where $u := \varphi_{s^-r^-}(s)$ and $v := \varphi_{t^-r^-}(t)$.

Whenever $\alpha < \kappa$ is a limit ordinal, we have no choice for the equivalence relation \equiv on T_{α} : For $s, t \in T_{\alpha}$ we let $s \equiv t$ if and only if $s \upharpoonright \gamma \equiv t \upharpoonright \gamma$ for all $\gamma < \alpha$.

Before defining the remaining tree isomorphisms we check, that this definition yields a nice t.e.r. up to level T_{α} . So fix $s \in T_{\alpha}$. For every $\gamma < \alpha$ and $u \equiv s | \gamma$ there is some $t \in T_{\alpha}$ equivalent to s and above u, namely $t = \varphi_{s | \gamma, u}(s)$. So niceness is maintained up to level α , and for equivalent pairs (s, t) of this kind we already have

the isomorphism $\varphi_{st} = \varphi_{s \upharpoonright \gamma, t \upharpoonright \gamma} \upharpoonright T(s)$ at hand. But there can be equivalent nodes s and t on level α , such that for all their pairs u, v of respective predecessors on the same level we have $\varphi_{uv}(s) \neq t$. However, each \equiv -class r/\equiv divides into a partition \mathcal{P} such that for every pair of nodes $s, t \equiv r$, both s and t are elements of the same member of \mathcal{P} if and only if they have such an inherited isomorphism $\varphi_{st} = \varphi_{s \upharpoonright \gamma, t \upharpoonright \gamma} \upharpoonright T(s)$.

After choosing a set of representatives J for the partition \mathcal{P} and fixing isomorphisms φ_{st} for representatives $s, t \in J$ we can construct the still missing isomorphisms in the same manner as above.

We finally choose a set I_{α} of representatives for the \equiv -classes of the limit level T_{α} without any further restriction.

This finishes the construction of \equiv , and we hope that it is clear that the result is a 2-nice t.e.r. on T.

4.2. **Hidden symmetries.** In Section 5.3 we will construct a κ -Souslin algebra with an ∞ -nice subalgebra but without large subalgebras, i.e., without automorphisms except for the identity. The next lemma and the subsequent theorem say that in such a situation there have to be other subalgebras, in particular subalgebras which do have symmetries.

This stands in sharp contrast to the subalgebra to be constructed in Section 6 which is essentially a unique subalgebra.

Lemma 4.8. Let \mathbb{A} be an ∞ -nice subalgebra of a κ -Souslin algebra \mathbb{B} . Then there is an ∞ -nice subalgebra \mathbb{C} of \mathbb{B} , such that \mathbb{C} admits a non-trivial automorphism φ and \mathbb{A} is the subalgebra of \mathbb{C} that consists of the fixed points under φ . So \mathbb{A} is a large subalgebra of \mathbb{C} , yet for all $a \in \mathbb{A}^+$ we have $\mathbb{A} \upharpoonright a \neq \mathbb{C} \upharpoonright a$.

Proof. Let T souslinize \mathbb{B} and let \equiv on T be ∞ -nice and represent \mathbb{A} in T. Choose any limit $\lambda < \kappa$ and let \simeq coincide with \equiv on $T \upharpoonright (\lambda + 1)$. Now we divide every \simeq -class a of level T_{λ} in two indexed parts $a = a_0 \dot{\cup} a_1$, such that for every pair $s \in a_0$ and $r <_T s$ there is a node $t \in a_1$ above r and vice versa, i.e., for $r < t \in a_1$ there exists $s \in a_0 \cap T(r)$. Another way to formulate this condition is to say that we consider T_{λ} as a subspace of $[T \upharpoonright \lambda]$ and require that each class a divided in two parts that lie densely in a. This can be done after choosing an enumeration of minimal length of the predecessor set $\bigcup \{s \in T \mid (\exists t \in a)s <_T t\}$. The family of these partitions gives rise to a map $i: T_{\lambda} \to \{0,1\}$, associating to every node s the index s with $s \in a_{i(s)}$.

Now let for $\alpha > \lambda$ and $s, t \in T_{\alpha}$

$$s \simeq t : \iff s \equiv t \text{ and } i(s \upharpoonright \lambda) = i(t \upharpoonright \lambda).$$

Then \simeq is clearly ∞ -nice when restricted to $T \upharpoonright C$, where $C = \{0\} \cup \kappa \setminus \lambda + 1$. This shows that the subalgebra $\mathbb{C} := \langle \sum s/\simeq | s \in T \rangle^{\mathrm{cm}}$ is nice and nowhere large in \mathbb{B} .

Furthermore, for every $s \in T$ above level $\lambda + 1$ the \equiv -class of s is divided into exactly two \simeq -classes. So we can define the automorphism φ of $(T \upharpoonright C)/\simeq$ that for each \equiv -class interchanges the two \simeq -classes. Then φ naturally extends to an automorphisms of $\mathbb C$ that has $\mathbb A$ as its fixed point algebra which is by Theorem 3.3 large in $\mathbb C$. (In fact $\mathbb A$ is 1-large in $\mathbb C$ as witnessed by $\{\sum f^{-1}, \{0\}\}$.)

4.3. Nowhere nice subalgebras. The main idea of the last proof, that of dividing the classes on a limit level in dense subsets, can be also used to construct nowhere nice subalgebras.

Theorem 4.9. If a κ -Souslin algebra \mathbb{B} has a nice and nowhere large subalgebra \mathbb{A} then there is a nowhere nice subalgebra \mathbb{C} of \mathbb{B} and \mathbb{A} is an ∞ -nice subalgebra of \mathbb{C} .

Proof. Let T souslinize \mathbb{B} and let \equiv represent \mathbb{A} in T. We inductively construct an almost nice, yet not nice refinement \simeq of \equiv , which represents \mathbb{C} as stated in the theorem. Up to level T_{ω} the new relation coincides with \equiv . Limit levels have to be treated canonically, and on double successor steps α as well as on successors α of limits with uncountable cofinality, we choose the minimal possible refinement by meeting

$$s \simeq t : \iff s \equiv t \text{ and } s^- \simeq t^-.$$

Let now $\alpha < \kappa$ be a limit of countable cofinality, cf $\alpha = \aleph_0$. Note that the (induced) \simeq -classes on the space $[T \upharpoonright \alpha]$ form closed subsets without isolated points. Therefore, regarding T_{α} as a subspace of $[T \upharpoonright \alpha]$, the former also divides into a partition whose members are closed subsets without isolated points (as T_{α} is suitable for \simeq on $T \upharpoonright \alpha$). To define \simeq on level $T_{\alpha+1}$ we first refine \simeq on T_{α} to the equivalence relation \sim in a way such that every \simeq class splits in two \sim -classes and for every $s \simeq t \in T_{\alpha}$ and $u <_T t$, there is a successor of u in s/\sim . (As in the proof of Lemma 4.8, one could also say that the \sim -classes lie densely in the sense of $[T \upharpoonright \alpha]$ in the \simeq -classes.) Then let for $s, t \in T_{\alpha+1}$:

$$s \simeq t : \iff s \equiv t \text{ and } s \upharpoonright \alpha \sim t \upharpoonright \alpha.$$

This procedure clearly refines \equiv to an almost nice t.e.r. \simeq .

Next we show, that no Souslinization S of \mathbb{B} admits a nice t.e.r. representing \mathbb{C} . By the Restriction Lemma 1.2 we only need to consider restrictions $S = T \upharpoonright C$ of T to a club $C \subset \kappa$. So let $\alpha \in C$ be a limit ordinal of countable cofinality and choose $s, t \in T_{\alpha}$, such that $s \simeq t$ but $s \not\sim t$. Then for every $r \in T \upharpoonright C$ above s there is no successor of t which is \simeq -equivalent to r. So s, t, r witness that \mathbb{C} is not nice.

If we now let C be the set of all limit ordinals below κ joined by 0, and defining on T/\simeq the t.e.r. \approx by

$$(s/\simeq) \approx (t/\simeq) : \iff s \equiv t,$$

then it is easy to see, that the ∞ -niceness descends from \equiv to \approx .

