# Understanding preservation theorems: Chapter VI of Proper and Improper Forcing, I 

Chaz Schlindwein<br>Department of Mathematics and Computing<br>Lander University<br>Greenwood, South Carolina 29649 USA<br>cschlind@lander.edu

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#### Abstract

We present an exposition of Section VI. 1 and most of Section VI. 2 from Shelah's book Proper and Improper Forcing. These sections offer proofs of the preservation under countable support iteration of proper forcing of various properties, including proofs that $\omega^{\omega}$-bounding, the Sacks property, the Laver property, and the $P$-point property are preserved by countable support iteration of proper forcing. Also, any countable support iteration of proper forcing that does not add a dominating real preserves "no Cohen reals."


## 1 Introduction

This paper is an exposition of some preservation theorems, due to Shelah [13, Chapter VI], for countable support iterations of proper forcing. These include the preservation of the ${ }^{\omega} \omega$-bounding property, the Sacks and Laver properties, the $P$-point property, and some others. Generalizations to revised countable support iterations of semi-proper forcings or even certain non-semi-proper forcings are given in [13, Chapter VI] but we do not address these more general iterations. The results of [13, Section VI.2] overlap the results of [3] and [4], but the methods are dissimilar. The article [1] covers similar ground.

This is the third in a sequence of expository papers covering parts of Shelah's book, Proper and Improper Forcing. The earlier papers were [11], which covers sections 2 through 8 of [13, Chapter XI] and [9], which covers sections 2 and 3 of [13, Chapter XV]. The fourth paper of this sequence is [12], which presents an exposition of [13, Sections VI. 3 and XVIII.3], including a proof of [13, Conclusion VI.2.15D]. Other papers by the author generalize certain other results in [13]; in no instance were we content to quote a result of Shelah without supplying a proof. Thus, [6] may be read, in part, as an exposition of [13, Sections V.6, IX.2, and IX.4]; [7] is, in part, an exposition of [13, Section V. 8 and Theorem III.8.5]; and [8] includes as a special case an alternative proof of [13, Theorem III.8.6]. Also, [6] answers [13, Question IX.4.9(1)]; [7] answers a question implicit in [13, Section IX.4]; [10] answers another such question and also may be read, in part, as an exposition of the results of Eisworth and Shelah [2] that weaken the assumption " $\alpha$-proper for every $\alpha<\omega_{1}$ " used in [13, Section V.6].

### 1.1 Notation

We write $p \leq q$ when $p$ is a stronger forcing condition than $q$.
When $\left\langle P_{\eta}: \eta \leq \beta\right\rangle$ is a forcing iterartion, and $\alpha<\beta$, we set $P_{\beta} / G_{P_{\alpha}}$ to be a $P_{\alpha}$-name characterized by

$$
V\left[G_{P_{\alpha}}\right] \models " P_{\beta} / G_{P_{\alpha}}=\left\{p \upharpoonright[\alpha, \beta): p \in P_{\beta} \text { and } p \upharpoonright \alpha \in G_{P_{\alpha}}\right\} . "
$$

Notice that when $p \in P_{\alpha}$ and $p \Vdash$ " $q \in P_{\beta} / G_{P_{\alpha}}$ " then $p \Vdash$ " $q \Vdash{ }^{\prime} \varphi^{\prime}$ "
makes sense. In contrast, $p^{\wedge} q \Vdash$ " $\varphi$ " makes sense only under the stronger assumption that $\hat{p^{\wedge} q \in P_{\beta} \text {. For example, it could be the case that } \operatorname{supp}\left(p^{\wedge}\right) ~}$ $q)=\beta$ yet $p \Vdash " \operatorname{supp}(q)$ is a singleton." For this reason we favor the former notation and eschew the latter.

## 2 Preservation of properness

The fact that properness is preserved under countable support iterations was proved by Shelah in 1978. The proof of this fact is the basis of all preservation theorems for countable support iterations.

Theorem 2.1 (Proper Iteration Lemma, Shelah). Suppose $\left\langle P_{\eta}\right.$ : $\eta \leq \kappa\rangle$ is a countable support forcing iteration based on $\left\langle Q_{\eta}: \eta<\kappa\right\rangle$ and for every $\eta<\kappa$ we have that $\mathbf{1} \Vdash_{P_{\eta}}$ " $Q_{\eta}$ is proper." Suppose also that $\alpha<\kappa$ and $\lambda$ is a sufficiently large regular cardinal and $N$ is a countable elementary submodel of $H_{\lambda}$ and $\left\{P_{\kappa}, \alpha\right\} \in N$ and $p \in P_{\alpha}$ is $N$-generic and $q$ is a $P_{\alpha}$-name and $p \Vdash$ " $q \in P_{\kappa} / G_{P_{\alpha}} \cap N\left[G_{P_{\alpha}}\right]$." Then there is $r \in P_{\kappa}$ such that $r$ is $N$-generic and $r \upharpoonright \alpha=p$ and $p \Vdash$ " $r \upharpoonright[\alpha, \kappa) \leq q$."

Proof. The proof proceeds by induction, so suppose that the Theorem holds for all iterations of length less than $\kappa$. Fix $\lambda$ and $N$ and $\alpha$ and $p$ and $q$ as in the assumption.

Case 1. $\kappa=\beta+1$ for some $\beta$.
Because $\beta \in N$ we may use the induction hypothesis to fix $p^{\prime} \in P_{\beta}$ such that $p^{\prime} \upharpoonright \alpha=p$ and $p^{\prime}$ is $N$-generic and $p \Vdash$ " $p^{\prime} \upharpoonright[\alpha, \beta) \leq q \upharpoonright \beta$." We have that $p^{\prime} \Vdash " q(\beta) \in N\left[G_{P_{\beta}}\right]$." Take $r \in P_{\kappa}$ such that $r \upharpoonright \beta=p^{\prime}$ and

$$
p^{\prime} \Vdash " r(\beta) \leq q(\beta) \text { and } r(\beta) \text { is } N\left[G_{P_{\beta}}\right] \text {-generic for } Q_{\beta} \text {." }
$$

Then $r$ is $N$-generic and we are done with the successor case.
Case 2. $\kappa$ is a limit ordinal.
Let $\beta=\sup (\kappa \cap N)$, and fix $\left\langle\alpha_{n}: n \in \omega\right\rangle$ an increasing sequence from $\kappa \cap N$ cofinal in $\beta$ such that $\alpha_{0}=\alpha$. Let $\left\langle\sigma_{n}: n \in \omega\right\rangle$ enumerate all the $P_{\kappa}$ names $\sigma \in N$ such that $\mathbf{1} \Vdash$ " $\sigma$ is an ordinal."

Using the induction hypothesis, build a sequence $\left\langle\left\langle p_{n}, q_{n}, \tau_{n}\right\rangle: n \in \omega\right\rangle$ such that $p_{0}=p$ and $p \Vdash$ " $q_{0} \leq q$ " and for each $n \in \omega$ we have all of the following:
(1) $p_{n} \in P_{\alpha_{n}}$ and $p_{n}$ is $N$-generic and $p_{n+1} \upharpoonright \alpha_{n}=p_{n}$ and $\tau_{n}$ is a $P_{\alpha_{n}}$-name.
(2) $p_{n} \Vdash$ " $q_{n} \in P_{\kappa} / G_{P_{\alpha_{n}}} \cap N\left[G_{P_{\alpha_{n}}}\right]$ and $\tau_{n} \in N\left[G_{P_{\alpha_{n}}}\right]$ and $q_{n} \Vdash{ }^{‘} \sigma_{n}=\tau_{n}$, and if $n>0$ then $q_{n} \leq q_{n-1} \upharpoonright\left[\alpha_{n}, \kappa\right)$."
(3) $p_{n} \Vdash$ " $p_{n+1} \upharpoonright\left[\alpha_{n}, \alpha_{n+1}\right) \leq q_{n} \upharpoonright \alpha_{n+1}$."

Define $r \in P_{\kappa}$ such that $(\forall n \in \omega)\left(r \upharpoonright \alpha_{n}=p_{n}\right)$ and $\operatorname{supp}(r) \subseteq \beta$. To see that $r$ is $N$-generic, suppose that $\sigma \in N$ is a $P_{\kappa}$-name for an ordinal. Fix $n$ such that $\sigma=\sigma_{n}$. Because $p_{n}$ is $N$-generic, we have

$$
p_{n} \Vdash " \operatorname{supp}\left(q_{n}\right) \subseteq \kappa \cap N\left[G_{P_{\alpha_{n}}}\right]=\kappa \cap N, "
$$

whence it is clear that

$$
p_{n} \Vdash " r \upharpoonright\left[\alpha_{n}, \kappa\right) \leq q_{n} . "
$$

We have

$$
p_{n} \Vdash " q_{n} \Vdash ' \sigma \in O r d \cap N\left[G_{P_{\alpha_{n}}}\right]=\operatorname{Ord} \cap N, ’ "
$$

where Ord is the class of all ordinals. Thus $r \Vdash$ " $\sigma \in N$." We conclude that $r$ is $N$-generic, and the Theorem is established.

Corollary 2.2 (Fundamental Theorem of Proper Forcing, Shelah). Suppose $\left\langle P_{\eta}: \eta \leq \kappa\right\rangle$ is a countable support forcing iteration based on $\left\langle Q_{\eta}: \eta<\kappa\right\rangle$ and for every $\eta<\kappa$ we have that $\mathbf{1} \Vdash_{P_{\eta}}$ " $Q_{\eta}$ is proper." Then $P_{\kappa}$ is proper.

Proof: Take $\alpha=0$ in the Proper Iteration Lemma.

## 3 Preservation of proper plus $\omega^{\omega}$-bounding

In this section we recount Shelah's proof of the preservation of "proper plus $\omega^{\omega}$-bounding." This is a special case of [13, Theorem VI.1.12] and is given as [13, Conclusion VI.2.8D]. Another treatment of this result can be found in [3] and [4], using different methods. Notably, the proof given in [3] assumes each forcing adds reals and [4] contains a patch for this deficiency, but in Shelah's proof presented here, the issue does not arise.
The following Lemma justifies the construction of $\left\langle p^{n}: n \in \omega\right\rangle$ and $\left\langle t_{n, m}\right.$ : $m \leq n\langle\omega\rangle$ and $\left\langle f_{m}: m \in \omega\right\rangle$ in [13, proof of Theorem VI.1.12], where Shelah's $p^{n}$ is our $p_{n} \upharpoonright n$. Our $p^{\prime}$ encapsulates the third paragraph of [13,
proof of Theorem VI.i.12], i.e., Shelah's assertion that w.l.o.g. $f(k)$ is a $P_{k}$-name of a natural number (see (2) below).

Lemma 3.1. Suppose $\left\langle P_{n}: n \leq \omega\right\rangle$ is a countable support iteration based on $\left\langle Q_{n}: n<\omega\right\rangle$. Suppose also that $f$ is a $P_{\omega}$-name for an element of ${ }^{\omega} \omega$, and suppose $p \in P_{\omega}$. Then there are $\left\langle p_{n}: n \in \omega\right\rangle$ and $\left\langle f_{n}: n \in \omega\right\rangle$ and $p^{\prime} \leq p$ such that $p_{0}=p^{\prime}$ and for every $n \in \omega$ we have that each of the following holds:
(1) $f_{n}$ is a $P_{n}$-name for an element of ${ }^{\omega} \omega$, and
(2) $p^{\prime} \upharpoonright n \Vdash$ " $p^{\prime} \upharpoonright[n, \omega) \Vdash ' f \upharpoonright(n+1)=f_{n} \upharpoonright(n+1)$,'" and
(3) $p_{n+1} \leq p_{n}$, and
(4) for all $m$, we have $p^{\prime} \upharpoonright n \Vdash$ " $p_{m}(n) \Vdash{ }^{\prime} f_{n} \upharpoonright(m+1)=f_{n+1} \upharpoonright(m+1)$. '"

Proof: Define $\left\langle q_{n}: n \in \omega\right\rangle$ and $\left\langle\sigma_{n}: n \in \omega\right\rangle$ such that $\sigma_{0} \in \omega$ and $q_{0} \leq p$ and $q_{0} \Vdash_{P_{\omega}} " f(0)=\sigma_{0} "$ and for every $n>0$ we have that $\sigma_{n}$ is a $P_{n}$-name for an integer and $p \upharpoonright n \Vdash_{P_{n}}$ " $q_{n} \in P_{\omega} / G_{P_{n}}$ and $q_{n} \leq q_{n-1} \upharpoonright[n, \omega)$ and $q_{n} \Vdash{ }^{\prime} \sigma_{n}=f(n) . ' "$

Define $p^{\prime} \in P_{\omega}$ by $(\forall n \in \omega)\left(p^{\prime}(n)=q_{n}(n)\right)$. We have that (2) holds.
We now define $\left\langle p_{n}: n>0\right\rangle$. Given $p_{n}$, build $\left\langle\left(q_{n}^{k}, \tau_{n}^{k}\right): k \leq n\right\rangle$ by downward induction by setting $\tau_{n}^{n+1}=\sigma_{n+1}$ and, given $\tau_{n}^{k+1}$, take $q_{n}^{k}$ and $\tau_{n}^{k}$ such that $\tau_{n}^{k}$ is a $P_{k}$-name for an integer and $p^{\prime} \upharpoonright k \Vdash " q_{n}^{k} \leq p_{n}(k)$ and $q_{n}^{k} \Vdash{ }^{\prime} \tau_{n}^{k}=\tau_{n}^{k+1}$.'" Choose $p_{n+1} \leq p_{n}$ such that for every $k \leq n$ we have $p_{n+1}(k)=q_{n}^{k}$.

For every $k \leq n$ let $f_{n}(k)$ be the $P_{n}$-name for $\sigma_{k}$ and for $k>n$ let $f_{n}(k)=\tau_{k}^{n}$.

The Lemma is established.
Definition 3.2. For $f$ and $g$ in ${ }^{\omega} \omega$ we say $f \leq g$ iff $(\forall n \in \omega)(f(n) \leq$ $g(n))$. We say that $P$ is ${ }^{\omega} \omega$-bounding iff $V\left[G_{P}\right] \models "\left(\forall f \in{ }^{\omega} \omega\right)(\exists g \in$ $\left.{ }^{\omega} \omega \cap V\right)(f \leq g) . "$

Lemma 3.3. Suppose $P$ is ${ }^{\omega} \omega$-bounding and $V\left[G_{P}\right] \models$ " $(\forall n \in \omega)\left(f_{n} \in\right.$ $\left.{ }^{\omega} \omega\right)$." Then $V\left[G_{P}\right] \models "\left(\exists\left\langle f_{n}^{\prime}: n \in \omega\right\rangle \in V\right)(\forall n \in \omega)\left(f_{n}^{\prime} \in{ }^{\omega} \omega\right.$ and $f_{n} \leq$ $f_{n}^{\prime}$."

Proof: Let $j$ be a one-to-one function from ${ }^{2} \omega$ onto $\omega$. In $V\left[G_{P}\right]$ define $h \in{ }^{\omega} \omega$ by $(\forall n \in \omega)(\forall m \in \omega)\left(h(j(n, m))=f_{n}(m)\right)$ and choose $h^{\prime} \in{ }^{\omega} \omega \cap V$
such that $h \leq h^{\prime}$. Define $\left\langle f_{n}^{\prime}: n \in \omega\right\rangle$ by $(\forall n \in \omega)(\forall m \in \omega)\left(h^{\prime}(j(n, m))=\right.$ $\left.f_{n}^{\prime}(m)\right)$. The Lemma is established.

The proof of the following Theorem is obtained from [13, proof of Theorem VI.1.12] by discarding all references to $x$ and replacing each tree with a function bounding its branches; both of these simplifications are justified by [12, Definition VI.2.8A]. Also we have removed any reference to non-proper forcings and we have made explicit the dependence of the functions $F_{0}, F_{1}$, and $F_{2}$ on the parameter $n$ (this dependence is suppressed in Shelah's proof).

Theorem 3.4. Suppose $\left\langle P_{\eta}: \eta \leq \kappa\right\rangle$ is a countable support iteration based on $\left\langle Q_{\eta}: \eta<\kappa\right\rangle$ and suppose $(\forall \eta<\kappa)\left(\mathbf{1} \Vdash_{P_{\eta}}\right.$ " $Q_{\eta}$ is proper and ${ }^{\omega} \omega$-bounding" $)$. Then $P_{\kappa}$ is ${ }^{\omega} \omega$-bounding.

Proof: By induction on $\kappa$. By standard arguments, taking into account the fact that a counterexample to ${ }^{\omega} \omega$-bounding cannot first appear in $V\left[G_{P_{\kappa}}\right]$ where $\kappa$ has uncountable cofinality, and the fact that the composition of two ${ }^{\omega} \omega$-bounding forcings is again ${ }^{\omega} \omega$-bounding, we only have to establish this for $\kappa=\omega$.

Fix $\lambda$ a sufficiently large regular cardinal and $N$ a countable elementary substructure of $H_{\lambda}$ such that $P_{\omega} \in N$ and suppose $p \in P_{\omega} \cap N$.

Let $\left\langle g_{j}: j<\omega\right\rangle$ list ${ }^{\omega} \omega \cap N$, with infinitely many repetitions.
Fix $p^{\prime}$ and $\left\langle\left(p_{n}, f_{n}\right): n \in \omega\right\rangle$ as in Lemma 3.1. We may assume that for every $n \in \omega$ we have $p^{\prime}$ and $p_{n}$ and $f_{n}$ are in $N$, and, furthermore, the sequence $\left\langle\left\langle p_{n}, f_{n}\right\rangle: n \in \omega\right\rangle$ is in $N$.

Define $g \in{ }^{\omega} \omega$ by

$$
g(i)=\max \left\{f_{0}(i), \max \left\{g_{k}(i): k \leq i\right\}\right\}
$$

For each $n \in \omega$, fix $P_{n}$-names $F_{n, 0}$ and $F_{n, 2}$ such that $V\left[G_{P_{n}}\right] \models " F_{n, 0}$ maps $Q_{n}$ into ${ }^{\omega} \omega$ and $F_{n, 2}$ maps $Q_{n}$ into $Q_{n}$ and for every $q^{\prime} \in Q_{n}$ we have $F_{n, 2}\left(q^{\prime}\right) \leq q^{\prime}$ and $F_{n, 2}\left(q^{\prime}\right) \Vdash{ }^{\prime} f_{n+1} \leq F_{n, 0}\left(q^{\prime}\right)$ '."

