THE FAN THEOREM, ITS STRONG NEGATION, AND THE DETERMINACY OF GAMES

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ABSTRACT. In the context of a weak formal theory called Basic Intuitionistic Mathematics BIM, we study Brouwer's Fan Theorem and a strong negation of the Fan Theorem, Kleene's Alternative (to the Fan Theorem). We prove that the Fan Theorem is equivalent to contrapositions of a number of intuitionistically accepted axioms of countable choice and that Kleene's Alternative is equivalent to strong negations of these statements. We discuss finite and infinite games and introduce a constructively useful notion of determinacy. We prove that the Fan Theorem is equivalent to the Intuitionistic Determinacy Theorem. This theorem says that every subset of Cantor space is, in our constructively meaningful sense, determinate. Kleene's Alternative is equivalent to a strong negation of a special case of this theorem. We also consider a uniform intermediate value theorem and a compactness theorem for classical propositional logic. The Fan Theorem is equivalent to each of these theorems and Kleene's Alternative is equivalent to strong negations of them. We end with a note on 'stronger' Fan Theorems. The paper is a sequel to [43].

1. Introduction

1.1. Intuitionistic reverse mathematics. L.E.J. Brouwer did not present his intuitionistic mathematics as a formal axiomatic theory. He did not like formalism and formalization and anxiously maintained the distinction between a mathematical proof and the linguistic expression that should help us to recover the proof but may fail to do so. The challenge to develop formal theories coming close to Brouwer's intentions was taken up by A. Heyting, G. Gentzen, S.C. Kleene, G. Kreisel, J. Myhill, A.S. Troelstra, and others.

Given a preferably weak formal basic theory Γ and a formal proof in Γ of a statement T from some extra assumption A, one may ask: is there also a formal proof in Γ of this extra assumption A from the statement T? The study of such questions, as far as they belong to the field of classical analysis or second-order arithmetic, is called *reverse mathematics*, see [26]. The weak basic theory there is RCA_0 .

1.2. The basic theory BIM. Our subject is intuitionistic reverse mathematics.

The weak basic theory we use is the two-sorted first-order intuitionistic theory BIM (Basic Intuitionistic Mathematics), introduced in [43]. The domain of discourse of BIM consists of two kinds of objects: natural numbers and infinite sequences of natural numbers. The axioms express some basic assumptions like the (full) principle of induction on the set ω of the natural numbers, and the fact that the set ω^{ω} of the infinite sequences of natural numbers is closed under the recursion-theoretic operations.

The reason that we use a basic theory different in spirit from the basic theory used in classical reverse mathematics is that, in intuitionistic analysis, one prefers the notion of an infinite sequence of natural numbers as a primitive notion above the notion of a subset of the set of the natural numbers, see [43, Section 5].

For the intuitionistic mathematician, the set ω^{ω} of all infinite sequences of natural numbers is not, as one sometimes says when explaining the notion of 'set' that lies at the basis of classical set theory, the result of taking together the earlier constructed and completed items that are to be the 'elements' of the set. The set ω^{ω} is a realm of possibilities: it is a framework for constructing, in the future, in all kinds of possibly as yet unforeseen ways, the objects that will be called the elements of the set. There are several kinds of infinite sequences $\alpha = (\alpha(0), \alpha(1), \ldots)$ of natural numbers. Sometimes, α is the result of executing a program, a finitely formulated algorithm. It is also possible that α is the result of a more or less free step-by-step construction that is not governed by a rule formulated at the start.

1.3. Two interpretations. The axioms of BIM hold for their intended interpretation, the interpretation given to them by the intuitionistic mathematician. The axioms of BIM also become true for her if she assumes that the elements of ω^{ω} are just the Turing-computable functions from ω to ω . Turing-computable functions may be represented by the natural number coding their program, and BIM may be seen to be a conservative extension of first-order intuitionistic arithmetic HA, Heyting arithmetic.

The model given by the computable functions thus is the *second interpretation* of BIM. Our study of this model is a contribution to *intuitionistic recursive analysis*.

In the weak context given by BIM one may study the further assumptions of the intuitionistic mathematician. They fall into three groups:

- (1) Axioms of Countable Choice,
- (2) Brouwer's Continuity Principle and the Axioms of Continuous Choice, and
- (3) Brouwer's Thesis on bars in Baire space ω^{ω} and the Fan Theorem.

The intuitionistic mathematican is prepared to argue the plausibility of these assumptions for her intended interpretation.

She defends the Axioms of Countable Choice, for instance, by explaining that the functions promised by the axioms may be constructed step by step.

She has no argument for the truth of the further assumptions under the second interpretation, where every function is assumed to be given by an algorithm. It is not clear to her if the Axioms of Countable Choice then are true.

