Comparison of fine structural mice via coarse iteration*

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Abstract Let \mathcal{M} be a fine structural mouse. Let \mathbb{D} be a fully backgrounded $L[\mathbb{E}]$ -construction computed inside an iterable coarse premouse S. We describe a process comparing \mathcal{M} with \mathbb{D} , through forming iteration trees on \mathcal{M} and on S. We then prove that this process succeeds.

 $\mathbf{Keywords}$ Inner model · Comparison · Background construction

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1 Introduction

Let \mathcal{M} be a fine structural mouse. Let $\mathbb{D} = \langle N_{\alpha} \rangle_{\alpha \leq \Lambda}$ be a fully backgrounded $L[\mathbb{E}]$ -construction ¹ computed inside an iterable coarse premouse S. In certain situations, one would like to compare \mathcal{M} with \mathbb{D} , carrying along the universe S. For example, one might want to form an iteration tree \mathcal{T} on \mathcal{M} , with last model \mathcal{M}' , and an iteration tree \mathcal{U} on S, also with a last model, such that either $i^{\mathcal{U}}(N_{\Lambda}) \leq \mathcal{M}'$ or $\mathcal{M}' = N_{\alpha}^{i^{\mathcal{U}}(\mathbb{D})}$ for some α . (Here \mathcal{T} is fine, as in [2, Chap. 5], and \mathcal{U} is coarse, as in [1].) We give details of such a comparison here,

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¹ That is, a background extender construction using total background extenders, similar to that defined in [2, §11].

making fairly minimal assumptions about the $L[\mathbb{E}]$ -construction. This sort of comparison is used (without explanation of the details) in [4]. ²

Notation & Definitions: Given a transitive structure M, we use both ord^M and ord(M) for $ord \cap M$. Likewise for other classes of M. See [1] for the definition of coarse premouse, and [5] for premouse. Let M be a premouse and let $\alpha \leq \operatorname{ord}^M$ be a limit. We write $M|\alpha$ for the P such that $\operatorname{ord}^P = \alpha$ and $P \subseteq M$, and we write $M|\alpha$ for its passive counterpart. We write F^M for the active extender of M, \mathbb{E}^M for the extender sequence of M, not including F^M , and \mathbb{E}^M for $\mathbb{E}^M \cap F^M$. We write lh for length, and $\nu(F)$ denotes the natural ral length of an extender F. We say M is typical iff condensation holds for the proper segments (i.e., proper initial segments) of M, and [3, 4.11, 4.12, 4.15] hold for M and its proper segments. (These properties are consequences of $(0, \omega_1, \omega_1 + 1)$ -iterability.) Given a squashed premouse N, we write N^{unsq} for the unsquash of N; if N is a premouse, we let $N^{\text{unsq}} = N$. Given an iteration tree \mathcal{T} of successor length $\theta + 1$, we write $b^{\mathcal{T}}$ for $[0, \theta]_{\mathcal{T}}$ and $\mathcal{I}^{\mathcal{T}}$ for $M_{\theta}^{\mathcal{T}}$. Given an extender E^* , we write $\rho(E^*)$ for the strength of E^* , i.e. the largest ρ such that $V_{\rho} \subseteq \text{Ult}(V, E^*)$. Let R be a coarse premouse and \mathcal{U} a putative iteration tree on R. We say \mathcal{U} is strictly strength increasing iff for every $\alpha + 1 < \beta + 1 < \text{lh}(\mathcal{U})$ we have $\rho^{M_{\alpha}^{\mathcal{U}}}(E_{\alpha}^{\mathcal{U}}) < \rho^{M_{\beta}^{\mathcal{U}}}(E_{\beta}^{\mathcal{U}})$; \mathcal{U} is nonoverlapping iff for every $\alpha + 1 < \text{lh}(\mathcal{U})$, $\mathcal{U}\text{-pred}(\alpha + 1)$ is the least $\gamma \leq \alpha$ such that $\operatorname{crit}(E_{\alpha}^{\mathcal{U}}) < \rho(E_{\delta}^{\mathcal{U}})$ for all $\delta \in [\gamma, \alpha)$; \mathcal{U} is normal iff it is strictly strength increasing and nonoverlapping. Given an iteration tree \mathcal{T} , we write $\mathcal{T} \cap \mathcal{P}$ for an extension of \mathcal{T} consisting only of padding $\mathcal{P} = \langle \emptyset, \emptyset, \ldots \rangle$; here $lh(\mathcal{P})$ is determined by context. We consider \emptyset as the trivial extender, with $\mathrm{Ult}(V,\emptyset)=V$ and $i_{\emptyset} = \text{id}$. We write $\nu(\emptyset) = \rho(\emptyset) = \infty$.

2 Main result

Definition 2.1 (Construction). Suppose $V = (|V|, \delta)$ is a coarse premouse. ⁴ Let $x \in \mathbb{R}$. We say \mathbb{C} is an $L[\mathbb{E}, x]$ -construction iff:

- (a) $\mathbb{C} = \langle N_{\alpha}, E_{\alpha}^* \rangle_{\alpha \leq \lambda}$ is a sequence of *x*-premice N_{α} and extenders $E_{\alpha}^* \in V_{\delta}$ (possibly $E_{\alpha}^* = \emptyset$);
- (b) $N_0 = \mathcal{J}_1(x);$
- (c) For limit $\eta \leq \lambda$, N_{η} is the lim inf of $\langle N_{\gamma} \rangle_{\gamma \leq n}$;
- (d) Let $\alpha < \lambda$. Either
 - (i) $N_{\alpha+1} = \mathcal{J}_1(\mathfrak{C}_{\omega}(N_{\alpha}));$ or
 - (ii) N_{α} is passive, $N_{\alpha+1}$ is active with $N_{\alpha+1} = (N_{\alpha}, F)$ for some F, and $F \upharpoonright \nu(F) \subseteq E_{\alpha+1}^*$.

² See the proofs of Corollary 14.3 and Theorem 16.1 of [4].

³ A related problem is that of comparing (the outputs of) two $L[\mathbb{E}]$ -constructions $\mathbb{C}^R, \mathbb{C}^S$, computed inside coarsely iterable universes R, S, through forming coarse iteration trees on R, S. This problem presents somewhat different challenges, and will be dealt with in a separate paper.

⁴ We don't assume $V \models \mathsf{ZFC}$ here; we're just working inside some coarse premouse.

Let $\mathbb{C} = \langle N_{\alpha}, E_{\alpha}^* \rangle_{\alpha \leq \lambda}$ be an $L[\mathbb{E}, x]$ -construction. Given $\alpha + 1 \leq \lambda$ such that $N = N_{\alpha+1}$ is active, $F = F^N$ and $E^* = E_{\alpha+1}^*$, we associate an extender $F_{\alpha+1}^* = E^* \upharpoonright \beta$ where β is least such that $\beta \geq \nu(F)$ and $\rho(E^* \upharpoonright \beta) \geq \min(\rho(E^*), \nu(F))$. Then in fact, $\rho(E^* \upharpoonright \beta) = \min(\rho(E^*), \nu(F))$. (We have $\nu(F) \leq \text{lh}(E^*)$ since $F \upharpoonright \nu(F) \subseteq E^*$.)

We say \mathbb{C} is (e) strongly reasonable, (f) reasonable, (g) normal iff for all $\alpha < \lambda$, if $N = N_{\alpha+1}$ is active then

- (e) Strong reasonableness: For all $\kappa < \nu(F^N)$, if $N \models "\kappa$ is inaccessible" then $\kappa < \rho(E_{\alpha+1}^*)$.
- (f) Reasonableness: Let λ be the largest limit cardinal of N, let $\eta = (\lambda^+)^N$ and $\kappa < \nu(F^N)$. Suppose κ is measurable in $U = \mathrm{Ult}(V, E^*_{\alpha+1})$ and for every $\xi < \eta$ there are $E', N' \in U$ such that $U \models \text{``E'}$ is an extender with $\mathrm{crit}(E') = \kappa$ ", N' is an active premouse, $\mathrm{ord}^{N'} > \xi$, either $N' \triangleleft N$ or $N' || \eta = N || \eta$, and $F^{N'} \upharpoonright \nu(F^{N'}) \subseteq E'$. Then $\kappa < \rho(E^*_{\alpha+1})$.
- (g) Normality: (i) $E_{\alpha+1}^* \in \mathcal{H}_{|V_{\rho}|+1}$, where $\rho = \rho(E_{\alpha+1}^*)$; (ii) For all κ such that $\nu(F^{N_{\alpha+1}}) < \kappa < \rho(E_{\alpha+1}^*)$, we have $\mathrm{Ult}(V, E_{\alpha+1}^*) \models \text{``}\kappa$ is not measurable''.

Remark 2.2. The reasonableness of \mathbb{C} is roughly what we need to prove that the comparison to be defined succeeds (it will be used to show that the coarse tree \mathcal{U} that we build does not move fine-structural generators); the definition is extracted from the proof. The assumption is probably not optimal, but it seems hard to get by with much less. In typical applications, an $L[\mathbb{E}, x]$ -construction is strongly reasonable or more; the proof that the comparison succeeds simplifies a little under this extra assumption (but only in one spot).

Remark 2.3. Given an active $N = N_{\alpha+1}$ and $E^* = E_{\alpha+1}^*$ as in 2.1, we have a canonical factor embedding $j : \text{Ult}(N, F^N) \to i_{E^*}(N)$, which is Σ_0 -elementary, preserves cardinals, and $\text{crit}(j) \geq \nu(F^N)$ and $j \circ i_{F^N}^N = i_{E^*} \upharpoonright N$. Using j, it's easy to see that if $\mathbb C$ is strongly reasonable then it is reasonable.

Remark 2.4. Our main theorem, 2.9, is used in the proofs of [4, 14.3, 16.1]. Given a real x, part (b) of the conclusion of the theorem can be used to ensure that for each limit $\lambda < \text{lh}(\mathcal{T})$, $(x, \mathcal{T} \upharpoonright \lambda)$ is (class) extender algebra generic over $M(\mathcal{T} \upharpoonright \lambda)$. This is used in the proof of [4, 16.1]. The first author wishes to thank Nam Trang for pointing out to him that the version of the theorem given in an earlier draft of the paper, which omitted (b), was insufficient for the proof of [4, 16.1]. In the construction of \mathcal{T} and \mathcal{U} , if one omits the use of extenders included specifically for the purposes of establishing (b), then one still obtains trees satisfying (a) and (c). The next two definitions relate to part (b).

Definition 2.5. A pair $(\mathcal{T}, \mathcal{U})$ of padded iteration trees is neat iff we have: (a) $lh(\mathcal{T}) = lh(\mathcal{U})$; (b) \mathcal{T} is on a premouse and is normal; (c) Let $\lambda \leq lh(\mathcal{T})$ be a limit. Then either $\mathcal{T} \upharpoonright \lambda$ is cofinally non-padded (i.e. $E_{\alpha}^{\mathcal{T}} \neq \emptyset$ for cofinally many $\alpha < \lambda$) or $\mathcal{U} \upharpoonright \lambda$ is cofinally non-padded. If both are cofinally non-padded then $\delta(\mathcal{T} \upharpoonright \lambda) = \delta(\mathcal{U} \upharpoonright \lambda)$. In any case, let δ_{λ} denote $\delta(\mathcal{T} \upharpoonright \lambda)$ or $\delta(\mathcal{U} \upharpoonright \lambda)$, whichever is defined. Then for all limits $\gamma < \lambda \leq lh(\mathcal{T})$ we have $\delta_{\gamma} < \delta_{\lambda}$.

Assume $(\mathcal{T}, \mathcal{U})$ is neat. The *neat code* for $(\mathcal{T}, \mathcal{U})$ is the set of triples (i, δ, γ) such that for some limit $\lambda < lh(\mathcal{T})$ we have $\delta = \delta_{\lambda}$ and either (i) $i = 0, \mathcal{T} \upharpoonright \lambda$ is cofinally non-padded and for some $\alpha \in [0,\lambda)_{\mathcal{T}}$ such that $E_{\alpha}^{\mathcal{T}} \neq \emptyset$, we have $\gamma = \text{lh}(E_{\alpha}^{\mathcal{T}})$; or (ii) i = 1 and $\mathcal{U} \upharpoonright \lambda$ is cofinally non-padded and $\gamma \in [0, \lambda)_{\mathcal{U}}$.

Definition 2.6. Let M be a premouse, let $\sigma < \operatorname{ord}^M$, $Y \subseteq \operatorname{ord}$ and $Z \subseteq \operatorname{ord}^3$. We say that M is (Y, Z)-valid at σ iff either σ is not a cardinal in M, or for all $E \in \mathbb{E}^M_+$, if $\sigma = \nu(E)$ and $M|\text{lh}(E) \models \text{``}\sigma$ is inaccessible" then $Y \subseteq \sigma$ and $(Y, Z \cap \sigma^3)$ satisfies all extender algebra⁵ axioms in M|lh(E) induced by E. We say that M is (Y, Z)-valid iff M is (Y, Z)-valid at σ for every $\sigma < \operatorname{ord}^M$.

Definition 2.7. A coarse iteration tree \mathcal{U} is normalizable iff it is nonoverlapping and for each $\alpha + 1 < \text{lh}(\mathcal{T}), M_{\alpha}^{\mathcal{T}} \models \text{``} E_{\alpha}^{\mathcal{T}} \in \mathcal{H}_{|V_{\rho}|+1} \text{ where } \rho = \rho(E_{\alpha}^{\mathcal{T}})\text{''}.$

The key property of a normalizable tree is the following:

Proposition 2.8. Let \mathcal{U} be a normalizable putative tree on a coarse premouse R. Then there is a unique normal padded putative tree \mathcal{U}' on R such that $lh(\mathcal{U}') = lh(\mathcal{U})$, and for each $\alpha + 1 < lh(\mathcal{U})$:

- $E_{\alpha}^{\mathcal{U}'} \neq \emptyset$ iff $\rho(E_{\alpha}^{\mathcal{U}}) < \rho(E_{\beta}^{\mathcal{U}})$ for all $\beta + 1 \in (\alpha + 1, \text{lh}(\mathcal{T}))$,
- if $E_{\alpha}^{\mathcal{U}'} \neq \emptyset$ then $E_{\alpha}^{\mathcal{U}'} = E_{\alpha}^{\mathcal{U}}$, for all limits $\lambda < \text{lh}(\mathcal{U})$, if \mathcal{U}' has non-padded stages cofinally in λ , then $[0,\lambda]_{\mathcal{U}'}=[0,\lambda]_{\mathcal{U}}.$

Proof. Omitted.

Theorem 2.9 (Main Theorem). Let $\mathcal{M}, S \in \mathcal{H}_{v^+}$ and $x \in \mathbb{R} \cap S$. Let

Suppose \mathcal{M} is an m-sound, typical⁶, normally $(m, v^+ + 1)$ -iterable (fine)

x-premouse. Let $\Sigma_{\mathcal{M}}$ be an $(m, v^+ + 1)$ -iteration strategy for \mathcal{M} . Suppose $S = (|S|, \delta^S)$ is a $(v^+ + 1)$ -iterable coarse premouse.⁷ Let Σ_S be a $(v^+ + 1)$ -iteration strategy for S. Let $\Lambda \leq \delta^S$ and let $\mathbb{D} = \langle N_\alpha \rangle_{\alpha \leq \Lambda} \in S$ be such that $S \models$ " \mathbb{D} is a reasonable $L[\mathbb{E}, x]$ -construction".

Let $A \subseteq v^+$ with A bounded in v^+ .

Then there is a padded m-maximal normal iteration tree \mathcal{T} on \mathcal{M} , via $\Sigma_{\mathcal{M}}$, and a padded iteration tree \mathcal{U} on S, via Σ_S , both of successor length $\langle v^+, \rangle$ such that:

- (a) Either:
 - (i) $i^{\mathcal{U}}(\mathfrak{C}_n(N_{\Lambda})) \leq \mathcal{I}^{\mathcal{T}}$; or
 - (ii) $b^{\mathcal{T}}$ does not drop in model or degree and $\mathcal{I}^{\mathcal{T}} = \mathfrak{C}_m(N_{\alpha}^{i^{\mathcal{U}}(\mathbb{D})})$ for some

⁵ Here we mean the " δ -generator" extender algebra. That is, for each $x \in \sigma^{<\omega}$ there is a corresponding atomic formula φ_x .

⁶ Typical is defined at the end of §1.

 $^{^{7}}$ It's not particularly important that S be a coarse premouse. We just need that iteration maps on S are sufficiently elementary, for iteration trees using extenders in V_{Λ}^{S} and its

- (b) $(\mathcal{T}, \mathcal{U})$ is neat. Let B be the neat code for $(\mathcal{T}, \mathcal{U})$.
 - (i) If $i^{\mathcal{U}}(\mathfrak{C}_n(N_{\Lambda})) \triangleleft \mathcal{I}^{\mathcal{T}}$ or $[b \text{ drops in model and } i^{\mathcal{U}}(\mathfrak{C}_n(N_{\Lambda})) = \mathcal{I}^{\mathcal{T}}]$ then let $P = i^{\mathcal{U}}(\mathfrak{C}_n(N_{\Lambda}))$ and $\rho = \rho_n^P$.