For each $n \in \omega$, use Lemma 3.3 to fix $F_{n, 1}$ such that $V\left[G_{P_{n}}\right] \models " F_{n, 1} \in V$ maps $\omega$ to ${ }^{\omega} \omega$ and for all $m \in \omega$ we have $F_{n, 0}\left(p_{m}(n)\right) \leq F_{n, 1}(m)$."

We may assume that for each $n \in \omega$ we have the name $F_{n, 1}$ is in $N$.
Claim 1. We may be build $\left\langle r_{n}: n \in \omega\right\rangle$ such that for every $n \in \omega$ we have that the following hold:
(1) $r_{n} \in P_{n}$ is $N$-generic.
(2) $r_{n+1} \upharpoonright n=r_{n}$.
(3) $r_{n} \Vdash$ " $f_{n} \leq g$."
(4) $r_{n} \Vdash$ " $r_{n+1}(n) \leq p^{\prime}(n)$."

Proof: Work by induction on $n$.
Case 1: $n=0$.
We have $f_{0} \leq g$.
Case 2: Otherwise.
Suppose we have $r_{n}$.
In $V\left[G_{P_{n}}\right]$, define $g_{n}^{*}$ by $(\forall i \in \omega)\left(g_{n}^{*}(i)=\max \left\{F_{n, 1}(m)(i): m \leq i\right\}\right)$.
We may assume the name $g_{n}^{*}$ is in $N$.
Notice that we have

$$
r_{n} \Vdash " g_{n}^{*} \in N\left[G_{P_{n}}\right] \cap V=N . "
$$

Therefore we may choose a $P_{n}$-name $k$ such that $r_{n} \Vdash$ " $g_{n}^{*}=g_{k}$ and $k>n "$ (in our notation we suppress the fact that $k$ depends on $n$ ).

Subclaim 1: $r_{n} \Vdash{ }^{\Vdash} F_{n, 2}\left(p_{k}(n)\right) \Vdash{ }^{\prime} f_{n+1} \leq g$.' "
Proof: For $i \geq k$ we have

$$
\begin{aligned}
& r_{n} \Vdash " F_{n, 2}\left(p_{k}(n)\right) \Vdash{ }^{`} f_{n+1}(i) \leq F_{n, 0}\left(p_{k}(n)\right)(i) \\
& \quad \leq F_{n, 1}(k)(i) \leq g_{n}^{*}(i)=g_{k}(i) \leq g(i) . " \text { " }
\end{aligned}
$$

The first inequality is by the definition of $F_{n, 0}$, the second inequality is by the definition of $F_{n, 1}$, the third inequality is by the definition of $g_{n}^{*}$ along with the fact that $i \geq k$, the equality is by the definition of $k$, and the last inequality is by the definition of $g$ along with the fact that $i \geq k$.

For $i<k$, we have

$$
r_{n} \Vdash{ }^{\prime} p_{k}(n) \Vdash{ }^{\prime} f_{n+1}(i)=f_{n}(i) \leq g(i) . ' "
$$

The equality is by the choice of $\left\langle\left(f_{m}, p_{m}\right): m \in \omega\right\rangle$ (see Lemma 3.1), and the inequality is by the induction hypothesis that Claim 1 holds for integers less than or equal to $n$.

Because $r_{n} \Vdash$ " $F_{n, 2}\left(p_{k}(n)\right) \leq p_{k}(n)$," we have that the Subclaim is established.

Choose $r_{n+1} \in P_{n+1}$ such that $r_{n+1}$ is $N$-generic and $r_{n+1} \upharpoonright n=r_{n}$ and

$$
r_{n} \Vdash " r_{n+1}(n) \leq F_{n, 2}\left(p_{k}(n)\right) . "
$$

This completes the proof of Claim 1.

Let $r=\bigcup\left\{r_{n}: n \in \omega\right\}$. We have that

$$
r \leq p \text { and } r \Vdash \text { " } f \leq g . "
$$

The Theorem is established.

## 4 The Sacks property

In this section we present Shelah's proof of the preservation of "proper plus Sacks property" under countable support iteration. The proof is a special case of [13, Theorem VI.1.12] and appears as [13, Conclusion VI.2.9D].

Definition 4.1. For $x$ and $y$ in ${ }^{\omega}(\omega-\{0\})$, we say that $x \ll y$ iff $(\forall n \in \omega)$ $(x(n) \leq y(n))$ and

$$
\lim _{n \rightarrow \infty} y(n) / x(n)=\infty
$$

In particular for $x \in^{\omega}(\omega-\{0\})$ we have $1 \ll x$ iff $x$ diverges to infinity.
The following Definition corresponds to [13, Definition VI.2.9A(b)].
Definition 4.2. For $T \subseteq{ }^{<\omega} \omega$ a tree and $x \in{ }^{\omega}(\omega-\{0\})$, we say that $T$ is an $x$-sized tree iff for every $n \in \omega$ we have that the cardinality of $T \cap^{n} \omega$ is at most $x(n)$, and $T$ has no terminal nodes.

Definition 4.3. For $T \subseteq{ }^{<\omega} \omega$ we set [ $T$ ] equal to the set of all $f \in{ }^{\omega} \omega$ such that every initial segment of $f$ is in $T$. That is, $[T]$ is the set of infinite branches of $T$.

Often the Sacks property is given as a property of pairs of models; however, because our focus is on forcing constructions, we define it to be a property of posets.

Definition 4.4. A poset $P$ has the Sacks property iff whenever $x \in$ ${ }^{\omega}(\omega-\{0\})$ and $1 \ll x$ then we have

$$
\mathbf{1} \Vdash_{P} "\left(\forall f \in{ }^{\omega} \omega\right)(\exists H \in V)(H \text { is an } x \text {-sized tree and } f \in[H]) . "
$$

Definition 4.5. Suppose $n \in \omega$. We say that $t$ is an $n$-tree iff $t \subseteq \leq^{n} \omega$ and $t$ is closed under initial segments and $t$ is non-empty and for every $\eta \in t$ there is $\nu \in t$ such that $\nu$ extends $\eta$ and $\operatorname{lh}(\nu)=n$.

The following Lemma shows that ( $D, R$ ) given in [13, Definition VI.2.9A] satisfies [13, Definition VI.2.2(3)( $\varepsilon)^{+}$]. The proof given here follows [13, proof of Claim VI.2.9B $\left.(\varepsilon)^{+}\right]$.

Lemma 4.6. $P$ has the Sacks property iff whenever $x$ and $z$ are elements of ${ }^{\omega}(\omega-\{0\})$ and $x \ll z$ we have that

$$
\begin{aligned}
& \mathbf{1} \Vdash_{P} \text { " }(\forall T)(\text { if } T \text { is an } x \text {-sized tree then } \\
& (\exists H \in V)(H \text { is a } z \text {-sized tree and } T \subseteq H)) .
\end{aligned}
$$

Proof: We prove the non-trivial direction. Suppose $P$ has the Sacks property and suppose $x$ and $z$ are given. Working in $V\left[G_{P}\right]$, suppose $T$ is given. For every $n \in \omega$ let

$$
\begin{aligned}
& \mathcal{T}_{n}(x)=\left\{t \subseteq \leq_{n} \omega: t \text { is an } n\right. \text {-tree } \\
& \text { and } \left.(\forall i \leq n)\left(\left|t \cap^{i} \omega\right| \leq x(i)\right)\right\} .
\end{aligned}
$$

Let

$$
\mathcal{T}(x)=\bigcup\left\{\mathcal{T}_{n}(x): n \in \omega\right\} .
$$

Under the natural order, $\mathcal{T}(x)$ is isomorphic to ${ }^{<\omega} \omega$.
Define $\zeta \in[\mathcal{T}(x)]$ by setting $\zeta(n)=T \cap{ }^{\leq n} \omega$ for all $n \in \omega$.
Define $y \in{ }^{\omega}(\omega-\{0\})$ by setting $y(n)$ equal to the greatest integer less than or equal to $z(n) / x(n)$ for every $n \in \omega$. Clearly $1 \ll y$, so we may choose a $y$-sized tree $H^{\prime} \subseteq \mathcal{T}(x)$ such that $\zeta \in\left[H^{\prime}\right]$ and $H^{\prime} \in V$.
Let $H=\bigcup H^{\prime}$. We have that $H$ is a $z$-sized tree and $H \in V$ and $T \subseteq H$. The Lemma is established.
The following Lemma is [13, Claim VI.2.4(1)] specialized to the case of Sacks property, and we follow the proof from [13].

Lemma 4.7. Suppose $y$ and $z$ are elements of ${ }^{\omega}(\omega-\{0\})$ and $y \ll z$. Suppose $P$ is a forcing such that $V\left[G_{P}\right] \models$ "for every countable $X \subseteq V$ there is a countable $Y \in V$ such that $X \subseteq Y$." Suppose in $V\left[G_{P}\right]$ we have that $\left\langle T_{n}: n \in \omega\right\rangle$ is a sequence of $y$-sized trees such that for every $n$ we have $T_{n} \in V$. Then in $V\left[G_{P}\right]$ there is a $z$-sized tree $T^{*} \in V$ and an increasing sequence of integers $\langle m(n): n \in \omega\rangle$ such that $m(0)=0$ and $(\forall n>0)(m(n)>n)$ and for every $\eta \in{ }^{<\omega} \omega$ we have

$$
(\forall n>0)(\exists i<n)\left(\eta \upharpoonright m(n+1) \in T_{m(i)}\right) \text { implies } \eta \in T^{*} \text {. }
$$

Proof: Fix $x \in{ }^{\omega}(\omega-\{0\})$ such that $y \ll x \ll z$. Fix $\left\langle x_{n}: n \in \omega\right\rangle$ a sequence of elements of ${ }^{\omega}(\omega-\{0\})$ such that $(\forall n \in \omega)\left(y \ll x_{n} \ll x_{n+1} \ll\right.$ $x)$.

Set $k(0)=0$ and for each $n>0$ set $k(n)$ equal to the least $k>n$ such that

$$
(\forall j \geq k)\left(2 x_{n}(j) \leq x_{n+1}(j) \text { and }(n+1) x(j) \leq z(j)\right) .
$$

Work in $V\left[G_{P}\right]$. Let $b \in V$ be a countable set of $y$-sized trees such that $\left\{T_{n}: n \in \omega\right\} \subseteq b$. Let $\left\langle S_{n}: n \in \omega\right\rangle \in V$ enumerate $b$ with infinitely many repetitions with $S_{0}=T_{0}$.

Build $\left\langle S_{n}^{\prime}: n \in \omega\right\rangle$ by setting $S_{0}^{\prime}=S_{0}$ and for every $n \in \omega$ let
(A) $S_{n+1}^{\prime}=\left\{\rho \in S_{n}: \rho \upharpoonright k(n) \in S_{n}^{\prime}\right\} \cup S_{n}^{\prime}$.

Claim 1. For all $n \in \omega$ we have that $S_{n}^{\prime}$ is an $x_{n}$-sized tree.
Proof: By induction on $n$. Clearly $S_{0}^{\prime}=T_{0}$ is an $x_{0}$-sized tree. For every $t<k(n)$ we have that $\left|S_{n+1}^{\prime} \cap^{t} \omega\right|=\left|S_{n}^{\prime} \cap{ }^{t} \omega\right| \leq x_{n}(t) \leq x_{n+1}(t)$. For every $t \geq k(n)$ we have $\left|S_{n+1}^{\prime} \cap^{t} \omega\right| \leq\left|S_{n}^{\prime} \cap t \omega\right|+\left|S_{n} \cap^{t} \omega\right| \leq x_{n}(t)+y(t) \leq x_{n+1}(t)$. The Claim is established.

Define $h \in{ }^{\omega} \omega$ by setting $h(0)=0$ and for every $n>0$ setting $h(n)$ equal to the least $m>n$ such that $T_{n}=S_{m}$.

Build $\left\langle n_{i}^{\prime}: i \in \omega\right\rangle$ an increasing sequence of integers such that $n_{0}^{\prime}=0$ and $n_{1}^{\prime}>k(1)$ and for every $i \in \omega$ we have
(B) $k\left(h\left(n_{i}^{\prime}\right)\right)<n_{i+1}^{\prime}$.

By (B) we have
(C) $(\forall i \in \omega)(\exists t)\left(n_{i}^{\prime}<k(t)<n_{i+1}^{\prime}\right)$.

Let $T^{*}=\left\{\eta \in{ }^{<\omega} \omega:(\forall n>0)(\exists i<n)\left(\eta \upharpoonright k(n) \in S_{k(i)}^{\prime}\right)\right\}$.
Claim 2. $T^{*}$ is a $z$-sized tree.
Proof. Given $t \geq k(1)$, choose $n \in \omega$ such that $k(n) \leq t<k(n+1)$. We have

$$
T^{*} \cap{ }^{t} \omega \subseteq\left\{\eta \in^{t} \omega:(\forall j \leq n+1)\left(\text { if } j>0 \text { then }(\exists i<j)\left(\eta \upharpoonright k(j) \in S_{k(i)}^{\prime}\right)\right)\right\}
$$

and so

$$
\left|T^{*} \cap{ }^{t} \omega\right| \leq \Sigma_{i \leq n}\left|S_{k(i)}^{\prime} \cap{ }^{t} \omega\right| \leq(n+1) x(t) \leq z(t)
$$

For $t<k(1)$ we have $T^{*} \cap{ }^{t} \omega=T_{0} \cap^{t} \omega$, so $\left|T^{*} \cap{ }^{t} \omega\right| \leq y(t) \leq z(t)$.
The Claim is established

For every $i \in \omega$ let $m_{i}=n_{4 i}^{\prime}$.
Fix $\eta \in{ }^{<\omega} \omega$ such that
(D) $(\forall i>0)(\exists j<i)\left(\eta \upharpoonright m_{i+1} \in T_{m_{j}}\right)$.

To establish the Lemma, it suffices to show
(E) $(\forall i>0)(\exists j<i)\left(\eta \upharpoonright k(i) \in S_{k(j)}^{\prime}\right)$,
since this implies $\eta \in T^{*}$.
Claim 3. $(\forall i>0)(\exists j<i)\left(\eta \upharpoonright n_{i+1}^{\prime} \in S_{n_{j}^{\prime}}^{\prime}\right)$.
We prove this by induction on $i$.
Case 1. $i<8$.
We have $n_{i+1}^{\prime} \leq n_{8}^{\prime}=m_{2}$ and by (D) we have $\eta \upharpoonright m_{2} \in T_{0}$. Therefore $\eta \upharpoonright n_{i+1}^{\prime} \in T_{0}=S_{0}=S_{0}^{\prime}$.

Case 2. $i \geq 8$,
Fix $i^{*}$ such that $4 i^{*} \leq i<4 i^{*}+4$.
By (D) we may fix $j^{*}<i^{*}$ such that
(F) $\eta \upharpoonright m_{i^{*}+1} \in T_{m_{j^{*}}}=S_{h\left(m_{j^{*}}\right)}$.

Using the fact that $4 j^{*}+1<i$ we have, by the induction hypothesis, that
(G) $\eta \upharpoonright n_{4 j^{*}+1}^{\prime} \in S_{n_{4 j^{*}}^{\prime}}^{\prime}=S_{m_{j^{*}}}^{\prime} \subseteq S_{h\left(m_{j^{*}}\right)}^{\prime}$.

By (B) we have
(H) $k\left(h\left(m_{j^{*}}\right)\right)<n_{4 j^{*}+1}^{\prime}$.

By (A), (F), (G), and (H) we have
(I) $\eta \upharpoonright m_{i^{*}+1} \in S_{h\left(m_{j^{*}}\right)+1}^{\prime}$.

We have
(J) $n_{i+1}^{\prime} \leq n_{4 i^{*}+4}^{\prime}=m_{i^{*}+1}$.

By (I) and (J) we have $\eta \upharpoonright n_{i+1}^{\prime} \in S_{h\left(m_{j^{*}}\right)+1}^{\prime} \subseteq S_{n_{4 j^{*}+2}^{\prime}}^{\prime}$.
Because $4 j^{*}+2<i$ the Claim is established.
To complete the proof of the Lemma, suppose $i>0$. By ( $\mathrm{E)} \mathrm{it} \mathrm{suffices}$ to show that there is $t<i$ such that $\eta \upharpoonright k(i) \in S_{k(t)}^{\prime}$.

Case 1: $k(i-1)<n_{0}^{\prime}$.
By (C) we have $n_{1}^{\prime} \geq k(i)$. By Claim 3 we have $\eta \upharpoonright n_{2}^{\prime} \in S_{0}$. Hence $\eta \upharpoonright n_{1}^{\prime} \in S_{0}$. Hence $\eta \upharpoonright k(i) \in S_{0}$.

Case 2: $n_{0}^{\prime} \leq k(i-1)$.
By (C) we know that there is at most one element of $\left\{n_{j}^{\prime}: j \in \omega\right\}$ strictly between $k(i-1)$ and $k(i)$. Hence we may fix $j>0$ such that $n_{j-1}^{\prime} \leq k(i-1)<k(i) \leq n_{j+1}^{\prime}$. If $\eta \upharpoonright n_{j+1}^{\prime} \in S_{0}$ then $\eta \upharpoonright k(i) \in S_{0}$ and we
are done, so assume otherwise. By Claim 3 we may fix $m<j$ such that $\eta \upharpoonright n_{j+1}^{\prime} \in S_{n_{m}^{\prime}}^{\prime}$. We have $\eta \upharpoonright k(i) \in S_{n_{m}^{\prime}}^{\prime} \subseteq S_{n_{j-1}^{\prime}}^{\prime} \subseteq S_{k(i-1)}^{\prime}$ and again we are done.

The Lemma is established.
The following Lemma shows that ( $D, R$ ) from [13, Definition VI.2.9A] satisfies [13, Definition VI.1.2( $\beta$ )(iv) and Remark VI.1.3(8)] The proof given here is [13, proof of Claim VI.2.9B $\left.\left(\gamma^{+}\right)\right]$.

Lemma 4.8. Suppose $x \in{ }^{\omega}(\omega-\{0\})$ and $z \in{ }^{\omega}(\omega-\{0\})$ and $x \ll z$. Suppose that $\left\langle T_{n}: n \in \omega\right\rangle$ is a sequence of $x$-sized trees and $T$ is an $x$-sized tree. Then there is a $z$-sized tree $T^{*}$ and a sequence of integers $\left\langle k_{i}: i \in \omega\right\rangle$ such that $T \subseteq T^{*}$ and for every $\eta \in T$ and $i \in \omega$ and every $\nu \in T_{m_{i}}$ extending $\eta$, if length $(\eta) \geq k_{i}$ then $\nu \in T^{*}$.