Brouwer's Continuity Principle and its extensions, the Axioms of Continuous Choice, certainly fail in the second interpretation.

The Thesis on bars in ω^{ω} was introduced by Brouwer for the sake of the Fan Theorem. The Fan Theorem itself, dating from 1924, see [5], might be called *the Thesis on Bars in Cantor space* C, see [39], [40], and [43, Subsection 2.3].

In 1950, see [11], Kleene saw that, in our second interpretation, also the Fan Theorem, and, a fortiori, the Thesis on bars in ω^{ω} , do not hold. Actually, a positively formulated *strong negation* of the Fan Theorem becomes true. In [43], we called this statement *Kleene's Alternative* (to the Fan Theorem).

1.4. **Strong negations.** The (strict) Fan Theorem, **FT**, see 2.2.4, is the statement:

$$\forall \alpha [Bar_{\mathcal{C}}(D_{\alpha}) \rightarrow \exists n [Bar_{\mathcal{C}}(D_{\overline{\alpha}n})]],$$

and Kleene's Alternative (to the Fan Theorem), KA, see 2.2.8, is the statement

$$\exists \alpha [Bar_{\mathcal{C}}(D_{\alpha}) \land \forall n [\neg Bar_{\mathcal{C}}(D_{\overline{\alpha}n})]].$$

We want to call $\mathbf{K}\mathbf{A}$ the strong negation $\neg !\mathbf{F}\mathbf{T}$ of $\mathbf{F}\mathbf{T}$. In general, if we decide to call a statement B the strong negation of a statement A, B will be a statement more positive than the negation of A that constructively implies the negation of A. We do not require that the statement B is completely positive in the sense that

the corresponding formula does not contain \neg and \rightarrow ¹. We do not introduce *strong negation* as a syntactical operation on formulas. It is important to realize that, once we have understood that statement A is equivalent to statement B, it may be the case that statements C, D, which have been chosen to be called the strong negations of A, B, respectively, fail to be equivalent.

If we have decided to call the formula (denoted by) B the strong negation of the formula (denoted by) A, we will write $B = \neg!A$, but note that this notation belongs to the meta-language of BIM. $\neg!$ is *neither* a connective belonging to the language of BIM *nor* a syntactical operation on formulas.

We shall prove a number of results of the form:

In BIM, A is equivalent to B and $\neg !B$ is equivalent to $\neg !A$.

When we do so, we try to explain that the conclusions $A \to B$ and $\neg!B \to \neg!A$ have a common ground and that also the conclusions $B \to A$ and $\neg!A \to \neg!B$ have a common ground.

1.5. Contrapositions or reversals. We may compare Weak König's Lemma, WKL, see 2.2.12:

$$\forall \alpha [\forall n [\neg Bar_{\mathcal{C}}(D_{\overline{\alpha}n})] \to \exists \gamma \in \mathcal{C} \forall n [\alpha(\overline{\gamma}n) = 0]].$$

with the (strict) Fan Theorem, FT:

$$\forall \alpha [Bar_{\mathcal{C}}(D_{\alpha}) \to \exists n [Bar_{\mathcal{C}}(D_{\overline{\alpha}n})]].$$

We like to say that **WKL** is a *reversal* or *contraposition* of **FT** and also that **FT** is a *reversal* or *contraposition* of **WKL**.

We like to write: $\mathbf{WKL} = \overleftarrow{\mathbf{FT}}$ and $\mathbf{FT} = \overleftarrow{\mathbf{WKL}}$.

In general, if we call a statement B the *contraposition* or *reversal* \overleftarrow{A} of a statement A, both A and B will be largely positively formulated statements and the classical mathematician would think A and B are equivalent, but, constructively, A and B will have quite different meanings.

This is clear from the above example as **FT** is intuitionistically true (under our first interpretation) and **WKL** is false (in both interpretations).

It may happen also that both A and B are intuitionistically true (at least under the first interpretation), although they make different sense. An important example of this phenomenon is given by Π_1^0 - $\mathbf{AC}_{\omega,2}$, see 4.7, and Σ_1^0 - $\mathbf{AC}_{\omega,2}$, see 5.4 and Lemma 5.3.

Note that *Kleene's Alternative* (to the Fan Theorem), **KA**, might be called the strong negation ¬!**WKL** of **WKL** as well as the strong negation ¬!**FT** of **FT**.

We do not claim that, given a statement A, there always is a unique candidate for being called the *contraposition* of A. We do not introduce *contraposition* as a syntactical operation on formulas and use the term only in certain specific cases. It is important to realize that, once we have understood that statement A is equivalent to statement B, it may be the case that statements C, D, which one would like to call contrapositions of A, B, respectively, fail to be equivalent.