 - (ii) If $\mathcal{I}^{\mathcal{T}} = \mathfrak{C}_m(N_{\alpha}^{i^{\mathcal{U}}(\mathbb{D})})$ for some $\alpha < i^{\mathcal{U}}(\Lambda)$ then let $P = \mathcal{I}^{\mathcal{T}}$ and $\rho = \rho_m^P$. (iii) Let $k = \min(m, n)$. If b does not drop in model and $\mathcal{I}^{\mathcal{T}} = i^{\mathcal{U}}(\mathfrak{C}_k(N_{\Lambda}))$ then let $P = \mathcal{I}^{\mathcal{T}}$ and $\rho = \rho_k^P$. Let $\tau = (\rho^+)^P$. Then $P|\tau$ is (A, B)-valid.
- (c) Let $\mathbb{C}^{\alpha} = i_{0,\alpha}^{\mathcal{U}}(\mathbb{D})$ for $\alpha < \text{lh}(\mathcal{U})$. We may take \mathcal{U} to satisfy condition (i) below; alternatively (ii) below. (But maybe not (i) and (ii) simultaneously.)
 - (i) \mathcal{U} is nonoverlapping, and for each $\alpha + 1 < lh(\mathcal{U})$, if $E_{\alpha}^{\mathcal{U}} \neq \emptyset$ then $M_{\alpha}^{\mathcal{U}} \models$ "There is $\gamma + 1 \leq i_{0,\alpha}^{\mathcal{U}}(\Lambda)$ such that $N = N_{\gamma+1}^{\mathbb{C}^{\alpha}}$ is active and $E_{\alpha}^{\widetilde{\mathcal{U}}} = F_{\gamma+1}^*$; or
 - (ii) For each $\alpha+1<\operatorname{lh}(\mathcal{U})$, if $E^{\mathcal{U}}_{\alpha}\neq\emptyset$ then $M^{\mathcal{U}}_{\alpha}\models$ "There is $\gamma+1\leq \Lambda$ such that $N=N^{\mathbb{C}^{\alpha}}_{\gamma+1}$ is active, and $E^{\mathcal{U}}_{\alpha}=E^*_{\gamma+1}$ ", and moreover, if $S\models$ " \mathbb{D} is normal" then \mathcal{U} is normalizable.

It seems we can't strengthen (c)(ii) by replacing "normalizable" with "normal", since the extraction of a normal tree \mathcal{U}' from a normalizable tree \mathcal{U} can change the model of origin for a given extender (e.g. we can have $E_{\alpha'}^{\mathcal{U}'} = E_{\alpha}^{\mathcal{U}}$ for some $\alpha' < \alpha$, and $M_{\alpha'}^{\mathcal{U}'} = M_{\alpha'}^{\mathcal{U}} \neq M_{\alpha}^{\mathcal{U}}$), so (c)(ii) might fail for \mathcal{U}' even if it held for \mathcal{U} . Also, conclusion (b) becomes somewhat unclear if we replace \mathcal{U} with \mathcal{U}' (at least, with regard to the genericity of the code for \mathcal{U}').

Proof (Theorem 2.9). We will first produce an m-maximal normal tree \mathcal{T} on \mathcal{M} , via $\Sigma_{\mathcal{M}}$, and a tree \mathcal{U} on S, via Σ_{S} , each of successor length $\langle v^{+}, \text{ such } \rangle$

- (a') Either:
 - (i) $i^{\mathcal{U}}(N_{\Lambda}) \leq \mathcal{I}^{\mathcal{T}}$; or
- (ii) $b^{\mathcal{T}}$ does not drop in model and $\mathcal{I}^{\mathcal{T}} = N_{\alpha}^{i^{\mathcal{U}}(\mathbb{D})}$ for some $\alpha \leq i^{\mathcal{U}}(\Lambda)$. (b') $(\mathcal{T}, \mathcal{U})$ is neat. Let B be the neat code for $(\mathcal{T}, \mathcal{U})$. If (a')(i) holds let P = $i^{\mathcal{U}}(N_{\Lambda})$; otherwise let $P = \mathcal{I}^{\mathcal{T}}$. Then P is (A, B)-valid.
- (c') \mathcal{U} satisfies 2.9(c)(i) (alternatively, at our will, 2.9(c)(ii)).

We will then find $\varepsilon < \text{lh}(\mathcal{T})$ such that $\mathcal{T} \upharpoonright \varepsilon + 1$ and $\mathcal{U} \upharpoonright \varepsilon + 1$ satisfy the requirements of 2.9.

To construct \mathcal{T} , we will define a sequence $\vec{\mathcal{T}} = \langle \mathcal{T}^{\alpha} \rangle_{\alpha < \zeta}$ of padded normal trees on \mathcal{M} , and will set $\mathcal{T} = \mathcal{T}^{\zeta}$. The trees \mathcal{T}^{α} approximate initial segments of \mathcal{T} ; we will have $lh(\mathcal{T}^{\alpha}) = \alpha + 1$. We simultaneously construct $\vec{\mathcal{T}}$ and \mathcal{U} , recursively through ordinal stages $\beta \leq \zeta$. The process is much like standard comparison, but is also significantly different.

When beginning stage β we will have already built $\mathcal{U} \upharpoonright \beta$ and $\mathcal{T} \upharpoonright \beta$. We will then define $\mathcal{U} \upharpoonright \beta + 1$ and \mathcal{T}^{β} . For limit β , the trees $\vec{\mathcal{T}} \upharpoonright \beta$ will be defined such that the sequence converges to a padded tree $\mathcal{T}^{<\beta}$ of length β with $(\mathcal{T}^{<\beta}, \mathcal{U} \upharpoonright \beta)$ neat; we then apply our iteration strategies to obtain \mathcal{T}^{β} and $\mathcal{U} \upharpoonright \beta + 1$. We

⁸ Here if $\rho = \operatorname{ord}^P$ or ρ is the largest cardinal of P then $(\rho^+)^P$ denotes ord^P . In particular, if n=0 and P is type 3 then $(\rho^+)^P=\operatorname{ord}^P$, not $\operatorname{ord}(\mathfrak{C}_0(P))$.

will show that for limit β , $\mathcal{T}^{\beta} \preceq \mathcal{T}^{\alpha}$ for all $\alpha \geq \beta$. (Applying this to the largest limit $\beta' \leq \alpha$, it follows that $(\mathcal{T}^{\alpha}, \mathcal{U} \upharpoonright \alpha + 1)$ is neat.) At successor stages $\alpha + 1$, we will choose extenders $E^{\mathcal{U}}_{\alpha} \in M^{\mathcal{U}}_{\alpha}$ and $E = E^{\mathcal{T}^{\alpha+1}}_{\alpha} \in \{\emptyset\} \cup \mathbb{E}_{+}(M^{\mathcal{T}^{\alpha}}_{\alpha})$, such that either $E^{\mathcal{U}}_{\alpha} \neq \emptyset = E$ or $E^{\mathcal{U}}_{\alpha} = \emptyset \neq E$. If $E \neq \emptyset$ then we will have $\mathrm{lh}(E) > \mathrm{lh}(F)$ for all extenders F used in \mathcal{T}^{α} , and we set $\mathcal{T}^{\alpha+1} = \mathcal{T}^{\alpha} \cap \langle E \rangle$, with $\mathcal{T}^{\alpha+1}$ -pred $(\alpha+1)$, etc, determined by m-maximality. In this case we are making a tentative decision to use E in the final tree \mathcal{T} ; this decision may be tentatively retracted later. If $E = \emptyset$ then we will set $\mathcal{T}^{\alpha+1} = \mathcal{T}^{\alpha} \upharpoonright (\gamma+1) \cap \mathcal{P}$, where $\gamma+1 \leq \mathrm{lh}(\mathcal{T}^{\alpha})$ and $\mathcal{P} = \langle \emptyset, \emptyset, \ldots \rangle$ consists of only padding. Here if $\gamma+1 < \mathrm{lh}(\mathcal{T}^{\alpha})$, we will have $E' = E^{\mathcal{T}^{\alpha}}_{\gamma} \neq \emptyset$, and we are tentatively retracting the use of E' from the final \mathcal{T} ; we may later change our mind about E' again. No such retractions occur in the construction of \mathcal{U} . Regarding padding, if $E^{\mathcal{U}}_{\alpha} = \emptyset$ we set \mathcal{U} -pred $(\alpha+1) = \alpha$ and $i^{\mathcal{U}}_{\alpha,\alpha+1} = \mathrm{id}$. If $E^{\mathcal{U}}_{\alpha} \neq \emptyset$ then we will ensure that $E^{\mathcal{U}}_{-\mathrm{pred}(\alpha+1)} \neq \emptyset$ also. Likewise for trees in \mathcal{T} .

We will simultaneously define various other objects in order to guide our selection of the extenders used in building $\vec{\mathcal{T}}, \mathcal{U}$, and in order to prove that the comparison succeeds.

We now begin the construction. We set \mathcal{T}^0 and $\mathcal{U} \upharpoonright 1$ to be the trivial trees on \mathcal{M} and S respectively.

Now consider stage $\alpha+1$. We are given trees \mathcal{T}^{α} and $\mathcal{U} \upharpoonright \alpha+1$, with last models $M^{\mathcal{T}^{\alpha}}_{\alpha}$ and $M^{\mathcal{U}}_{\alpha}$ respectively. Define $M^{\alpha}=M^{\mathcal{T}^{\alpha}}_{\alpha}$, $R^{\alpha}=M^{\mathcal{U}}_{\alpha}$, $\mathbb{C}^{\alpha}=i^{\mathcal{U}}_{0,\alpha}(\mathbb{D})$, $\Lambda^{\alpha}=i^{\mathcal{U}}_{0,\alpha}(\Lambda)$ and $N^{\alpha}_{\beta}=N^{\mathbb{C}^{\alpha}}_{\beta}$.

We will analyse M^{α} and $(R^{\alpha}, \mathbb{C}^{\dot{\alpha}})$. This will culminate in either a proof that our comparison has already succeeded (i.e., $\mathcal{T}^{\alpha}, \mathcal{U} \upharpoonright \alpha + 1$ are as in (a')-(c')), or else in a selection of extenders $E_{\alpha}^{\mathcal{T}^{\alpha+1}}$, $E_{\alpha}^{\mathcal{U}}$, chosen by finding the earliest roots of disagreement between M^{α} and \mathbb{C}^{α} , or the first extenders we reach that, if ignored, could be an obstacle to validity. The analysis is related to resurrection (see [2, §12]). We will in fact define the analysis a little more generally. After this, we will explain how we determine $\mathcal{U}-\text{pred}(\alpha+1)$ and any retraction of extenders required to form $\mathcal{T}^{\alpha+1}$.

Definition 2.10 ((Y,Z)-descent). Let M be an x-premouse. Let $R=(|R|,\delta)$ be a coarse premouse with $x \in R$. Let $\Gamma \leq \delta$ and let $\mathbb{C}=\langle N_{\alpha} \rangle_{\alpha \leq \Gamma} \in R$ be such that $R \models$ " \mathbb{C} is a reasonable $L[\mathbb{E},x]$ -construction". Let $Y,Z \subseteq$ ord. The (Y,Z)-descent of $(M,(R,\mathbb{C}))$ is a quadruple (c,d,e,θ) , defined as follows.

We will first define $k < \omega$ and $c = \langle \gamma_i, \xi_i, \mu_i, \theta_i \rangle_{i \leq k}$, with $\gamma_i, \xi_i, \mu_i \in \text{ord}$ for all $i \leq k$, $\theta_i \in \text{ord}$ for i < k, and $\theta_k \in \text{ord} \cup \{\dagger\}$. We will also say " θ_k is undefined" to mean " $\theta_k = \dagger$ ". 9

We will have $k \geq 0$. Let $\gamma_0 = \operatorname{ord}^M$ and $\xi_0 = \Gamma$.

Suppose that for some $i < \omega$, we have determined that $k \geq i$, and have defined $\gamma_i \leq \mathsf{ord}^M$ and $\xi_i \leq \Gamma$.

Let μ_i be the largest ordinal μ such that $(*)_{\mu}$ holds, where $(*)_{\mu}$ asserts:

⁹ Usually $\theta_k \in \text{ord}$. If in the descent of $(M^{\alpha}, (R^{\alpha}, \mathbb{C}^{\alpha}))$ we get $\theta_k = \dagger$ then we will show that the comparison has already been successful, i.e. $\mathcal{T}^{\alpha}, \mathcal{U} \upharpoonright \alpha + 1$ are as required. Moreover, this is the only manner in which the comparison can terminate.

- (i) $\mu \leq \gamma_i \cap \operatorname{ord}(N_{\xi_i});$
- (ii) $M|\mu = N_{\xi_i}|\mu;$
- (iii) if $\mu < \gamma_i$ then μ is a cardinal of $M|\gamma_i$;
- (iv) if $\mu < \operatorname{ord}(N_{\xi_i})$ then μ is a cardinal of N_{ξ_i} ;
- (v) $M|\mu$ is (Y, Z)-valid.

Note μ_i is well defined, as $(*)_0$ holds, and if μ is a limit of ordinals μ' such that $(*)_{\mu'}$ holds then both $M|\mu$, $N_{\xi_i}|\mu$ are passive, and $(*)_{\mu}$ holds.

Let $(\dagger)_i$ be the statement " $\mu_i = \min(\gamma_i, \operatorname{ord}(N_{\xi_i}))$ ".

Suppose $(\dagger)_i$ holds. Then we set k = i, stop the analysis, and do not define θ_k . Note that here if $\gamma_k < \operatorname{ord}(N_{\xi_k})$ then $\mu_k = \gamma_k$ is a cardinal of N_{ξ_k} and $M|\gamma_k \triangleleft N_{\xi_k}$, so $M|\gamma_k$ is passive. Likewise if $\operatorname{ord}(N_{\xi_k}) < \gamma_k$.

Now suppose $(\dagger)_i$ fails. So μ_i is a cardinal of $M|\gamma_i$ and of N_{ξ_i} . Let θ_i be the sup of all ordinals $\delta + \omega$ such that $\delta \in \gamma_i \cap N_{\xi_i}$ and $M|\delta = N_{\xi_i}|\delta$ and $M|\delta$ projects to μ_i and is (Y, Z)-valid at μ_i .

We consider two cases. In the following if μ_i is the largest cardinal of $M|\gamma_i$ then $(\mu_i^+)^{M|\gamma_i}$ denotes $\gamma_i = \operatorname{ord}^{M|\gamma_i}$, and likewise for N_{ξ_i} .

Case 1. Either (i) $\theta_i = (\mu_i^+)^{M|\gamma_i} = (\mu_i^+)^{N_{\xi_i}}$, or (ii) $M|\theta_i = N_{\xi_i}|\theta_i$ is not (Y, Z)-valid.

Then let k = i; we have finished defining c.

Case 2. Otherwise.

Then we will have k > i; we have not yet finished defining c.

If $\theta_i < (\mu_i^+)^{M|\gamma_i}$ then let $\gamma_{i+1} < \gamma_i$ be least such that $\theta_i \leq \gamma_{i+1}$ and $\rho_\omega^{M|\gamma_{i+1}} = \mu_i$.

If $\theta_i = (\mu_i^+)^{M|\gamma_i|}$ then let $\gamma_{i+1} = \gamma_i$.

If $\theta_i < (\mu_i^+)^{N_{\xi_i}}$ then let $q \triangleleft N_{\xi_i}$ be least such that $N_{\xi_i} | \theta_i \leq q$ and $\rho_{\omega}^q = \mu_i$, and let $\xi_{i+1} < \xi_i$ be such that $\mathfrak{C}_{\omega}(N_{\xi_{i+1}}) = \mathfrak{C}_0(q)$.

If $\theta_i = (\mu_i^+)^{N_{\xi_i}}$ then let $\xi_{i+1} = \xi_i$.