Remark 4.10. By Theorem 4.9 and since niceness and largeness are local properties, if the κ -Souslin algebra $\mathbb B$ has a non-large subalgebra of $\mathbb B$, then there is also one which is not nice. In particular, if $\mathbb B$ is homogeneous or if $\mathbb B$ has a pair of independent Souslin subalgebras, then $\mathbb B$ has nowhere nice subalgebras.

Part 2. Some constructions of κ -Souslin algebras with certain subalgebras

5. T.E.R.S AND TOPOLOGY

In this section we develop topological tools which we use to construct Souslin algebras with nowhere large subalgebras. For these tools to be applicable also in cases where $\kappa > \aleph_1$ we have to generalize a few notions and facts concerning Baire Category.

We then formulate and prove the Reduction Lemma for t.e.r.s which roughly states that (under favorable circumstances) for a given tree of limit height with a t.e.r. on it, there is an extension of the tree such that the t.e.r. remains honest, i.e. is preserved.

The constructions carried out in the subsequent sections use \diamondsuit hypotheses. The Reduction Lemma and the surrounding lemmata can of course also be applied in forcing constructions of generic κ -Souslin trees once the hypothesis on cardinal arithmetic is satisfied in the ground model.

5.1. Some basic descriptive set theory for weight μ . We introduce some variants of several classical topological notions that we will use in the Souslin tree constructions in subsequent sections. The spaces of interest are all homeomorphic to ${}^{\mu}\mu$, the analog of Baire space $\mathbb N$ for some regular cardinal μ . Furthermore, the generalizations of some classical results as formulated here only hold in case that $\mu^{<\mu} = \mu$. (While in the case of $\mu = \aleph_0$ this follows from the axiom of choice, it is an extra assumption extending ZFC + \diamondsuit_{μ^+} if μ is uncountable.)

So in this section (and also in the remainder of the article) μ denotes a regular cardinal satisfying $\mu^{<\mu} = \mu$. Letting $\kappa := \mu^+$, the letter T is used in this section for a μ -closed and κ -normal tree of height $<\kappa$ which has an isomorphic copy of $<\mu$ densely embedded onto a club set of its levels. Thus the spaces [T] and μ are homeomorphic.

For a topological space \mathcal{X} and a subset M of \mathcal{X} we say that M is μ - G_{δ} if M is the intersection over a family of size μ of open subsets of \mathcal{X} . The notion of μ - F_{σ} is defined analogously. We start with the analog of the Baire Category Theorem.

Theorem 5.1 (Baire Category Theorem for weight μ). Assume that μ is a regular cardinal. For each $\nu < \mu$ let U_{ν} be a dense open subset of ${}^{\mu}\mu$. Then the intersection $\bigcap_{\nu < \mu} U_{\mu}$ is dense in ${}^{\mu}\mu$.

Proof. First note that the intersection of less than μ open subsets of ${}^{\mu}\mu$ is open. From this it is easy to see, that $\bigcap_{\nu<\mu}U_{\mu}$ intersects every non-empty open subset of ${}^{\mu}\mu$.

Because of Theorem 5.1, we say that a set $M \subset [T]$ is μ -comeagre if it contains a μ - G_{δ} set which is the intersection over a family of μ dense open sets, and we say that it is μ -meagre, if its complement in [T] is μ -comeagre, i.e., if it is the union of a family of up to μ nowhere dense sets. We furthermore call a topological space \mathcal{X} μ -Baire if every μ -comeagre subset of \mathcal{X} is dense in \mathcal{X} . So e.g., the above theorem simply states that μ is μ -Baire. On the other hand, every discrete space is μ -Baire as well.

Proposition 5.2. Let S be a μ -closed κ -Souslin tree carrying the t.e.r. \equiv . Then there is a club $C \subset \kappa$ such that for every element α of C with cofinality $\operatorname{cf}(\alpha) = \mu$ letting $T = S \upharpoonright \alpha$ every \equiv -classe of [T] is μ -Baire (when equipped with the topology inherited from [T]).

Proof. Set $\mathbb{B} := \mathrm{RO}(S)$ and let \mathbb{A} denote the subalgebra represented by \equiv . Recall from Remark 4.2 that there is a unique element $b \in \mathbb{B}$ with the property that $b\mathbb{A}$ is nowhere large in $\mathbb{B} \upharpoonright b$ and $(-b)\mathbb{A}$ is large $\mathbb{B} \upharpoonright (-b)$. We will split up S with the aid of b and gain a decomposition of $[T \upharpoonright \alpha]$ and its \equiv -classes in a discrete (and therefore μ -Baire) part and and one that is μ -Baire by resemblance to μ .

Let Y be a set of optimal witnesses of largeness for $(-b)\mathbb{A}$ in $\mathbb{B} \upharpoonright (-b)$ as found in the proof of Lemma 3.4. Let $\beta < \kappa$ be large enough such that $Y \subset \langle S_{\beta} \rangle^{\mathrm{cm}}$, i.e. X is contained in the subalgebra completely generated by the β th level of S. Now with

$$G := \{ s \in T_{\beta} \mid s \leq_{\mathbb{B}} -b \}$$

and

 $H := T_{\beta} \setminus G = \{ s \in T_{\beta} \mid s \leq_{\mathbb{B}} b \}$

let

$$S^+ := \bigoplus_{s \in G} T(s)$$

and

$$S^{-} := \bigoplus_{s \in H} T(s).$$

In the part S^+ (where \mathbb{A} is large) the limit classes of \equiv are discrete subsets $[S^+ \upharpoonright \alpha]$. Now we consider the part S^- (where \mathbb{A} is nowhere large). Pick the club $C' \subset \kappa$ such that for all subsequent members $\gamma < \delta$ of C' and all pairs of nodes s < t of S^- with $\operatorname{ht}_{S^-}(s) = \gamma$ and $\operatorname{ht}_{S^-}(t) = \delta$ there are μ further successors of s equivalent to t, i.e., on $S^- \upharpoonright C$ the t.e.r. \equiv "splits" μ -ary immediately above every node. Our final club set is $C := \{\beta + \gamma \mid \gamma \in C'\}$. It is routine to check that for every member α of C' with $\operatorname{cf}(\alpha) = \mu$ and $x \in [S^- \upharpoonright \alpha]$ we have $(x/\equiv) \approx {}^{\mu}\mu$ using the fact that S is μ -closed. But then for every $\alpha \in C$ of cofinality μ and $x \in [S \upharpoonright \alpha]$ the class x/\equiv is decomposed in a discrete S^+ -part and a continuous S^- -part. If now for $\nu < \mu$ the set U_{ν} is open dense in x/\equiv , then it contains the whole discrete part and an open dense subset of the S^- -part. Then Theorem 5.1 immediately states that the μ -comeagre subset $\bigcap U_{\nu}$ of x/\equiv is dense.

A subset $M \subset [T]$ has the μ -Baire Property if there is an open set $U \subset [T]$, such that differences $M \setminus U$ and $U \setminus M$ are both μ -meagre. We need to show that the μ -Baire Property is shared by somewhat more complicated sets which appear to be the analogue of analytic subsets of a Polish space. For this we use the fact that the class of subsets of [T] having the μ -Baire Property contains all open sets and is closed under the following modification of the Souslin Operation \mathcal{A} : Assign to every sequence $s \in {}^{<\mu}\mu$ a subset P_s of [T]. Call this family (P_s) a μ -Souslin scheme. Then the image of this μ -Souslin scheme under our operation \mathcal{A}^{μ} is given by

$$\mathcal{A}^{\mu}_{\sigma}(P_s) := \bigcup_{f \in {}^{\mu}\mu} \bigcap_{\nu < \mu} P_{f \upharpoonright \nu}.$$

Here we can assume that the Souslin scheme is regular (i.e. $s \subset t \Rightarrow P_t \subseteq P_s$) and continuous (i.e. $P_t = \bigcap_{s < t} P_s$ for all limit nodes t).