Proof. Choose $y \in{ }^{\omega}(\omega-\{0\})$ such that $x \ll y \ll z$.
Fix $n^{*}$ such that $\left(\forall i \geq n^{*}\right)(2 x(i) \leq y(i))$.
For every $n \geq n^{*}$ let $T_{n}^{\prime}=\left\{\eta \in T_{n}: \eta \upharpoonright n \in T\right\} \cup T$, and for $n<n^{*}$ let $T_{n}^{\prime}=T$.

We have $T_{n}^{\prime}$ is a $y$-sized tree for every $n \in \omega$.
For each $n \in \omega$ set $k_{n}$ equal to the least $k \geq n^{*}$ such that $(\forall j \geq k)$ $((n+2) y(j) \leq z(j))$.

Let $T^{*}=\left\{\eta \in{ }^{\omega} \omega:(\forall i>0)(\exists j<i)\left(\eta \upharpoonright k_{i} \in T_{k_{j}}^{\prime}\right)\right\}$.
Clearly $T \subseteq T^{*}$.
Claim: $T^{*}$ is a $z$-sized tree.
Proof: Like Claim 2 of the proof of Lemma 4.7.
Now suppose that $\eta \in T$ and $i \in \omega$ and length $(\eta) \geq k_{i}$ and $\nu$ extends $\eta$ and $\nu \in T_{k_{i}}$. We show $\nu \in T^{*}$.

Because $\nu$ extends an element of $T$ of length at least $k_{i}$, we have that $\nu \in T_{k_{i}}^{\prime}$. Choose $h \in\left[T_{k_{i}}^{\prime}\right]$ extending $\nu$. It suffices to show that $h \in\left[T^{*}\right]$. Therefore it suffices to show that for every $n \in \omega$ we have
$(*)_{n}(\exists j<n)\left(h \upharpoonright k_{n} \in T_{k_{j}}^{\prime}\right)$.
Fix $n \in \omega$.
Case 1: $i<n$.
Because $h \in\left[T_{k_{i}}^{\prime}\right]$ we have $h \upharpoonright k_{n} \in T_{k_{i}}^{\prime}$, so $(*)_{n}$ holds.
Case 2: $n \leq i$.

We have $h \upharpoonright k_{n}=\nu \upharpoonright k_{n}=\eta \upharpoonright k_{n} \in T \subseteq T_{k_{0}}^{\prime}$. Therefore $(*)_{n}$ holds.
The Lemma is established.
The proof of the following Lemma is "proof of (d)" in [13, proof of Claim VI.1.8]. Thus, it shows that $(D, R)$ from [13, Definition VI.2.9A] is a strong covering model [13, Definition VI.1.6(4)]. Indeed, we (and Shelah) show something stronger insofar as the quantifier "there exists $\left\langle x_{n}: n \in \omega\right\rangle$ " in [13, Definition VI.1.6(4)(d)] is replaced with the quantifier "for all increasing $\left\langle x_{n}: n \in \omega\right\rangle$ bounded below $z$."

Lemma 4.9. Suppose $y \in{ }^{\omega}(\omega-\{0\})$ and $z \in{ }^{\omega}(\omega-\{0\})$, and suppose $\left\langle x_{n}: n \in \omega\right\rangle$ is a sequence of elements of ${ }^{\omega}(\omega-\{0\})$ such that $(\forall n \in \omega)$ $\left(x_{n} \ll x_{n+1} \ll y \ll z\right)$. Suppose for every $n \in \omega$, we have $x_{n}^{*} \in{ }^{\omega}(\omega-\{0\})$ and $x_{n}^{*} \ll x_{n} \ll x_{n+1}^{*}$, and we have $\left\langle x_{n, j}: n \in \omega, j \in \omega\right\rangle$ is a sequence of elements of ${ }^{\omega}(\omega-\{0\})$ such that for every $j \in \omega$ we have $x_{n} \ll x_{n, j} \ll$ $x_{n, j+1} \ll x_{n+1}^{*}$. Suppose that $P$ is a forcing such that $V\left[G_{P}\right] \models$ "for every countable $X \subseteq V$ there is a countable $Y \in V$ such that $X \subseteq Y$." Suppose in $V\left[G_{P}\right]$ we have that $T \in V$ is an $x_{0}$-sized tree and $\left\langle T_{n, j}: n \in \omega, j \in \omega\right\rangle$ is a sequence such that for every $n \in \omega$ and $j \in \omega$ we have that $T_{n, j} \in V$ is an $x_{n, j}$-sized tree (but the sequence need not be in $V$ ). Then in $V\left[G_{P}\right]$ there is a sequence $\left\langle T^{n}: n \in \omega\right\rangle$ and $T^{*} \in V$ such that $T^{*}$ is a $z$-sized tree and $T \subseteq T^{*}$ and for every $n \in \omega$ we have
(i) $T^{n} \subseteq T^{n+1}$ and $T^{n} \in V$ is an $x_{n}$-sized tree, and
(ii) for every $j \in \omega$ and every $g \in\left[T_{n, j}\right]$ there is $k \in \omega$ such that for every $\eta \in T_{n, j}$ extending $g \upharpoonright k$, if $\eta \upharpoonright k \in T^{n} \cap T^{*}$ then $\eta \in T^{n+1} \cap T^{*}$.

Proof: Work in $V\left[G_{P}\right]$. Let $T^{0}=T$. Given $T^{n} \in V$, build $\left\langle T_{n, j}^{\prime}: j \in \omega\right\rangle$ as follows. Let $T_{n, 0}^{\prime}=T^{n}$. Given $T_{n, j}^{\prime}$ take $m(n, j) \in \omega$ such that

$$
(\forall t \geq m(n, j))\left(2 x_{n, j}(t) \leq x_{n, j+1}(t)\right) .
$$

Let $T_{n, j+1}^{\prime}=\left\{\eta \in T_{n, j}: \eta \upharpoonright m(n, j) \in T_{n, j}^{\prime}\right\} \cup T_{n, j}^{\prime}$.
Claim 1. Whenever $i \leq j<\omega$ we have $T_{n, i}^{\prime} \subseteq T_{n, j}^{\prime}$.
Proof. Clear.
Claim 2. Suppose $T^{n}$ is an $x_{n}$-sized tree. Then $(\forall j \in \omega)\left(T_{n, j}^{\prime}\right.$ is an $x_{n, j}$-sized tree).

Proof: It is clear that $T_{n, 0}^{\prime}$ is an $x_{n, 0}$-sized tree. Assume that $T_{n, j}^{\prime}$ is an $x_{n, j}$-sized tree. Fix $t \in \omega$.

Case 1: $t<m(n, j)$.
We have that

$$
T_{n, j+1}^{\prime} \cap^{t} \omega=T_{n, j}^{\prime} \cap^{t} \omega
$$

and so

$$
\left|T_{n, j+1}^{\prime} \cap^{t} \omega\right| \leq x_{n, j}(t) \leq x_{n, j+1}(t)
$$

Case 2: $t \geq m(n, j)$.
We have

$$
T_{n, j+1}^{\prime} \cap{ }^{t} \omega \subseteq\left(T_{n, j}^{\prime} \cap^{t} \omega\right) \cup\left(T_{n, j} \cap^{t} \omega\right) .
$$

Therefore we have

$$
\left|T_{n, j+1}^{\prime} \cap^{t} \omega\right| \leq 2 x_{n, j}(t) \leq x_{n, j+1}(t)
$$

The Claim is established.
For each $n \in \omega$, using Claim 2 and Lemma 4.7 with $k_{n, j}$ here equal to $m(j)$ there, we my find an increasing sequence of integers $\left\langle k_{n, j}: j \in \omega\right\rangle$ and $T^{n+1} \in V$ such that $k_{n, 0}=0$ and $(\forall j>0)\left(k_{n, j}>j\right)$ and if $T^{n}$ is an $x_{n}$-sized tree, then $T^{n+1}$ is an $x_{n+1}$-sized tree such that for all $\eta \in{ }^{<\omega} \omega$, we have

$$
(\forall j>0)(\exists i<j)\left(\eta \upharpoonright k_{n, j+1} \in T_{n, k_{n, i}}^{\prime}\right) \text { implies } \eta \in T^{n+1}
$$

This completes the construction of $\left\langle T^{n}: n \in \omega\right\rangle$ and $\left\langle T_{n, j}^{\prime}: j \in \omega, n \in \omega\right\rangle$. Applying mathematical induction, we have that each $T^{n}$ is in fact an $x_{n}$-sized tree.

Claim 3. $T^{n} \subseteq T^{n+1}$ for every $n \in \omega$.
Proof: By Claim 1 we have that $T^{n} \subseteq T_{n, i}^{\prime}$ for every $i \in \omega$. By the definition of $T^{n+1}$ we have that

$$
T^{n+1} \supseteq \bigcap\left\{T_{n, k_{n, i}}^{\prime}: i \in \omega\right\} \supseteq \bigcap\left\{T_{n, i}^{\prime}: i \in \omega\right\} \supseteq T^{n}
$$

The Claim is established.
Applying Lemma 4.7 again, with $k_{n}$ here equal to $m(n)$ there, we obtain an increasing sequence of integers $\left\langle k_{n}: n \in \omega\right\rangle$ and a $z$-sized tree $T^{*} \in V$ such that $(\forall n \in \omega)\left(n<k_{n}\right)$ and for every $\eta \in^{<\omega} \omega$, we have that

$$
(\forall n>0)(\exists i<n)\left(\eta \upharpoonright k_{n+1} \in T^{k_{i}}\right) \text { implies } \eta \in T^{*}
$$

Notice that $T^{0} \subseteq \bigcap\left\{T^{n}: n \in \omega\right\} \subseteq T^{*}$.

Now we verify that $\left\langle T^{n}: n \in \omega\right\rangle$ and $T^{*}$ satisfy the remaining conclusions of the Lemma. Accordingly, fix $n \in \omega$ and $j \in \omega$ and $g \in\left[T_{n, j}\right]$. Let

$$
k=\max \left(k_{n}, \max \left\{k_{n, j^{\prime}}: j^{\prime} \leq j+1\right\}, \max \left\{m\left(n, j^{\prime}\right): j^{\prime} \leq j+1\right\}\right) .
$$

Fix $\eta \in T_{n, j}$ extending $g \upharpoonright k$ and assume that $\eta \upharpoonright k \in T^{n} \cap T^{*}$.
Claim 4. $\eta \in T^{n+1}$.
Proof: It suffices to show

$$
\left(\forall j^{\prime}>0\right)\left(\exists i<j^{\prime}\right)\left(\eta \upharpoonright k_{n, j^{\prime}+1} \in T_{n, k_{n, i}}^{\prime}\right) .
$$

Fix $j^{\prime} \in \omega$ and let $i=\min \left(j, j^{\prime}\right)$.
Case 1: $j^{\prime} \leq j$.
Because $k_{n, j^{\prime}+1} \leq k$ we have that $\eta \upharpoonright k_{n, j^{\prime}+1} \in T^{n} \subseteq T_{n, k_{n, i}}^{\prime}$, as required.
Case 2: $j<j^{\prime}$.
It suffices to show that $\eta \upharpoonright k_{n, j^{\prime}+1} \in T_{n, k_{n, j}^{\prime}}^{\prime}$. Because $g \upharpoonright k=\eta \upharpoonright k \in T^{n}$ and $m(n, j) \leq k$, we have that $g \upharpoonright m(n, j) \in T^{n} \subseteq T_{n, j}^{\prime}$. Because we have $\eta \in T_{n, j}$ and $\eta \upharpoonright m(n, j)=g \upharpoonright m(n, j) \in T_{n, j}^{\prime}$, we know by the definition of $T_{n, j+1}^{\prime}$ and Claim 1 that $\eta \in T_{n, j+1}^{\prime} \subseteq T_{n, k_{n, j}}^{\prime}$.

Claim 4 is established.
Claim 5. $\eta \in T^{*}$.
Proof: It suffices to show $(\forall i>0)\left(\exists i^{\prime}<i\right)\left(\eta \upharpoonright k_{i+1} \in T^{k_{i}}\right)$. Towards this end, fix $i \in \omega$.

Case 1: $i \leq n$.
Because $\eta \upharpoonright k \in T^{*}$ and $\eta$ extends $g \upharpoonright k$, we have $g \upharpoonright k_{i+1} \in T^{*}$ and hence we may take $i^{\prime}<i$ such that $g \upharpoonright k_{i+1} \in T^{k_{i^{\prime}}}$. But we also have $\eta \upharpoonright k_{i+1}=g \upharpoonright k_{i+1}$, so we finish Case 1.

Case 2: $n<i$.
We let $i^{\prime}=i-1$. By Claim 4 we have $\eta \upharpoonright k_{i+1} \in T^{n+1}$, and by Claim 3 we have that $T^{n+1} \subseteq T^{k_{n}} \subseteq T^{k_{i^{\prime}}}$.

Claim 5 and the Lemma are established.
Using Shelah's terminology, we have by Lemmas 4.7, 4.8, and 4.9 that $(D, R)$ from [13, Definition VI.2.9A] is a smooth strong covering model [13, Definition VI.1.7(2)] (see [13, Claim VI.1.8(1)].

The proof of the following Theorem is [13, proof of Theorem VI.1.12] for the case of [13, Definition VI.2.9A].

Theorem 4.10. Suppose $\left\langle P_{\eta}: \eta \leq \kappa\right\rangle$ is a countable support iteration based on $\left\langle Q_{\eta}: \eta<\kappa\right\rangle$ and suppose $(\forall \eta<\kappa)\left(\mathbf{1} \Vdash_{P_{\eta}}\right.$ " $Q_{\eta}$ is proper and has the Sacks property"). Then $P_{\kappa}$ has the Sacks property.

Proof: The proof proceeds by induction on $\kappa$. Since no counterexample can first appear when $\operatorname{cf}(\kappa)$ is uncountable, we may, using standard arguments, assume $\kappa$ is either 2 or $\omega$. The case $\kappa=2$ is easily handled using Lemma 4.6, so assume $\kappa=\omega$.

Suppose that $\lambda$ is a sufficiently large regular cardinal and $N$ is a countable elementary sustructure of $H_{\lambda}$ and $1 \ll z$ and $\mathbf{1} \Vdash_{P_{\omega}}$ " $\zeta \in{ }^{\omega} \omega$," and $\left\{P_{\omega}, z\right\} \in N$ and the $P_{\omega}$-name $\zeta$ is in $N$ and $p \in P_{\omega} \cap N$.

Fix $p^{\prime}$ and $\left\langle\left(p_{n}, \zeta_{n}\right): n \in \omega\right\rangle \in N$ as in Lemma 3.1, using $\zeta$ for $f$ and $\zeta_{n}$ for $f_{n}$.

Fix $y \ll z$ such that $1 \ll y$ and $y \in N$.
Let $\Omega=\{x \in N: 1 \ll x \ll y\}$. Let $\left\langle y_{n}: n \in \omega\right\rangle$ enumerate $\Omega$. Build $\left\langle x_{n}^{*}: n \in \omega\right\rangle$ as follows. Fix $x_{0}^{*} \in \Omega$, and for each $n \in \omega$ choose $x_{n+1}^{*} \in \Omega$ such that $x_{n}^{*} \ll x_{n+1}^{*}$ and $y_{n} \ll x_{n+1}^{*}$. Also build $\left\langle x_{n}: n \in \omega\right\rangle,\left\langle y_{n}^{*}: n \in \omega\right\rangle$, $\left\langle x_{n}^{\prime}: n \in \omega\right\rangle$, and $\left\langle y_{n}^{\prime}: n \in \omega\right\rangle$ sequences of elements of $\Omega$ such that for each $n \in \omega$ we have $x_{n}^{*} \ll x_{n} \ll y_{n}^{*} \ll x_{n}^{\prime} \ll y_{n}^{\prime} \ll x_{n+1}^{*}$.

For each $n \in \omega$ let $\left\langle T_{n, j}: j \in \omega\right\rangle$ list, with infinitely many repetitions, all $T^{\prime} \in N$ such that there is some $y^{*} \ll x_{n+1}^{*}$ such that $T^{\prime}$ is a $y^{*}$-sized tree, and build $\left\langle x_{n, j}: j \in \omega\right\rangle$ a sequence of elements of $\Omega$ such that for every $j \in \omega$ we have that $x_{n} \ll x_{n, j} \ll x_{n, j+1} \ll x_{n+1}^{*}$ and $T_{n, j}$ is an $x_{n, j}$-sized tree.

Working in $V\left[G_{P_{\omega}}\right]$, use Lemma 4.9 to choose $T^{*}$ a $z$-sized tree and $\left\langle T^{n}: n \in \omega\right\rangle \in V\left[G_{P_{\omega}}\right]$ a sequence such that $T^{*} \in V$ and $T^{0} \subseteq T^{*}$ and $\zeta_{0} \in\left[T^{0}\right]$ and for every $n \in \omega$ we have that $T^{n} \in V$ is an $x_{n}$-sized tree and $T^{n} \subseteq T^{n+1}$ and for every $j \in \omega$ and every $g \in\left[T_{n, j}\right]$ there is $k \in \omega$ such that for every $\eta \in T_{n, j}$ extending $g \upharpoonright k$, if $\eta \upharpoonright k \in T^{n} \cap T^{*}$ then $\eta \in T^{n+1} \cap T^{*}$. We may assume the $P_{\omega}$-names $T^{*}$ and $T^{n}$ are in $N$ for each $n$.

Note that the reason we worked in $V\left[G_{P_{\omega}}\right]$ rather than in $V$ in the previous paragraph is because we wish to allow $g$ to range over $\left[T_{n, j}\right]$ with the brackets interpreted in $V\left[G_{P_{\omega}}\right]$ (i.e., $g$ need not be in $V$ ).