Suppose Case 2 attains at stage i. Then:

- (a) $(\gamma_{i+1}, \xi_{i+1}) <_{\text{lex}} (\gamma_i, \xi_i)$.
- (b) $\theta_i \leq \gamma_{i+1} \leq \gamma_i$ and $M||\theta_i = N_{\xi_i}||\theta_i = N_{\xi_{i+1}}||\theta_i$.
- (c) Either θ_i is a cardinal of $M|\gamma_{i+1}$ or $\theta_i = \gamma_{i+1}$, and either θ_i is a cardinal of $N_{\xi_{i+1}}$ or $\theta_i = \operatorname{ord}(N_{\xi_{i+1}})$. The latter follows from the universality of standard parameters and condensation of the models of \mathbb{C} .
- (d) $\mu_i \le \mu_{i+1}$.
- (e) Suppose $\mu_i = \mu_{i+1}$. Then $\mu_{i+1} < \min(\gamma_{i+1}, \operatorname{ord}(N_{\xi_{i+1}}))$, $(\dagger)_{i+1}$ fails, $\theta_{i+1} = \theta_i$, and k = i + 1, but Case 1(ii) fails at stage i + 1. (If Case 1(ii) attained at stage i + 1 then it would in fact attain at stage i, by universality.)

There must be a stage i at which Case 1 attains, by (a) above. This defines k and c. We next define d and e.

Let $\vec{\rho}$ ($\vec{\sigma}$ resp.) be the set of all ρ such that for some i < k, $\rho = \mu_i$ and $\gamma_{i+1} < \gamma_i$ ($\xi_{i+1} < \xi_i$ resp.). For such ρ, i with $\rho \in \vec{\rho}$, let $\gamma_{\rho} = \gamma_{i+1}$ and $P_{\rho} = M | \gamma_{\rho}$ (so $\rho_{\omega}(P_{\rho}) = \rho$). For such $\sigma = \rho, i$ with $\sigma \in \vec{\sigma}$, let $\xi_{\sigma} = \xi_{i+1}$, $Q_{\sigma} = N_{\xi_{\sigma}}$

and q_{σ} be such that $\mathfrak{C}_0(q_{\sigma}) = \mathfrak{C}_{\omega}(Q_{\sigma})$. Note that $\vec{\rho} \cup \vec{\sigma} = \{\mu_0, \dots, \mu_{k-1}\}$. Finally, let $P_0 = M$, $Q_0 = N_{\Gamma}$, and let q_0 be undefined.

Let $d = \langle \gamma_{\rho}, P_{\rho} \rangle_{\rho \in \{0\} \cup \vec{\rho}}$, $e = \langle q_{\sigma}, \xi_{\sigma}, Q_{\sigma} \rangle_{\sigma \in \{0\} \cup \vec{\sigma}}$ and $\theta = \theta_k$. This completes the definition of descent.

Remark 2.11. We continue with the same notation. Let $\rho \in \vec{\rho}$ and $\sigma \in \vec{\sigma}$. We claim that $P_{\rho} \neq q_{\sigma}$. For suppose $P_{\rho} = q_{\sigma}$. Let $P = P_{\rho}$. Then $\rho = \rho_{\omega}^{P} = \sigma$. We have i < k such that $\rho = \mu_{i}$. Also, $\theta_{i} = (\rho^{+})^{P} \leq \operatorname{ord}^{P}$. But $P \triangleleft P_{i}$ and $P \triangleleft Q_{i}$. Therefore P is not (Y, Z)-valid at ρ , so $P|\theta_{i}$ is not (Y, Z)-valid. But then k = i, contradiction.

Remark 2.12. Suppose $(\dagger)_k$ fails.

We have $P_k = M|\gamma_k$ and $Q_k = N_{\xi_k}$. We have $M||\theta_k = Q_k||\theta_k$ and this model is (Y, Z)-valid. Note that either $M|\theta_k$ is active or $Q_k|\theta_k$ is active.

Suppose that $M|\theta_k = Q_k|\theta_k$. Let $P = M|\theta_k$. Then P is not (Y, Z)-valid. For otherwise, by Case 1(i), we have $(*)_{\theta_k}$, so $\mu_k \geq \theta_k$, contradiction. So P is type 3, and μ_k is a limit cardinal of P_k and of Q_k . Moreover, we claim that $P = N_{\xi}$ for some ξ . For suppose $P \triangleleft Q_k$, and let ξ be such that $\mathfrak{C}_{\omega}(N_{\xi}) = \mathfrak{C}_0(P)$. Then $\rho_{\omega}^P = \mu_k$, and the core embedding $\mathfrak{C}_0(P) \to \mathfrak{C}_0(N_{\xi})$ is in fact the identity. So if $P \neq N_{\xi}$ then by the initial segment condition, $P \in N_{\xi}$, but then by universality, $P \in P$, contradiction.

Now suppose that Case 1(i) attains at stage k. Then if $M|\theta_k$ is active then $\theta_k = \gamma_k$, and if $Q_k|\theta_k$ is active then $\theta_k = \operatorname{ord}(Q_k)$.

So in any case, $Q_k|\theta_k = N_{\xi}$ for some ξ . (If $Q_k|\theta_k$ is passive then this is because μ_k is a cardinal of $Q_k = N_{\xi_k}$ and $Q_k|\theta_k$ is a limit of levels projecting to μ_k .)

We now proceed with the construction. Let B be the neat code for $(\mathcal{T}^{\alpha}, \mathcal{U} \upharpoonright \alpha + 1)$. Consider the (A, B)-descent of $(M, (R, \mathbb{C})) = (M^{\alpha}, (R^{\alpha}, \mathbb{C}^{\alpha}))$; we use notation as in 2.10.

Suppose that $(\dagger)_k$ fails. Then the comparison has not yet succeeded. We will specify $E_{\alpha}^{\mathcal{T}^{\alpha+1}}$ and $E_{\alpha}^{\mathcal{U}}$. Exactly one of these extenders will be non-empty, with $E_{\alpha}^{\mathcal{U}} \neq \emptyset$ if it's reasonable. This helps to organize the analysis. ¹⁰

Let $E = F^{M|\theta_k}$ and $F = F^{N_{\xi_k}|\theta_k}$. If $E \neq \emptyset = F$ then set $E_{\alpha}^{\mathcal{T}^{\alpha+1}} = E$ and $E_{\alpha}^{\mathcal{U}} = \emptyset$. Otherwise $F \neq \emptyset$; in this case set $E_{\alpha}^{\mathcal{T}^{\alpha+1}} = \emptyset$ (even if $E \neq \emptyset$), let ξ be such that $N_{\xi_k}|\theta_k = N_{\xi}$ (see 2.12) and set $E_{\alpha}^{\mathcal{U}}$ to be either $E^* = (E_{\xi}^*)^{\mathbb{C}}$ or $F^* = (F_{\xi}^*)^{\mathbb{C}}$, depending on what properties we want for \mathcal{U} . For 2.9(c)(i) we use F^* ; for 2.9(c)(ii) we use E^* . In all cases also define $F_{\alpha}^{\mathcal{U}} = F$.

If $E_{\alpha}^{\mathcal{U}} \neq \emptyset$ then set $\mathcal{U}-\operatorname{pred}(\alpha+1)$ to be the least $\gamma \leq \alpha$ such that $E_{\gamma}^{\mathcal{U}} \neq \emptyset$ and for all $\delta \in [\gamma, \alpha)$, $\operatorname{crit}(E_{\alpha}^{\mathcal{U}}) < \rho(E_{\delta}^{\mathcal{U}})$ and $\operatorname{crit}(E_{\alpha}^{\mathcal{U}}) \leq \nu(F_{\delta}^{\mathcal{U}})$. Note that if we are following the prescription for 2.9(c)(i), then we always have $\rho(E_{\delta}^{\mathcal{U}}) \leq \nu(F_{\delta}^{\mathcal{U}})$, so \mathcal{U} will be non-overlapping. If $R \models$ " \mathbb{C} is normal" and we are following

We might have organized the comparison such that if both $M|\theta_k$ and $N_{\xi_k}|\theta_k$ are active, then $E_{\alpha}^{\mathcal{T}^{\alpha+1}} = F^{M|\theta_k}$ and $E_{\alpha}^{\mathcal{U}} = E_{\xi_k}^*$ (or $E_{\alpha}^{\mathcal{U}} = F_{\xi_k}^*$). However, then we may get $M|\theta_k \leq i_{E_{\alpha}^{\mathcal{U}}}(N_{\xi_k})$. If this occurs and $M|\theta_k \neq N_{\xi_k}|\theta_k$ we would want to retract our use of $F^{M|\theta_k}$ when defining $\mathcal{T}^{\alpha+2}$. This is one motivation to wait longer before using an extender in $\vec{\mathcal{T}}$.

the prescription for 2.9(c)(ii), then $E^* = E^{\mathcal{U}}_{\alpha}$ is such that $E^* \in \mathcal{H}_{|V_{\rho(E^*)}|+1}$, and for any $\beta \leq \alpha$, if $\operatorname{crit}(E^{\mathcal{U}}_{\alpha}) < \rho(E^{\mathcal{U}}_{\delta})$ for all $\delta \in [\beta, \alpha)$, we automatically have $\operatorname{crit}(E^{\mathcal{U}}_{\alpha}) \leq \nu(F^{\mathcal{U}}_{\beta})$. So in this case, \mathcal{U} will be normalizable.¹¹

If $E_{\alpha}^{\mathcal{T}^{\alpha+1}} \neq \emptyset$ then we set $\mathcal{T}^{\alpha+1} = \mathcal{T}^{\alpha} \setminus \left\langle E_{\alpha}^{\mathcal{T}^{\alpha+1}} \right\rangle$; normality and m-maximality determine the remaining structure of $\mathcal{T}^{\alpha+1}$.

Suppose $E^{\mathcal{U}}_{\alpha} \neq \emptyset$, so $E^{\mathcal{T}^{\alpha+1}}_{\alpha} = \emptyset$. Suppose there is $\gamma + 1 < \text{lh}(\mathcal{T}^{\alpha})$ such that $E^{\mathcal{T}^{\alpha}}_{\gamma} \neq \emptyset$, and $M^{\alpha}||\text{lh}(E^{\mathcal{T}^{\alpha}}_{\gamma}) \not \geq i \frac{M^{\mathcal{U}}_{\alpha}}{E^{\mathcal{U}}_{\alpha}}(N_{\xi_{k}})$. Let γ be the least such. We set $\mathcal{T}^{\alpha+1} = (\mathcal{T}^{\alpha} \upharpoonright (\gamma+1)) \upharpoonright \mathcal{P}$, where $\mathcal{P} = \langle \emptyset, \emptyset, \ldots \rangle$ is only padding, such that $\text{lh}(\mathcal{T}^{\alpha+1}) = \alpha + 2$. If there is no such $\gamma + 1$ we set $\mathcal{T}^{\alpha+1} = \mathcal{T}^{\alpha} \upharpoonright \langle \emptyset \rangle$.

This completes stage $\alpha + 1$ of the comparison, given that $(\dagger)_k$ fails.

Remark 2.13. Suppose now that $(\dagger)_k$ holds. We set $\zeta = \alpha$, and claim that the comparison has succeeded, i.e. that $\mathcal{T} = \mathcal{T}^{\zeta}$ and $\mathcal{U} = \mathcal{U} \upharpoonright \zeta + 1$ satisfy (a')-(c'). We have either $M|\gamma_k \leq N_{\xi_k}$ or $N_{\xi_k} \leq M|\gamma_k$. First observe that either $\gamma_k = \gamma_0 = \operatorname{ord}^M$ or $\xi_k = \xi_0 = \Gamma$. Suppose not, so k > 0 and $\gamma_k < \operatorname{ord}^M$ and $\xi_k < \Gamma$. By 2.10(e), $\mu_{k-1} < \mu_k$, so $\rho_{\omega}(M|\gamma_k) < \mu_k$ and $\rho_{\omega}(N_{\xi_k}) < \mu_k$. But μ_k is a cardinal of both models. Therefore $M|\gamma_k = N_{\xi_k}$, so $\mu = \rho_{\omega}(M|\gamma_k) \in \vec{\rho} \cap \vec{\sigma}$ and $M|\gamma_k = P_{\mu} = q_{\mu}$, contradicting 2.11.

If $M|\gamma_k \triangleleft N_{\xi_k}$ then $\mu_k = \gamma_k$ (by $(\dagger)_k$) so $M|\gamma_k$ is a cardinal proper segment of N_{ξ_k} . This gives that $M|\gamma_k = N_{\xi}$ for some $\xi < \xi_k$, and $\rho_{\omega}^{M|\gamma_k} = \gamma_k = \gamma_0$, so in fact $M = N_{\xi}$, and $b^{\mathcal{T}}$ does not drop in model or degree. This completes the proof in this case. So assume $N_{\xi_k} \trianglelefteq M|\gamma_k$. If $\xi_k = \xi_0$ we are done, and this follows if $N_{\xi_k} \triangleleft M|\gamma_k$, as in the previous case. So we are left with the case that $\xi_k < \xi_0$ and $N_{\xi_k} = M|\gamma_k = M$. We must prove that $b^{\mathcal{T}}$ does not drop in model. We will do this later, because to do so, and to prove that the comparison terminates, we first need to analyse the comparison in detail.

This completes stage $\alpha + 1$ of the comparison.

Given $\langle \mathcal{T}^{\alpha} \rangle_{\alpha < \eta}$, η a limit, let $\mathcal{T}^{<\eta} = \lim_{\alpha \to \eta} \mathcal{T}^{\alpha}$. That is, $\operatorname{lh}(\mathcal{T}^{<\eta}) = \eta$ and for all $\gamma < \eta$, $E_{\gamma}^{\mathcal{T}^{<\eta}} = \lim_{\alpha \to \eta} E_{\gamma}^{\mathcal{T}^{\alpha}}$. (Note that the sequence $\left\langle E_{\gamma}^{\mathcal{T}^{\beta}} \right\rangle_{\beta \in [\gamma+1,\eta)}$ is of the form $\mathcal{E} \cap \mathcal{P}$, where $\mathcal{E} = \langle E, E, \ldots \rangle$ is constant with $E \neq \emptyset$ (possibly $\operatorname{lh}(\mathcal{E}) = 0$), and $\mathcal{P} = \langle \emptyset, \emptyset, \ldots \rangle$ (possibly $\operatorname{lh}(\mathcal{P}) = 0$).) We may have that $\mathcal{T}^{<\eta}$ is eventually only padding, but note that in this case, $\mathcal{U} \upharpoonright \eta$ is cofinally non-padded. Finally, let $\mathcal{T}^{\eta} = \mathcal{T}^{<\eta} \cap \Sigma_{\mathcal{M}}(\mathcal{T}^{<\eta})$ and $\mathcal{U} \upharpoonright \eta + 1 = \mathcal{U} \upharpoonright \eta \cap \Sigma_{S}(\mathcal{U} \upharpoonright \eta)$.

This completes the definition of the comparison.

We now work toward a proof that the comparison succeeds. For this we need to establish some agreement conditions, by induction through $\mathrm{lh}(\vec{\mathcal{T}},\mathcal{U})$. First we establish some notation.

Fix $\alpha < \text{lh}(\tilde{\mathcal{T}}, \mathcal{U})$. With notation as in the definition of the descent of $(M^{\alpha}, (R^{\alpha}, \mathbb{C}^{\alpha}))$, let $c^{\alpha} = c$, $\gamma_i^{\alpha} = \gamma_i$, $(\dagger)_i^{\alpha} = (\dagger)_i$, etc. Also let $\gamma_i^{\alpha}, (\dagger)_i^{\alpha}$ denote $\gamma_{k_{\alpha}}^{\alpha}, (\dagger)_{k_{\alpha}}^{\alpha}$, etc. If $(\dagger)^{\alpha}$ fails and the stage α descent terminates through Case

¹¹ If $R \models$ "C is not normal" and we are aiming for 2.9(c)(ii), then the clause "and $\operatorname{crit}(E^{\mathcal{U}}_{\alpha}) \leq \nu(F^{\mathcal{U}}_{\delta})$ " in the definition of $\mathcal{U}-\operatorname{pred}(\alpha+1)$ might prevent \mathcal{U} from being nonoverlapping, but it is needed for our proof to work.

1(ii), let P^{α}_* denote $P^{\alpha}|\theta^{\alpha}$. Otherwise let P^{α}_* denote P^{α} . Define Q^{α}_* similarly, and also let ξ^{α}_* be the ξ such that $Q^{\alpha}_* = N^{\alpha}_{\xi}$. So if $(\dagger)^{\alpha}$ fails then $\mu^{\alpha} < \theta^{\alpha} = ((\mu^{\alpha})^+)^{P^{\alpha}_*} = ((\mu^{\alpha})^+)^{Q^{\alpha}_*}$, and $P^{\alpha}_*||\theta^{\alpha} = Q^{\alpha}_*||\theta^{\alpha}$. Let λ^{α} be the largest $\lambda \leq \mu^{\alpha}$ such that λ is a limit of cardinals of P^{α} (equivalently, of Q^{α} , P^{α}_* , or Q^{α}_*).