Theorem 5.3 (Nikodym's Theorem for weight μ). Let μ be a regular cardinal that satisfies $\mu^{<\mu} = \mu$ and let T be a normal μ -splitting and μ -closed tree of height $\alpha < \mu^+$, cf(α) = μ . Then the class of subsets of [T] that possess the μ -Baire Property is closed under the operation \mathcal{A}^{μ} .

Proof. Check that the proof as carried out for the case $\mu = \omega$ in [15, Section 29.C] including all references also works under our circumstances.

When constructing a homogeneous Souslin tree, it is convenient to have arbitrarily many symmetries in the initial segments of the tree. The following generalization of a lemma of Kurepa (cf. [18]) provides this. We will also apply it in the proof of the Reduction Lemma.

Theorem 5.4 (Kurepa Lemma for regular cardinals). Let μ be a regular cardinal that satisfies $\mu^{<\mu} = \mu$, and let S, T be normal μ -splitting and μ -closed trees of height $\alpha < \mu^+$, cf(α) = μ . Then S and T are isomorphic.

Proof. We show that the classical back-and-forth argument of Kurepa also works in the general context of a regular, possibly uncountable cardinal $\mu = \mu^{<\mu}$.

Let μ , α , S and T be as stated in the lemma. Then $|S| = |T| = \mu$. So we can pick dense sets $X \subset [S]$ and $Y \subset [T]$, both of cardinality μ . Enumerate X and Y by $(x_{\nu})_{\nu<\mu}$ and $(y_{\nu})_{\nu<\mu}$ respectively. We construct two maps, a bijection $\overline{\varphi}: X \to Y$

and the closely related tree isomorphism $\varphi: S \to T$, such that for all $x \in X$ we will have that $\overline{\varphi}(x) = \varphi^*x$. The even ordinals $< \mu$ will count the "forth" steps while the "back" steps will have odd numbers. For any ordinal $\nu < \mu$ let $\overline{\varphi}: X_{\nu} \to Y_{\nu}$ be the bijection constructed after stage ν , i.e., X_{ν} and Y_{ν} are the subsets of X and Y respectively that contain the elements which have been considered in the construction stages $0 \dots, \nu$.

We start the construction by assigning $\overline{\varphi}(x_0) = y_0$ and $\varphi(x_0 \upharpoonright \gamma) = y_0 \upharpoonright \gamma$ for all $\gamma < \alpha$.

We only describe the odd successor steps of the construction, the even successor steps being symmetric. So let $\nu < \mu$ by odd (and $\nu - 1$ even). Let ξ be minimal such that $y_{\xi} \in Y \setminus Y_{\nu-1}$ and set

$$\gamma := \sup \{ \beta < \alpha \mid (\exists y \in Y_{\nu-1}) \, y \upharpoonright \beta = y_{\xi} \upharpoonright \beta \}.$$

Since α is of cofinality $\mu > |Y_{\nu-1}|$, we have that $\gamma < \alpha$. In order to choose a $\overline{\varphi}$ -pre-image of y_{ξ} we have to pick some branch in X going through $s := \varphi^{-1}(y_{\xi} \upharpoonright \gamma)$. But we furthermore have to ensure that no successor of s is an element of some $x \in X_{\nu-1}$ that is already occupied. But, as our trees are μ -splitting and we have $|X_{\nu-1}| < \nu < \mu$, this is not a problem. So let $\zeta < \mu$ be minimal such that $x_{\zeta} \upharpoonright \gamma = s$ and for all $x \in X_{\nu-1}$ we have $x_{\zeta} \upharpoonright (\gamma + 1) \neq x \upharpoonright (\gamma + 1)$. We assign $\overline{\varphi}(x_{\zeta}) = y_{\xi}$, and $\varphi(x_{\zeta} \upharpoonright \beta) = y_{\xi} \upharpoonright \beta$ for all $\beta < \alpha$ and define $X_{\nu} = X_{\nu-1} \cup \{x_{\zeta}\}$ and $Y_{\nu} = Y_{\nu-1} \cup \{y_{\xi}\}$.

If $\lambda < \mu$ is a limit ordinal we just collect what has been fixed so far and set $X_{\lambda} = \bigcup_{\nu < \lambda} X_{\nu}$ and $Y_{\lambda} = \bigcup_{\nu < \lambda} Y_{\nu}$.

Finally, it is easy to check, that this construction does not break down and yields a bijective map $\overline{\varphi}: X \to Y$ and an associated tree isomorphism $\varphi: S \to T$.

By now we have collected enough facts from descriptive set theory to prove the Reduction Lemma 5.7 and carry out the constructions in Sections 5.3 and 6.

The final two lemmata of this section will be used in Section 7.2 to design a more involved interplay between the subalgebra structure and the endomorphisms of the κ -Souslin algebra.

Proposition 5.5. If $X \subseteq [T]$ is μ -comeagre and $\varphi : [T] \to X$ is continuous, onto and open, then the images of μ -comeagre subsets of [T] under φ are μ -comeagre.

Proof. It is clear that every dense subset D of [T] has a dense φ -image as the map is onto and continuous. It follows by openness of φ that nowhere dense subsets of X have nowhere dense inverse images. Since the operations of taking unions and taking pre-images commute, we also have μ -meagre inverse images for μ -meagre subsets of X. Form this we deduce the claim of the proposition.

So let $M \subset [T]$ be comeagre. Without loss of generality we can even assume that M is μ - G_{δ} . So there is a regular and continuous Souslin scheme (P_s) consisting of closed sets $P_s \subset [T]$ such that $\mathcal{A}_s^{\mu} = \varphi^{\circ}M$. By the Nikodym's Theorem 5.3 we

then know that φ " M has the μ -BP. So assume towards a contradiction that there is some $U \subset [T]$ open such that the intersection of U and φ " M is μ -meagre. For then φ^{-1} " $(U \cap \varphi$ " $M) \supset M \cap \varphi^{-1}$ " U is μ -meagre, contradicting the fact that M is μ -comeagre.

The proposition just proven in conjunction with the following lemma will be used to implement an isomorphism between the κ -Souslin algebra under construction and one of its ∞ -nice subalgebras.

Lemma 5.6. Let \equiv be a nice t.e.r. on T. Then the canonical mapping $\pi: T \to T/\equiv$ induces a continuous map $\overline{\pi}: [T] \to [T/\equiv]$ and $\overline{\pi}$ is onto and open.

Proof. This a straight forward application of the niceness of \equiv .

5.2. The Reduction Lemma. The Reduction Lemma for t.e.r.s stated below is a simple observation, but it will be crucial in κ -Souslin algebra constructions that implement nowhere large subalgebras. It asserts that we can reduce any comeagre subset M of $[T \mid \alpha]$ to a comeagre subset from which we can choose the new level T_{α} in a way that a given t.e.r. extends to $T \mid (\alpha + 1)$. This formulation makes it very flexible, e.g., it is no problem to combine the construction of subalgebras with that of endomorphisms as performed in Section 7.2.

Recall that given an equivalence relation \equiv on a topological space \mathcal{X} we say that a subset $M \subset \mathcal{X}$ is suitable for \equiv if for every equivalence class x/\equiv the intersection with M is either empty or dense in x/\equiv (viewed as a subspace of \mathcal{X}). The central idea of the proof will be to sort out those classes which are not hit by M in a dense subset and then check that the remaining classes still form a μ -comeagre set.

Lemma 5.7 (Reduction Lemma). Assume that μ is regular such that $\mu^{<\mu} = \mu$ and $\kappa = \mu^+$. Let T be a κ -normal and μ -closed tree of height $\alpha < \kappa$ with $\mathrm{cf}(\alpha) = \mu$ carrying a t.e.r. \equiv . We denote the induced equivalence relation on [T] also by \equiv and assume that for all branches $x \in [T]$ the space x/\equiv is μ -Baire. Furthermore let M be a μ -comeagre subset of [T]. Then the set

$$M' := \{x \in M \mid x/\equiv \cap M \text{ is dense in } x/!\equiv \}$$

is μ -comeagre in [T] and suitable for \equiv .

Note that by Proposition 5.2 the hypothesis that the \equiv -classes be μ -Baire is no restriction.