Using the induction hypothesis, for every $n \in \omega$, let $F_{n, 0}$ and $F_{n, 1}$ and $F_{n, 2}$ be $P_{n}$-names such that
(A) $\mathbf{1} \Vdash \vdash_{P_{n}}$ " $F_{n, 0}$ and $F_{n, 1}$ and $F_{n, 2}$ are functions each of whose domains is equal to $Q_{n}$, such that

$$
\begin{aligned}
& \left(\forall q^{\prime} \in Q_{n}\right)\left(F_{n, 0}\left(q^{\prime}\right) \text { is an } F_{n, 1}\left(q^{\prime}\right)\right. \text {-sized tree } \\
& \text { and } F_{n, 1}\left(q^{\prime}\right) \ll y_{n}^{*} \text { and } F_{n, 2}\left(q^{\prime}\right) \leq q^{\prime} \\
& \text { and } \left.\left.F_{n, 2}\left(q^{\prime}\right) \Vdash \zeta_{n+1} \in\left[F_{n, 0}\left(q^{\prime}\right)\right]\right]^{\prime}\right) . "
\end{aligned}
$$

We may assume that the names $F_{n, 0}$ and $F_{n, 1}$ and $F_{n, 2}$ are in $N$.
For each $n \in \omega$ we may, in $V\left[G_{P_{n}}\right]$, use Lemma 4.8 to choose $T_{n}^{\#}$ a $x_{n}^{\prime}$-sized tree and $\left\langle k_{i}^{n}: i \in \omega\right\rangle$ an increasing sequence of integers such that $T^{n} \subseteq T_{n}^{\#}$ and for every $\eta \in T^{n}$ and every $i \in \omega$ and every $\nu \in F_{n, 0}\left(p_{k_{i}^{n}}(n)\right)$, if length $(\eta) \geq k_{i}^{n}$ and $\nu$ extends $\eta$, then $\nu \in T_{n}^{\#}$.

Using the induction hypothesis and Lemma 4.6, we may take $\tilde{T}_{n}$ such that $\tilde{T}_{n} \in V$ is $y_{n}^{\prime}$-sized tree and $T_{n}^{\#} \subseteq \tilde{T}_{n}$.

We may assume the $P_{n}$-names $\tilde{T}_{n}$ and $\left\langle k_{i}^{n}: i \in \omega\right\rangle$ are in $N$.
Claim 1. We may be build $\left\langle r_{n}: n \in \omega\right\rangle$ such that for every $n \in \omega$ we have that the following hold:
(1) $r_{n} \in P_{n}$ is $N$-generic, and
(2) $r_{n+1} \upharpoonright n=r_{n}$, and
(3) $r_{n} \Vdash$ " $\zeta_{n} \in\left[T^{n}\right] \cap\left[T^{*}\right]$," and
(4) $r_{n} \leq p^{\prime} \upharpoonright n$."

Proof: By induction on $n$. For $n=0$ we have nothing to prove. Suppose we have $r_{n}$.

By (A) and the definition of $\tilde{T}_{n}$ we have that
(B) $r_{n} \Vdash{ }^{\Vdash} T^{n} \subseteq \tilde{T}_{n} "$
and
(C) $r_{n} \Vdash$ "for every $\eta \in T^{n}$ and every $i \in \omega$ and every $\nu \in F_{n, 0}\left(p_{k_{i}^{n}}(n)\right)$, if length $(\eta) \geq k_{i}^{n}$ and $\nu$ extends $\eta$ then $\nu \in \tilde{T}_{n}$."

By (C) and the fact that, by the induction hypothesis, we know $r_{n} \Vdash$ " $\zeta_{n} \in\left[T^{n}\right]$," we have that
(D) $r_{n} \Vdash "(\forall j \in \omega)\left(F_{n, 2}\left(p_{k_{j}^{n}}(n)\right) \Vdash{ }^{`}\left(\forall \nu \in F_{n, 0}\left(p_{k_{j}^{n}}(n)\right)\right)(\right.$ if $\nu$ extends $\zeta_{n} \upharpoonright k_{j}^{n}$ then $\left.\left.\nu \in \tilde{T}_{n}\right)\right)^{\prime} . "$

We have

$$
r_{n} \Vdash " \tilde{T}_{n} \in N\left[G_{P_{n}}\right] . "
$$

We also have

$$
r_{n} \Vdash " \tilde{T}_{n} \in V . "
$$

Therefore, because $r_{n}$ is $N$-generic, we have

$$
r_{n} \Vdash " \tilde{T}_{n} \in N . "
$$

Therefore there is a $P_{n}$-name $m$ such that

$$
r_{n} \Vdash " \tilde{T}_{n}=T_{n, m} \text { and } m>n . "
$$

Using this fact along with the fact that $\left\langle T^{n}: n \in \omega\right\rangle$ and $T^{*}$ were chosen as in the conclusion of Lemma 4.9 and also using the fact that $r_{n} \Vdash$ " $\zeta_{n} \in$ $\left[T^{n}\right] \subseteq\left[\tilde{T}_{n}\right]$," we may choose $k$ to be a $P_{n}$-name for an integer such that
(E) $r_{n} \Vdash "\left(\forall \eta \in \tilde{T}_{n}\right)\left(\right.$ if $\eta$ extends $\zeta_{n} \upharpoonright k$ and $\eta \upharpoonright k \in T^{n} \cap T^{*}$ then $\left.\eta \in T^{n+1} \cap T^{*}\right)$."

Choose $j$ to be a $P_{n}$-name for an integer such that $r_{n} \Vdash$ " $k_{j}^{n} \geq k$."
Subclaim 1. $r_{n} \Vdash " F_{n, 2}\left(p_{k_{j}^{n}}(n)\right) \Vdash ' \zeta_{n+1} \in\left[\tilde{T}_{n}\right]$.'"
Proof. It suffices to show
$r_{n} \Vdash " F_{n, 2}\left(p_{k_{j}^{n}}(n)\right) \Vdash{ }^{‘}\left(\forall j^{\prime}>j\right)\left(\zeta_{n+1} \upharpoonright k_{j^{\prime}}^{n} \in \tilde{T}_{n}\right) . ' "$
Fix $j^{\prime}$ a $P_{n+1}$-name for an integer such that
$r_{n} \Vdash " F_{n, 2}\left(p_{k_{j}^{n}}(n)\right) \Vdash{ }^{\prime} j^{\prime}>j$.'"
We know by the induction hypothesis that $r_{n} \Vdash$ " $\zeta_{n} \in\left[T^{n}\right]$." Therefore
(F) $r_{n} \Vdash{ }^{\Vdash} \zeta_{n} \upharpoonright k_{j}^{n} \in T^{n}$."

By the definition of $\left\langle p_{i}: i \in \omega\right\rangle$, we have
(G) $r_{n} \Vdash " p_{k_{j}^{n}}(n) \Vdash{ }^{\prime} \zeta_{n} \upharpoonright k_{j}^{n}=\zeta_{n+1} \upharpoonright k_{j}^{n}$." "

By (A) we have
(H) $r_{n} \Vdash " F_{n, 2}\left(p_{k_{j}^{n}}(n)\right) \Vdash ‘ \zeta_{n+1} \in\left[F_{n, 0}\left(p_{k_{j}^{n}}(n)\right]\right) . ' "$

Combining (F), (G), (H), and the definition of $\tilde{T}_{n}$, we have that

$$
r_{n} \Vdash " F_{n, 2}\left(p_{k_{j}^{n}}(n)\right) \Vdash ‘ \zeta_{n+1} \upharpoonright k_{j^{\prime}}^{n} \in \tilde{T}_{n} . ’ "
$$

The Subclaim is established.
Subclaim 2. $r_{n} \Vdash{ }^{\Vdash} F_{n, 2}\left(p_{k_{j}^{n}}(n)\right) \Vdash$ ' $\zeta_{n+1} \in\left[T^{n+1}\right] \cap\left[T^{*}\right] . " "$
Proof: By (E) we have
(I) $r_{n} \Vdash$ " $\left(\forall \eta \in \tilde{T}_{n}\right)$ (if $\eta$ extends $\zeta_{n} \upharpoonright k_{j}^{n}$ and $\eta \upharpoonright k_{j}^{n} \in T^{n} \cap T^{*}$ then $\left.\eta \in T^{n+1} \cap T^{*}\right)$."

Work in $V\left[G_{P_{n}}\right]$ with $r_{n} \in G_{P_{n}}$. Fix $\eta \in \tilde{T}_{n}$ and suppose
$F_{n, 2}\left(p_{k_{j}^{n}}(n)\right) \Vdash$ " $\eta$ is an initial segment of $\zeta_{n+1}$ with $\operatorname{lh}(\eta) \geq k_{j}^{n}$."
To establish the Subclaim, it suffices to show
(J) $F_{n, 2}\left(p_{k_{j}^{n}}(n)\right) \Vdash " \eta \in T^{n+1} \cap T^{*}$."

By the definition of $\left\langle p_{i}: i \in \omega\right\rangle$ we have

$$
p_{k_{j}^{n}}(n) \Vdash " \eta \upharpoonright k_{j}^{n}=\zeta_{n+1} \upharpoonright k_{j}^{n}=\zeta_{n} \upharpoonright k_{j}^{n} . "
$$

Hence by the fact that Claim 1 holds for the integer $n$ we have
(K) $p_{k_{j}^{n}}(n) \Vdash " \eta \upharpoonright k_{j}^{n} \in T^{n} \cap T^{*}$."

By Subclaim 1, (I), (K), and the fact that $F_{n, 2}\left(p_{k_{j}^{n}}(n)\right) \leq p_{k_{j}^{n}}(n)$ we obtain (J).

Subclaim 2 is established.
To complete the induction establishing Claim 1, we take $r_{n+1} \in P_{n+1}$ such that $r_{n+1} \upharpoonright n=r_{n}$ and $r_{n+1}$ is $N$-generic and $r_{n} \Vdash$ " $r_{n+1}(n) \leq$ $F_{n, 2}\left(p_{k_{j}^{n}}(n)\right) . "$

Claim 1 is established.
Define $q^{\prime}$ by

$$
q^{\prime}=\bigcup\left\{r_{n} \upharpoonright n: n \in \omega\right\} .
$$

By Claim 1 we have that
$q^{\prime} \Vdash$ "for every $n \in \omega$ we have $\zeta_{n} \in\left[T^{*}\right]$ and $\zeta_{n} \upharpoonright n=\zeta \upharpoonright n$, and therefore $\zeta \in\left[T^{*}\right]$."

The Theorem is established.

## 5 The Laver Property

In this section, we present Shelah's proof that the Laver property is preserved by countable support iteration of proper forcing.

Definition 5.1. Suppose $f \in{ }^{\omega}(\omega-\{0\})$ and $\mathbf{1} \ll f$. We say that $T$ is an $f$-tree iff $T$ is a tree and $(\forall \eta \in T)(\forall n \in \operatorname{dom}(\eta))(\eta(n)<f(n))$.

Definition 5.2. We say that $P$ is $f$-preserving iff whenever $z$ is in ${ }^{\omega}(\omega-$ $\{0\})$ and $1 \ll z$ then
$\mathbf{1} \Vdash_{P}$ " $\left(\forall g \in{ }^{\omega} \omega\right)(g \leq f$ implies there exists $H \in V$ such that $H$ is a $z$-sized $f$-tree and $g \in[H])$."

Definition 5.3. We say that $P$ has the Laver property iff for every $f \in$ ${ }^{\omega}(\omega-\{0\})$ such that $\mathbf{1} \ll f$ we have that $P$ is $f$-preserving.

The following Theorem is [13, Claim VI.2.10C(2)].
Theorem 5.4. $P$ has the Sacks property iff $P$ has the Laver property and $P$ is ${ }^{\omega} \omega$-bounding.

Proof: We first assume that $P$ has the Sacks property and we show that $P$ is ${ }^{\omega} \omega$-bounding. Given $p \in P$ and a name $f$ such that $p \Vdash$ " $f \in{ }^{\omega} \omega$," take $z \in{ }^{\omega}(\omega-\{0\})$ such that $\mathbf{1} \ll z$ and use the fact that $P$ has the Sacks property to obtain $q \leq p$ and a $z$-sized tree $H$ such that $q \Vdash$ " $f \in[H]$." For every $n \in \omega$ let

$$
g(n)=\max \{\eta(n): \eta \in H \text { and } \operatorname{lh}(\eta)>n\} .
$$

Then we have $q \Vdash$ " $f \leq g$." This establishes the fact that $P$ is ${ }^{\omega} \omega$ bounding.

It is clear that if $P$ has the Sacks property, then it has the Laver property.

Finally we assume that $P$ has the Laver property and is ${ }^{\omega} \omega$-bounding, and we show that $P$ has the Sacks property. So suppose that $p \in P$ and $\mathbf{1} \ll z$ and $p \Vdash$ " $g \in{ }^{\omega} \omega$." Using the fact that $P$ is ${ }^{\omega} \omega$-bounding, take $p^{\prime} \leq p$ and $f \in{ }^{\omega} \omega$ such that $p^{\prime} \Vdash " g \leq f$." Using the fact that $P$ has the Laver property, take $q \leq p^{\prime}$ and $H$ a $P$-name such that

$$
q \Vdash \text { " } H \text { is a } z \text {-sized } f \text {-tree and } g \in[H] \text { and } H \in V . "
$$

The Theorem is established.
The following is [13, Conclusion VI.2.10D].
Theorem 5.5. Suppose $\left\langle P_{\eta}: \eta \leq \kappa\right\rangle$ is a countable support iteration based on $\left\langle Q_{\eta}: \eta<\kappa\right\rangle$ and suppose $(\forall \eta<\kappa)\left(\mathbf{1} \Vdash_{P_{\eta}}\right.$ " $Q_{\eta}$ is proper and has the Laver property.") Then $P_{\kappa}$ has the Laver property.

Proof: Fix $f \in{ }^{\omega} \omega$ such that $\mathbf{1} \ll f$. Repeat the proofs of Lemma 4.6 through Theorem 4.10 with "tree" replaced by " $f$-tree." The Theorem is established.

## 6 ( $f, g$ )-bounding

In this section we establish the preservation of $(f, g)$-bounding forcing. For an exact formulation, see Theorem 6.5 below. This proof is due to Shelah, of course; see [13, Conclusion VI.2.11F].

The following Definition corresponds to [13, Definition VI.2.11A].
Definition 6.1. We say that $T$ is an $(f, g)$-corseted tree iff
(0) $T \subseteq{ }^{<\omega} \omega$ is a tree with no terminal nodes, and
(1) $f$ and $g$ are functions with domain $\omega$, and
(2) $(\forall n \in \omega)(f(n) \in\{r \in \mathbf{R}: 1<r\} \cup\{\omega\})$, and
(3) $(\forall n \in \omega)\left(g(n) \in\{r \in \mathbf{R}: 1<r\} \cup\left\{\aleph_{0}\right\}\right)$, and
(4) $f$ and $g$ diverge to infinity, and
(5) $(\forall \eta \in T)(\forall i \in \operatorname{dom}(\eta))(\eta(i)<f(i))$, and
(6) $(\forall n \in \omega)(\mid\{\eta(n): \eta \in T$ and $n \in \operatorname{dom}(\eta)\} \mid \leq g(n))$.

Definition 6.2. Suppose that $f$ and $g$ are functions as in Definition 6.1. We say that $P$ is $(f, g)$-bounding iff $\mathbf{1} \Vdash_{P} "\left(\forall h \in{ }^{\omega} \omega\right)[(\forall n \in \omega)(h(n)<$ $f(n))$ implies $(\exists T \in V)(T$ is an $(f, g)$-corseted tree and $h \in[T])]$."

The following Lemma is the analogue of Lemma 4.6.
Lemma 6.3. $P$ is $\left(f^{g^{k}}, g^{1 / k}\right)$-bounding for infinitely many $k \in \omega$ iff whenever $x<z$ are positive rational numbers and $\gamma \in \omega$ then $\mathbf{1} \Vdash$ " if $T$ is an $\left(f^{g^{\gamma}}, g^{x}\right)$-corseted tree then $(\exists H \in V)\left(H\right.$ is an $\left(f^{g^{\gamma}}, g^{z}\right)$-corseted tree and $T \subseteq H)$."

Proof. We prove the non-trivial direction. Fix an integer $k$ such that $k>x$ and $P$ is $\left(f^{g^{\gamma+k}}, g^{1 /(\gamma+k)}\right)$-bounding and $k>1 /(z-x)$. Let $X=$ $\left\{n \in \omega: g(n)=\aleph_{0}\right\}$.
For every $m \in \omega-X$ define

$$
\mathcal{T}_{m}=\left\{S \subseteq \omega: \sup (S) \leq f(m)^{g(m)^{\gamma}} \text { and }|S|<g(m)^{x}\right\}
$$

For every $m \in \omega-X$ define

$$
\mathcal{T}_{m}^{\prime}=\left\{i \in \omega: i \leq\left(f(m)^{g(m)^{\gamma}}+1\right)^{g(m)^{k}}\right\} .
$$

Because $x<k$ we may choose, for each integer $m$ not in $X$, a one-to-one mapping $h_{m}$ from $\mathcal{T}_{m}$ into $\mathcal{T}_{m}^{\prime}$.

Define

$$
\begin{gathered}
\mathcal{T}=\left\{\xi \in{ }^{<\omega} \omega:(\forall m \in \omega-X)\left(\xi(m) \in \mathcal{T}_{m}^{\prime}\right)\right. \\
\text { and }(\forall m \in X)(\xi(m)=1)\} .
\end{gathered}
$$

In $V\left[G_{P}\right]$ let $\zeta \in[\mathcal{T}]$ denote the function defined by

$$
\begin{gathered}
(\forall m \in \omega-X)\left(\zeta(m)=h_{m}(\{\eta(m): \eta \in T \text { and } m \in \operatorname{dom}(\eta)\})\right) \\
\text { and }(\forall m \in X)(\zeta(m)=1) .
\end{gathered}
$$

Because $P$ is $\left(f^{g^{\gamma+k}}, g^{1 /(\gamma+k)}\right)$-bounding, we may take $H^{\prime} \in V$ such that $H^{\prime}$ is an $\left(f^{g^{\gamma+k}}, g^{1 /(\gamma+k)}\right)$-corseted tree and $\zeta \in\left[H^{\prime}\right]$. Define $H$ by $H(m)=\bigcup\left\{h_{m}^{-1}(t):\left(\exists \eta \in H^{\prime}\right)(t=\eta(m))\right.$ and $\left.t \in \operatorname{range}\left(h_{m}\right)\right\}$ for $m \in \omega-X$, and $H(m)=\omega$ for $m \in X$.
When $g(m)$ is finite, we have

$$
\begin{aligned}
|H(m)| & \leq\left|H^{\prime}(m)\right| \cdot \max \left\{\left|h_{m}^{-1}(t): t \in \operatorname{range}\left(h_{m}\right)\right|\right\} \\
& \leq g^{x}(m) \cdot g^{1 /(\gamma+k)}(m)<g^{z}(m) .
\end{aligned}
$$

We have that $H$ is an $\left(f^{g^{\gamma}}, g^{z}\right)$-corseted tree and $\mathbf{1} \Vdash$ " $T \subseteq H$." The Lemma is established.