Let $\eta < \text{lh}(\vec{\mathcal{T}}, \mathcal{U})$ be a limit. When $\mathcal{T}^{<\eta}$ is cofinally non-padded, let $M(\vec{\mathcal{T}} \upharpoonright \eta)$ denote $M(\mathcal{T}^{<\eta})$; otherwise, let $M(\vec{\mathcal{T}} \upharpoonright \eta)$ denote $M_{\eta}^{\mathcal{T}^{\eta}} \upharpoonright \delta$, where $\delta = \delta(\mathcal{U} \upharpoonright \eta)$. These coincide when $\mathcal{T}^{<\eta}$ and $\mathcal{U} \upharpoonright \eta$ are both cofinally non-padded.)

If the comparison runs to stage $v^+ + 1$, then we stop it there, producing $\mathcal{T}^{v^+}, \mathcal{U} \upharpoonright (v^+ + 1)$; in this case set $\zeta = v^+$. Otherwise we stop at the first stage $\zeta + 1 < v^+$ such that $(\dagger)^{\zeta}$ holds, producing final trees $\mathcal{T}^{\zeta}, \mathcal{U} \upharpoonright (\zeta + 1)$.

Before beginning the analysis we make a couple more observations.

Remark 2.14. Suppose $E_{\varepsilon}^{\mathcal{T}^{\varepsilon+1}} \neq \emptyset$. We have $\ln(E_{\varepsilon}^{\mathcal{T}^{\varepsilon+1}}) = \theta^{\varepsilon}$, and the stage ε descent does not terminate through Case 1(ii). Therefore $\theta^{\varepsilon} = \gamma^{\varepsilon}$. Let $N = M^{\varepsilon}$. Then $\theta^{\varepsilon} = \operatorname{ord}^{N}$ iff $\bar{\rho}^{\varepsilon} = \emptyset$. Suppose $\bar{\rho}^{\varepsilon} \neq \emptyset$, so $k_{\varepsilon} > 0$. Let $\bar{\rho}^{\varepsilon} = \{\mu_{i_{0}}^{\varepsilon} < \ldots < \mu_{i_{n}}^{\varepsilon}\}$, with $i_{n} < k_{\varepsilon}$. (If $\mu^{\varepsilon} \in \bar{\rho}^{\varepsilon}$ then $\mu^{\varepsilon} = \mu_{k_{\varepsilon}-1}^{\varepsilon}$.) Then $\langle \lambda_{0}, \ldots, \lambda_{n} \rangle = \langle \gamma_{i_{n}+1}^{\varepsilon}, \ldots, \gamma_{i_{0}+1}^{\varepsilon} \rangle$ is the $\gamma_{i_{n}+1}^{\varepsilon}$ -model-dropdown sequence for N below ord^{N} . That is, $\lambda_{0} = \gamma_{i_{n}+1}^{\varepsilon}$ and λ_{i+1} is the least $\lambda > \lambda_{i}$ such that $\lambda < \operatorname{ord}^{N}$ and $\rho_{\omega}^{N|\lambda} < \rho_{\omega}^{N|\lambda_{i}}$, with n as large as possible. Moreover, $\mu_{i_{j}}^{\varepsilon} = \rho_{\omega}(N|\gamma_{i_{j}+1}^{\varepsilon})$ for each $j \leq n$.

Similar remarks apply when $E_{\varepsilon}^{\mathcal{U}} \neq \emptyset$, but things can be a little different, as it is possible for the stage ε descent to terminate through Case 1(ii).

Remark 2.15. We will prove that \mathcal{U} does not move fine structural generators. That is, if $\alpha+1<_{\mathcal{U}}\beta+1$ then $\nu(F_{\alpha}^{\mathcal{U}})\leq \mathrm{crit}(E_{\beta}^{\mathcal{U}})$. The proof of this depends on other properties of \mathcal{U} , to be established inductively, by Claim 1 below. However, if $R\models \text{``C'}$ is strongly reasonable" then we can prove the fact right now; the more general case is an elaboration of this argument. Suppose otherwise. For simplicity, we assume \mathcal{U} has no padding. Let $\beta+1<\mathrm{lh}(\mathcal{U})$ be least such that for some $\alpha+1<_{\mathcal{U}}\beta+1$, we have $\kappa=\mathrm{crit}(E_{\beta}^{\mathcal{U}})<\nu(F_{\alpha}^{\mathcal{U}})$. Let $\gamma=\mathcal{U}-\mathrm{pred}(\beta+1)$. We claim that γ is a successor. For otherwise we have $\alpha+1$ as above with $\alpha+1<_{\mathcal{U}}\gamma$. By minimality of $\beta+1$, $\nu(F_{\alpha}^{\mathcal{U}})\leq \mathrm{crit}(F_{\beta}^{\mathcal{U}})$ for all $\beta'+1\in(\alpha+1,\gamma)_{\mathcal{U}}$. This implies $\nu(F_{\alpha}^{\mathcal{U}})<\rho(E_{\delta}^{\mathcal{U}})$ and $\nu(F_{\alpha}^{\mathcal{U}})\leq\nu(F_{\delta}^{\mathcal{U}})$ for all $\delta\in[\alpha+1,\gamma)$. But then $\mathcal{U}-\mathrm{pred}(\beta+1)<\gamma$, contradiction. So let $\alpha+1=\gamma$. By minimality of $\beta+1$, we have $\kappa<\nu(F_{\alpha}^{\mathcal{U}})$. Since $\mathcal{U}-\mathrm{pred}(\beta+1)>\alpha$, $\rho(E_{\alpha}^{\mathcal{U}})\leq\kappa$. We claim that (*) $Q_*^{\alpha}\models "\kappa$ is inaccessible". But then since $M_{\alpha}^{\mathcal{U}}\models "\mathbb{C}^{\alpha}$ is strongly reasonable", $\kappa<\rho(E_{\alpha}^{\mathcal{U}})$, contradiction.

So we prove (*). We have $\kappa < \rho(E^{\mathcal{U}}_{\delta})$ for all $\delta \in [\alpha+1,\beta)$, so κ is measurable in $M^{\mathcal{U}}_{\alpha+1}$, and therefore inaccessible in $i^{\mathcal{U}}_{\varepsilon,\alpha+1}(Q^{\alpha}_{*}|\mu) = i^{M^{\mathcal{U}}_{\alpha}}_{E^{\mathcal{U}}_{\alpha}}(Q^{\alpha}_{*}|\mu)$, where $\mu = \mathrm{crit}(E^{\mathcal{U}}_{\alpha})$ and $\varepsilon = \mathcal{U}\mathrm{-pred}(\alpha+1)$. Moreover, μ is a cardinal of Q^{α}_{*} , so κ is inaccessible in $U' = i^{M^{\mathcal{U}}_{\alpha}}_{E^{\mathcal{U}}_{\alpha}}(Q^{\alpha}_{*})$. But therefore κ is inaccessible in $U = \mathrm{Ult}_{0}(Q^{\alpha}_{*}, F^{\mathcal{U}}_{\alpha})$, since $\kappa < \nu(F^{\mathcal{U}}_{\alpha})$ and using the factor embedding $j: U \to U'$ (see 2.3). Therefore κ is inaccessible in Q^{α}_{*} , as required.

 $^{^{12}}$ This might involve a slight abuse of notation, as δ need not be determined by $\vec{\mathcal{T}} \upharpoonright \eta$ alone.

The following claim lists various facts about the comparison, particularly how different stages are related. Most of our work will be in giving its statement and proof. It is proved by induction on ι . Probably the most central fact is (2).

Claim 1. For all $\iota < \zeta + 1 = lh(\vec{\mathcal{T}}, \mathcal{U})$:

- (1) Suppose $\eta \leq \alpha \leq \iota$, η is a limit, and either α is a limit or $\alpha < \iota$. Then $\mathcal{T}^{\eta} = \mathcal{T}^{\alpha} \upharpoonright \eta + 1$ and $(\mathcal{T}^{\alpha}, \mathcal{U} \upharpoonright \alpha + 1)$ is neat. Let B^{α} denote the neat code for $(\mathcal{T}^{\alpha}, \mathcal{U} \upharpoonright \alpha + 1)$. Then $B^{\alpha} \subseteq (\lambda^{\alpha} + 1)^3$. For all $\alpha_1 < \alpha_2 < \iota$ we have $B^{\alpha_1} = B^{\alpha_2} \cap (\lambda^{\alpha_1} + 1)^3.$
- (2) Let $\alpha, \beta < \iota$, let $\rho \in \vec{\rho}^{\alpha}$ and $\sigma \in \vec{\sigma}^{\beta}$. Then $P_{\rho}^{\alpha} \neq q_{\sigma}^{\beta}$.
- (3) Let $\alpha \leq \beta < \iota$. Then:
 - (i) Suppose $\alpha < \beta$. Then $\lambda^{\alpha} \leq \lambda^{\beta}$, the models P^{α} , P^{β} , Q^{α} , Q^{β} agree strictly below $((\lambda^{\alpha})^{+})^{P_{*}^{\alpha}} = ((\lambda^{\alpha})^{+})^{Q_{*}^{\alpha}}$, and λ^{α} is a limit cardinal of each of these models.
 - (ii) \mathcal{T}^{α} agrees with \mathcal{T}^{β} in terms of use and indexing of extenders E such that $lh(E) < \lambda^{\alpha}$. That is, $\mathcal{T}^{\alpha} \upharpoonright \gamma + 1 = \mathcal{T}^{\beta} \upharpoonright \gamma + 1$ where γ is least such that either $\gamma = \alpha$ or $E_{\gamma}^{\mathcal{T}^{\alpha}} \neq \emptyset$ and $\lambda^{\alpha} < \text{lh}(E_{\gamma}^{\mathcal{T}^{\alpha}})$; and if $\gamma \in [\alpha, \beta)$ and
 $$\begin{split} E_{\gamma}^{\mathcal{T}^{\beta}} \neq \emptyset \text{ then } \lambda^{\alpha} < \text{lh}(E_{\gamma}^{\mathcal{T}^{\beta}}). \\ \text{(iii) } \text{If } \alpha + \omega \leq \iota \text{ then there is } n < \omega \text{ such that } \lambda^{\alpha} < \lambda^{\alpha+n}. \end{split}$$
- (4) Suppose $\alpha < \beta < \iota$ and $E = E_{\alpha}^{\mathcal{T}^{\beta}} \neq \emptyset$. Then: (i) $\vec{\mu}^{\alpha+1} \cap \text{lh}(E) = \vec{\sigma}^{\alpha+1} \cap \text{lh}(E)$ and $e^{\alpha+1} \upharpoonright \text{lh}(E) = e^{\alpha}$
 - (ii) $lh(E) < \mu^{\beta}$
 - (iii) $\vec{\rho}^{\beta} \cap lh(E) = \emptyset$
 - (iv) $M^{\alpha}||\mathrm{lh}(E) \triangleleft P_*^{\beta}$ and $M^{\alpha}||\mathrm{lh}(E) \trianglelefteq Q_*^{\beta}$ and $\mathrm{lh}(E)$ is a cardinal of P_*^{β} . If $\begin{array}{l} \operatorname{lh}(E) = \operatorname{ord}(Q_*^\beta) \text{ then } \beta = \alpha + 1 \text{ and } (\dagger)^\beta. \\ (\text{v}) \text{ If } F_\beta^\mathcal{U} \neq \emptyset \text{ then } E \upharpoonright \nu(E) \not\subseteq F_\beta^\mathcal{U}. \end{array}$
- (5) Suppose $\alpha < \beta < \beta + 1 < \iota$ and $E = E_{\alpha}^{\mathcal{T}^{\beta}} \neq \emptyset$. If E is retracted at stage $\beta + 1$, i.e. $E_{\alpha}^{\mathcal{T}^{\beta+1}} = \emptyset$, then E is the last non-empty extender used in \mathcal{T}^{β} , $lh(E) = \mu^{\beta} = \nu(F_{\beta}^{\mathcal{U}}), \ \lambda^{\alpha} = \lambda^{\beta} \text{ and } \beta < \alpha + \omega.$
- (6) Suppose $\alpha <_{\mathcal{U}} \beta < \iota$ and $\kappa = \operatorname{crit}(i_{\alpha,\beta}^{\mathcal{U}}) < \infty$. Then $e^{\beta} \upharpoonright \kappa = i_{\alpha,\beta}^{\mathcal{U}}(e^{\alpha} \upharpoonright \kappa)$,
- and for all $\rho \in \vec{\sigma}^{\alpha} \cap \kappa$, $q_{\rho}^{\beta} = q_{\rho}^{\alpha}$. (7) Suppose $\chi + 1 < \iota$ and $E_{\chi}^{\mathcal{U}} \neq \emptyset$. Let $\alpha = \mathcal{U} \operatorname{pred}(\chi + 1)$, $\kappa = \operatorname{crit}(E_{\chi}^{\mathcal{U}})$ and $\rho = \max(\{0\} \cup (\vec{\sigma}^{\alpha} \cap \kappa))$. Then:
 - (i) $\kappa < \operatorname{ord}(Q_{\rho}^{\alpha}),$
 - (ii) $\kappa \leq \lambda^{\alpha}$ and Q^{α} , Q^{α}_{ρ} , Q^{χ} agree through κ , which is a limit cardinal of
 - (iii) $E_{\chi}^{\mathcal{U}}$ does not move fine structural generators. That is, given any $\gamma+1<_{\mathcal{U}}$

 $\chi + 1$ such that $E_{\gamma}^{\mathcal{U}} \neq \emptyset$, we have $\nu(F_{\gamma}^{\mathcal{U}}) \leq \operatorname{crit}(E_{\chi}^{\mathcal{U}})$. Let μ' be the largest cardinal μ'' of $N = i \frac{M_{\chi}^{\mathcal{U}}}{E_{\chi}^{\mathcal{U}}}(Q_{*}^{\chi})$ such that $\mu'' \leq \mu^{\chi}$. Let $\theta' = ((\mu')^+)^{Q_*^{\chi}}$. Then:

- (iv) $N||\theta' = Q^{\chi}||\theta'$.
- (v) Suppose $\mu' < \mu^{\chi}$. Then $F_{\chi}^{\mathcal{U}}$ is type 1 or type 3 and $\theta' = \mu^{\chi}$.

- (vi) $Q_{\rho}^{\chi+1}$ and N agree through $i_{\alpha,\chi+1}^{\mathcal{U}}(\kappa)=i_{E_{\chi}^{\mathcal{U}}}^{M_{\chi}^{\mathcal{U}}}(\kappa)$, a cardinal of these
- (vii) $\lambda^{\chi} \leq \mu' \leq \mu^{\chi+1}$ and μ' is a cardinal of $Q_{\rho}^{\chi+1}$ and of $Q^{\chi+1}$,
- (viii) $\mu' \leq \min(\vec{\sigma}^{\chi+1} \cap [\kappa, \infty)),$ (ix) The models $Q_*^{\chi+1}$, $Q^{\chi+1}$, $Q_\rho^{\chi+1}$, Q^χ agree strictly below θ' .
- (8) Suppose $\chi+1 < \iota$ and $E_{\chi}^{\mathcal{U}} \neq \emptyset$. We use the notation of (7) and let $\mu = \mu^{\chi}$. If there is no retraction at stage $\chi+1$, i.e. if $\mathcal{T}^{\chi+1}=\mathcal{T}^{\chi} \cap \langle \emptyset \rangle$, then let $\beta=\chi$. If there is retraction, let $\beta = \gamma$, where $E_{\gamma}^{\mathcal{T}^{\chi}}$ is the retracted extender. So
 - (i) $\vec{\rho}^{\chi+1} \cap \mu' = \vec{\rho}^{\beta} \cap \mu'$ and $P_{\rho'}^{\chi+1} = P_{\rho'}^{\beta}$ for all $\rho' \in \{0\} \cup (\vec{\rho}^{\chi+1} \cap \mu')$.
 - (ii) Let $\rho' = \max(\{0\} \cup (\vec{\rho}^{\chi+1} \cap \mu'))$. Then μ' is a cardinal of $P_{\sigma'}^{\chi+1}$.
- (9) Suppose $\eta \leq \iota$ is a limit such that $\mathcal{U} \upharpoonright \eta$ is eventually only padding. So $M_{\eta}^{\mathcal{U}} = \lim_{\beta < \eta} M_{\beta}^{\mathcal{U}}$. Let $\delta = \delta(\vec{\mathcal{T}} \upharpoonright \eta)$. Let (c, d, e, θ) be the (A, B^{η}) -descent of $(M(\vec{\mathcal{T}} \upharpoonright \eta), (M_{\eta}^{\mathcal{U}}, \mathbb{C}^{\eta}))$. Let $\delta = \delta(\vec{\mathcal{T}} \upharpoonright \eta)$. Then:
 - (i) If $\gamma \leq \beta < \eta$ are such that $\mathcal{U} \upharpoonright \eta = \mathcal{U} \upharpoonright (\gamma + 1) \ \mathcal{P}$, then $e^{\gamma} \subseteq e^{\beta}$.
 - (ii) $e = e^{\alpha}$ for all sufficiently large $\alpha < \eta$. Let $\alpha < \eta$ be such that $\mathcal{U} \upharpoonright \eta =$ $\mathcal{U} \upharpoonright (\alpha + 1) \ \mathcal{P}$ and $e^{\alpha} = e$. Let $\rho = \max(\{0\} \cup \vec{\sigma}^{\alpha})$. Then $\mathcal{T}^{<\eta} \upharpoonright [\alpha, \eta)$ is given by the standard comparison of the phalanx $\Phi(\mathcal{T}^{\alpha})$ with $Q_{\rho}^{\alpha} = Q_{\rho}^{e}$.
 - (iii) δ is a limit of cardinals of Q_{ρ}^{e} .
- (10) Assume the hypotheses and notation of (9) and also that $\eta < \iota$. Then:
 - (i) $e^{\eta} \upharpoonright \delta = e$, so $Q^{\eta}_{\rho} = Q^{e}_{\rho}$;
 - (ii) $\vec{\rho}^{\eta} \cap \delta = \emptyset$;
 - (iii) δ is a limit of cardinals of M^{η} , Q^{η}_{ρ} , Q^{η} , these models agree through δ , and $\delta \leq \lambda^{\eta}$;
 - (iv) If $\delta = \min(\operatorname{ord}(Q_{\rho}^{\alpha}), \operatorname{ord}(M^{\eta}))$ then $(\dagger)^{\eta}$.
- (11) Suppose $\eta \leq \iota$ is a limit and $\mathcal{U} \upharpoonright \eta$ is cofinally non-padded. Let

$$\vec{\sigma}^{<\eta} = \bigcup_{\xi < \iota \eta} \vec{\sigma}^{\xi} \cap \operatorname{crit}(i^{\mathcal{U}}_{\xi,\eta}). \tag{1}$$

Then there is $\xi <_{\mathcal{U}} \eta$ such that $\vec{\sigma}^{<\eta} = \vec{\sigma}^{\xi} \cap \operatorname{crit}(i_{\xi,\eta}^{\mathcal{U}})$. Let ξ be such, let $\rho = \max(\{0\} \cup \vec{\sigma}^{<\eta})$ and $\delta = \delta(\mathcal{U} \upharpoonright \eta)$. Then $M(\vec{\mathcal{T}} \upharpoonright \eta) = i_{\xi,\eta}^{\mathcal{U}}(Q_{\rho}^{\xi}) | \delta$ and δ is a limit of cardinals of $i_{\xi,\eta}^{\mathcal{U}}(Q_{\rho}^{\xi})$.