Proof. Without loss of generality we assume that M is a μ - G_{δ} set, i.e. $M = \bigcap_{\nu < \mu} U_{\nu}$, where all the U_{ν} are dense open in [T]. Define for $\nu < \mu$

$$X_{\nu} = \bigcup \left\{ x/\equiv \mid (x/\equiv) \cap \ U_{\nu} \text{ is not dense in } x/\equiv \right\}.$$

Note that for $x \notin X_{\nu}$ the set $(x/\equiv) \cap U_{\nu}$ is then open and dense in x/\equiv . To prove the Reduction Lemma, we show that X_{ν} is μ -meagre for every $\nu < \mu$, for then $M' = M \setminus \bigcup_{\nu \in \mu} X_{\nu}$ is as desired: For every member $x \in M'$ we then have $(x/\equiv) \cap M' = (x/\equiv) \cap M$, which is by construction of M' a μ -comeagre subset of the μ -Baire set x/\equiv .

In order to show that the sets X_{ν} are all μ -meagre fix $\nu < \mu$ and define for every node $s \in T$

$$Y_s := \bigcup \{x/\equiv \mid x \in \hat{s} \text{ and } (x/\equiv) \cap U_{\nu} \cap \hat{s} = \emptyset \}.$$

For every $x \in Y_s$ the basic open set \hat{s} is the witness of the fact that $(x/\equiv) \cap U_{\nu}$ is not dense in x/\equiv . Because of $X_{\nu} = \bigcup_{s \in T} Y_s$ and $|T| = \mu$, it is enough to show that Y_s is μ -meagre for every $s \in T$.

If we fix $s \in T_{\beta}$ then of course $Y_s = \bigcup_{r \in T_{\beta}} Y_s \cap \hat{r}$. For all nodes $r \in T_{\beta}$ with $r \not\equiv s$ we have $Y_s \cap \hat{r} = \emptyset$. On the other hand the set $Y_s \cap \hat{s} \subset \hat{s} \setminus U_{\nu}$ is nowhere dense by the definition of Y_s . To prove that the intersections $Y_s \cap \hat{r}$ are μ -meagre also for $r \equiv s$ we claim that Y_s has the μ -Baire Property, i.e. there is an open set $V \subset [T]$ such that the differences $V \setminus Y_s$ and $V \setminus Y_s$ are both μ -meagre. The proof of this claim follows below. We first apply it to prove the Reduction Lemma.

Along with Y_s the set $Y_s \cap \hat{r}$ has the μ -Baire Property as well, so either (i) it is μ -meagre or else (ii) there is a node t > r, such that $Y_s \cap \hat{t}$ is μ -comeagre and therefore dense in \hat{t} . Towards a contradiction, we assume that the second case holds and fix t. Then every node u above t is equivalent to $x \upharpoonright \operatorname{ht}(u)$ for some $x \in Y_s \cap \hat{s}$. Our task is to exhibit a node u^* above t that is equivalent to $y \upharpoonright \operatorname{ht}(u^*)$ for some $y \in \hat{s} \setminus Y_s$. This will give the desired contradiction.

Let u be any immediate successor of t. By our assumption there is w above s and equivalent to u. Letting $v := w \upharpoonright \operatorname{ht}(t)$ be the immediate predecessor of w we get $v \equiv t$. Now $Y_s \cap \hat{w}$ is also nowhere dense. So there certainly is a node w^* above w such that $\widehat{w^*} \cap Y_s$ is empty, i.e., letting $\gamma = \operatorname{ht} w^*$ we have

$$(\bigstar) \qquad (\forall x \in \hat{s} \cap Y_s) \quad w^* \not\equiv x \upharpoonright \gamma.$$

Now the honesty of \equiv for the triple (v, w^*, t) (which is not a dispute as (v, w, t) is not –by the existence of $u \equiv w$ above t) gives us the node $u^* \equiv w^*$ on level T_{γ} above u.

Now let $z \in \widehat{u}^*$ be any branch. We show that $z \notin Y_s$, contradicting our assumption that $Y_s \cap \widehat{r}$ is not μ -meagre. If z was in Y_s , then there would be a branch $x \equiv z$ in $Y_s \cap \widehat{s}$. This in turn would imply that

$$x \upharpoonright \gamma \equiv z \upharpoonright \gamma = u^* \equiv w^*,$$

which is impossible by (\bigstar) .

Finally we prove that Y_s has the μ -Baire Property. For this we give a μ -Souslin scheme which consists of open sets and yields Y_s under the operation \mathcal{A}^{μ} . Fix a club $C \subset \alpha$ of order type μ . To simplify notation we replace $^{<\mu}\mu$ by $T \upharpoonright C$ as index set. (Theorem 5.4 gives us the necessary tree isomorphism.)

Let $\tilde{U} := \{t \in T \mid \hat{t} \subseteq U_{\nu}\}$. For a node $r \in T \upharpoonright C$ of height $\operatorname{ht}(r) \leq \operatorname{ht}(s)$ simply set $P_r = [T]$. For nodes r higher up define

$$P_r := \begin{cases} \bigcup \left\{ \hat{t} \mid r \equiv t \right\}, & r \upharpoonright \operatorname{ht}(s) \equiv s \text{ and } (r/\equiv) \cap \tilde{U} = \emptyset; \\ \emptyset, & \text{otherwise.} \end{cases}$$

For every $x \in Y_s$ we easily have $x \in (x/\equiv) = \bigcap_{\beta \in C} P_{x \upharpoonright \beta}$. If on the other hand $x \in \bigcap_{\beta \in C} P_{y \upharpoonright \beta}$ for $x, y \in [T]$, then x and y are equivalent, $x \equiv y$, and thus $P_{x \upharpoonright \beta} = P_{y \upharpoonright \beta}$ for all $\beta \in C$. In this case we have

$$x \in \bigcap_{\beta \in C} P_{x \upharpoonright \beta} = (x/\equiv) \subset Y_s.$$

This finishes the proof.

5.3. A rigid Souslin algebra with non-rigid subalgebras. Our first application of the Reduction Lemma is a relatively simple construction of a rigid κ -Souslin algebra $\mathbb B$ that has a nice and nowhere large subalgebra $\mathbb A$. By Lemma 4.8 and Theorem 4.9 this algebra $\mathbb B$ also has non-rigid and nowhere nice subalgebras. This is opposed to the construction in the following section, where the explicitly construed subalgebra is nowhere nice and no non-rigid neither ∞ -nice subalgebras occur.

Theorem 5.8. Assume that μ is a regular cardinal such that $\mu^{<\mu} = \mu$ and $\diamondsuit_{\kappa}(\mathrm{CF}_{\mu})$ hold, where $\kappa := \mu^{+}$. Then there is a rigid κ -Souslin algebra which has an ∞ -nice subalgebra.

Proof. We aim at constructing a κ -Souslin tree T with a μ -nice t.e.r. \equiv . The rigidity of $\mathbb{B} = \operatorname{RO} T$ is obtained by designing T such that for all club sets C of κ the restricted tree $T \upharpoonright C$ is rigid by a standard argument. Then by the Restriction Lemma \mathbb{B} will also be rigid.

Let $(R_{\nu})_{\nu \in \mathrm{CF}_{\mu}}$ be a $\diamondsuit_{\kappa}(\mathrm{CF}_{\mu})$ -sequence. We inductively construct T as a κ -normal, μ -closed and μ -splitting tree on the supporting set κ along with the t.e.r. \equiv . In successor steps we appoint to each maximal node μ direct successor nodes and extend \equiv in any way that maintains the μ -niceness of \equiv .

In the limit step $\alpha < \kappa$ we have so far constructed $T \upharpoonright \alpha$ and \equiv on this tree. If $\operatorname{cf}(\alpha) < \mu$ we extend every cofinal branch through $T \upharpoonright \alpha$ to level T_{α} . The new level T_{α} then has cardinality $\mu^{<\mu} = \mu < \kappa$. The t.e.r. on level T_{α} is completely determined by its behavior on the levels below.