The following Lemma is the analogue of Lemma 4.7.
Lemma 6.4. Suppose $\left\langle r_{n}: n \in \omega\right\rangle$ is a bounded sequence of positive rational numbers and $y \in \mathbf{Q}$ and $\sup \left\{r_{n}: n \in \omega\right\}<y$. Suppose $P$ is a forcing such that $V\left[G_{P}\right] \models$ "for every countable $X \subseteq V$ there is a countable $Y \in V$ such that $X \subseteq Y$.". Suppose in $V\left[G_{P}\right]$ we have $(\forall n \in \omega)$ ( $T_{n} \in V$ is an $\left(f, g^{r_{n}}\right)$-corseted tree). Then in $V\left[G_{P}\right]$ there is an $\left(f, g^{y}\right)$ corseted tree $T^{*} \in V$ and an increasing sequence of integers $\left\langle k_{n}: n \in \omega\right\rangle$ such that $k_{0}=0$ and $(\forall i>0)\left(i<k_{i}\right)$ and for every $\eta \in{ }^{<\omega} \omega$ we have

$$
\begin{gathered}
(\forall t \in \operatorname{dom}(\eta))(\exists j \in \omega)\left(\exists \nu \in T_{k_{j}}\right)\left(k_{j} \leq t \text { and } \nu(t)=\eta(t)\right) \\
\text { implies } \left.\eta \in T^{*}\right) .
\end{gathered}
$$

Proof: The proof is similar to the proof of Lemma 4.7. We note the following modifications. We must choose $x \in \mathbf{Q}$ such that $\sup \left\{r_{n}: n \in\right.$
$\omega\}<x<y$. By recursion choose $\left\langle k_{n}: n \in \omega\right\rangle$ an increasing sequence of integers such that $k_{0}=0$ and $(\forall n>0)(\exists j \in \omega)\left(k_{n} \leq j\right.$ implies $(n+$ 1) $\left.g(j)^{x} \leq g(j)^{y}\right)$.

The definition of $T^{*}$ is changed to $T^{*}=\left\{\eta \in{ }^{<\omega} \omega:(\forall t \in \operatorname{dom}(\eta))(\exists j \in\right.$ $\omega)\left(\exists \nu \in S_{k_{j}}^{\prime}\right)\left(k_{j} \leq t\right.$ and $\left.\left.\nu(t)=\eta(t)\right)\right\}$.

Clearly $T^{*}$ is a tree.
Claim. $T^{*}$ is an $\left(f, g^{y}\right)$-corseted tree.
Proof: Fix $t \in \omega$.
Case 1: $t \geq k_{1}$.
Choose $m \in \omega$ such that $k_{m} \leq t<k_{m+1}$. We have that

$$
\begin{gathered}
\mid\left\{\eta(t): \eta \in T^{*} \text { and } t \in \operatorname{dom}(\eta)\right\}\left|=\Sigma_{j \leq m}\right|\left\{\eta(t): \eta \in T_{k_{j}} \text { and } t \in \operatorname{dom}(\eta)\right\} \mid \\
\leq(m+1) g^{x}(t) \leq g^{y}(t)
\end{gathered}
$$

Case 2: $t<k_{1}$.
We have $\left\{\eta(t): \eta \in T^{*}\right\}=\{\eta(t): \eta \in T\}$, so it follows that $|H(t)| \leq$ $g^{y}(t)$.

The Claim is established.
The other requirements of the Lemma are the same as in the proof of Lemma 4.7. The Lemma is established.

The following is [13, Conclusion VI.2.11F].
Theorem 6.5. Suppose $\left\langle P_{\eta}: \eta \leq \kappa\right\rangle$ is a countable support iteration based on $\left\langle Q_{\eta}: \eta<\kappa\right\rangle$ and suppose that for every $\eta<\kappa$ we have that $\mathbf{1} \Vdash$ "for infinitely many $k \in \omega$ we have that $Q_{\eta}$ is proper and $\left(f^{g^{k}}, g^{1 / k}\right)$ bounding." Then $P_{\kappa}$ is $\left(f^{g^{k}}, g^{1 / k}\right)$-bounding for every positive $k \in \omega$.

Proof: The same as Theorem 4.10, with ${ }^{\omega}(\omega-\{0\})$ replaced by $\mathbf{Q}$, and $\ll$ replaced by $<$, and $x$-sized tree replaced by $\left(f^{g^{\gamma}}, g^{x}\right)$-corseted tree.

## $7 \quad P$-point property

In this section we define the $P$-point property and prove that it is preserved by countable support iteration of proper forcings. This is due to Shelah [13, Conclusion VI.2.12G].
Definition 7.1. Suppose $n \in \omega$ and $x \in{ }^{\omega}(\omega-\{0\})$ is strictly increasing. We say that $(j, k, m)$ is an $x$-bound system above $n$ iff each of the following holds:
(0) $k \in \omega$, and
(1) $j$ and $m$ are functions from $k+1$ into $\omega$, and
(2) $j(0)>x(n+m(0)+1)$, and
(3) $(\forall l<k)(j(l+1)>x(j(l)+m(l+1)+1))$.

Definition 7.2. Suppose $n \in \omega$ and $x \in{ }^{\omega}(\omega-\{0\})$ is strictly increasing and $(j, k, m)$ is an $x$-bound system above $n$ and $T$ is a tree. We say that $T$ is a $(j, k, m, \eta)$-squeezed tree iff $T$ has no terminal nodes and each of the following holds:
(1) $\operatorname{dom}(\eta)=\left\{(l, t) \in{ }^{2} \omega: l \leq k\right.$ and $\left.t \leq m(l)\right\}$, and
(2) $(\forall(l, t) \in \operatorname{dom}(\eta))\left(\eta(l, t) \in{ }^{j(l)} \omega\right)$, and
(3) $(\forall \nu \in T)(\exists(l, t) \in \operatorname{dom}(\eta))(\nu$ is comparable with $\eta(l, t))$.

It is easy to see that the following Definition is equivalent to [13, Definition VI.2.12A].

Definition 7.3. We say that $T$ is $x$-squeezed iff for every $n \in \omega$ there is some $x$-bound system $(j, m, k)$ above $n$ such that $T$ is $(j, k, m, \eta)$-squeezed for some $\eta$.

In other words, $T$ is $x$-squeezed when, living above any given level of $T$, say $\{\xi \in T: \operatorname{lh}(\xi)=n+1\}$, there is a maximal antichain $\mathcal{A}$ of $T$ that can be decomposed as $\mathcal{A}=\bigcup\left\{\mathcal{A}_{l}: l \leq k\right\}$ where each $\mathcal{A}_{l}$ is a subset of $\{\xi \in T: \operatorname{lh}(\xi)=j(l)\}$ of cardinality at most $m(l)+1$, such that the levels of $\mathcal{A}$ are stratified so sparsely that conditions (2) and (3) of Definition 7.1 hold. Notice that for any given $l \leq k$ we may have that $\{\eta(l, t): t \leq m(l)\}$ is a proper superset of $\mathcal{A}_{l}$; indeed, it need not even be a subset of $T$. We could modify Definition 7.2 to require this, but there is no need to do so.

Lemma 7.4. Suppose $1 \ll x \ll y$ and both $x$ and $y$ are strictly increasing and $T$ is a $y$-squeezed tree. Then $T$ is an $x$-squeezed tree.

Proof: Every $y$-bound system is an $x$-bound system.
Definition 7.5. We say that $P$ has the $P$-point property iff for every $x \in{ }^{\omega}(\omega-\{0\})$ strictly increasing, we have

$$
\mathbf{1} \Vdash \text { " }\left(\forall f \in{ }^{\omega} \omega\right)(\exists H \in V)(f \in[H] \text { and } H \text { is an } x \text {-squeezed tree)." }
$$

Lemma 7.6. $P$ has the $P$-point property iff for every $x \in{ }^{\omega}(\omega-\{0\})$ strictly increasing and every $p \in P$, if $p \Vdash_{P} " f \in{ }^{\omega} \omega$ " there are $q \leq p$ and an $x$-squeezed tree $H$ such that $q \Vdash$ " $f \in[H]$."

Proof: Assume that $P$ has the $P$-point property. Given $x, p$, and $f$, there is $q \leq p$ and $H \subseteq{ }^{<\omega} \omega$ such that $q \Vdash$ " $f \in[H]$ and $H$ is an $x$ squeezed tree." By the Shoenfield Absoluteness Theorem we have that $H$ is an $x$-squeezed tree.

The other direction is immediate, and so the Lemma is established.
Lemma 7.7. Suppose $T$ is an $x$-squeezed tree and $n \in \omega$. Then $T \cap{ }^{n} \omega$ is finite.

Proof. Fix $(j, k, m)$ an $x$-bound system above $n$ and fix $\eta$ such that $T$ is a $(j, k, m, \eta)$-squeezed tree. We have $T \cap^{n} \omega \subseteq\{\eta(s, t) \upharpoonright n: t \leq j(k)$ and $s \leq m(t)\}$.

The following Lemma is [13, Claim VI.2.12B(1)].
Lemma 7.8. Suppose that $P$ has the $P$-point property. Then $P$ is ${ }^{\omega} \omega$ bounding.

Proof: Suppose $p \in P$ and $p \Vdash$ " $f \in{ }^{\omega} \omega$." Pick $x \in{ }^{\omega}(\omega-\{0\})$ such that $1 \ll x$, and take $q \leq p$ and $H$ an $x$-squeezed tree such that $q \Vdash$ " $f \in[H]$." By Lemma 7.7 we may define $h \in{ }^{\omega} \omega$ by $(\forall n \in \omega)(h(n)=\max \{\nu(n)$ : $\nu \in H$ and $n \in \operatorname{dom}(\nu)\})$. Clearly $q \Vdash " f \leq h$," and the Lemma is established.
The following Lemma is [13, Claim VI.2.12B(2)].
Lemma 7.9. Suppose $P$ has the Sacks property. Then $P$ has the $P$-point property.

Proof. Suppose $x \in{ }^{\omega}(\omega-\{0\})$ is strictly increasing and $p \in P$ and $p \Vdash$ " $f \in{ }^{\omega} \omega$." Choose $y \in{ }^{\omega}(\omega-\{0\})$ monotonically non-decreasing such that for $n>x(3)$ we have that $y(n)$ is the greatest $t \in \omega$ such that $x(3 t)<n$. Using the Sacks property, choose $q \leq p$ and $H$ a $y$-sized tree such that $q \Vdash$ " $f \in[H]$."

Notice that for all $t>0$ we have $y(x(3 t))$ is less than or equal to the greatest integer $k$ satisfying $x(3 k)<x(3 t)$, and therefore we have
$(*)(\forall t \in \omega)(y(x(3 t))<t)$.

Suppose $n>x(3)$. Let $j$ be such that $\operatorname{dom}(j)=\{0\}$ and $j(0)=$ $x(2 n)+1$ and let $k=0$; and let $m$ be such that $\operatorname{dom}(m)=\{0\}$ and $m(0)=\left|H \cap^{j(0)} \omega\right|$.

Claim: $(j, k, m)$ is an $x$-bound system above $n$.
Proof: We have $x(n+m(0)+1) \leq x(n+1+y(x(2 n)+1)) \leq x(n+$ $1+y(x(3 n))) \leq x(n+1+n-1)<x(2 n)+1=j(0)$. The first inequality is because $m(0)=\left|H \cap{ }^{j(0)} \omega\right| \leq y(j(0))=y(x(2 n)+1)$. The second inequality is because $x$ is strictly increasing and $y$ is monotonically nondecreasing. The third inequality is by $(*)$.

The Claim is established.
Define $\eta$ with domain equal to $\{(0 . i): i<m(0)\}$ and such that $\langle\eta(0, i)$ : $i<m(0)\rangle$ enumerates $H \cap^{j(0)} \omega$. Clearly $H$ is a $(j, k, m, \eta)$-squeezed tree, so the Lemma is established.

Lemma 7.10. Suppose $y \in{ }^{\omega}(\omega-\{0\})$ is strictly increasing and $T$ and $T^{\prime}$ are $y$-squeezed trees. Then $T \cup T^{\prime}$ is a $y$-squeezed tree.

Proof: Given $n \in \omega$, choose $(j, k, m, \eta)$ such that $(j, k, m)$ is a $y$-bound systems above $n$ and $T$ is $(j, k, m, \eta)$-squeezed. Let $h=j(k)$. Choose $\left(j^{\prime}, m^{\prime}, k^{\prime}\right)$ a $y$-bound system above $h$ and choose $\eta^{\prime}$ such that $T^{\prime}$ is $\left(j^{\prime}\right.$, $\left.k^{\prime}, m^{\prime}, \eta^{\prime}\right)$-squeezed. We proceed to fuse $(j, k, m, \eta)$ with $\left(j^{\prime}, k^{\prime} m^{\prime}, \eta^{\prime}\right)$. For every $l \leq k$ let $j^{*}(l)=j(l)$ and for every $l$ such that $k<l \leq k+k^{\prime}+1$ let $j^{*}(l)=j^{\prime}\left(l-k_{n}-1\right)$. Let $k^{*}=k+k^{\prime}+1$. For every $l \leq k$ let $m^{*}(l)=m(l)$ and for every $l$ such that $k<l \leq k+k^{\prime}+1$ let $m^{*}(l)=m^{\prime}(l-k-1)$. For every $l \leq k$ and $\beta \leq m(l)$ let $\eta^{*}(l, \beta)=\eta(l, \beta)$ and for every $l$ such that $k<l \leq k+k^{\prime}+1$ and every $\beta \leq m^{\prime}(l-k-1)$ let $\eta^{*}(l, \beta)=\eta^{\prime}(l-k-1, \beta)$. It is straightforward to verify that $\left(j^{*}, k^{*}, m^{*}\right)$ is a $y$-bound system above $n$ and that $T \cup T^{\prime}$ is $\left(j^{*}, k^{*}, m^{*}, \eta^{*}\right)$-squeezed.

The Lemma is established.
Definition 7.11. Suppose $n \in \omega$ and $h \in{ }^{\omega} \omega$ and $y \in{ }^{\omega}(\omega-\{0\})$ is strictly increasing. Suppose $(j, m, k)$ is a $y$-bound system above $n$. We say that $(j, m, k)$ is $h$-tight iff $j(k)<h(n)$. For $T$ a $y$-squeezed tree, we say that $T$ is $h$-tight iff for every $n \in \omega$ there is an $h$-tight $y$-bound system $(j, m, k)$ above $n$ such that for some $\eta$ we have that $T$ is $(j, m, k, \eta)$-squeezed for some $\eta$.

The following Lemma should be compared with Lemma 4.6. Notice the
fact that what we prove here is stronger in that the same $y \in{ }^{\omega}(\omega-\{0\})$ is used in both the hypothesis and the conclusion. This strengthening is possible by the use of Lemma 7.10. The proof of Lemma 7.12 is [13, proof of Claim VI.2.12C( $\varepsilon)^{+}$], except that we have $x=y$. Thus we are proving a stronger statement than [13, Claim VI.2.12C $\left.(\varepsilon)^{+}\right]$, but in fact Shelah likewise proves this stronger statement without saying so.

Lemma 7.12. $P$ has the $P$-point property iff whenever $y \in{ }^{\omega}(\omega-\{0\})$ is strictly increasing then $\mathbf{1} \mid$ "whenever $T$ is a $y$-squeezed tree then $(\exists H \in V)$ ( $H$ is a $y$-squeezed tree and $T \subseteq H$ )."

Proof. We prove the non-trivial direction. Suppose $p \in P$ and $p \Vdash$ " $T$ is a $y$-squeezed tree." Fix $q^{\prime} \leq p$. By Lemma 7.8 we may choose $h \in{ }^{\omega} \omega$ and $q \leq q^{\prime}$ such that $q \Vdash$ " $T$ is $h$-tight."

Define $z \in{ }^{\omega}(\omega-\{0\})$ by $z(0)=0$ and

$$
(\forall n \in \omega)(z(n+1)=h(z(n)) .
$$

For every $n \in \omega$ let $\mathcal{T}_{n}=\left\{t \subseteq{ }^{<h(n)} \omega: t=T \cap{ }^{<h(n)} \omega\right.$ for some $h$-tight $y$-squeezed tree $T\}$.

Let $\mathcal{T}=\bigcup\left\{\mathcal{T}_{n}: n \in \omega\right\}$. We implicitly fix an isomorphism from ${ }^{<\omega} \omega$ onto $\mathcal{T}$.

Using the fact that $P$ satisfies the $P$-point property, fix $q^{*} \leq q$ and $\mathcal{C} \subseteq \mathcal{T}$ such that $\mathcal{C}$ is a $z$-squeezed tree and $q^{*} \Vdash{ }^{\prime \prime}(\forall n \in \omega)\left(T \cap^{<h(n)} \omega \in \mathcal{C}\right)$."

Define $H^{*}=\bigcup \mathcal{C}$ and let $H=\left\{\nu \in H^{*}:(\forall n \in \omega)\left(\exists \eta \in{ }^{n} \omega \cap H^{*}\right)(\eta\right.$ is comparable with $\nu)\}$.

Pick a $z$-bound system $\left(j^{*}, k^{*}, m^{*}\right)$ above $n$ and $\eta^{*}$ such that $\mathcal{C}$ is a $\left(j^{*}, k^{*}, m^{*}, \eta^{*}\right)$-squeezed tree.

Fix $n \in \omega$. We show that there is a $y$-bound system $(j, m, k)$ above $n$ such that for some $\eta$ we have that $H$ is $(j, m, k, \eta)$-squeezed.

Claim 1. For every $\beta \leq m^{*}(0)$ we have $\mathrm{ht}\left(\eta^{*}(0, \beta)\right) \geq h(z(n+\beta+1))$. For every non-zero $\alpha \leq k^{*}$ and every $\beta \leq m^{*}(\alpha)$ we have $\operatorname{ht}\left(\eta^{*}(\alpha, \beta)\right) \geq$ $h\left(z\left(j^{*}(\alpha-1)+\beta+1\right)\right)$.

Proof: For every $\beta \leq m^{*}(0)$ we have $\operatorname{ht}\left(\eta^{*}(0, \beta)\right)=h\left(\operatorname{rk}_{\mathcal{T}}\left(\eta^{*}(0, \beta)\right)\right)=$ $h\left(j^{*}(0)\right) \geq h\left(z\left(n+m^{*}(0)+1\right)\right) \geq h(z(n+\beta+1))$. For every non-zero $\alpha \leq k^{*}$ and every $\beta \leq m^{*}(\alpha)$ we have $\operatorname{ht}\left(\eta^{*}(\alpha, \beta)\right)=h\left(\operatorname{rk}_{\mathcal{T}}\left(\eta^{*}(\alpha, \beta)\right)\right)=$ $h\left(j^{*}(\alpha)\right) \geq h\left(z\left(j^{*}(\alpha-1)+m^{*}(\alpha)+1\right)\right) \geq h\left(z\left(j^{*}(\alpha-1)+\beta+1\right)\right)$.