- (12) Assume the hypotheses of (11) and also that $\eta < \iota$. Fix δ, ξ, ρ as there.

 - $\begin{array}{ll} \text{(i)} \ \ \vec{\sigma}^{\eta} \cap \delta = \vec{\sigma}^{<\eta} \ \ \text{and} \ \ e^{\eta} \upharpoonright \delta = i^{\mathcal{U}}_{\xi,\eta}(e^{\xi} \upharpoonright \kappa) \ \ \text{where} \ \kappa = \mathrm{crit}(i^{\mathcal{U}}_{\xi,\eta}). \\ \text{(ii)} \ \ \delta \leq \mathsf{ord}(Q^{\eta}_{\rho}), \ \delta \ \ \text{is a limit of cardinals of} \ \ Q^{\eta}_{\rho} \ \ \text{and} \ \ \delta \leq \lambda^{\eta}. \ \ \text{If} \ \ \delta = \mathsf{ord}(Q^{\eta}_{\rho}) \end{array}$ then $(\dagger)^{\eta}$.
 - (iii) $Q^{\eta} | \delta = Q^{\eta}_{\rho} | \delta = M(\tilde{\mathcal{T}} | \eta) = \liminf_{\gamma < \eta} Q^{\gamma}.$
 - (iv) Suppose $\mathcal{T}^{<\eta}$ is cofinally non-padded. Then δ is a limit of cardinals of M^{η} , and $\vec{\rho}^{\eta} \cap \delta = \emptyset$. If $\delta = \operatorname{ord}(M^{\eta})$ then $(\dagger)^{\eta}$.
 - (v) Suppose $\mathcal{T}^{<\eta}$ is eventually only padding. Then there is $\gamma < \eta$ such that: $\mathcal{T}^{<\eta}=\mathcal{T}^{\gamma} \cap \mathcal{P}$ (so $M^{\eta}=M^{\gamma}$), $d^{\gamma} \upharpoonright \lambda^{\gamma}=d^{\eta} \upharpoonright \delta$, and letting $\rho' = \max(\{0\} \cup (\vec{\rho}^{\gamma} \cap \lambda^{\gamma})), \, \delta \text{ is a limit of cardinals of } P_{\rho'}^{\gamma} = P_{\rho'}^{\eta} \text{ and of } P_{\rho'}^{\gamma} = P_{\rho'}^{\eta}$ P^{η} . Moreover, $M(\vec{\mathcal{T}} \upharpoonright \eta) \leq P^{\eta}_{\rho'}$. If $\delta = \gamma^{\eta}_{\rho'}$ then $(\dagger)^{\eta}$.

Proof. We proceed by induction on ι . We write, for example, "(3i)($\iota < 5$)" for (3i) for values of $\iota < 5$.

Case 1. $\iota \leq 1$

For $\iota = 0$ the claim is trivial. For $\iota = 1$ the only non-trivial item is (2), which follows 2.11.

Case 2. $\iota = \chi + 2$

We must prove (1) for $\alpha = \chi + 1$, (2) for $\max(\alpha, \beta) = \chi + 1$, (3),(4),(6) for $\beta = \chi + 1$, (5) for $\beta + 1 = \chi + 1$, and (7),(8) for $\chi + 1$.

(7i),(7ii): We have that κ is a limit cardinal of $M_{\alpha}^{\mathcal{U}}$ and $Q^{\alpha}, Q_{\rho}^{\alpha} \in M_{\alpha}^{\mathcal{U}}$, and $\kappa \leq \nu(F_{\alpha}^{\mathcal{U}}) < \operatorname{ord}(Q_{*}^{\alpha})$, so $\kappa \leq \lambda^{\alpha}$, and by choice of ρ , $Q^{\alpha}|\kappa = Q_{\rho}^{\alpha}|\kappa$ and $\kappa < \operatorname{ord}(Q_{\rho}^{\alpha})$. Applying (3i) to α, χ , we get that $Q^{\alpha}|\kappa = Q^{\chi}|\kappa$ and κ is a limit of cardinals of Q^{χ} ; in fact $\kappa < \operatorname{ord}(Q_{*}^{\chi})$ since $\kappa = \operatorname{crit}(F_{\chi}^{\mathcal{U}})$.

(7iii): Let $\gamma + 1 \leq_{\mathcal{U}} \alpha$ with $E_{\gamma}^{\mathcal{U}} \neq \emptyset$. Suppose that $\kappa = \operatorname{crit}(E_{\chi}^{\mathcal{U}}) < \nu(F_{\gamma}^{\mathcal{U}})$. By (7iii)($\iota = \chi + 1$) and part of 2.15, we may assume $\gamma + 1 = \alpha$, and $\rho(E_{\gamma}^{\mathcal{U}}) \leq \kappa$. This will lead to a contradiction with the reasonableness of \mathbb{C}^{γ} in $M_{\gamma}^{\mathcal{U}}$.

We need to establish the hypotheses on κ given in 2.1(f). We will first establish the appropriate facts about $U = \text{Ult}(M_{\gamma}^{\mathcal{U}}, E_{\gamma}^{\mathcal{U}})$, and then if $E_{\gamma}^{\mathcal{U}} \neq E^* = (E_{\mathcal{E}_{\gamma}}^*)^{\mathbb{C}^{\gamma}}$, deduce them about $U' = \text{Ult}(M_{\gamma}^{\mathcal{U}}, E^*)$.

As in 2.15, κ is measurable in $M_{\gamma+1}^{\mathcal{U}}$ and so in $\mathrm{Ult}(M_{\gamma}^{\mathcal{U}}, E_{\gamma}^{\mathcal{U}})$. Let $\xi < \eta = ((\lambda^{\gamma})^{+})^{Q_{*}^{\gamma}}$. By (3i) $(\iota = \chi + 1)$, $Q^{\chi}||\eta = Q^{\gamma}||\eta = Q_{*}^{\gamma}||\eta$ and λ^{γ} is a limit cardinal of Q_{*}^{χ} . If $\lambda^{\gamma} = \nu(F_{\chi}^{\mathcal{U}})$ let $F = F_{\chi}^{\mathcal{U}}$ and $E = E_{\chi}^{\mathcal{U}} \upharpoonright \lambda^{\gamma}$. If $\lambda^{\gamma} < \nu(F_{\chi}^{\mathcal{U}})$ then there is $\varsigma < \eta$ such that $\xi < \varsigma$ and $F = F_{\chi}^{\mathcal{U}} \upharpoonright \varsigma + 1$ is non-type Z. Then F and $E = E_{\chi}^{\mathcal{U}} \upharpoonright \varsigma + 1$ are both generated by $\lambda^{\gamma} \cup \{\varsigma\}$. Moreover, by the initial segment condition, there is δ with $F = F_{*}^{Q_{*}^{\chi}|\delta}$. Moreover, letting $N = Q_{*}^{\chi}|\delta$, either $N \triangleleft Q^{\gamma}||\eta$ or $N||\eta = Q^{\gamma}||\eta$.

Now we claim that $N, E \in \text{Ult}(M_{\gamma}^{\mathcal{U}}, F_{\gamma}^{\mathcal{U}})$ and E is an extender there. If $\kappa < \lambda^{\gamma}$ let i = 1; if $\kappa = \lambda^{\gamma}$ let i = 2. Then (N, E) is coded by an element of

$$V_{\lambda^{\gamma}+i}^{M_{\chi}^{\mathcal{U}}} \subseteq V_{\lambda^{\gamma}+i}^{M_{\gamma+1}^{\mathcal{U}}} = V_{\lambda^{\gamma}+i}^{\mathrm{Ult}(M_{\gamma}^{\mathcal{U}}, E_{\gamma}^{\mathcal{U}})}. \tag{2}$$

To see line (2), suppose first i=1. Then for every $\delta \in [\gamma, \chi)$, $\lambda^{\gamma} \leq \lambda^{\delta} \leq \nu(F_{\delta}^{\mathcal{U}})$, and $F_{\delta}^{\mathcal{U}} \upharpoonright \nu(F_{\delta}^{\mathcal{U}}) \subseteq E_{\delta}^{\mathcal{U}}$, so $i_{E_{\delta}^{\mathcal{U}}}(\operatorname{crit}(E_{\delta}^{\mathcal{U}})) \geq \lambda^{\gamma}$. This gives (2) in this case. If i=2 then $\lambda^{\gamma} < i_{E_{\delta}^{\mathcal{U}}}(\operatorname{crit}(E_{\delta}^{\mathcal{U}}))$ for every such δ , which suffices. Now in either case, $V_{\kappa+1}(M_{\gamma+1}^{\mathcal{U}}) = V_{\kappa+1}(M_{\chi}^{\mathcal{U}})$, so E is an extender in $M_{\gamma+1}^{\mathcal{U}}$ and in $\operatorname{Ult}(M_{\gamma}^{\mathcal{U}}, F_{\gamma}^{\mathcal{U}})$, as required.

Finally, suppose $E^* \neq E^{\mathcal{U}}_{\gamma}$. So we are following the prescription for 2.9(c)(i), and $E^{\mathcal{U}}_{\gamma} = E^* \upharpoonright \beta$ for some $\beta \geq \nu(F^{\mathcal{U}}_{\gamma})$. So we have a fully elementary $j: U \rightarrow U' = \mathrm{Ult}(M^{\mathcal{U}}_{\gamma}, E^*)$ with $\mathrm{crit}(j) \geq \beta$. So $j(\kappa) = \kappa$ and κ is measurable in U'. Moreover, fixing ξ, N, E as above, N' = j(N) and E' = j(E) witness 2.1(f) with respect to ξ .

Now since $M_{\gamma}^{\mathcal{U}} \models \text{``C}^{\gamma}$ is reasonable", we have that $\kappa < \rho(E^*)$, so $\kappa < \rho(E_{\gamma}^{\mathcal{U}})$, contradiction.

(7iv),(7v): Let $i=i_\chi^{M_\chi^U}$ and $j: \mathrm{Ult}_0(Q_*^\chi,F_\chi^U) \to N=i(Q_*^\chi)$ the factor map. Now μ^χ is the largest cardinal of Q_*^χ , so is a cardinal of $\mathrm{Ult}_0(Q_*^\chi,F_\chi^U)$. So if $\mathrm{crit}(j) \geq \mathrm{lh}(F_\chi^U)$ or μ^χ is a limit cardinal of Q_*^χ then $\mu'=\mu^\chi$, and condensation (and that $Q_*^\chi \leq Q^\chi$) gives (7iv). Suppose $\mathrm{crit}(j)=\mu^\chi$ is a successor cardinal of Q^χ . Then $Q_*^\chi = Q^\chi$ and μ^χ is not a cardinal of N, so $\mu' < \mu^\chi$ and μ' is the largest cardinal of N which is $<\mu^\chi$. Since $\mathrm{crit}(j) \geq \nu(F_\chi^U)$, F_χ^U is either type 1 or type 3. Moreover, $Q^\chi||\mathrm{ord}(Q^\chi)=U|\mathrm{ord}(Q^\chi)$ and $U|\mu^\chi=N||\mu^\chi$, so μ' is the largest cardinal of Q^χ which is $<\mu^\chi$, so $\mu^\chi=((\mu')^+)^{Q^\chi}=\theta'$.

(5): Assume $\beta+1=\chi+1$ and $\alpha<\chi$ is such that $E=E_{\alpha}^{\mathcal{T}^{\chi}}\neq\emptyset$ is retracted at stage $\chi+1$, i.e. $E_{\alpha}^{\mathcal{T}^{\chi+1}}=\emptyset$. We use (4)($\iota=\chi+1$). By (4ii), $\mathrm{lh}(E)\leq\mu^{\chi}$. By (7iv), with N as there, $M^{\chi}||\mu^{\chi}=N||\mu^{\chi}$. But $N|\mathrm{lh}(E)\neq M^{\alpha}||\mathrm{lh}(E)=M^{\chi}||\mathrm{lh}(E)$, since E is being retracted. So $\mathrm{lh}(E)=\mu^{\chi}$ and $N|\mu^{\chi}$ is active. By (7iv), $\mu'<\mu^{\chi}$, so $\mu^{\alpha}=\mu'$, and by (4ii), E is the last extender used in \mathcal{T}^{χ} . By (7v), $\mu'=\nu(F_{\chi}^{\mathcal{U}})$.

The fact that $\lambda^{\chi} = \lambda^{\alpha}$ follows by (4iv) and since $\mu^{\chi} = lh(E)$. By (3iii)($\iota = \chi$) this implies $\chi < \alpha + \omega$.

- (1): We may assume that $\alpha = \chi + 1$ and η is the largest limit $\leq \chi$. Let $\delta = \delta(\mathcal{T}^{\eta} \upharpoonright \eta)$. Then $\delta \leq \lambda^{\eta} \leq \lambda^{\chi}$ by (12ii),(10iii),(3)($\iota = \chi + 1$). So the property follows from (5).
 - (6): We may assume $\beta = \chi + 1$ and $\alpha = \mathcal{U} \text{pred}(\chi + 1)$.

It suffices to prove (*) $\vec{\sigma}^{\chi+1} \cap \kappa = \vec{\sigma}^{\alpha} \cap \kappa$ and for all $\rho \in \vec{\sigma}^{\chi+1} \cap \kappa$, $\xi_{\rho}^{\chi+1} = i_{\alpha,\chi+1}^{\mathcal{U}}(\xi_{\rho}^{\alpha})$; and letting $\gamma \leq \chi$ be such that $\mathcal{T}^{\chi+1} = \mathcal{T}^{\gamma} \cap \mathcal{P}$, we have $M^{\chi+1} = M^{\gamma}$, $\vec{\rho}^{\chi+1} \cap \kappa = \vec{\rho}^{\gamma} \cap \kappa$, and for all $\rho \in \vec{\rho}^{\chi+1} \cap \kappa$, $\gamma_{\rho}^{\chi+1} = \gamma_{\rho}^{\gamma}$.