Let now α be of cofinality μ . We consider the induced equivalence relation \equiv on the space $[T \upharpoonright \alpha]$. The \equiv -classes are perfect (closed and without isolated points) and non-empty subsets of $[T \upharpoonright \alpha]$. The level under construction, T_{α} , corresponds to a dense subset Q of $[T \upharpoonright \alpha]$ of cardinality μ . In order to obtain a nice extension of \equiv to the new level we have to choose this subset $Q \subset [T \upharpoonright \alpha]$ such that it is suitable for \equiv , i.e., that for every \equiv -class $a \subset [T \upharpoonright \alpha]$ the set $a \cap Q$ is either empty or dense in a.

Every automorphism φ of $T \upharpoonright C$ for some club $C \subset \alpha$ induces an autohomeomorphism $\overline{\varphi}$ on $[T \upharpoonright \alpha]$. In order to achieve a rigid algebra we have to choose some limit levels in a way that prevents the potential automorphisms (proposed by the \diamondsuit -sequence) from extending to the next level. This is done by first choosing a branch $x \in [T \upharpoonright \alpha]$ and then the dense set $Q \subset [T \upharpoonright \alpha]$ such that $x \in Q$ but $\overline{\varphi}(x) \notin Q$. (This is a standard argument.)

Now for the choice of Q in the following three cases:

- 1) If $\alpha < \mu \alpha$ or R_{α} is neither a maximal antichain of $T \upharpoonright \alpha$ nor does it code an automorphism of $T \upharpoonright C$ for some club C of α , then we first choose a dense set Q_0 of $[T \upharpoonright \alpha]$ with cardinality μ . Then let for $x \in Q_0$ be Q_x a dense subset of x/\equiv of size μ . Finally set $Q = \bigcup_{x \in Q_0} Q_x$.
- 2) If $\alpha = \mu \alpha$ and R_{α} codes an automorphism φ , then we and start as in the first case and get $Q' = \bigcup_{x \in Q_0} Q_x$. Choose $x_0 \in Q'$ and set $Q := Q' \setminus \{\overline{\varphi}(x_0)\}$. Then Q is easily seen to be suitable for \equiv while at the same time preventing φ from extending to T_{α} .
- 3) If R_{α} is a maximal antichain of $T \upharpoonright \alpha$, we want, as in classical Souslin tree constructions under \diamondsuit , that every node of T_{α} lies above some node of R_{α} . The set

$$M := \{ x \in [T \upharpoonright \alpha] \mid (\exists s \in R_{\alpha}) s \in x \}$$

of cofinal branches that pass through nodes in R_{α} is open dense in $[T \upharpoonright \alpha]$. We thus can apply the Reduction Lemma 5.7 and get a μ -comeagre subset $N \subset M$ which is suitable for \equiv . Then we proceed as above, only that all members of Q are chosen from N.

Note that we can arrange the coding such that we do not have to consider a coincidence of cases 2) and 3) (which would no longer pose a problem anyway).

The result of this recursive construction is a rigid κ -Souslin tree carrying the μ -nice t.e.r. \equiv which represents the subalgebra \mathbb{A} .

6. A Lonely nowhere nice subalgebra

We use the Reduction Lemma to produce a κ -Souslin algebra \mathbb{B} that essentially has only one subalgebra \mathbb{A} which is furthermore nowhere nice. In particular \mathbb{B} and all its subalgebras are rigid. Compare this to the phenomenon of hidden symmetries of Section 4.2 which occur whenever there is an ∞ -nice subalgebra: While the latter

support the paradigm that subalgebras witness some form of symmetry, the following construction shows that this is not true for nowhere nice subalgebras. This can also be seen in relation to [17, Theorem 2] which exhibits a similar phenomenon in presence of homogeneity.

Theorem 6.1. Assume that μ is a regular cardinal such that $\mu^{<\mu} = \mu$ and $\diamondsuit_{\kappa}(\mathrm{CF}_{\mu})$ hold, where $\kappa := \mu^{+}$. Then there is a κ -Souslin algebra \mathbb{B} with a nowhere nice subalgebra \mathbb{A} and the following holds: For every subalgebra \mathbb{C} of \mathbb{B} there is an antichain F of \mathbb{B} such that

$$\mathbb{C} = \mathbb{B} \upharpoonright (-\sum F) \times \prod_{a \in F} (a\mathbb{A}).$$

Moreover, \mathbb{B} is rigid and does not admit ∞ -nice subalgebras.

Proof. This construction resembles the previous one, the main differences being

- (1) the repeated destruction of niceness for \equiv by picking up the key idea from the proof of Theorem 4.9 and
- (2) the more delicate choice procedure on limit levels in order to maintain \equiv as an almost nice t.e.r. while destroying almost all the others.

Let again $(R_{\nu})_{\nu \in \mathrm{CF}_{\mu}}$ denote our $\Diamond_{\kappa}(\mathrm{CF}_{\mu})$ -sequence. We will use it to kill unwanted antichains, t.e.r.s and tree automorphisms. The tree T order will again be defined on the supporting set κ , and it will be μ -splitting and μ -closed. The almost nice t.e.r. to represent the nowhere nice subalgebra \mathbb{A} will be denoted by \equiv .

If α is a successor of a successor ordinal or of a limit ordinal γ with $\operatorname{cf}(\gamma) < \mu$ (or if $\alpha = 1$), then we extend \equiv in any way that maintains almost μ -niceness. If α is a limit ordinal whose cofinality is below μ , then we extend all cofinal branches of $T \upharpoonright \alpha$ to nodes in T_{α} .

We come to the choice of the limit levels for the case where $cf(\alpha) = \mu$. Our diamond set R_{α} proposes either a map on T as described below or it codes a pair (r, \simeq) , where $r \in T \upharpoonright \alpha$ is a node and \simeq is a t.e.r. on $(T \upharpoonright C)(r)$ for some C club in α with min C = ht(r).

We first describe how to choose T_{α} whenever R_{α} codes an unwanted symmetry of T or of T/\equiv . If R_{α} is an isomorphism between normal cofinal subtrees of $T \upharpoonright C$ or between normal cofinal subtrees of $(T/\equiv) \upharpoonright C$ then we choose the new level as in Section 5.3 and destroy the isomorphism while maintaining the honesty of the t.e.r. \equiv . By this precaution we guarantee, that both $\mathbb B$ and $\mathbb A$ and also all of their non-trivial relative algebras won't have any symmetries, and therefore they will not have any non-trivial large subalgebras.

If R_{α} proposes a t.e.r. \simeq , which differs from \equiv in an essential manner (see below), then we have to choose T_{α} such that \simeq is no longer honest above r when extended to T_{α} . The node r is introduced to prevent the fatal situation in which by accident

locally some of the unwanted t.e.r.s survive. In the end r will vary over all nodes of T.

We distinguish between three cases in the manner how \simeq differs from \equiv , the first being the one of *negligible* difference in which we (cannot and therefore) will not destroy \simeq , while in the other two cases we can and will prevent \simeq from extending to the limit level T_{α} . We list the three cases and describe how T_{α} is to be chosen. When mentioning \simeq or \equiv we always mean the equivalence relation induced on \hat{r} .

(1) The proposed relation \simeq refines \equiv on \hat{r} in a way such that there is a maximal antichain E of T(r) such that for all $s \in E$ the restriction of \simeq to \hat{s} coincides with either \equiv (restricted to \hat{s}) or with =, the identity, and the following holds for all nodes $s, t \in E$, such that \equiv and \simeq are equal above s and above t: If there are branches $x \in \hat{s}$ and $y \in \hat{t}$ with $x \simeq y$ (and therefore also $x \equiv y$, as \simeq refines \equiv) then the two equivalence relations also coincide on the set $\hat{s} \cup \hat{t}$. (Note that this last condition is not a requirement put on \simeq but on the choice of the antichain E; it has to be quite fine.)

We then let

$$M:=([T\!\!\upharpoonright\!\!\alpha]\setminus \hat{r})\cup\bigcup_{s\in E}\hat{s}$$

and apply the Reduction Lemma 5.7 to get a μ -comeagre subset M' of M from which we choose T_{α} such that it is suitable for \equiv .

(2) Here \equiv on \hat{r} is again refined by \simeq but there is no antichain as in case (1). Then there must be a node s above r and branch x through s for which $\hat{s} \cap x/\simeq$ is a nowhere dense subset of $\hat{s} \cap x/\equiv$.