By Claim 1 we may construct $y$-bound systems as follows. For every $\beta \leq$ $m^{*}(0)$, fix an $h$-tight $y$-bound system $\left(j^{0, \beta}, m^{0, \beta}, k^{0, \beta}\right)$ above $z(n+\beta+1)$ along with $\eta^{0, \beta}$ such that for some $\left(j^{0, \beta}, m^{0, \beta}, k^{0, \beta}, \eta^{0, \beta}\right)$-squeezed tree $T$ we have $\eta^{*}(0, \beta)=\left\langle h(z(n+\beta+1)) \omega \cap T\right.$, and for every non-zero $\alpha \leq k^{*}$ and $\beta \leq m^{*}(\alpha)$, fix an $h$-tight $y$-bound system $\left(j^{\alpha, \beta}, m^{\alpha, \beta}, k^{\alpha, \beta}\right)$ above $z\left(j^{*}(\alpha-\right.$ $1)+\beta+1$ ) along with $\eta^{\alpha, \beta}$ such that for some ( $j^{\alpha, \beta}, m^{\alpha, \beta}, k^{\alpha, \beta}, \eta^{\alpha, \beta}$ )squeezed tree $T$ we have $\eta^{*}(\alpha, \beta)=\left\langle h\left(z\left(j^{*}(\alpha-1)+\beta+1\right)\right) \omega \cap T\right.$.

We define

$$
\hat{\jmath}(\alpha, \beta, \gamma)=j^{(\alpha, \beta)}(\gamma)
$$

and

$$
\hat{k}(\alpha, \beta)=k^{(\alpha, \beta)}
$$

and

$$
\hat{m}(\alpha, \beta, \gamma)=m^{(\alpha, \beta)}(\gamma)
$$

and for $t \leq \hat{m}(\alpha, \beta, \gamma)$ let

$$
\hat{\eta}(\alpha, \beta, \gamma, t)=\eta^{(\alpha, \beta)}(\gamma, t) .
$$

Claim 2. Suppose $n \in \omega$. Then we have the following:
(1) $\hat{\jmath}(0,0,0)>y(n+\hat{m}(0,0,0)+1)$, and
(2) For every $\alpha \leq k^{*}$ and $\beta \leq m^{*}(\alpha)$ and $\gamma<\hat{k}(\alpha, \beta)$ we have $\hat{\jmath}(\alpha, \beta, \gamma+$ 1) $>y(\hat{\jmath}(\alpha, \beta, \gamma)+\hat{m}(\alpha, \beta, \gamma+1)+1)$, and
(3) For every $\alpha \leq k^{*}$ and every $\beta<m^{*}(\alpha)$ we have $\hat{\jmath}(\alpha, \beta+1,0)>$ $y(\hat{\jmath}(\alpha, \beta, \hat{k}(\alpha, \beta))+\hat{m}(\alpha, \beta+1,0)+1)$, and
(4) For every $\alpha<k^{*}$ we have $\hat{\jmath}(\alpha+1,0,0)>y\left(\hat{\jmath}\left(\alpha, m^{*}(\alpha), \hat{k}\left(\alpha, m^{*}(\alpha)\right)\right)+\right.$ $\hat{m}(\alpha+1,0,0)+1)$.

Proof: Clause (1) holds because $j^{(0,0)}(0)>y\left(n+m^{(0,0)}(0)+1\right)$.
Clause (2) holds because $j^{(, \alpha, \beta)}(\gamma+1)>y\left(j^{(\alpha, \beta)}(\gamma)+m^{(\alpha, \beta)}(\gamma+1)+1\right)$.
We verify clause (3) as follows.
Case A: $\alpha=0$.
Notice that $j^{0, \beta)}\left(k^{(0, \beta)}\right)<h(z(n+\beta+1))$ becuase the system $\left(j^{(0, \beta)}\right.$, $\left.m^{(0, \beta)}, k^{(0, \beta)}\right)$ is $h$-tight above $z(n+\beta+1)$. Notice also that $j^{(0, \beta+1)}(0)>$ $y\left(z(n+\beta+2)+m^{(0, \beta+1)}(0)+1\right)$ because the system $\left(j^{(0, \beta+1)}, m^{(0, \beta+1)}\right.$, $\left.k^{(0, \beta+1)}\right)$ is above $z(n+\beta+2)$. Hence we have $\hat{\jmath}(0, \beta+1,0)=j^{(0, \beta+1)}(0)>$
$y\left(z(n+\beta+2)+m^{(0, \beta+1)}(0)+1\right) \geq y\left(h(z(n+\beta+1))+m^{(0, \beta+1)}(0)+1\right) \geq$ $y\left(j^{(0, \beta)}\left(k^{(0, \beta)}\right)+m^{(0, \beta+1)}(0)+1\right)=y(\hat{\jmath}(0, \beta, \hat{k}(0, \beta))+\hat{m}(0, \beta+1,0)+1)$.

Case B: $\alpha>0$.
Notice that $j^{\alpha, \beta)}\left(k^{(\alpha, \beta)}\right)<h\left(z\left(j^{*}(\alpha-1)+\beta+1\right)\right)$ becuase the system $\left(j^{(\alpha, \beta)}, m^{(\alpha, \beta)}, k^{(\alpha, \beta)}\right)$ is $h$-tight above $z\left(j^{*}(\alpha-1)+\beta+1\right)$. Notice also that $j^{(\alpha, \beta+1)}(0)>y\left(z\left(j^{*}(\alpha-1)+\beta+2\right)+m^{(\alpha, \beta+1)}(0)+1\right)$ because the system $\left(j^{(\alpha, \beta+1)}, m^{(\alpha, \beta+1)}, k^{(\alpha, \beta+1)}\right)$ is above $z\left(j^{*}(\alpha-1)+\beta+2\right)$. Hence we have $\hat{\jmath}(\alpha, \beta+1,0)=j^{(\alpha, \beta+1)}(0)>y\left(z\left(j^{*}(\alpha-1)+\beta+2\right)+m^{(\alpha, \beta+1)}(0)+1\right) \geq$ $y\left(h\left(z\left(j^{*}(\alpha-1)+\beta+1\right)\right)+m^{(\alpha, \beta+1)}(0)+1\right) \geq y\left(j^{(\alpha, \beta)}\left(k^{(\alpha, \beta)}\right)+m^{(\alpha, \beta+1)}(0)+\right.$ 1) $=y(\hat{\jmath}(\alpha, \beta, \hat{k}(\alpha, \beta))+\hat{m}(\alpha, \beta+1,0)+1)$.

To see that clause (4) holds, we have $\hat{\jmath}(\alpha+1,0,0)=j^{(\alpha+1,0)}(0) \geq$ $y\left(z\left(j^{*}(\alpha)+1\right)+m^{(\alpha+1,0)}(0)+1\right) \geq y\left(h\left(z\left(j^{*}(\alpha)\right)\right)+m^{(\alpha+1,0)}(0)+1\right) \geq$ $y\left(h\left(j^{*}(\alpha)\right)+m^{(\alpha+1,0)}(0)+1\right) \geq y\left(h\left(z\left(j^{*}(\alpha-1)\right)+m^{*}(\alpha)+1\right)\right)+m^{(\alpha+1,0)}(0)+$ 1) $\geq y\left(j^{\left(\alpha, m^{*}(\alpha)\right)}\left(k^{\left(\alpha, m^{*}(\alpha)\right)}\right)+m^{(\alpha+1,0)}(0)+1\right)$.

The first inequality is because the system $\left(j^{(\alpha+1,0)}, m^{(\alpha+1,0)}, k^{(\alpha+1,0)}\right)$ is above $z\left(j^{*}(\alpha)+1\right)$ whence by clause (2) of Definition 7.1 we have the first inequality. The second inequality is by definition of the function $z$. The third inequality is by the fact that $z$ is an increasing function. The fourth inequality is because $\left(j^{*}, m^{*}, k^{*}\right)$ satisfies clause (2) of Definition 7.1. The fifth inequallity is because the system $\left(j^{\left(\alpha, m^{*}(\alpha)\right)}, m^{\left(\alpha, m^{*}(\alpha)\right)}, k^{\left(\alpha, m^{*}(\alpha)\right)}\right)$ is $h$-tight above $z\left(j^{*}(\alpha-1)+m^{*}(\alpha)+1\right)$.

The Claim is established.
Claim 3. Suppose $n \in \omega$ and $\nu \in H$. Then there are $\alpha \leq k^{*}$ and $\beta \leq m^{*}(\alpha)$ and $\gamma \leq \hat{k}(\alpha, \beta)$ and $\delta \leq \hat{m}(\alpha, \beta, \gamma)$ such that $\nu$ is comparable with $\hat{\eta}(\alpha, \beta, \gamma, \delta)$.

Proof. Pick $t \in \mathcal{C}$ such that $\nu \in t$. Take $\alpha$ and $\beta$ such that $t$ is comparable with $\eta^{*}(\alpha, \beta)$.

Case 1: $\nu \in \eta^{*}(\alpha, \beta)$.
Take $\gamma \leq k^{(\alpha, \beta)}$ and $\delta \leq m^{(\alpha, \beta)}(\gamma)$ such that $\nu$ is comparable with $\eta^{(\alpha, \beta)}(\gamma, \delta)$.

Case 2: $\nu \notin \eta^{*}(\alpha, \beta)$.
If $\alpha=0$ then let $\zeta=z(n+\beta+1)$ and if $\alpha>0$ then let $\zeta=z\left(j^{*}(\alpha-1)+\right.$ $\beta+1)$. Let $\nu^{\prime}=\nu \upharpoonright h(\zeta)$. Choose $\gamma \leq k^{(\alpha, \beta)}$ and $\delta \leq m^{(\alpha, \beta)}(\gamma)$ such that $\nu^{\prime}$ is comparable with $\eta^{(\alpha, \beta)}(\gamma, \delta)$. Because the system $\left(j^{\alpha, \beta}, m^{\alpha, \beta)}, k^{(\alpha, \beta)}\right)$ is $h$-tight above $\zeta$ we have $\eta^{(\alpha, \beta)}(\gamma, \delta) \leq \nu^{\prime}$. Therefore $\eta^{(\alpha, \beta)}(\gamma, \delta) \leq \nu$.

The Claim is established.
For each $n \in \omega$ define $\zeta(\alpha, \beta, \gamma)$ by the following recursive formulas:

$$
\zeta(0,0,0)=0 .
$$

For $\alpha \leq k^{*}$ and $\beta \leq m^{*}(\alpha)$ and $\gamma<\hat{k}(\alpha, \beta)$ we have

$$
\zeta(\alpha, \beta, \gamma+1)=\zeta(\alpha, \beta, \gamma)+1
$$

For $\alpha \leq k^{*}$ and $\beta<m^{*}(\alpha)$ we have

$$
\zeta(\alpha, \beta+1,0)=\zeta(\alpha, \beta, \hat{k}(\alpha, \beta))+1
$$

For $\alpha<k^{*}$ we have

$$
\zeta(\alpha+1,0,0)=\zeta\left(\alpha, m^{*}(\alpha), \hat{k}\left(\alpha, m^{*}(\alpha)\right)\right)+1
$$

Define $\tilde{\jmath}(\zeta(\alpha, \beta, \gamma))=\hat{\jmath}(\alpha, \beta, \gamma)$, and $\tilde{m}(\zeta(\alpha, \beta, \gamma))=\hat{m}(\alpha, \beta, \gamma)$, and $\tilde{k}(\zeta(\alpha, \beta))=\hat{k}(\alpha, \beta)$, and $\tilde{\eta}(\zeta(\alpha, \beta, \gamma, \delta))=\hat{\eta}(\alpha, \beta, \gamma, \delta)$.

Claim 4. $(\tilde{\jmath}, \tilde{k}, \tilde{m})$ is a $y$-bound system above $n$ and $H$ is a $(\tilde{\jmath}, \tilde{k}, \tilde{m}, \tilde{\eta})$ squeezed tree.

Proof. By Claims 2 and 3.
The Lemma is established.
Lemma 7.13. Suppose $x \in{ }^{\omega}(\omega-\{0\})$ is strictly increasing and suppose that for each $n \in \omega$ we have that $T_{n}$ is an $x$-squeezed tree. Then there are $T^{*}$ and $\left\langle\gamma_{t}: t \in \omega\right\rangle$ an increasing sequence of integers such that $T^{*}$ is an $x$-squeezed tree and $\gamma_{0}=0$ and $(\forall t>0)\left(t<\gamma_{t}\right)$ and for every $f \in{ }^{<\omega} \omega$ we have

$$
(\forall t>0)(\exists s<t)\left(f \upharpoonright \gamma_{t} \in T_{\gamma_{s}}\right) \text { iff } f \in T^{*} .
$$

Proof: For each $n \in \omega$ choose $h_{n} \in{ }^{\omega} \omega$ such that $T_{n}$ is $h_{n}$-tight.
We build as follows. Let $\gamma_{0}=0$. Given $\gamma_{t}$, define $g_{t}(0)=\gamma_{t}$. For $0 \leq s \leq t$ let $g_{t}(s+1)=h_{\gamma_{s}}\left(g_{t}(s)\right)$. Let $\gamma_{t+1}=g_{t}(t+1)$.

Let $T^{*}=\left\{\eta \in{ }^{<\omega} \omega:(\forall t>0)(\exists s<t)\left(\eta \upharpoonright \gamma_{t} \in T_{\gamma_{s}}\right)\right\}$.
Now fix $n \in \omega$. We build an $x$-bound system ( $j, m, k$ ) above $n$ and we build $\eta$ so that $(j, m, k)$ and $\eta$ witness the fact that $T^{*}$ is $x$-squeezed.

For every $t \in \omega$ and $s \leq t$ choose an $h_{\gamma_{s}}$-tight $x$-bound system $\left(j_{t}^{s}, m_{t}^{s}, k_{t}^{s}\right)$ above $g_{t}(s)$ along with $\eta_{t}^{s}$ such that $T_{\gamma_{s}}$ is $\left(j_{t}^{s}, m_{t}^{s}, k_{t}^{s}, \eta_{t}^{s}\right)$-squeezed.

We define $\zeta$ such that for $\alpha \geq n$ and $\beta \leq \alpha$ and $\gamma \leq k_{\beta}^{\alpha}$ we have

- $\zeta(n, 0,0)=0$, and
- if $\gamma<k_{\beta}^{\alpha}$ then $\zeta(\alpha, \beta, \gamma+1)=\zeta(\alpha, \beta, \gamma)+1$, and
- if $\beta<\alpha$ then $\zeta(\alpha, \beta+1,0)=\zeta\left(\alpha, \beta, k_{\beta}^{\alpha}\right)+1$, and
- if $\alpha \geq n$ then $\zeta(\alpha+1,0,0)=\zeta\left(\alpha, \alpha, k_{\alpha}^{\alpha}\right)+1$.

We define $(j, m, k)$ such that for every $\alpha \geq n$ and $\beta \leq \alpha$ and $\gamma \leq k_{\beta}^{\alpha}$ we have

- $j(\zeta(\alpha, \beta, \gamma))=j_{\beta}^{\alpha}(\gamma)$, and
- $m(\zeta(\alpha, \beta, \gamma))=m_{\beta}^{\alpha}(\gamma)$, and
- $k=k_{\beta}^{\alpha}$.

Claim 1. $(j, m, k)$ is an $x$-bound system above $n$.
Proof: Clause (1) of Definition 7.1 is immediate.
Clause (2) of Definition 7.1 holds because $j(0)=j_{n}^{0}(0)>x\left(g_{n}^{0}(0)+\right.$ $\left.m_{n}^{0}(0)+1\right) \geq x(n+m(0)+1)$. The first inequality holds because the system $\left(j_{n}^{0}, m_{n}^{0}, k_{n}^{0}\right)$ is above $g_{n}^{0}(0)$ and it satisfies clause (2) of Definition 7.1.

We have $j(\zeta(\alpha, \beta, \gamma+1))=j_{\alpha}^{\beta}(\gamma+1)>x\left(j_{\alpha}^{\beta}(\gamma)+m_{\alpha}^{\beta}(\gamma+1)+1\right)=$ $x(j(\zeta(\alpha, \beta, \gamma))+m(\zeta(\alpha, \beta, \gamma+1))+1)$.

We have $j(\zeta(\alpha, \beta+1,0))=j_{\alpha}^{\beta+1}(0)>x\left(g_{\alpha}(\beta+1)+m_{\alpha}^{\beta+1}(0)+1\right) \geq$ $x\left(h_{\gamma_{\beta}}\left(g_{\alpha}(\beta)\right)+m_{\alpha}^{\beta+1}(0)+1\right) \geq x\left(j_{\alpha}^{\beta}\left(k_{\alpha}^{\beta}\right)+m_{\alpha}^{\beta+1}(0)+1\right)=x\left(j\left(\zeta\left(\alpha, \beta, k_{\alpha}^{\beta}\right)\right)+\right.$ $m(\zeta(\alpha, \beta+1,0))+1)$.

The first inequality is clause (2) of Definition 7.1 applied to the system $\left(j_{\alpha}^{\beta+1}, m_{\alpha}^{\beta+1}, k_{\alpha}^{\beta+1}\right)$. The second inequality is by the definition of $g_{\alpha}$. The third inequality is because the system $\left(j_{\alpha}^{\beta}, m_{\alpha}^{\beta}, k_{\alpha}^{\beta}\right)$ is $h_{\gamma_{\beta}}$-tight above $g_{\alpha}(\beta)$.

The Claim is established.
We define $\eta$ such that for every $\alpha \geq n$ and $\beta \leq \alpha$ and $\gamma \leq k_{\beta}^{\alpha}$ and $\delta \leq m_{\alpha}^{\beta}(\gamma)$ we have $\eta(\zeta(\alpha, \beta, \gamma), \delta)=\eta_{\alpha}^{\beta}(\gamma, \delta)$.

Claim 3: $T^{*}$ is a $(j, k, m, \eta)$-squeezed tree.
Proof: It is straightforward to verify that $T^{*}$ is a tree and that clause (1) and clause (2) of Definition 7.2 hold.