Since $\kappa < \mu^{\chi}$, by (5) and (3i)($\iota = \chi + 1$): ignoring padding, either $\mathcal{T}^{\chi+1} = \mathcal{T}^{\chi}$ or $\mathcal{T}^{\chi+1} \cap \langle E \rangle = \mathcal{T}^{\chi}$ for some E such that $\kappa < \text{lh}(E)$; and $\kappa \leq \lambda^{\gamma}, \lambda^{\alpha}$, and $P^{\gamma}|_{\kappa} = Q^{\chi}|_{\kappa} = Q^{\alpha}|_{\kappa}$ and κ is a cardinal of P^{γ} , Q^{χ} and Q^{α} . Therefore also $M^{\chi+1}|_{\kappa} = M^{\chi}|_{\kappa} = Q^{\chi}|_{\kappa}$. Moreover, by (1), $B^{\chi+1}$, B^{χ} , B^{α} and B^{γ} have the same intersection with κ^3 , and $Q^{\chi}|_{\kappa}$ is (A, B^{χ}) -valid.

Since $\kappa < \operatorname{ord}(P^{\gamma})$, for all $\rho \in \vec{\rho}^{\gamma}$, we have $\kappa < \gamma_{\rho}^{\gamma}$. Now by $(2)(\iota = \chi + 1)$, for all $\rho \in \vec{\rho}^{\gamma} \cap \vec{\sigma}^{\alpha}$, $P_{\rho}^{\gamma} \neq q_{\rho}^{\alpha}$. This will give the claim, by induction through $(d,e)^{\chi+1} \upharpoonright \kappa$. That is, we have $P_0^{\chi+1} = P_0^{\gamma}$ and $\xi_0^{\chi+1} = i_{\alpha,\chi+1}^{\mathcal{U}}(\xi_0^{\alpha})$. Since $\kappa = \operatorname{crit}(E_{\chi}^{\mathcal{U}})$, $Q_0^{\chi+1}$ and Q_0^{α} agree below κ , and have the same cardinals below κ . Assume $\langle \mu_0, \ldots, \mu_j \rangle = (\vec{\sigma}^{\alpha} \cup \vec{\rho}^{\gamma}) \cap \kappa \neq \emptyset$; the contrary case is simpler. Note that $\mu_0 < \kappa$ is a cardinal in both $P_0^{\chi+1}$ and $P_0^{\chi+1}$. If $\mu_0 \in \vec{\rho}^{\gamma} \setminus \vec{\sigma}^{\alpha}$ then $P_0^{\chi+1}$, $P_0^{\chi} \in P_0^{\chi}$ and $P_{\mu_0}^{\gamma}$ agree beyond their common value $p_0^{\chi+1} \in P_0^{\chi}$, and $p_0^{\chi+1} \in P_0^{\chi+1}$ and $p_0^{\chi+1} \in P_0^{\chi+1}$ and $p_0^{\chi+1} \in P_0^{\chi+1} \in P_0^{\chi+1}$ and $p_0^{\chi+1} \in P_0^{\chi+1} \in P_0^{\chi+1}$, and $p_0^{\chi+1} \in P_0^{\chi+1} \in P_0^{\chi+1}$. If $p_0 \in \vec{\rho}^{\chi+1} \setminus \vec{\sigma}^{\chi+1}$ and $p_0^{\chi+1} \in P_0^{\chi+1} \in P_0^{\chi+1}$, and $p_0^{\chi+1} \in P_0^{\chi+1} \in P_0^{\chi+1}$. If $p_0 \in \vec{\rho}^{\chi+1} \in P_0^{\chi+1}$ and $p_0^{\chi+1} \in P_0^{\chi+1} \in P_0^{\chi+1}$. If $p_0 \in \vec{\rho}^{\chi+1} \in P_0^{\chi+1}$ and $p_0^{\chi+1} \in P_0^{\chi+1}$, we cardinal of $p_0^{\chi+1} \in P_0^{\chi+1} \in P_0^{\chi+1}$. If $p_0 \in \vec{\rho}^{\chi+1} \in P_0^{\chi+1}$ and $p_0^{\chi+1} \in P_0^{\chi+1}$, we cardinal of $p_0^{\chi+1} \in P_0^{\chi+1}$. If $p_0^{\chi+1} \in P_0^{\chi+1}$ and $p_0^{\chi+1} \in P_0^{\chi+1}$, we cardinal of $p_0^{\chi+1} \in P_0^{\chi+1}$. If $p_0^{\chi+1} \in P_0^{\chi+1}$ it is similar, noting that $p_0^{\chi+1} \in P_0^{\chi+1}$. Now iterate this argument through to $p_0^{\chi+1}$, resulting in $p_0^{\chi+1} \in P_0^{\chi+1}$ where $p_0^{\chi+1} \in P_0^{\chi+1}$. Now iterate this argument through to $p_0^{\chi+1}$, where

 $\sigma = \max(\{0\} \cup (\vec{\sigma}^{\alpha} \cap \kappa)). \text{ Then } P_{j+1}^{\chi+1} | \kappa = Q^{\chi} | \kappa = Q_{j+1}^{\chi+1} | \kappa, \text{ and } \kappa \text{ is a cardinal of both } P_{j+1}^{\chi+1} \text{ and } Q_{j+1}^{\chi+1}, \text{ so } \kappa \leq \mu_{j+1}^{\chi+1}. \text{ This proves (*)}.$

(7vi): This follows from the fact that $Q_{\rho}^{\chi+1}=i_{\alpha,\chi+1}^{\mathcal{U}}(Q_{\rho}^{\alpha})$ (just shown), and that $Q_{\rho}^{\alpha}|\kappa=Q_{*}^{\chi}|\kappa$, and κ is a cardinal of Q_{*}^{χ} and of Q_{ρ}^{α} .

(7vii)-(7ix); (8): By (7iv),(7vi), we have that μ' is a cardinal of $Q_{\rho}^{\chi+1}$ and $Q_{\rho}^{\chi+1}||\theta'=Q^{\chi}||\theta'$ (this gives part of (7ix)). We proved, in the argument for (6), that (8i) holds with " κ " replacing " μ ". If there is no retraction things are easier; assume otherwise, so $\gamma<\chi$, and by (5), $\text{lh}(E)=\mu^{\chi}$ where $E=E_{\gamma}^{T^{\chi}}$, and $\mu^{\gamma}=\mu'<\mu^{\chi}=\theta'$. Let $\rho'=\max(\{0\}\cup(\bar{\rho}^{\gamma}\cap\mu'))$. So E is active on $P_{\gamma}^{*}\leq P_{\rho'}^{\gamma}$ and μ' is a cardinal of $P_{\rho'}^{\gamma}$ (which will give (8ii)). Moreover, $P_{\rho'}^{\gamma}||\text{lh}(E)$ is a cardinal segment of Q^{χ} , so $P_{\rho'}^{\gamma}||\mu^{\chi}=Q_{\rho}^{\chi+1}||\mu^{\chi}$ (which will give (7ix)). Also, $B^{\chi+1}=B^{\chi}$ and $Q^{\chi}|\mu^{\chi}$ is (A,B^{χ}) -valid. Now an induction through $(d,e)^{\chi+1}\mid \mu'$ like for (6) gives (8i) and (7viii), and the observations above give (8ii) and (7vii),(7ix).

(4i): Assume $\alpha = \chi$ for non-triviality. An argument like for (6) works, using the facts: $B^{\chi+1} = B^{\chi}$; and $E^{\mathcal{U}}_{\chi} = \emptyset$, so $M^{\mathcal{U}}_{\chi+1} = M^{\mathcal{U}}_{\chi}$ and $\mathbb{C}^{\chi+1} = \mathbb{C}^{\chi}$; and $E = E^{\mathcal{T}^{\chi+1}}_{\chi} \neq \emptyset$, $\theta^{\chi} = \mathrm{lh}(E)$ is a cardinal of $N = M^{\chi+1}$, and $N|\theta^{\chi} = Q^{\chi}|\theta^{\chi}$ is passive.

(4ii)-(4iv): If $\alpha=\chi$, use (4i) and the facts above. Suppose $\alpha<\chi$. If $E_\chi^{\mathcal{T}^{\chi+1}}\neq\emptyset$ then $\mathrm{lh}(E_\chi^{\mathcal{T}^{\chi+1}})>\mathrm{lh}(E)$, and (4)($\iota=\chi+1$) implies the result. Suppose instead $E_\chi^{\mathcal{U}}\neq\emptyset$. By (4ii)($\iota=\chi+1$), $\mathrm{lh}(E)\leq\mu^\chi$. By (7), defining μ',ρ as there, $\mu'\leq\mu^{\chi+1},\,Q_\rho^{\chi+1}|\mu'=Q^\chi|\mu'$ and μ' is a cardinal of $Q_\rho^{\chi+1}$ and Q^χ . Let $N=M^{\chi+1}$. Since $\mathcal{T}^{\chi+1}$ uses E, $\mathrm{lh}(E)$ is a cardinal of N. By (4iv)($\iota=\chi+1$), $N|\mathrm{lh}(E)\vartriangleleft Q_*^\chi$ and $\mathrm{lh}(E)$ is a cardinal of Q^χ . If $\mathrm{lh}(E)\leq\mu'$ this suffices. Assume $\mu'<\mathrm{lh}(E)=\mu^\chi$. Since E was not retracted, by (7), $Q_\rho^{\chi+1}|\mathrm{lh}(E)=N|\mathrm{lh}(E)$, these are passive, but since $\mu'<\mathrm{lh}(E)$, $\mathrm{lh}(E)$ is not a cardinal of $Q_\rho^{\chi+1}$. Therefore $\mu'\in\vec{\sigma}^{\chi+1}\backslash\vec{\rho}^{\chi+1}$, and $((\mu')^+)^{q_{\mu'}^{\chi+1}}=\mathrm{lh}(E)\in q_{\mu'}^{\chi+1}$, which gives the result.

(4v): This follows from (4iv) and the initial segment condition.

(3): For (3i) we may assume $\alpha = \chi$; use (4ii),(4iv) and (7vii),(7ix). For (3ii) we may assume $\alpha = \chi$. If $E = E_{\chi}^{\mathcal{T}^{\chi+1}} \neq \emptyset$ then $\mathcal{T}^{\chi+1} = \mathcal{T}^{\chi} \cap \langle E \rangle$ and $\lambda^{\chi} < \text{lh}(E)$. If E is an extender retracted at $\chi + 1$ then by (5), $\lambda^{\chi} < \text{lh}(E)$. Part (3iii) is trivial by induction.

(2): Suppose otherwise. We may assume $\max(\alpha, \beta) = \chi + 1$. By 2.11, $\alpha \neq \beta$. Let $P = P_{\rho}^{\alpha} = q_{\sigma}^{\beta}$. We have $\rho = \rho_{\omega}^{P} = \sigma$. Let $(\mathcal{T}^{\alpha})', (\mathcal{T}^{\beta})'$ be the non-padded trees equivalent to $\mathcal{T}^{\alpha}, \mathcal{T}^{\beta}$. We claim that $(\mathcal{T}^{\alpha})' = (\mathcal{T}^{\beta})' \upharpoonright \gamma + 1$ for some $\gamma + 1 < \text{lh}((\mathcal{T}^{\beta})')$, and in fact γ is least such that $(\rho^{+})^{P} \leq \text{lh}(E_{\gamma}^{(\mathcal{T}^{\beta})'}) \leq \text{ord}^{P}$.

To see this, first note that $P \triangleleft M^{\alpha}$, and $(\mathcal{T}^{\alpha})'$ is m-maximal and via $\Sigma_{\mathcal{M}}$, and by (4iii), $\operatorname{lh}(E) \leq \rho$ for each E used by $(\mathcal{T}^{\alpha})'$, and $P|\rho \leq \mathcal{I}^{(\mathcal{T}^{\alpha})'}$. Now $(\mathcal{T}^{\alpha})'$ is the unique non-padded tree satisfying these conditions. But since $q_{\rho}^{\beta} = P$, $M^{\beta}||(\rho^{+})^{P} = P||(\rho^{+})^{P}$, so $(\mathcal{T}^{\alpha})' \leq (\mathcal{T}^{\beta})'$, and for any $E \neq \emptyset$ used by \mathcal{T}^{β} but not by \mathcal{T}^{α} , we have $(\rho^{+})^{P} \leq \operatorname{lh}(E)$. Now let us show that $(\mathcal{T}^{\beta})' \neq (\mathcal{T}^{\alpha})'$, and letting γ be least such that $E = E_{\gamma}^{(\mathcal{T}^{\beta})'}$ is not used in $(\mathcal{T}^{\alpha})'$, we have

lh(E) \leq ord^P. Suppose otherwise. Then $P \triangleleft M^{\beta} = P_0^{\beta}$, and letting $i < k_{\beta}$ be least such that $\mu_i^{\beta} = \rho$, we have $\gamma_i^{\beta} \geq (\rho^+)^P$, but then $P \triangleleft P_i^{\beta}$ (if $j \leq i$ then either $P_j^{\beta} = P_0^{\beta}$ or $\rho_{\omega}(P_j^{\beta}) < \mu_i^{\beta}$). But $P = q_{\rho}^{\beta} \triangleleft Q_i^{\beta}$, and $\theta_i^{\beta} = (\rho^+)^P$. Therefore P is not (A, B^{β}) -valid. So $k_{\beta} = i$, contradiction.

Now let γ be least such that $\rho < \text{lh}(E_{\gamma}^{\mathcal{T}^{\beta}})$. Then $\rho \in \vec{\rho}^{\gamma}$ and $P_{\rho}^{\gamma} = P$. This is by similar reasoning to that in the previous paragraph.

So we may assume $\gamma = \alpha < \beta = \chi + 1$. Now suppose $\mathcal{U} \upharpoonright \chi + 2 = \mathcal{U} \upharpoonright (\gamma+1)^{\widehat{\rho}}\mathcal{P}$, so $M^{\mathcal{U}}_{\chi+1} = M^{\mathcal{U}}_{\gamma}$. By (4i) and (9),(10)($\iota = \chi+1$), $\vec{\sigma}^{\gamma} = \vec{\sigma}^{\chi+1} \cap \text{lh}(E)$, and $\xi^{\chi+1}_{\rho} = \xi^{\gamma}_{\rho}$ for all $\rho \in \vec{\sigma}^{\gamma}$. But $\rho < \text{lh}(E)$ and if $\rho \in \vec{\sigma}^{\gamma}$ then $q^{\gamma}_{\rho} \neq P^{\gamma}_{\rho}$, but $q^{\chi+1}_{\rho} = P^{\gamma}_{\rho}$, contradiction. So $E^{\mathcal{U}}_{\varepsilon} \neq \emptyset$ for some $\varepsilon \in (\gamma, \chi]$. Let ε be least such that $\gamma < \varepsilon$, $E^{\mathcal{U}}_{\varepsilon} \neq \emptyset$, $\varepsilon+1 \leq_{\mathcal{U}} \chi+1$, and let $\delta = \mathcal{U}-\text{pred}(\varepsilon+1)$. So either $\delta < \gamma$ or $\delta = \gamma'$, where $\gamma' > \gamma$ is least such that $E^{\mathcal{U}}_{\gamma'} \neq \emptyset$. Now $\kappa = \text{crit}(i^{\mathcal{U}}_{\delta,\chi+1}) \leq \rho$, by (6) and (2)($\iota = \chi + 1$). Also $\rho < \text{lh}(E) \leq \mu^{\varepsilon} \leq \nu(F^{\mathcal{U}}_{\varepsilon})$. So if $\varepsilon < \chi$ then $\rho < \text{crit}(i^{\mathcal{U}}_{\varepsilon+1,\chi+1})$ by (7iii), so $\rho \in \vec{\sigma}^{\varepsilon+1}$ and $q^{\varepsilon+1}_{\rho} = q^{\chi+1}_{\rho}$, contradicting (2)($\iota = \chi + 1$). So $\varepsilon = \chi$. So by (7), since $\kappa \leq \rho$ and $\rho \in \vec{\sigma}^{\chi+1}$, we have $\rho \geq \mu'$, where μ' is defined as there. But $\rho < \text{lh}(E) \leq \mu^{\chi}$, so $\rho = \mu' < \mu^{\chi}$. Now let $\rho' = \max(\{0\} \cup \vec{\sigma}^{\delta} \cap \kappa\}$. Then by (7), $Q^{\chi+1}_{\rho'} = i^{\mathcal{U}}_{\delta,\chi+1}(Q^{\delta}_{\rho'})$ and $q^{\chi+1}_{\rho} \triangleleft Q^{\chi+1}_{\rho'}$, and so on $\mathbb{E}^{N'}$. Therefore E should have been retracted and not used in $\mathcal{T}^{\chi+1}$, contradiction.

It is particularly in order to deal with the preceding situation that we use retraction of extenders.

Case 3. ι is a limit η .

We must prove (3iii)($\iota = \alpha + \omega$) and (1),(9),(11)($\iota = \eta$).

(1): We omit the proof.