In this case choose a node t above s such that $(x/\simeq) \cap \hat{t} \neq \emptyset$ yet $x/\simeq \not\subset \hat{t}$. Let

$$M:=[T\!\!\upharpoonright\!\!\alpha]\setminus(\hat t\cap x/\!\!\simeq)$$

and note that M is already suitable for \equiv . Choose T_{α} suitable for \equiv such that at least one branch in $(x/\simeq) \setminus \hat{t}$ is extended. Then \simeq is no longer honest as witnessed by the dispute $(x|\operatorname{ht}(t),x,t)$.

(3) In the last case, \equiv is not refined by \simeq . This means that there is a branch x through r such that $(x/\cong) \cap \hat{r}$ is not contained in $(x/\equiv) \cap \hat{r}$. Since both sets are closed in \hat{r} , their intersection cannot be a dense subset of $(x/\cong) \cap \hat{r}$. This means that we can find a node s above r such that there is a branch

$$y \in (\hat{s} \cap x/\cong) \setminus x/\equiv$$
.

Let

$$M:=(\,[T\!\!\upharpoonright\!\!\alpha]\setminus\,x/\!\!\simeq\,)\cup\,x/\!\!\equiv$$

and apply the Reduction Lemma to get M'. As M contains all of x/\equiv , this class will still be present in M'. If we now choose (the branches extended to

nodes in) T_{α} from M' and make sure that x is extended, then the dispute (x|ht(s), s, x) shows by the choice of s that \simeq is no longer honest.

Note that the first case also seals maximal antichains.

Now let $\alpha = \gamma + 1$ where $\operatorname{cf}(\gamma) = \mu$. By the inductive hypothesis and our convention on successor levels of T we have so far constructed the tree $T \upharpoonright (\alpha + 1)$ and the t.e.r. \equiv on $T \upharpoonright \alpha$. Regarding the \equiv -classes of level T_{γ} as subspaces of the space $[T \upharpoonright \gamma]$ we divide each class a in two parts $a = a_{\operatorname{red}} \cup a_{\operatorname{green}}$ in way such that both parts are dense subsets of a. This is analogous to the proofs of Lemma 4.8 and Theorem 4.9 and gives us a coloring of the whole limit level T_{γ} with colors "red" and "green". We then extend \equiv to level T_{α} such that

- for each limit node $s \in T_{\gamma}$ its set succ(s) of direct successors is partitioned by \equiv into μ sets of size μ and
- if $s, t \in T_{\gamma}$ are equivalent and of the same color, then every successor of s has an equivalent successor of t and
- if $s, t \in T_{\gamma}$ are not equivalent or of different colors, then none of their successors are equivalent.

This assures that \equiv is almost nice, yet $\mathbb{A} = \langle \sum s/\equiv \rangle^{\mathrm{cm}}$ will be nowhere nice in $\mathbb{B} = \mathrm{RO}\,T$.

Having completed the construction of T and \equiv , we now prove that every subalgebra of \mathbb{B} is of the form described in the statement of the theorem.

Let \mathbb{C} be any atomless subalgebra of \mathbb{B} and let \simeq be a t.e.r. on $T \upharpoonright D'$ for some club $D' \subset \kappa$ representing \mathbb{C} . Let D be the set of μ th order limit points of D', i.e., let $D = D_{\mu}$ where

$$D_0 := D', \quad D_{\nu+1} := \{ \text{limit points of } D_{\nu} \}, \quad D_{\lambda} := \bigcap_{\nu < \lambda} \text{ for limit } \lambda.$$

Let S be the stationary set of those ordinals $\alpha \in D$ of cofinality $cf(\alpha) = \mu$, such that \simeq on $T \upharpoonright \alpha$ is coded together with the node $r = \mathbf{root}$ by R_{α} .

By our case split in the construction of T_{α} , \simeq must always have fallen under case (1) above. Furthermore there are an ordinal $\alpha^* < \kappa$ and a maximal antichain E of $T \upharpoonright \alpha^*$, such that E is the antichain which is referred to in case (1) for all $\alpha > \alpha^*$. (If this was not true, then at some limit stage of S we would have dropped out of case (1), thereby destroying \simeq .) We finally have to assemble the elements of E as follows. Let

$$e:=\sum\{s\in E\mid \simeq \text{ is equality on } T(s)\},$$

define on the set $E' := \{s \in E \mid se = 0\}$ of nodes disjoint from e the equivalence relation

$$s \sim t : \iff ((\forall c \in \mathbb{C})sc \neq 0 \iff tc \neq 0)$$

and let F be the set of sums $\sum s/\sim$ for all $s\in E$ disjoint from e. As $\mathbb{B}\upharpoonright e$ has no large subalgebras but $\mathbb{B}\upharpoonright e$ itself, we already know that $e\mathbb{C}=\mathbb{B}\upharpoonright e$. By a similar argument we have $f\mathbb{C}=f\mathbb{A}$ for all $f\in F$. And as there are no further symmetries between relative algebras of \mathbb{A} and \mathbb{B} , these pieces are relative algebras of \mathbb{C} , i.e., the proof is finished.

7. The Schröder-Bernstein Property for Souslin Algebras

Say for a class \mathcal{C} of complete Boolean algebras that it has the *Schröder-Bernstein Property* if all pairs $A, B \in \mathcal{C}$ are isomorphic whenever they are regularly embeddable into each other. The classical Schröder-Bernstein Theorem then states that the class of power set algebras has the Schröder-Bernstein Property.

In his well-known list of set theoretic problems D. H. Fremlin [8] asked among others the question **FW**, whether or not the class of homogeneous, c.c.c. Boolean algebras has the Schröder-Bernstein Property.

Farah⁶ has solved this problem by pointing out that (assuming ZFC only) the Cohen algebra C_{\aleph_2} , that adjoins \aleph_2 generic Cohen reals, has a subalgebra $B \not\cong C_{\aleph_2}$, in which C_{\aleph_2} can be regularly embedded (cf. [7, p.93]). By [7, Proposition 4.1] it is easy to see that B is weakly homogeneous and by a theorem of Koppelberg and Solovay (cf. [24, Theorem 18.4.1]) it is therefore homogeneous.

Here we consider the question whether the class S_{\aleph_1} of \aleph_1 -Souslin algebras (and more generally, the class S_{κ} of κ -Souslin algebras for a regular uncountable cardinal κ) has the Schröder-Bernstein Property. Of course, under Souslin's Hypothesis we have $S_{\aleph_1} = \emptyset$ and the answer is trivially affirmative. But what if there are Souslin algebras?

Theorem 7.1. The Schröder-Bernstein Theorem for \aleph_1 -Souslin algebras is independent theory ZFC + \neg SH.

The proof of this theorem stretches over the remaining two sections of the paper. In fact we will show that the Schröder-Bernstein Theorem for κ -Souslin algebras consistently fails for every successor cardinal κ .

7.1. A model of $\neg SH$ where the Schröder-Bernstein Theorem for \aleph_1 -Souslin algebras holds. The model we use for the first part of our independence proof was constructed by Abraham and published in [1, Section 4]. It is a Jensen-style forcing iteration with the modification that not all \aleph_1 -Souslin trees are killed, but one is preserved. Similar models have also been obtained using the P_{max} -forcing method of Woodin, cf. [19, Section 8] and [23, Section 4.0].

This preserved tree has a certain property which frequently appears in the literature on Souslin trees under various names: Jensen ([14]) and Todorčević [25] call

⁶another solution of problem **FW** is due to S. Geschke

these Souslin trees full trees, Abraham and Shelah ([2, 1]) use Souslin trees with all derived trees Souslin, Fuchs and Hamkins ([9]) denote them as $(\omega$ -fold) Souslin off the generic branch, while Larson ([19]), Shelah and Zapletal ([23]) simply say free trees.