To verify clause (3), suppose we have $\nu \in T^{*}$. We show that $\nu$ is comparable to some $\eta(l, i)$ with $(l, i) \in \operatorname{dom}(\eta)$. Choose $\nu^{\prime} \in T^{*}$ such that $\nu \leq \nu^{\prime}$ and $\operatorname{lh}\left(\nu^{\prime}\right) \geq \gamma_{n+1}$. It suffices to show that $\nu^{\prime}$ is comparable with some $\eta(l, i)$ with $(l, i) \in \operatorname{dom}(\eta)$. Because $\nu^{\prime} \in T^{*}$ we may choose $s \leq n$ such that $\nu^{\prime} \upharpoonright \gamma_{n+1} \in T_{\gamma_{s}}$. We may select $(l, i) \in \operatorname{dom}\left(\eta_{n}^{s}\right)$ such that $\nu^{\prime} \upharpoonright \gamma_{n+1}$ is comparable with $\eta_{n}^{s}(l, i)$. We have $\eta_{s}^{n}(l, i)=\eta(\zeta(n, s, l), i)$, so $\operatorname{lh}\left(\eta_{n}^{s}(l, i)\right)=j(\zeta(n, s, l))=j_{n}^{s}(l) \leq h_{\gamma_{s}}\left(g_{n}(s)\right)=g_{n}(s+1) \leq \gamma_{n+1}$. Therefore $\eta(\zeta(n, s, l), i) \leq \nu^{\prime} \upharpoonright \gamma_{n+1}$ and therefore $\eta(\zeta(n, s, l), i)$ is comparable with $\nu^{\prime}$.

The Claim and the Lemma are established.
The following Lemma is the analogue of Lemma 4.7. The fact that the Lemma is stronger reflects the fact that Lemma 7.10 holds.

Lemma 7.14. Suppose $x \in{ }^{\omega}(\omega-\{0\})$ is strictly increasing, and suppose $P$ is a forcing notion such that $V\left[G_{P}\right] \models$ "for all countable $X \subseteq V$ there is a countable $Y \in V$ such that $X \subseteq Y$ and $\left\langle T_{n}: n \in \omega\right\rangle$ is a sequence of $x$-squeezed trees and $(\forall n \in \omega)\left(T_{n} \in V\right)$." Then $V\left[G_{P}\right] \models$ "there is a strictly increasing sequence of integers $\left\langle m_{i}: i \in \omega\right\rangle$ and an $x$-squeezed tree $T^{*} \in V$ such that $m_{0}=0$ and $(\forall i>0)\left(m_{i}>i\right)$ and for every $\eta \in^{<\omega} \omega$, if $(\forall i>0)(\exists j<i)\left(\eta \upharpoonright m_{i+1} \in T_{m_{j}}\right)$ then $\eta \in T^{*}$."
Proof: Work in $V\left[G_{P}\right]$. Let $b \in V$ be a countable set such that $\left\{T_{n}\right.$ : $n \in \omega\} \subseteq b \in V$ and $(\forall x \in b)\left(x\right.$ is an $x$-squeezed tree). Let $\left\langle S_{n}: n \in \omega\right\rangle \in$ $V$ enumerate $b$ with infinitely many repetitions such that $S_{0}=T_{0}$. Build $\left\langle S_{n}^{\prime}: n \in \omega\right\rangle$ by setting $S_{0}^{\prime}=S_{0}$ and for every $n>0$ set $S_{n}^{\prime}=S_{n} \cup S_{n-1}^{\prime}$. Build $h$ mapping $\omega$ into $\omega$ by setting $h(0)=0$ and for every $n>0$ set $h(n)$ equal to the least integer $m>n$ and $T_{n}=S_{m}$.

Using Lemma 7.13, take $T^{*} \in V$ an $x$-squeezed tree and $\left\langle k_{i}: i \in \omega\right\rangle$ such that for every $\eta \in{ }^{<\omega} \omega$ we have $\eta \in T^{*}$ iff $(\forall n>0)(\exists i<n)\left(\eta \upharpoonright k_{n} \in S_{k_{i}}^{\prime}\right)$.

Build $\left\langle n_{i}^{\prime}: i \in \omega\right\rangle$ an increasing sequence of integers such that $n_{0}^{\prime}=0$ and $n_{1}^{\prime}>k_{1}$ and for every $i \in \omega$ we have $h\left(n_{i}^{\prime}\right)<n_{i+1}^{\prime}$ and
$\left(^{*}\right)(\exists t \in \omega)\left(n_{i}^{\prime}<k_{t}<n_{i+1}^{\prime}\right)$.
For every $i \in \omega$ let $m_{i}=n_{2 i}^{\prime}$.
Fix $\eta \in{ }^{<\omega} \omega$ such that $(\forall i>0)(\exists j<i)\left(\eta \upharpoonright m_{i+1} \in T_{m_{j}}\right)$. To establish the Lemma, it suffices to show $\eta \in T^{*}$. By choice of $T^{*}$, it suffices to show $(\forall n>0)(\exists i<n)\left(\eta \upharpoonright k_{n} \in S_{k_{i}}^{\prime}\right)$.

Claim 1. $(\forall i>0)(\exists j<i)\left(\eta \upharpoonright n_{i+1}^{\prime} \in S_{n_{j}^{\prime}}^{\prime}\right)$.
Proof: The proof breaks into two cases.
Case 1: $i<4$.
We have $n_{i+1}^{\prime} \leq n_{4}^{\prime}=m_{2}$, and $\eta \upharpoonright m_{2} \in T_{0}$, so $\eta \upharpoonright n_{i+1}^{\prime} \in S_{0}^{\prime}$.
Case 2: $i \geq 4$.
Fix $i^{*}>0$ such that $2 i^{*} \leq i \leq 2 i^{*}+1$.
We may fix $j^{*}<i^{*}$ such that $\eta \upharpoonright m_{i^{*}+1} \in T_{m_{j^{*}}}$.
Now, we have
(*) $i+1 \leq 2 i^{*}+2$ so
$\left(^{* *}\right) n_{i+1}^{\prime} \leq m_{i^{*}+1}$.
We also have
$(* * *) \eta \upharpoonright m_{i^{*}+1} \in T_{m_{j^{*}}} \subseteq S_{h\left(m_{j^{*}}\right)}^{\prime}$.
By (**) and (***) we have
$(* * * *) \eta \upharpoonright n_{i+1}^{\prime} \in S_{h\left(m_{j^{*}}\right)}^{\prime}$.
Note that
$(* * * * *) h\left(m_{j^{*}}\right)=h\left(n_{2 j^{*}}^{\prime}\right) \leq n_{2 j^{*}+1}^{\prime} \leq n_{2 i^{*}-1}^{\prime} \leq n_{i-1}^{\prime}$.
By (****) and (*****) we have $\eta \upharpoonright n_{i+1}^{\prime} \in S_{h\left(m_{j^{*}}\right)}^{\prime} \subseteq S_{n_{i-1}^{\prime}}^{\prime}$.
The Claim is established.
To complete the proof of the Lemma, suppose $i>0$. We must show that there is $t<i$ such that $\eta \upharpoonright k_{i} \in S_{k_{t}}^{\prime}$.

Case 1: $k_{i-1}<n_{0}^{\prime}$.
By (*) we have $n_{1}^{\prime} \geq k_{i}$. By Claim 1 we have $\eta \upharpoonright n_{1}^{\prime} \in S_{0}$. Hence $\eta \upharpoonright k_{i} \in S_{0}$.

Case 2: $n_{0}^{\prime} \leq k_{i-1}$.
By $\left(^{*}\right)$ we know that there is at most one element of $\left\{n_{j}^{\prime}: j \in \omega\right\}$ strictly between $k_{i-1}$ and $k_{i}$. Hence we may fix $j>0$ such that $n_{j-1}^{\prime} \leq k_{i-1}<$ $k_{i} \leq n_{j+1}^{\prime}$. If $\eta \upharpoonright n_{j+1}^{\prime} \in S_{0}$ then $\eta \upharpoonright k_{i} \in S_{0}$ and we are done, so assume otherwise. By Claim 1 we may fix $m<j$ such that $\eta \upharpoonright n_{j+1}^{\prime} \in S_{n_{m}^{\prime}}^{\prime}$. We have $\eta \upharpoonright k_{i} \in S_{n_{m}^{\prime}}^{\prime} \subseteq S_{n_{j-1}^{\prime}}^{\prime} \subseteq S_{k_{i-1}}^{\prime}$ and again we are done.

The Lemma is established.
The following Lemma is the analogue of Lemma 4.9.
Lemma 7.15. Suppose $y \in{ }^{\omega}(\omega-\{0\})$ is strictly increasing and $P$ is a forcing notion such that $V\left[G_{P}\right] \models$ "for all countable $X \subseteq V$ there is a countable $Y \in V$ such that $X \subseteq Y$ and $\left\langle T_{n}: n \in \omega\right\rangle$ is a sequence of
$y$-squeezed trees, each of which is in $V$." Then in $V\left[G_{P}\right]$ there is a $y$ squeezed tree $T^{*} \in V$ such that for every $n \in \omega$ and every $j \in \omega$ and every $g \in\left[T_{j}\right]$ there is $k \in \omega$ such that for every $\eta \in T_{j}$ extending $g \upharpoonright k$, if $\eta \upharpoonright k \in T^{*}$ then $\eta \in T^{*}$.

Proof: In $V[G]$, build a sequence of $y$-squeezed trees $\left\langle T_{j}^{\prime}: j \in \omega\right\rangle$, each in $V$, such that $T_{0}^{\prime}=T_{0}$ and for every $j \in \omega$ we have $T_{j+1}^{\prime}=T_{j}^{\prime} \cup T_{j+1}$. By Lemma 7.14 we may find an increasing sequence of integers $\left\langle k_{n}: n \in \omega\right\rangle$ and a $y$-squeezed tree $T^{*} \in V$ such that $(\forall n>0)\left(k_{n}>n\right)$ and for every $\eta \in{ }^{<\omega} \omega$ we have

$$
(\forall n>0)(\exists i<n)\left(\eta \upharpoonright k_{n} \in T_{k_{i}}^{\prime}\right) \text { iff } \eta \in T^{*} .
$$

Fix $j \in \omega$ and $g \in\left[T_{j}\right]$. Let $k=\max \left\{k_{j^{\prime}}: j^{\prime} \leq j\right\}$. Fix $\eta \in T_{j}$ extending $g \upharpoonright k$ and assume $\eta \upharpoonright k \in T^{*}$. It suffices to show that $\eta \in T^{*}$. If $j=0$ then $\eta \in T_{0}=T_{0}^{\prime} \subseteq T^{*}$. Therefore, we assume that $j>0$. It suffices to show that

$$
(\forall i>0)\left(\exists i^{\prime}<i\right)\left(\eta \upharpoonright k_{i} \in T_{k_{i^{\prime}}}^{\prime}\right) .
$$

Towards this end, fix $i>0$.
Case 1: $i \leq j$.
Because $k_{i} \leq k$ we have $\eta \upharpoonright k_{i} \in T^{*}$. Therefore we may take $i^{\prime}<i$ such that $\eta \upharpoonright k_{i} \in T_{k_{i^{\prime}}}^{\prime}$.

Case 2: $0<j<i$.
Because $\eta \in T_{j}$ we have $\eta \upharpoonright i \in T_{j} \subseteq T_{j+1}^{\prime} \subseteq T_{k_{j}}^{\prime}$.
The Lemma is established.
The following is the analogue of Lemma 4.8.
Lemma 7.16. Suppose $y \in{ }^{\omega}(\omega-\{0\})$ is strictly increasing and $\zeta \in{ }^{\omega} \omega$ and $\left\langle T_{n}: n \in \omega\right\rangle$ is a sequence of $y$-squeezed trees. Then there is a $y$ squeezed tree $T^{*}$ and a sequence of integers $\left\langle m_{i}: i \in \omega\right\rangle$ such that $\zeta \in\left[T^{*}\right]$ and $T \subseteq T^{*}$ and for every $i \in \omega$ and every $j>m_{i}$ and every $\nu \in T_{m_{i}}$ extending $\zeta \upharpoonright j$ we have $\nu \in T^{*}$.
Proof. Define $\left\langle T_{k}^{\prime}: k \in \omega\right\rangle$ by setting $T_{0}^{\prime}=T_{0} \cup\{\zeta \upharpoonright n: n \in \omega\}$ and for every $k \in \omega$ set $T_{k+1}^{\prime}=T_{k}^{\prime} \cup T_{k+1}$.

By Lemma 7.14 we may choose $T^{*}$ a $y$-squeezed tree and $\left\langle m_{i}: i \in \omega\right\rangle$ an increasing sequence of integers such that

$$
\left(\forall g \in{ }^{\omega} \omega\right)\left((\forall n>0)(\exists i<n)\left(g \upharpoonright m_{n} \in T_{m_{i}}^{\prime}\right) \text { implies } g \in\left[T^{*}\right]\right)
$$

Now suppose that $\eta \in T$ and $i \in \omega$ and length $(\eta) \geq m_{i}$ and $\nu$ extends $\eta$ and $\nu \in T_{m_{i}}$. We show $\nu \in T^{*}$.

Choose $h \in\left[T_{m_{i}}\right]$ extending $\nu$. It suffices to show that $h \in\left[T^{*}\right]$. Therefore it suffices to show that $(\forall k>0)(\exists j<k)\left(h \upharpoonright m_{k} \in T_{m_{j}}^{\prime}\right)$.

Fix $k \in \omega$. If $i<k$ then because $h \in\left[T_{m_{i}}\right]$ we have that $h \upharpoonright m_{k} \in T_{m_{i}} \subseteq$ $T_{m_{i}}^{\prime}$ and we are done. If instead $k \leq i$ then $h \upharpoonright m_{k}=\eta \upharpoonright m_{k} \in T_{0}^{\prime}$ and again we are done.

The Lemma is established.
The following Theorem is [13, Theorem VI.1.12] for the case of the $P$ point property. Rather than simply referring to the proof of Theorem 4.10, we give the complete argument to demonstrate the simplifications afforded us by the fact that Lemma 7.10 holds.

Theorem 7.17. Suppose $\left\langle P_{\eta}: \eta \leq \kappa\right\rangle$ is a countable support iteration based on $\left\langle Q_{\eta}: \eta<\kappa\right\rangle$ and suppose $(\forall \eta<\kappa)\left(\mathbf{1} \Vdash_{P_{\eta}}\right.$ " $Q_{\eta}$ is proper and has the $P$-point property"). Then $P_{\kappa}$ has the $P$-point property.

Proof: By induction on $\kappa$. No counterexample can first appear at a stage of uncountable cofinality, and the successor case is easily handled using Lemma 7.12 , so we may assume $\kappa=\omega$.

Suppose $\lambda$ is a sufficiently large regular cardinal and $\zeta$ is a $P_{\omega}$-name and $y \in{ }^{\omega}(\omega-\{0\})$ is strictly increasing and $\mathbf{1} \Vdash_{P_{\omega}} " \zeta \in{ }^{\omega} \omega$." Suppose $N$ is a countable elementary submodel of $H_{\lambda}$ and $\left\{P_{\omega}, y, \zeta\right\} \in N$.

Fix $p$ in $P_{\omega} \cap N$.
Fix $p^{\prime} \in N$ and $\left\langle\left(p_{n}, \zeta_{n}\right): n \in \omega\right\rangle \in N$ as in Lemma 3.1.
Let $\left\langle T_{j}^{\prime}: j \in \omega\right\rangle$ list all $T^{\prime} \in N$ such that we have that $T^{\prime}$ is a $y$-squeezed tree, with infinitely many repetitions.

Working in $V\left[G_{P_{\omega}}\right]$, use Lemma 7.15 to choose $T^{*} \in V$ a $y$-squeezed tree such that for every $n \in \omega$ and every $j \in \omega$ and every $g \in\left[T_{j}^{\prime}\right]$ there exists $k \in \omega$ such that for every $\eta \in T_{j}^{\prime}$ extending $g \upharpoonright k$, if $\eta \upharpoonright k \in T^{*}$ then $\eta \in T^{*}$.

In the preceding paragraph, we worked in $V\left[G_{P_{\omega}}\right]$ so that the brackets about $T_{j}^{\prime}$ would be interpreted in $V\left[G_{P_{\omega}}\right]$; i.e., $g$ need not be in $V$.

Claim 1. We may be build $\left\langle r_{n}: n \in \omega\right\rangle$ such that for every $n \in \omega$ we have that the following hold:
(1) $r_{n} \in P_{n}$ is $N$-generic, and
(2) $r_{n+1} \upharpoonright n=r_{n}$, and
(3) $r_{n} \Vdash{ }^{\Vdash} \zeta_{n} \in\left[T^{*}\right]$," and
(4) $r_{n} \leq p^{\prime} \upharpoonright n$."

Proof: By induction on $n$. For $n=0$ we have nothing to prove. Suppose we have $r_{n}$.

Let $F_{0}$ and $F_{2}$ be $P_{n}$-names such that
$(*) \mathbf{1} \Vdash{ }^{*} F_{0}$ and $F_{2}$ are functions and each of whose domains is equal to $Q_{n}$, such that
$\left(\forall q^{\prime} \in Q_{n}\right)\left(F_{0}\left(q^{\prime}\right)\right.$ is a $y$-squeezed tree and $F_{2}\left(q^{\prime}\right) \leq q^{\prime}$ and $\left.F_{2}\left(q^{\prime}\right) \Vdash{ }^{\prime} \zeta_{n+1} \in\left[F_{0}\left(q^{\prime}\right)\right]^{\prime}\right) . "$

We may assume that the names $F_{0}$ and $F_{2}$ are in $N$. Notice that $F_{0}$ and $F_{2}$ depend on $n$, although this dependence is suppressed in our notation.

Working in $V\left[G_{P_{n}}\right]$, use Lemma 7.16 to choose $T_{n}^{\#}$ a $y$-squeezed tree in $V$ and $\left\langle k_{i}: i \in \omega\right\rangle$ an increasing sequence of integers (this sequence depends on $n$ but this fact is suppressed in our notation) such that $\zeta_{n} \in\left[T_{n}^{\#}\right]$ and for every $\eta$ and every $i \in \omega$ and every $\nu \in F_{0}\left(p_{k_{i}}(n)\right)$, if $\eta$ is a proper initial segment of $\zeta_{n}$ and length $(\eta) \geq k_{i}$ and $\nu$ extends $\eta$, then $\nu \in T_{n}^{\#}$.

We may assume the $P_{n}$-name $T_{n}^{\#}$ is in $N$.
Using the induction hypothesis and Lemma 7.12, fix $\tilde{T}_{n} \in V$ a $y$ squeezed tree such that $T_{n}^{\#} \subseteq \tilde{T}_{n}$.