(3iii): Let $\delta = \delta(\vec{\mathcal{T}} \upharpoonright \alpha + \omega)$. Then δ is a limit of limit cardinals of $M(\vec{\mathcal{T}} \upharpoonright \alpha + \omega)$, since either $\mathcal{T}^{<\alpha+\omega}$, or $\mathcal{U} \upharpoonright \alpha + \omega$, is cofinally non-padded, and in the case that $\mathcal{T}^{<\alpha+\omega}$ is eventually only padding, if $\gamma <_{\mathcal{U}} \alpha + \omega$ and $\kappa = \mathrm{crit}(i^{\mathcal{U}}_{\gamma,\alpha+\omega})$, then $\kappa \leq \lambda^{\gamma}$ is a limit cardinal of $M^{\mathcal{U}}_{\gamma}$ and Q^{γ} , and so by (3i)($\iota < \eta$), $Q^{\gamma} \upharpoonright \kappa \leq M(\vec{\mathcal{T}} \upharpoonright \alpha + \omega)$ and κ is a limit cardinal there. Now, $\lambda^{\alpha} \leq \lambda^{\alpha+n} \leq \mu^{\alpha+n}$ for all $n < \omega$, by (3i). Now suppose $\mathcal{U} \upharpoonright \alpha + \omega$ is cofinally non-padded. Then for every $\chi \in [\alpha, \alpha + \omega)$ such that $E^{\mathcal{U}}_{\chi} \neq \emptyset$, we have $\lambda^{\alpha} \leq \mathrm{lh}(F^{\mathcal{U}}_{\chi})$. Since $M^{\mathcal{U}}_{\alpha+\omega}$ is wellfounded, there is such a $\chi < \alpha + \omega$ with $\lambda^{\alpha} < \mathrm{crit}(F^{\mathcal{U}}_{\chi}) \leq \lambda^{\chi}$. It's similar if $\mathcal{T}^{<\alpha+\omega}$ is cofinally non-padded.

(9): This follows $(4),(9),(10)(\iota < \eta)$ and $(1)(\iota = \eta)$. Prove (9i) first; the others follow. (Note any descent has finite length.)

(11): This follows $(6),(3i)(\iota < \eta)$.

Case 4. $\iota = \eta + 1$ for a limit η .

We must prove (3),(4),(6) with $\beta = \eta$, (2) with $\max(\alpha, \beta) = \eta$, and (10),(12).

(10): Let α be as in (9ii). Then δ is a limit of cardinals of M^{η} , and of $Q^{\alpha}_{\rho} = Q^{\alpha}$, since lh(E) is a cardinal of Q^{α} for each extender E used by $\mathcal{T}^{<\eta}$, by (4)($\iota = \eta$) and (9). Moreover, $M^{\eta} | \delta = Q^{\alpha} | \delta$ by (9). This gives the result.

(12): We assume that $\mathcal{T}^{<\eta}$ is eventually only padding as the contrary case is easier. However, there still may be cofinally many $\alpha < \eta$ such that $E_{\alpha}^{\mathcal{T}^{\alpha+1}} \neq \emptyset$. We prove most of (12v) and omit the rest. Let γ_0 be least such that $\mathcal{T}^{<\eta} = \mathcal{T}^{<\eta} \upharpoonright (\gamma_0 + 1) \curvearrowright \mathcal{P}$. Let A_0 be the set of all $\beta \in [\gamma_0, \eta)$ such that $\mathcal{T}^\beta = \mathcal{T}^{\gamma_0} \curvearrowright \mathcal{P}$. Then A_0 is cofinal in η . Let $N = M^{\gamma_0}$. For each $\beta \in A_0$, $M^\beta = N$. For $\beta_1 < \beta_2$ with $\beta_1, \beta_2 \in A_0$ we have $d^{\beta_1} \upharpoonright \lambda^{\beta_1} = d^{\beta_2} \upharpoonright \lambda^{\beta_2}$. This follows by induction on β_2 , using (8i),(3i),(12v)($\iota < \eta$) (note (12v) applies at every limit $\eta' \in (\gamma_0, \eta)$ as $\mathcal{T}^{<\eta'} = \mathcal{T}^{\gamma_0} \curvearrowright \mathcal{P}$). So there is $\gamma \in A_0$ such that $\vec{\rho}^\beta \cap \lambda^\beta \subseteq \lambda^\gamma$ for all $\beta \in A_0 \cap [\gamma, \eta)$. But $\delta = \sup_{\beta < \eta} \lambda^\beta$, by (3i),(3ii). It follows that γ is as required. Now use an argument like that for (6); we omit the details.

(2): Suppose $P_{\rho}^{\alpha} = q_{\rho}^{\beta}$. By the argument for (2) in the " $\iota = \chi + 2$ " case, we may assume $\alpha < \beta = \eta$, and that argument shows that \mathcal{T}^{η} uses some extender E such that $\rho < \text{lh}(E)$. Therefore $\rho < \delta(\vec{\mathcal{T}} \upharpoonright \eta)$. But then by (10i),(12i), $q_{\rho}^{\eta} = q_{\rho}^{\xi}$ for some $\xi < \eta$. So $P_{\rho}^{\alpha} = q_{\rho}^{\xi}$, contradicting (2)($\iota = \max(\alpha + 1, \xi + 1)$).

(3),(4),(6): We omit the proof.

This completes the proof of Claim 1.

We can now show that the construction works.

Claim 2. The comparison terminates at some stage $\zeta < \theta^+$.

Proof. Suppose not. Then we reach $\mathcal{T} = \mathcal{T}^{\theta^+}$ and $\mathcal{U} = \mathcal{U} \upharpoonright \theta^+ + 1$. Since \mathcal{M}, S have cardinality $< \theta^+$, both $\mathcal{T} \upharpoonright \theta^+$ and $\mathcal{U} \upharpoonright \theta^+$ are cofinally non-padded. Let η be some large ordinal and $\pi: H \to V_{\eta}$ elementary with H transitive, H of cardinality θ , $\operatorname{crit}(\pi) > \theta$, and $\vec{\mathcal{T}} \upharpoonright (\theta^+ + 1), \mathcal{U}$, etc, in $\operatorname{range}(\pi)$. Let $\kappa = \operatorname{crit}(\pi)$. As usual, $i_{\kappa,\theta^+}^{\mathcal{T}}$ and $i_{\kappa,\theta^+}^{\mathcal{U}}$ both exist, have critical point κ , send κ to θ^+ , and agree over $M_{\kappa}^{\mathcal{T}} \cap M_{\kappa}^{\mathcal{U}}$. Let $\alpha + 1 \in (\kappa, \theta^+]_{\mathcal{T}}$ be such that $\operatorname{crit}(E_{\alpha}^{\mathcal{T}}) = \kappa$ and let $\beta + 1 \in (\kappa, \theta^+]_{\mathcal{U}}$ be such that $\operatorname{crit}(E_{\beta}^{\mathcal{U}}) = \kappa$. Since \mathcal{T} is normal and since \mathcal{U} does not move fine structural generators, by Claim 1(7iii), the extenders $E_{\alpha}^{\mathcal{T}}$ and $E_{\beta}^{\mathcal{U}}$ are compatible over $\mathcal{P}(\kappa) \cap M_{\kappa}^{\mathcal{T}} \cap M_{\kappa}^{\mathcal{U}}$, through $\nu = \min(\nu(E_{\alpha}^{\mathcal{T}}), \nu(F_{\beta}^{\mathcal{U}}))$, and $\operatorname{crit}(i_{\beta+1,\theta^+}^{\mathcal{U}}) \geq \nu(F_{\beta}^{\mathcal{U}})$.

For all $\gamma \in [\kappa, \theta^+]$ we have $\mathcal{P}(\kappa) \cap M_{\kappa}^{\mathcal{T}} = \mathcal{P}(\kappa) \cap M_{\gamma}^{\mathcal{T}}$ and $\mathcal{P}(\kappa) \cap M_{\kappa}^{\mathcal{U}} = \mathcal{P}(\kappa) \cap M_{\gamma}^{\mathcal{U}}$. Also, letting $\kappa' \geq \kappa$ be least such that $E_{\kappa'}^{\mathcal{T}} \neq \emptyset$, we have $M_{\kappa'}^{\mathcal{T}} = M_{\kappa}^{\mathcal{T}}$ and $\text{lh}(E_{\kappa'}^{\mathcal{T}}) \geq (\kappa^+)^{M_{\kappa}^{\mathcal{T}}}$. So $\mathcal{P}(\kappa) \cap M_{\kappa}^{\mathcal{T}} \subseteq M_{\kappa'}^{\mathcal{T}} || \text{lh}(E_{\kappa'}^{\mathcal{T}}) = Q^{\kappa'} || \theta^{\kappa'} \in M_{\kappa'}^{\mathcal{U}}$, so $\mathcal{P}(\kappa) \cap M_{\kappa}^{\mathcal{T}} \subseteq M_{\kappa}^{\mathcal{U}}$.

Let $Q=M(\mathcal{T}\upharpoonright\theta^+)$. So $Q,M_\kappa^\mathcal{T},M_\alpha^\mathcal{T}$ all compute the same value for κ^+ and agree strictly below that point. Also $E_\alpha^\mathcal{T}\notin Q$. By Claim 1(6) we have that $Q_\rho^{\beta+1}=i^\mathcal{U}_{\kappa,\beta+1}(Q_\rho^\kappa)$, where $\rho=\max(\{0\}\cup(\vec{\sigma}^\kappa\cap\kappa))$. Also, $Q=i^\mathcal{U}_{\kappa,\theta+}(Q_\rho^\kappa|\kappa)$, which, again by Claim 1(6), implies that $\rho=\max(\{0\}\cup(\vec{\sigma}^{\beta+1}\cap\operatorname{crit}(i^\mathcal{U}_{\beta+1,\theta+}))$. Since $\operatorname{crit}(i^\mathcal{U}_{\beta+1,\theta+})\geq\nu(F_\beta^\mathcal{U})$, we have $Q|\nu(F_\beta^\mathcal{U})=Q_\rho^{\beta+1}|\nu(F_\beta^\mathcal{U})$ and $Q||\nu(F_\beta^\mathcal{U})=Q^\beta|\nu(F_\beta^\mathcal{U})$. In particular, $Q||(\kappa^+)^{Q^\beta}=Q^\beta|(\kappa^+)^{Q^\beta}$. However, we might have $(\kappa^+)^{Q^\beta}<(\kappa^+)^Q$.

Since $F_{\beta}^{\mathcal{U}} \upharpoonright \nu(F_{\beta}^{\mathcal{U}}) \subseteq E_{\beta}^{\mathcal{U}}$, the compatibility of $E_{\alpha}^{\mathcal{T}}$ with $E_{\beta}^{\mathcal{U}}$ implies that if $\nu(E_{\alpha}^{\mathcal{T}}) \leq \nu(F_{\beta}^{\mathcal{U}})$ then $E_{\alpha}^{\mathcal{T}} \upharpoonright \nu(E_{\alpha}^{\mathcal{T}}) \subseteq F_{\beta}^{\mathcal{U}}$, and if $\nu(F_{\beta}^{\mathcal{U}}) \leq \nu(E_{\alpha}^{\mathcal{T}})$ then

 $F_{\beta}^{\mathcal{U}} \upharpoonright \nu(F_{\beta}^{\mathcal{U}}) \subseteq E_{\alpha}^{\mathcal{T}}$. But maybe $\nu(F_{\beta}^{\mathcal{U}}) < (\kappa^{+})^{Q}$, in which case $F_{\beta}^{\mathcal{U}}$ is not total over Q.

Subclaim 1. $\nu(E_{\alpha}^{\mathcal{T}}) \geq \nu(F_{\beta}^{\mathcal{U}})$ and $\alpha > \beta$.

Proof. Suppose $\nu(E_{\alpha}^{\mathcal{T}}) < \nu(F_{\beta}^{\mathcal{U}})$. Then $(\kappa^+)^{M_{\alpha}^{\mathcal{T}}} \leq \nu(E_{\alpha}^{\mathcal{T}}) < \nu(F_{\beta}^{\mathcal{U}})$ and $E_{\alpha}^{\mathcal{T}} \upharpoonright \nu(E_{\alpha}^{\mathcal{T}})$ is a proper, non-type Z initial segment of $F_{\beta}^{\mathcal{U}}$. So $E_{\alpha}^{\mathcal{T}} \in i_{\kappa,\beta+1}^{\mathcal{U}}(Q_{\rho}^{\kappa})$, by Claim 1(7vi). But $\operatorname{crit}(i_{\beta+1,\theta^+}^{\mathcal{U}}) \geq \nu(F_{\beta}^{\mathcal{U}})$, so $\operatorname{crit}(i_{\beta+1,\theta^+}^{\mathcal{U}}) > \operatorname{lh}(E_{\alpha}^{\mathcal{T}})$, so $E_{\alpha}^{\mathcal{T}} \in Q$, contradiction.

Now $\alpha \neq \beta$ by construction. If $\alpha < \beta$ then by Claim 1(4ii), $\nu(E_{\alpha}^{\mathcal{T}}) < \text{lh}(E_{\alpha}^{\mathcal{T}}) \leq \nu(F_{\beta}^{\mathcal{U}})$, a contradiction, which proves the subclaim.

So
$$\nu = \nu(F_{\beta}^{\mathcal{U}})$$
. Let $N = M_{\alpha}^{\mathcal{T}} | \text{lh}(E_{\alpha}^{\mathcal{T}}) = M^{\alpha} | \text{lh}(E_{\alpha}^{\mathcal{T}})$.

Subclaim 2. Either (a) $F^{\mathcal{U}}_{\beta} \in \mathbb{E}^{N}_{+}$ or else (b) $F^{\mathcal{U}}_{\beta}$ is either type 1 or type 3, $N|\nu$ is active and $F^{\mathcal{U}}_{\beta} \in \mathbb{E}^{\mathrm{Ult}(N||\nu,\mathbb{E}^{N}_{\nu})}$.

Proof. We know that $F_{\beta}^{\mathcal{U}} \upharpoonright \nu \subseteq E_{\alpha}^{\mathcal{T}}$. So if $(\kappa^+)^Q \leq \nu$ then the desired conclusion follows the initial segment condition.

Suppose $\nu < (\kappa^+)^Q$. Then $F^{\mathcal{U}}_{\beta}$ is type 1. For otherwise, $\gamma = (\kappa^+)^{Q^\beta} < \nu$, so γ is a cardinal of $Q^{\beta+1}_{\rho}$, contradicting that $(\kappa^+)^{Q^{\beta+1}_{\rho}} = (\kappa^+)^Q$. So $F^{\mathcal{U}}_{\beta}$ is a partial normal measure derived from $E^{\mathcal{T}}_{\alpha}$, inducing the type 1 premouse $R = Q^\beta$ such that $\nu = (\kappa^+)^R < (\kappa^+)^N$ and $R|\nu = N||\nu$. We now use [3, 4.11, 4.12, 4.15] to yield the conclusion of the subclaim. Since \mathcal{M} is typical, these apply to N. However, if $N|\nu$ is active with a type 3 extender, then we must verify that [3, 4.15] applies; that is, we must verify that $R||\operatorname{ord}^R = \operatorname{Ult}(N||\nu, \mathbb{E}^N_{\nu})||\operatorname{ord}^R$. Well, \mathcal{T}^β and \mathcal{T}^α use the same extenders E such that $\operatorname{lh}(E) < \nu$. However, $N|\nu$ is active while $M^\beta|\nu$ is not, so \mathcal{T}^β uses \mathbb{E}^N_{ν} . Moreover, ν is the largest cardinal of R, and $R||\operatorname{ord}^R = M^\beta||\operatorname{ord}^R$. Therefore \mathcal{T}^β uses no extenders E such that $\nu < \operatorname{lh}(E) < \operatorname{ord}^R$ and $M^\beta||\operatorname{ord}^R = \operatorname{Ult}(N||\nu, \mathbb{E}^N_{\nu})||\operatorname{ord}^R$. So [3, 4.15] applies.

This completes the proof of the subclaim.

Subclaim 3. $F_{\beta}^{\mathcal{U}} \notin \mathbb{E}_{+}(M^{\beta})$.

Proof. Suppose $F^{\mathcal{U}}_{\beta} \in \mathbb{E}_{+}(M^{\beta})$. Then $P^{\beta}|\theta^{\beta} = M^{\beta}|\text{lh}(F^{\mathcal{U}}_{\beta}) = Q^{\beta}|\theta^{\beta}$ is active, but $(\dagger)^{\beta}$ fails, so by 2.12, $M^{\beta}|\text{lh}(F^{\mathcal{U}}_{\beta})$ is not (A, B^{β}) -valid. But A is bounded in κ . So $F^{\mathcal{U}}_{\beta}$ induces an extender algebra axiom which is not satisfied by (A, B^{β}) , which gives a contradiction as usual, proving the subclaim.

Subclaim 4. \mathcal{T}^{β} uses $F_{\beta}^{\mathcal{U}}$.