We follow the last three authors and call an \aleph_1 -normal tree T of height ω_1 free, if for every finite antichain $A = \{s_0, \ldots, s_n\}$ the product tree

$$\bigotimes_{k \le n} T(s_k)$$

is an \aleph_1 -Souslin tree. Abraham calls a product tree like this *a tree derived from* T (of dimension n+1). In his model, call it V, there is a free tree R such that every \aleph_1 -Souslin tree contains a copy of one R's derived trees as a subtree. But this means that every \aleph_1 -Souslin tree in V is the tree sum of countably many trees which are all derived from R.

Furthermore, it is easily checked that for a free tree R the class \mathcal{D}_R of countable Cartesian products of the regular open algebras of all trees derived from R, i.e. the class $(\mathcal{S}_{\aleph_1})^V$ of all \aleph_1 -Souslin algebras in V, has the Schröder-Bernstein Property. So the Schröder-Bernstein Theorem for \mathcal{S}_{\aleph_1} holds in this model and is therefore consistent to the theory $\mathsf{ZFC} + \neg \mathsf{SH}$.

7.2. No Schröder-Bernstein Theorem under \diamondsuit . To prove the complementary consistency statement for our independence result, we perform a last \diamondsuit_{κ} -construction of a κ -Souslin tree.

Theorem 7.2. Assume that μ is a regular cardinal such that $\mu^{<\mu} = \mu$ and $\diamondsuit_{\mu^+}(\mathrm{CF}_{\mu})$ hold. Let $\kappa := \mu^+$. Then there is a homogeneous κ -Souslin algebra $\mathbb B$ that has a pair of ∞ -nice subalgebras $\mathbb A$ and $\mathbb C$ such that $\mathbb C$ is a subalgebra of $\mathbb A$ and isomorphic to $\mathbb B$ yet $\mathbb A$ and $\mathbb B$ are not isomorphic:

$$\mathbb{B} \cong \mathbb{C} \leq \mathbb{A} \leq \mathbb{B}, \quad yet \quad \mathbb{A} \ncong \mathbb{B}.$$

Proof. Let μ and κ be as in the statement of the theorem. We will construct the μ -closed and μ -normal Souslinization $T \subset {}^{<\kappa}\mu$ of $\mathbb B$ along with

- μ -nice t.e.r.s \equiv and \sim representing $\mathbb C$ and $\mathbb A$ respectively such that \sim refines \equiv in an μ -nice fashion, and
- a family (φ_{st}) of tree automorphisms $\varphi_{st}: T \to T$ satisfying $\varphi_{st}(s) = t$ for all pairs s, t of nodes of the same height in T, and
- a tree isomorphism $\varphi: T/\equiv \to T$; we will define φ as a map $T\to T$ which is invariant under \equiv .

Furthermore we diagonalize every potential isomorphism between \mathbb{A} and \mathbb{B} , i.e., between trees $T \upharpoonright C$ and $(T/\sim) \upharpoonright C$ for every club $C \subset \kappa$.

We start our construction with the trivial level $T_0 = \{\emptyset\}$ carrying only trivial relations. For the definition of a successor level $T_{\alpha+1}$ out of level T_{α} we attach to every node $s \in T_{\alpha} \subseteq {}^{\alpha}\mu$ all possible successors $s^{\smallfrown}(\nu) = s \cup \{(\alpha, \nu)\} \in {}^{\alpha+1}\mu$, where ν ranges over all ordinals less than μ .

In order to extend the t.e.r.s to the new level fix a partition $\mathcal{P} = (P_{\xi,\eta})$ of μ into μ sets of size μ indexed by pairs ξ, η of ordinals less than μ . Set $P_{\xi} := \bigcup_{\eta < \mu} P_{\xi,\eta}$ and define the extensions of \equiv and \sim given on $T \upharpoonright (\alpha + 1)$ to $T_{\alpha+1}$ by letting for $r, t \in T_{\alpha}$

$$r^{\hat{}}(\nu) \equiv t^{\hat{}}(\lambda) : \iff r \equiv t \text{ and } (\exists \xi < \mu)\nu, \ \lambda \in P_{\xi}$$

and

$$r^{\hat{}}(\nu) \sim t^{\hat{}}(\lambda)$$
 : $\iff r \sim t \text{ and } (\exists \xi, \eta < \mu)\nu, \ \lambda \in P_{\xi,\eta}$.

To extend φ to the next level let

$$\varphi(r^{\hat{}}(\nu)) = \varphi(r)^{\hat{}}(\xi)$$
 if and only if $\nu \in P_{\varepsilon}$.

The tree automorphisms $\varphi_{st}: T \upharpoonright (\alpha + 1) \to T \upharpoonright (\alpha + 1)$ (for $\operatorname{ht}(s) = \operatorname{ht}(t) \leq \alpha$) are extended to the next level by the simple rule $\varphi_{st}(r) = \varphi_{st}(r) = \varphi_{st}(r)$. To install for $s = r \cap (\nu), v = t \cap (\lambda) \in T_{\alpha+1}$ a new tree automorphism of T we extend the initial segment $\varphi_{rt} \upharpoonright (T \upharpoonright \alpha + 1)$ to $T_{\alpha+1}$ in another direction: For $w \in T_{\alpha}$ and $\varepsilon < \mu$ set:

$$\varphi_{sv}(w^{\smallfrown}(\varepsilon)) = \begin{cases} \varphi_{rt}(w)^{\smallfrown}(\varepsilon), & \text{if } \varepsilon \neq \lambda, \nu \\ \varphi_{rt}(w)^{\smallfrown}(\mu), & \text{if } \varepsilon = \lambda \\ \varphi_{rt}(w)^{\smallfrown}(\lambda), & \text{if } \varepsilon = \nu. \end{cases}$$

We have finished the successor stage of the construction of the tree T and the additional structure. Note that for every γ the final segments of the images $\varphi(x)$ and $\varphi_{st}(x)$ of $x \in [T \upharpoonright \alpha]$ beyond γ only depend on the final segment of x beyond γ .

From now on we consider the limit stage α . Once the set of limit nodes on level α is chosen all mappings and t.e.r.s extend to the new level in a unique way.

We have to find a set $Q \subset [T \upharpoonright \alpha]$ such that Q is

- (i) of cardinality μ ,
- (ii) dense in $[T \upharpoonright \alpha]$,
- (iii) closed under the application of the (homeomorphism of $[T \upharpoonright \alpha]$ induced by the) tree automorphism $\overline{\varphi_{st}}$ (and its inverse mapping $\overline{\varphi}_{ts}$) for all nodes $s, t \in T \upharpoonright \alpha$ of the same height,
- (iv) suitable for \equiv and \sim ,
- (v) and closed under the application of the continuous map $\overline{\varphi}$ and its inverse mapping in the sense that for every $x \in Q$ there are $u, y \in Q$ such that we have $\overline{\varphi}(u/\equiv) = x$ and $\overline{\varphi}(x/\equiv) = y$.

We take the members R_{ν} of our \diamondsuit_{κ} -sequence to be subsets or binary relations on the initial segments $T \upharpoonright \alpha$ of our tree by virtue of some pre-fixed bijection between κ

and ${}^{\kappa}\mu \cup {}^{\kappa}(\mu \times \mu)$. Let H be the monoid of maps acting on $[T \upharpoonright \alpha]$ generated by the maps $\overline{\varphi}$ and $\overline{\varphi_{st}}$ for $s, t \in T \upharpoonright \alpha$ of the same height. For a point $x \in [T \upharpoonright \alpha]$ let $\operatorname{orb}(x) := \operatorname{orb}_H(x) := \{h(x) \mid h \in H\}$ be its orbit under the action of H.

If $\alpha < \kappa$ is a limit ordinal such that either $\operatorname{cf}(\alpha) < \mu$ or the set R_{α} neither is a maximal antichain of $T \upharpoonright \alpha$ nor does it induce a tree isomorphism between $(T \upharpoonright C) / \sim$ and $T \upharpoonright C$ for some club set C of α , then we find a subset $Q \subset [T \upharpoonright \alpha]$ satisfying points (i-v) above as follows. Choose any branch $y \in [T \upharpoonright \alpha]$ and let $Q_0 := \operatorname{orb}(y)$. Note that Q_0 is already closed under the action of H, dense in $[T \upharpoonright \alpha]$ and suitable for \sim and \equiv . (This follows from the construction as $\operatorname{orb}(x) \cap (x/\equiv)$ is dense in (x/\equiv) and similarly for \sim .) To provide inverse images under the maps from H it suffices to care about $\overline{\varphi}$ as all the other generators have their inverse in H. For every $x \in [T \upharpoonright \alpha]$ fix a branch z_x such that $\varphi(z_x) = x$. For $n \in \omega$ let $Q_{n+1} := Q_n \cup \bigcup_{x \in Q_n} \operatorname{orb}(z_x)$ and finally $Q = \bigcup Q_n$.