Because $\tilde{T}_{n}$ is a $P_{n}$-name in $N$ forced to be in $V$, we conclude that by the $N$-genericity of $r_{n}$ that

$$
r_{n} \Vdash " \tilde{T}_{n} \in N . "
$$

Therefore there is a $P_{n}$-name $m$ such that

$$
r_{n} \Vdash " \tilde{T}_{n}=T_{m}^{\prime} \text { and } m>n . "
$$

Because $T^{*}$ was chosen as in the conclusion of Lemma 7.15 , we may choose $k$ to be a $P_{n}$-name for an integer such that
$\left(^{* *}\right) r_{n} \Vdash$ " $\left(\forall \eta \in \tilde{T}_{n}\right)$ (if $\eta$ extends $\zeta_{n} \upharpoonright k$ and $\eta \upharpoonright k \in T^{*}$ then $\left.\eta \in T^{*}\right)$."
Choose $j$ to be a $P_{n}$-name such that $r_{n} \Vdash " k_{j} \geq k$."
Subclaim 1. $r_{n} \Vdash{ }^{\prime} F_{2}\left(p_{k_{j}}(n)\right) \Vdash ' \zeta_{n+1} \in\left[\tilde{T}_{n}\right]$.'"
Proof. It suffices to show
$r_{n} \Vdash " F_{2}\left(p_{k_{j}}(n)\right) \Vdash{ }^{\prime}\left(\forall j^{\prime}>j\right)\left(\zeta \upharpoonright k_{j^{\prime}} \in \tilde{T}_{n}\right) .{ }^{\prime} "$
Fix $j^{\prime}$ a $P_{n+1}$-name for an integer such that
$r_{n} \Vdash{ }^{\Vdash} F_{2}\left(p_{k_{j}}(n)\right) \Vdash{ }^{\prime} j^{\prime}>j, ’ "$
By the definition of $\left\langle p_{i}: i \in \omega\right\rangle$ we have
$\left({ }^{* * *}\right) r_{n} \Vdash{ }^{\Vdash} p_{k_{j}}(n) \Vdash{ }^{\prime} \zeta_{n} \upharpoonright k_{j}=\zeta_{n+1} \upharpoonright k_{j}$. ."
By $\left({ }^{*}\right)$ we have
$(* * * *) r_{n} \Vdash{ }^{\Vdash} F_{2}\left(p_{k_{j}}(n)\right) \Vdash{ }^{\prime} \zeta_{n+1} \in\left[F_{0}\left(p_{k_{j}}(n)\right)\right] . ' "$
Combining $\left({ }^{* * *}\right),\left({ }^{(* * * *}\right)$, and the definition of $\tilde{T}_{n}$, we have that

$$
r_{n} \Vdash{ }^{\Downarrow} F_{2}\left(p_{k_{j}}(n)\right) \Vdash{ }^{\prime} \zeta_{n+1} \upharpoonright k_{j^{\prime}} \in \tilde{T}_{n} ., "
$$

The Subclaim is established.
Subclaim 2. $r_{n} \Vdash{ }^{-} F_{2}\left(p_{k_{j}}(n)\right) \Vdash{ }^{\prime} \zeta_{n+1} \in\left[T^{*}\right] . ’ "$
Proof: By ( ${ }^{* *}$ ) we have
$(\dagger) r_{n} \Vdash$ " $\left(\forall \eta \in \tilde{T}_{n}\right)\left(\eta \upharpoonright k_{j} \in T^{*}\right.$ implies $\left.\eta \in T^{*}\right)$."
Work in $V\left[G_{P_{n}}\right]$ with $r_{n} \in G_{P_{n}}$. Fix $\eta \in \tilde{T}_{n}$ and suppose $F_{2}\left(p_{k_{j}}(n)\right) \Vdash$ " $\eta$ is an initial segment of $\zeta_{n+1}$ and $\operatorname{lh}(\eta) \geq k_{j}$." To establish the Subclaim it suffices to show
(\#) $F_{2}\left(p_{k_{j}}(n)\right) \Vdash " \eta \in T^{*}$."
By the definition of $\left\langle p_{i}: i \in \omega\right\rangle$ we have

$$
p_{k_{j}}(n) \Vdash " \eta \upharpoonright k_{j}=\zeta_{n} \upharpoonright k_{j} . "
$$

Hence by the fact that Claim 1 holds for the integer $n$ we have
$(\dagger \dagger) p_{k_{j}}(n) \Vdash " \eta \upharpoonright k_{j} \in T^{*}$."
By Subclaim 1, $(\dagger),(\dagger \dagger)$, and the fact that $F_{2}\left(p_{k_{j}}(n)\right) \leq p_{k_{j}}(n)$ we obtain

$$
F_{2}\left(p_{k_{j}}(n)\right) \Vdash " \eta \in T^{*} . "
$$

Subclaim 2 is established.
To complete the induction establishing Claim 1, we take $r_{n+1} \in P_{n+1}$ such that $r_{n+1} \upharpoonright n=r_{n}$ and $r_{n+1}$ is $N$-generic and $r_{n} \Vdash$ " $r_{n+1}(n) \leq$ $F_{2}\left(p_{k_{j}}(n)\right)$."

Claim 1 is established.

Define $q$ by

$$
q=\bigcup\left\{r_{n}: n \in \omega\right\}
$$

We have $q \leq p$ and $q \Vdash$ " $\zeta \in\left[T^{*}\right]$." The Theorem is established.

## 8 On adding no Cohen reals

In [13, Conclusion VI.2.13D(1)], Shelah states that a countable support iteration of proper forcings, each of which adds no Cohen reals, either adds no Cohen reals or adds a dominating real. However, according to Jakob Kellner, Shelah has stated that this is an error, and the result holds only at limit stages. In this section, we prove the limit case.

Definition 8.1. A nowhere dense tree $T \subseteq{ }^{<\omega} \omega$ is a non-empty tree such that for every $\eta \in T$ there is some $\nu$ extending $\eta$ such that $\nu \notin T$. A perfect tree $T \subseteq{ }^{<\omega} \omega$ is a non-empty tree such that for every $\eta \in T$, the set of successors of $\eta$ in $T$ is not linearly ordered.

Lemma 8.2. $P$ does not add any Cohen reals iff $1 \Vdash_{P} "\left(\forall f \in{ }^{\omega} \omega\right)(\exists H \in$ $V)(H$ is a nowhere dense perfect tree and $f \in[H]) . "$

Proof: This is a tautological consequence of the definition of Cohen real.
Lemma 8.3. Suppose $\operatorname{cf}(\kappa)=\omega$ and $\left\langle\alpha_{n}: n \in \omega\right\rangle$ is an increasing sequence of ordinals cofinal in $\kappa$ such that $\alpha_{0}=0$. Suppose $\left\langle P_{\eta}: \eta \leq \kappa\right\rangle$ is a countable support forcing iteration based on $\left\langle Q_{\eta}: \eta<\kappa\right\rangle$ and for every $\eta<\kappa$ we have $V\left[G_{P_{\eta}}\right] \models " Q_{\eta}$ is proper." Suppose $p \in P_{\kappa}$ and $p \Vdash$ " $f \in{ }^{\omega} \omega$." Then there are $p^{\prime} \leq p$ and $\left\langle\eta_{n}: n \in \omega\right\rangle$ such that for every $n \in \omega$ we have that $\eta_{n}$ is a $P_{\alpha_{n}}$ name and $p^{\prime} \upharpoonright \alpha_{n} \Vdash{ }^{\prime \prime} p^{\prime} \upharpoonright\left[\alpha_{n}, \kappa\right) \Vdash{ }^{\prime} \eta_{n}=f \upharpoonright n$.'"

Proof: Let $\lambda$ be a sufficiently large regular cardinal and $N$ a countable elementary substructure of $H_{\lambda}$ such that $\left\{P_{\kappa}, p, f,\left\langle\alpha_{n}: n \in \omega\right\rangle\right\} \in N$.

Using the Proper Iteration Lemma, build $\left\langle\left(p_{n}, q_{n}, \eta_{n}\right): n \in \omega\right\rangle$ such that $q_{0}=p$ and for every $n \in \omega$ we have the following:
(1) $p_{n} \in P_{\alpha_{n}}$ is $N$-generic and $\eta_{n}$ is a $P_{\alpha_{n}}$-name, and
(2) $p_{n} \Vdash{ }^{\prime} q_{n+1} \leq q_{n} \upharpoonright\left[\alpha_{n}, \kappa\right)$ and $q_{n+1} \in N\left[G_{P_{\alpha_{n}}}\right]$ and $q_{n+1} \Vdash{ }^{\prime} \eta_{n}=f \upharpoonright n, '$ " and
(3) $p_{n+1} \upharpoonright \alpha_{n}=p_{n}$ and $p_{n} \Vdash$ " $p_{n+1} \upharpoonright\left[\alpha_{n}, \alpha_{n+1}\right) \leq q_{n+1} \upharpoonright \alpha_{n+1}$."

Letting $p^{\prime}=\bigcup\left\{p_{n}: n \in \omega\right\}$ establishes the Lemma.
The proof of the following Theorem is [13, proofs of Claims VI.2.5(2) and VI.2.13C].

Theorem 8.4. Suppose $\operatorname{cf}(\kappa)=\omega$ and $\left\langle P_{\eta}: \eta \leq \kappa\right\rangle$ is a countable support forcing iteration based on $\left\langle Q_{\eta}: \eta<\kappa\right\rangle$ and for every $\eta<\kappa$ we have $V\left[G_{P_{\eta}}\right] \models$ " $Q_{\eta}$ is proper" and $P_{\eta}$ does not add any Cohen reals. Suppose $P_{\kappa}$ does not add any dominating reals. Then $P_{\kappa}$ does not add any Cohen reals.

Proof. Fix $\left\langle\alpha_{n}: n \in \omega\right\rangle$ cofinal in $\kappa$ with $\alpha_{0}=0$. Also in $V\left[G_{P_{\kappa}}\right]$ fix $f \in{ }^{\omega} \omega$. Suppose $p \in P_{\kappa}$ and $p \Vdash$ " $f$ is a Cohen real."

Let $p^{\prime} \leq p$ and $\left\langle\eta_{n}: n \in \omega\right\rangle$ be as in Lemma 8.3.
Working in $V\left[G_{P_{\kappa}}\right]$ with $p^{\prime} \in G_{P_{\kappa}}$, let $\left\langle T_{n}: n \in \omega\right\rangle$ be a sequence of nowhere dense perfect trees such that $(\forall n \in \omega)\left(T_{n} \in V\right.$ and $\left.\eta_{n} \in T_{n}\right)$.

Let $B \in V$ be a countable set of nowhere dense perfect trees such that for every $n \in \omega$ we have $T_{n} \in B$. Let $\left\langle S_{n}: n \in \omega\right\rangle \in V$ enumerate $B$ with infinitely many repetitions such that $T_{0}=S_{0}$.

Build inductively $\left\langle S_{n}^{\prime}: n \in \omega\right\rangle$ such that $S_{n+1}^{\prime}=S_{n+1} \cup S_{n}^{\prime}$ and $S_{0}^{\prime}=S_{0}$.
Define $h \in{ }^{\omega} \omega$ by setting $h(k)$ equal to the least $m>k$ such that $T_{k} \subseteq S_{m}^{\prime}$, for every $k \in \omega$. Because $P_{\kappa}$ adds no dominating reals we may choose $g \in{ }^{\omega} \omega \cap V$ and $A \subseteq \omega$ such that $A=\{n \in \omega: g(n)>h(n)\}$ and $A$ is infinite.

Choose $\left\langle k_{i}: i \in \omega\right\rangle \in V$ an increasing sequence of integers as follows. Let $k_{0}=0$. Given $k_{n}$, choose $k_{n+1} \geq \max \left(k_{n}+1,2\right)$ such that $\left(\forall \nu \in \leq k_{n} k_{n}\right)$ $\left[\left(\exists \nu^{\prime} \in{ }^{k_{n+1}} \omega\right.\right.$ extending $\left.\nu\right)\left(\forall i \leq k_{n}\right)\left(\nu^{\prime} \notin S_{g(i)}^{\prime}\right)$ and $\left(\forall i \leq k_{n}\right)\left(\exists \nu_{1} \in S_{g(i)}^{\prime}\right)$ $\left(\exists \nu_{2} \in S_{g(i)}^{\prime}\right)\left(\nu_{1}\right.$ and $\nu_{2}$ are distinct extensions of $\nu$ and $\operatorname{lh}\left(\nu_{1}\right)=\operatorname{lh}\left(\nu_{2}\right)=$ $\left.k_{n+1}\right)$ ].

Let $T^{0}=\left\{\eta \in{ }^{<\omega} \omega:(\exists s \in \omega)(\exists j \in \omega)\left(k_{2 s} \leq j<k_{2 s+1}\right.\right.$ and $\eta \upharpoonright j \in S_{0}^{\prime}$ and $\left.\left.\eta \in S_{g(j)}^{\prime}\right)\right\}$.

Let $T^{1}=\left\{\eta \in{ }^{<\omega} \omega:(\exists s \in \omega)(\exists j \in \omega)\left(k_{2 s+1} \leq j<k_{2 s+2}\right.\right.$ and $\eta \upharpoonright j \in S_{0}^{\prime}$ and $\left.\left.\eta \in S_{g(j)}^{\prime}\right)\right\}$.

Claim 1: $T^{0}$ is a nowhere dense tree.
Proof. Suppose $\eta \in T^{0}$. Choose $s$ and $j$ witnessing this. Also take $n \geq s$ so large that $\eta \in \leq k_{2 n} k_{2 n}$.

We choose $\nu$ extending $\eta$ such that $\operatorname{lh}(\nu)=k_{2 n+2}$ and $\left(\forall i \leq k_{2 n+1}\right)(\nu \notin$ $\left.S_{g(i)}^{\prime}\right)$. In particular we have $\nu \notin S_{0}^{\prime}$. We show that $\nu \notin T^{0}$. So suppose, towards a contradiction, that $s^{\prime} \in \omega$ and $j^{\prime} \in \omega$ and $k_{2 s^{\prime}} \leq j^{\prime}<k_{2 s^{\prime}+1}$ and $\nu \upharpoonright j^{\prime} \in S_{0}^{\prime}$ and $\nu \in S_{g\left(j^{\prime}\right)}^{\prime}$. Because $\nu \in S_{g\left(j^{\prime}\right)}^{\prime}$ we know $j^{\prime} \geq k_{2 n+1}$. Necessarily, then, $j^{\prime} \geq k_{2 n+2}$. But then $\nu=\nu \upharpoonright j^{\prime} \in S_{0}^{\prime}$. This contradiction establishes the Claim.

Claim 2. $T^{0}$ is a perfect tree.
Proof: Given $\eta \in T^{0}$, let $s \in \omega$ and $j \in \omega$ be witnesses.
Case 1: $\operatorname{lh}(\eta) \geq j$.
Let $\nu$ and $\nu^{\prime}$ be incomparable elements of $S_{g(j)}^{\prime}$ extending $\eta$. We have that $\nu$ and $\nu^{\prime}$ are in $T^{0}$; this is witnessed by the integers $s$ and $j$.

Case 2: $\operatorname{lh}(\eta)<j$.
Take $\nu$ and $\nu^{\prime}$ distinct extensions of $\eta$ such that $\nu \in S_{0}$ and $\nu^{\prime} \in S_{0}$ and $\operatorname{lh}(\nu)=\operatorname{lh}\left(\nu^{\prime}\right)=j$. We have $\nu \in S_{g(j)}^{\prime}$ and $\nu^{\prime} \in S_{g(j)}^{\prime}$ because $S_{0} \subseteq S_{g(j)}^{\prime}$. We have that $\nu$ and $\nu^{\prime}$ are in $T^{0}$; this is witnessed by the integers $s$ and $j$.

Claim 3: $T^{1}$ is a nowhere dense perfect tree.
Proof: Similar to Claims 1 and 2.
Let $B_{0}=\bigcup\left\{\left[k_{2 i}, k_{2 i+1}\right): i \in \omega\right\}$ and let $B_{1}=\bigcup\left\{\left[k_{2 i+1}, k_{2 i+2}\right): i \in \omega\right\}$.
Claim 4: $\left(\forall n \in A \cap B_{0}\right)\left(\eta_{n} \in T^{0}\right)$.
Proof: Given $n \in A \cap B_{0}$ choose $s \in \omega$ such that $k_{2 s} \leq n<k_{2 s+1}$. We have $\eta_{n} \in T_{n} \subseteq S_{h(n)}^{\prime} \subseteq S_{g(n)}^{\prime}$ and $\eta_{n} \in S_{0}^{\prime}$. Hence $\eta_{n} \in T^{0}$. The Claim is established.

Claim 5: $\left(\forall n \in A \cap B_{1}\right)\left(\eta_{n} \in T^{1}\right)$.
Proof: Similar to Claim 4.
We have that $T^{0}$ and $T^{1}$ are elements of $V$. Furthermore, if $A \cap B_{0}$ is infinite, we have by Claim 4 that for infinitely many $n$ we have $\eta_{n} \in T^{0}$ and hence $f \in\left[T^{0}\right]$. Otherwise by Claim 5 it follows that for infinitely many $n$ we have $\eta_{n} \in T^{1}$ and hence $f \in\left[T^{1}\right]$. The Theorem is established.

## 9 On not adding reals not belonging to any closed null sets of $V$

In this section we give Shelah's proof that the property " $P$ does not add any real not belonging to any closed set of measure zero of the ground
model" is preserved at limit stages by countable support iterations of proper forcings assuming the iteration does not add dominating reals.

Theorem 9.1. Suppose $\left\langle P_{\eta}: \eta \leq \kappa\right\rangle$ is a countable support iteration based on $\left\langle Q_{\eta}: \eta<\kappa\right\rangle$ and suppose $\kappa$ is a limit ordinal and $(\forall \eta<\kappa)\left(P_{\eta}\right.$ does not add reals not in any closed measure zero set of $V$ ). Suppose also that $P_{\kappa}$ does not add any dominating reals. Then $P_{\kappa}$ does not add any real not in any closed measure zero set of $V$.

Proof: Repeat the proof of Theorem 8.3 with "nowhere dense perfect tree" replace by "perfect tree with Lebesgue measure zero" throughout, and choosing $k_{n+1}$ so large that $2^{-k_{n+1}} \cdot \mid\left\{\nu \in{ }^{k_{n+1}} \omega:\left(\exists i \leq k_{n}\right)\right.$ $\left.\left(\nu \in S_{g(i)}^{\prime}\right)\right\} \mid<1 / n$.

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