¹³ It seems one might try to deduce [3, 4.11, 4.12, 4.15] from the n=0 condensation given in [2, pp.87,88]. That is, let $E=\mathbb{E}_{\gamma}^{N}$ be the type 1 initial segment of $E_{\alpha}^{\mathcal{T}}$. Using a restriction of the factor map $j: \mathrm{Ult}_0(Q|\nu,F^Q) \to \mathrm{Ult}_0(N|(\kappa^+)^N,E)$, we get a Σ_0 -elementary $\pi:Q\to N|\gamma$, with $\mathrm{crit}(\pi)=\nu$ and $\pi(\nu)=(\kappa^+)^N$. Moreover, $\rho_1^Q\le\nu$. However, π need not be Σ_1 -elementary, even for formulas without parameters, so π might not even be a weak 0-embedding (for instance, if $F=F^Q$ is the least partial measure derived from E such that F is on \mathbb{E}^N). So the n=0 condensation of [2, pp.87,88] does not apply.

Proof. Suppose Subclaim 2(a) holds. Then \mathcal{T}^{α} and \mathcal{T}^{β} use the same extenders E such that $lh(E) < lh(F_{\beta}^{\mathcal{U}})$. Since $F_{\beta}^{\mathcal{U}} \in \mathbb{E}_{+}^{N}$ but $F_{\beta}^{\mathcal{U}} \notin \mathbb{E}_{+}(M^{\beta})$, \mathcal{T}^{β} uses $F_{\beta}^{\mathcal{U}}$. If Subclaim 2(b) holds it is similar but there are $\delta_{0} < \delta_{1} < \beta$ such that $E_{\delta_{0}}^{\mathcal{T}^{\beta}} = \mathbb{E}_{\nu}^{N}$, $E_{\delta_{1}}^{\mathcal{T}^{\beta}} = F_{\beta}^{\mathcal{U}}$, and $E_{\delta}^{\mathcal{T}^{\beta}} = \emptyset$ for all $\delta \in (\delta_{0}, \delta_{1})$. This completes the proof of the subclaim.

But Subclaim 4 contradicts Claim 1(4ii) at stage $\beta + 1$, proving Claim 2.

By Claim 2 we have $\zeta < \theta^+$ such that $(\dagger)^{\zeta}$ holds. Let $\mathcal{T} = \mathcal{T}^{\zeta}$ and $\mathcal{U} = \mathcal{U} \upharpoonright$

Claim 3. Either $b^{\mathcal{T}}$ does not drop in model or $i^{\mathcal{U}}(N_A) \leq \mathcal{I}^{\mathcal{T}}$.

Proof. We first relate cores of models on \mathcal{T} to the structures arising in the comparison.

Subclaim. Let $\alpha + 1 < \text{lh}(\mathcal{T}^{\beta})$ and let $\varepsilon = \mathcal{T}^{\beta} - \text{pred}(\alpha + 1)$. Let $\kappa = \text{crit}(E_{\alpha}^{\mathcal{T}^{\beta}})$. If $\kappa < \min(\bar{\rho}^{\varepsilon})$ then \mathcal{T}^{β} does not drop in model at $\alpha + 1$ (here $\min(\emptyset) = \infty$). If $\min(\vec{\rho}^{\varepsilon}) \leq \kappa$ then $M_{\alpha+1}^{*\mathcal{T}^{\beta}} = P_{\rho}^{\varepsilon}$ where $\rho = \max(\vec{\rho}^{\varepsilon} \cap (\kappa+1))$.

Proof. This follows 2.14.

Now suppose the claim fails. So $b^{\mathcal{T}}$ drops in model, and by 2.13, we may assume that $\mathcal{I}^{\mathcal{T}} = M^{\zeta} = Q^{\zeta}$ and $\xi^{\zeta} < \xi_0^{\zeta}$. Let $\varepsilon < \text{lh}(\mathcal{T})$ be such that $\mathfrak{C}_{\omega}(\mathcal{I}^{\mathcal{T}})^{\mathrm{unsq}} \triangleleft M_{\varepsilon}^{\mathcal{T}}$. Let $\rho = \rho_{\omega}(\mathcal{I}^{\mathcal{T}})$. By the Subclaim, $\rho \in \vec{\rho}^{\varepsilon}$ and $\mathfrak{C}_{\omega}(\mathcal{I}^{\mathcal{T}}) = \mathfrak{C}_{0}(P_{\rho}^{\varepsilon})$. We have $\mathcal{I}^{\mathcal{T}} = Q^{\zeta} = Q_{\sigma}^{\zeta}$ for some $\sigma \in \vec{\sigma}^{\zeta}$ (since $\xi^{\zeta} < \xi_{0}^{\zeta}$). So $\mathfrak{C}_0(q_{\sigma}^{\zeta}) = \mathfrak{C}_{\omega}(Q_{\sigma}^{\zeta}) = \mathfrak{C}_0(P_{\rho}^{\varepsilon})$. Therefore $q_{\sigma}^{\zeta} = P_{\rho}^{\varepsilon}$, contradicting Claim 1(2), and completing the proof of Claim 3.

We have shown that \mathcal{T}, \mathcal{U} satisfy conditions (a')-(c'). We now refine this to complete the proof of 2.9:

Claim 4. There is $\varepsilon \leq \zeta$ such that $(\mathcal{T} \upharpoonright \varepsilon + 1, \mathcal{U} \upharpoonright \varepsilon + 1)$ satisfies the requirements of 2.9.

Proof. If $i^{\mathcal{U}}(N_{\Lambda}) \triangleleft \mathcal{I}^{\mathcal{T}}$ then N_{Λ}^{ζ} is ω -sound, and we just use $\varepsilon = \zeta$. So assume that $\mathcal{I}^{\mathcal{T}} = N_{\alpha}^{\zeta}$ for some $\alpha \leq \Lambda^{\zeta}$. Let $b = b^{\mathcal{T}}$. If b does not drop in model or degree, again we use $\varepsilon = \zeta$. So assume that b drops in model or degree. We have two cases to deal with: (i) either b drops in model or $[\alpha = \Lambda^{\zeta}$ and m > n]; (ii) otherwise.

We assume we are in case (ii), but the proof in case (i) is almost the same. So b drops in degree but not in model, and $(\alpha, m) \leq_{\text{lex}} (\Lambda^{\zeta}, n)$. Now $\mathfrak{C}_m(N_\alpha^\zeta) = \mathfrak{C}_m(\mathcal{I}^\mathcal{T}) = M_\gamma^\mathcal{T}$ for some $\gamma < \mathrm{lh}(\mathcal{T})$. Let γ be least such and γ' greatest such (so $\gamma' \geq \gamma$ is least such that $E = E_{\gamma'}^{\mathcal{T}} \neq \emptyset$). Let $\rho = \rho_m^{\mathcal{T}^{\mathcal{T}}}$ and let $\tau = (\rho^+)^{M_{\gamma}^{\mathcal{T}}}$. Let $\beta \in c = b^{\mathcal{U}}$ be least such that either $i_{\beta,\zeta}^{\mathcal{U}} = \mathrm{id}$ or $\mathrm{crit}(i_{\beta,\zeta}^{\mathcal{U}}) > \rho$. Let β' be largest such that $M_{\beta'}^{\mathcal{U}} = M_{\beta}^{\mathcal{U}}$. Let $\varepsilon = \max(\gamma,\beta)$. We will show that this works.

Since b does not drop in model and $\operatorname{crit}(i_{\gamma,\zeta}^{\mathcal{T}}) \geq \rho$, we have $\operatorname{lh}(E) \geq \tau$, and if $lh(E) = \tau$ then E is type 2. Since type 2 extenders are not relevant to validity, therefore $M_{\gamma}^{\mathcal{T}}|\tau$ is $(A, B^{\gamma'})$ -valid.

By choice of β and elementarity, $M_{\gamma}^{\mathcal{T}} = \mathfrak{C}_m(N_{\alpha'}^{\beta})$ for some $\alpha' \leq \Lambda^{\beta}$.

Subclaim. $\varepsilon = \min(\beta', \gamma')$.

Proof. Since $\beta \leq \beta'$ and $\gamma \leq \gamma'$, we just have to rule out the possibility that either $\beta \leq \beta' < \gamma$ or $\gamma \leq \gamma' < \beta$.

Suppose $\beta \leq \beta' < \gamma$. In particular, $\gamma \neq 0$ and $lh(\mathcal{U}) > \beta' + 1$, so $E^{\mathcal{U}}_{\beta'} \neq \emptyset$. Now $\rho < \operatorname{crit}(i^{\mathcal{U}}_{\beta,\zeta}) \leq \lambda^{\beta'}$ by Claim 1(7ii), but because $\beta' < \gamma$, Claim 1(3i) gives that $\lambda^{\beta'} < \rho_m(M_{\gamma}^T)$, contradiction.

Now suppose $\gamma \leq \gamma' < \beta$. Let $\xi \geq \gamma'$ be least such that $E^{\mathcal{U}}_{\xi} \neq \emptyset$ and $\xi + 1 \in c$. Then $\xi > \gamma'$ since $E^{\mathcal{T}}_{\gamma'} \neq \emptyset$. Since $\tau \leq \text{lh}(E)$, therefore by Claim 1, $\tau \leq \nu(F^{\mathcal{U}}_{\xi}) \leq \text{crit}(i^{\mathcal{U}}_{\xi+1,\zeta})$. So $\beta = \xi + 1$. Let $\sigma = \mathcal{U} - \text{pred}(\xi + 1)$, let $j = i^{\mathcal{U}}_{\sigma,\xi+1}$ and let $\kappa = \text{crit}(j)$. Let η be such that $N^{\sigma}_{\eta} = M^{\mathcal{T}}_{\gamma} | \kappa$. Let $\eta' > \eta$ be least such that either $\eta' = \infty$ or for some k, $(\eta', k) <_{\text{lex}} (\Lambda^{\sigma}, n)$ and $\rho_{k+1}(N^{\sigma}_{\eta'}) < \kappa$. So $j(\eta')$ is defined similarly in $M^{\mathcal{U}}_{\xi+1}$. Now $M^{\mathcal{T}}_{\gamma} | |\tau = N^{\xi+1}_{\alpha'}| |\tau$ and $\rho_m(N^{\xi+1}_{\alpha'}) = \rho$. By Claim 1, and since E was never retracted after stage γ' , $j(M^{\mathcal{T}}_{\gamma} | \kappa) | \text{lh}(E) = M^{\mathcal{T}}_{\gamma} | |\text{lh}(E)$, and ρ is a cardinal in $j(M^{\mathcal{T}}_{\gamma} | \kappa)$.

Suppose $\tau = (\rho^+)^{j(M_{\gamma}^{\mathcal{T}}|\kappa)}$. Then $N_{\alpha'}^{\xi+1}$ witnesses that $j(\eta') \neq \infty$, so $\eta' \neq \infty$. But $j(\eta') < \alpha'$, because $\rho \notin \text{range}(j)$. Moreover, $\rho_{\omega}(N_{\eta'}^{\sigma}) < \kappa$. But $N_{\alpha'}^{\xi+1}|\kappa = j(N_{\eta'}^{\sigma})|\kappa$, which leads to contradiction.

So $\tau < (\rho^+)^{j(M_{\gamma}^{\mathcal{T}}|\kappa)}$. But then the properties of $N_{\alpha'}^{\xi+1}$, and that ρ is a cardinal of $j(M_{\gamma}^{\mathcal{T}}|\kappa)$, give that $M_{\gamma}^{\mathcal{T}} \triangleleft j(M_{\gamma}^{\mathcal{T}}|\kappa)$, contradicting the fact that $M_{\gamma}^{\mathcal{T}}||\text{lh}(E) \trianglelefteq j(M_{\gamma}^{\mathcal{T}}|\kappa)$. This proves the subclaim.

Now by the subclaim, $B^{\varepsilon} = B^{\gamma'} \cap (\rho + 1)^3$, so $M_{\gamma}^{\mathcal{T}} | \tau$ is (A, B^{ε}) -valid, and the claim, and properties 2.9(a),(b), follow.

This completes the proof of the theorem.

We finish with one corollary to the foregoing proof, which answers a question of Nam Trang and Martin Zeman. For simplicity we assume that m=n=0.

Corollary 2.16. Let \mathcal{M} , etc, be as in the statement of 2.9, and assume m=n=0. Let \mathcal{T},\mathcal{U} be constructed as in its proof. Let $\zeta+1=\operatorname{lh}(\mathcal{T})=\operatorname{lh}(\mathcal{U})$. Let $\hat{S}=M^{\mathcal{U}}_{\zeta}$ and $\hat{\mathbb{D}}=i^{\mathcal{U}}_{\zeta}(\mathbb{D})$. Let $\hat{\mathcal{T}},\hat{\mathcal{U}}$ be given by applying the same construction to $(\mathcal{M},(\hat{S},\hat{\mathbb{D}}))$. Then $\hat{\mathcal{T}}$ is the non-padded tree \mathcal{T}' equivalent to \mathcal{T} and $\hat{\mathcal{U}}$ is only padding.

Proof. We adopt the notation of the proof of 2.9 regarding the construction of \mathcal{T}, \mathcal{U} . Let \hat{M}^{α} , \hat{R}^{α} , etc, be the corresponding notation regarding the construction of $\hat{\mathcal{T}}, \hat{\mathcal{U}}$. Note that since m = n = 0, Claim 4 of the proof of 2.9 is trivial and its proof does nothing.

Claim. For each $\alpha < \text{lh}(\mathcal{T}')$, $\hat{\mathcal{T}}^{\alpha} = \mathcal{T}' \upharpoonright \alpha + 1$ and $\hat{\mathcal{U}} \upharpoonright \alpha + 1$ is pure padding.

Proof. The proof is by induction on α . Suppose it holds for α , and $\operatorname{lh}(\mathcal{T}') > \alpha+1$. Let B be the neat code for $(\mathcal{T},\mathcal{U})$. Let $P = \mathcal{I}^{\mathcal{T}}$ or $P = i^{\mathcal{U}}(N_{\Lambda}^{\mathbb{D}})$, whichever is smaller. Because P is (A, B)-valid, and because of the inductive hypothesis,

and the fact that $\hat{\mathcal{T}}^{\alpha}$ and \mathcal{T} are coded (via their neat codes) in a manner that ignores padding, the proof that $\hat{\mathcal{T}}^{\alpha+1} = \mathcal{T}' \upharpoonright \alpha + 2$ will not break down due to (A, \hat{B}^{α}) -invalidity. (Since $\hat{\mathcal{U}} \upharpoonright \alpha + 1$ is pure padding, this portion of \hat{B}^{α} is also not a problem.)

Now let $\gamma = \text{lh}(E_{\alpha}^{\mathcal{T}'})$. Let β be such that $E_{\beta}^{\mathcal{T}} = E_{\alpha}^{\mathcal{T}'}$. Then $\hat{d}^{\alpha} = d^{\beta}$ and $\hat{e}^{\alpha} = e^{\zeta} \upharpoonright \gamma$ and $\hat{P}^{\alpha} = P^{\beta}$. (Recall $\zeta + 1 = \text{lh}(\mathcal{T})$; see 2.10 for the definition of d^{β} , e^{ζ} , etc.) This follows by an argument like in the proof of Claim 1(6), combined with the above observations regarding validity, and using that $M_{\beta}^{\mathcal{T}}||\mathrm{lh}(E_{\beta}^{\mathcal{T}})$ is a cardinal segment of $\mathcal{I}^{\mathcal{T}}$ and Q^{ζ} (by Claim 1(4)). Also, $\hat{Q}^{\alpha}|\hat{\theta}^{\alpha}$ is passive since $\hat{Q}^{\alpha}|\hat{\theta}^{\alpha} \leq Q^{\zeta}$. So $E_{\alpha}^{\hat{T}} = E_{\alpha}^{T'}$, as required. The claim easily follows.

So we reach stage $\hat{\zeta}$, at which we have $\hat{M}^{\hat{\zeta}} = M^{\zeta}$ and $\hat{\mathcal{U}} \upharpoonright \hat{\zeta} + 1$ is pure padding. But then $(\hat{\uparrow})^{\hat{\zeta}}$ holds since $(\dot{\uparrow})^{\zeta}$ does. This completes the proof.

References

- 1. Donald Martin and John R. Steel. Iteration trees. Journal of the American Mathematical Society, 7(1):1-73, January 1989.
- 2. William Mitchell and John R. Steel. Fine structure and iteration trees. Number 3 in Lectures Notes in Logic. Springer-Verlag, 1994.
- 3. Farmer Schlutzenberg. Measures in mice. PhD thesis, University of California, Berkeley,
- 4. John R. Steel. Derived models associated to mice. In C.T. Chong, editor, Computational Prospects of Infinity - Part I: Tutorials, Lecture Notes. World Scientific Publishing Company, Incorporated, 2008.
- 5. John R. Steel. An outline of inner model theory. In Matthew Foreman and Akihiro Kanamori, editors, Handbook of set theory, volume 3, chapter 19. Springer, first edition,