If R_{α} guesses a maximal antichain A of $T \upharpoonright \alpha$ and a tree isomorphism ψ between $(T \upharpoonright C)/\sim$ and $T \upharpoonright C$ for some club $C \subseteq \alpha$ (again we take ψ to be a \sim -invariant map $T \upharpoonright C \to T \upharpoonright C$), then Q also has to satisfy:

- (vi) every branch $x \in Q$ passes through a node s in A and
- (vii) there is a branch $y \in Q$ such that $\overline{\psi}(x) \neq y$ for all $x \in Q$.

(Of course, in case that R_{α} only guesses one out of antichain and tree isomorphism the other corresponding condition is void.) We satisfy point (vi) by simply restricting our set of potential nodes to

$$K = \{ x \in [T \upharpoonright \alpha] \mid (\exists \ s \in A) s \in x \}$$

which is an open dense subset of $[T \upharpoonright \alpha]$. To fulfill the last requirement is a more subtle task. The set Q has to be designed around a special branch y which on one hand must not be reached by $\overline{\psi}$ from within Q and on the other hand brings its orbit and also some necessary inverse images under $\overline{\varphi}$. For every branch $y \in K$ we define its forbidden set

$$F_y := \bigcup_{h \in H} (\overline{\psi} \circ h)^{-1}(y).$$

First we have to check that there are enough branches that lie outside of their forbidden sets. We claim that the set

$$L := \{ y \in K \mid y \notin F_y \}$$
$$= \bigcap_{h \in H} \{ y \in K \mid \overline{\psi}(h(y)) \neq y \}$$

is μ -comeagre in $[T \upharpoonright \alpha]$. For $h \in H$ the set $\{y \in K \mid \overline{\psi}(h(y)) \neq y\}$ is clearly open in K and, as $\overline{\psi}$ is nowhere 1-to-1, also dense.

Our next step provides h-pre-images for all $h \in H$. Again, it suffices to care about $h = \varphi$. For a μ -comeagre subset of $[T \upharpoonright \alpha]$, such as our set L defined above, the subset of branches x such that the whole orbit of x is in L, i.e., $\operatorname{orb}(x) \subset L$, is again μ -comeagre. Also the set of $\overline{\varphi}$ -images of the latter branches is μ -comeagre by Proposition 5.5 and Lemma 5.6; and therefore also the set $M := \bigcap M_n$, where $M_0 := L$ and

$$M_{n+1} := \overline{\varphi}^{"} \{ x \in M_n \mid \operatorname{orb}(x) \subset M_n \}.$$

M and all the sets M_n are clearly closed under the action of H, so they contain $\operatorname{orb}(x)$ along with x and are, as a consequence, suitable for \equiv . On the other hand, if we are given $y \in M$, then for every n the set $\overline{\varphi}^{-1}(y) \cap M_n$ is μ -comeagre in the space $\overline{\varphi}^{-1}(y)$ (which is homeomorphic to $[T \upharpoonright \alpha]$ by Kurepas Lemma). So the set $\overline{\varphi}^{-1}(y) \cap M$ is also μ -comeagre in $\overline{\varphi}^{-1}(y)$ and thus non-empty.

Finally we choose any member $y \in M$ and let $N := M \setminus F_y$. If we can show that h"N = N for all $h \in H$, then N is suitable for \sim and \equiv and we can proceed as in the default case above with $Q_0 := \operatorname{orb}(y)$ and the only further restriction that the inverse images z_x of x under $\overline{\varphi}$ are to be chosen from N.

For every $x \in M$ and $h \in H$ we clearly have that $h(x) \in F_y$ implies $x \in F_y$, so $h"N \subseteq N$ holds for all $h \in H$.

Concerning the converse inclusion, it is again enough to consider $h = \overline{\varphi}$, because $\overline{\varphi}_{st}$ is invertible in H and therefore done by the first inclusion. We show that $\overline{\varphi}(x) = z$ and $x \in F_y$ imply that either $z \in F_y$ or there is a branch $x' \notin F_y$ with $x' \equiv x$ and thereby $\overline{\varphi}(x') = z$ as well. So let $\overline{\varphi}(x) = z$ and $\overline{\psi}(h(x)) = y$ for some $h \in H$. There are three possible types for h. In the first case let h be a homeomorphism, i.e., a concatenation of maps of type $\overline{\varphi}_{st}$. Then h locally maps $\overline{\equiv}$ to $\overline{\equiv}$ and \sim to \sim . As every \sim -class is nowhere dense in its corresponding $\overline{\equiv}$ -class, $F_y \cap \overline{\varphi}^{-1}(z)$ is μ -comeagre in $\overline{\varphi}^{-1}(z)$. So there must be some $x' \in N \cap \overline{\varphi}^{-1}(y) \setminus F_y$. If in the second case $h = g \circ \overline{\varphi}$ then we are already done, for then we have $\overline{\varphi}(g(z)) = y$, and z lies in the forbidden set. Finally let $h = h' \circ \overline{\varphi} \circ g$ where g is a homeomorphism. Here we exploit the fact that g leaves a final segment of x unchanged and that the components of $\overline{\varphi}(x)$ only depend on the corresponding components of x. So let $s, t, u \in T_\alpha$ and v, w be such that $x = s \circ v$ and $g(x) = t \circ v$ while $z = \overline{\varphi}(x) = r \circ w$ and $\overline{\varphi}(g(x)) = u \circ w$. Then easily

$$\overline{\varphi}(g(x)) = u^{\hat{}}w = \overline{\varphi}_{ru}(r^{\hat{}}w) = \overline{\varphi}_{ru}(z).$$

But then $\overline{\psi} \circ h' \circ \overline{\varphi}_{ru}(z) = y$, so z again is a member of the forbidden set.

This finishes the construction and it can easily be seen that with $\mathbb{B} := \operatorname{RO} T$, $\mathbb{A} := \langle T/\sim \rangle$ and $\mathbb{C} := \langle T/\equiv \rangle$ we have that \mathbb{C} is ∞ -nice in \mathbb{A} which in turn is ∞ -nice in \mathbb{B} , and that \mathbb{B} and \mathbb{C} are isomorphic via φ while \mathbb{B} and \mathbb{A} are not, because all potential isomorphisms between T and T/\sim have been diagonalized away.

8. Concluding remarks

Concerning the representation theory of Souslin algebras, we have not touched here how it can be used to analyze independent subalgebras and (free) product Souslin algebras. This can fruitfully be applied, e.g., to strongly homogeneous and to free Souslin algebras, cf. [21, Sections 1.5-6].

Of course, most if not all of the constructions in Part 2 of the present paper could be carried out without any recourse to topology and using sophisticated or involved combinatorial arguments. But, as we hope the constructions performed in the preceding sections demonstrate, the topological view substantially simplifies the diagonalization procedures once the basic notions have been established. In most cases, it is not hard to see that consistent "conditions" imposed on the branches to be extended are comeagre sets which therefore can freely be combined (up to μ conditions at a time).

In [22] (resp. in [21, Chapter 2]) we give a further construction of an \aleph_1 -Souslin algebra with aboundant homogeneity properties and with many subalgebras. We hope that such a construction can in the end be used to construct a model of ZFC with a unique Souslin line (up to isomorphism).

ACKNOWLEDGMENTS

I sincerely thank my thesis advisor Sabine Koppelberg, in particular for her scrutiny in pointing out the gaps in my naïve first arguments.

I would also like to thank Peter Krautzberger for proof reading large parts of this text and for his valuable suggestions on how to improve the presentation.

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