1.6. Non-intuitionistic assumptions. The reader may wonder why we pay attention to statements that fail in both our models, like Weak König's Lemma, WKL, and Bishop's Omniscience Principles, LPO, see 2.2.15, and LLPO, see 2.2.14. Doing so, however, we come to understand that certain other statements, being equivalent, in BIM, to one of them, also do not make sense in either one of our two interpretations.

¹See, for instance, the sentences 7.3 and 7.5 and Subsection 9.3.

- 1.7. Our aim. As in [42] and [43], it is our aim, in this paper, to find statements that are, in BIM, equivalent to either \mathbf{FT} or $\neg !\mathbf{FT} = \mathbf{KA}$.
- 1.8. **The contents of the paper.** Apart from this introduction, the paper contains Sections 2-13.

Section 2 is divided into two Subsections. In Subsection 2.1 we introduce the formal system BIM. Subsection 2.2 lists a number of assumptions one might study in the context of BIM.

In Section 3 we prove that, in BIM, the Σ_1^0 -Separation Principle Σ_1^0 -Sep is equivalent to WKL.

In Section 4 we formulate some special cases of the First Axiom of Choice $\mathbf{AC}_{\omega,\omega}$, among them Π_1^0 - $\mathbf{AC}_{\omega,2}$, the Π_1^0 -Axiom of Countable Binary Choice.

In Section 5 we introduce Σ_1^0 - $\overrightarrow{AC}_{\omega,2}$, a contraposition of Π_1^0 - $AC_{\omega,2}$, and we prove: in BIM, Σ_1^0 - $\overrightarrow{AC}_{\omega,2}$ is equivalent to FT, and a strong negation of Σ_1^0 - $\overrightarrow{AC}_{\omega,2}$ is equivalent to $\neg!FT = KA$.

Section 5 thus shows that a contraposition of Π_1^0 - $\mathbf{AC}_{\omega,2}$ fails in our second interpretation. This gives us no conclusion about the validity of Π_1^0 - $\mathbf{AC}_{\omega,2}$ itself in our second interpretation.

In Section 6 we formulate some special cases of the Second Axiom Scheme of Countable Choice $\mathbf{AC}_{\omega,\omega^{\omega}}$, among them $\mathbf{\Pi}_{1}^{0}$ - $\mathbf{AC}_{\omega,\mathcal{C}}$, the $\mathbf{\Pi}_{1}^{0}$ -Axiom of Countable Compact Choice.

In Section 7 we introduce $\Sigma_1^0 - \overleftarrow{\mathbf{AC}_{\omega,\mathcal{C}}}$, a contraposition of $\Pi_1^0 - \mathbf{AC}_{\omega,\mathcal{C}}$, and we prove: in BIM, $\Sigma_1^0 - \overleftarrow{\mathbf{AC}_{\omega,\mathcal{C}}}$ is equivalent to \mathbf{FT} , and a strong negation of $\Sigma_1^0 - \overleftarrow{\mathbf{AC}_{\omega,\mathcal{C}}}$ is equivalent to $\neg !\mathbf{FT}$.

In Section 8 we introduce Σ_1^0 - $\overrightarrow{AC}_{2,\mathcal{C}}$, a contraposition of a statement provable in BIM, to wit, the Π_1^0 - "axiom" of Twofold Compact Choice.

We prove: in BIM + Π_1^0 -AC_{$\omega,2$}, Σ_1^0 -AC_{2,C} is equivalent to FT. There is no companion result for \neg !FT.

In Section 9 we consider finite and infinite games. We explain in what sense we want to call such games I-determinate or II-determinate. We see that Σ_1^0 - $\overrightarrow{\mathbf{AC}}_{\omega,2}$ can be read as the statement that certain 2-move games are I-determinate. We prove: in BIM, \mathbf{FT} is equivalent to the statement that every subset of Cantor space $\mathcal C$ is (weakly) I-determinate, and $\neg !\mathbf{FT}$ is equivalent to the statement that there exists an open subset of $\mathcal C$ that positively fails to be I-determinate.

In Section 10 we consider a *Uniform Contrapositive Intermediate Value Theorem* UIVT and we prove: in BIM, FT is equivalent to UIVT and ¬!FT is equivalent to a strong negation of UIVT.

In Section 11 we see that, if one formulates the compactness theorem for classical propositional logic carefully and contrapositively, one obtains a statement that, in BIM, is equivalent to **FT**. ¬!**FT** is equivalent to a strong negation of this statement.

In Section 12 we ask the reader's attention for the *Approximate-Fan Theorem* **AppFT**, a statement stronger than **FT**. We did so already in [43, Subsection 10.2], see also [44]. **AppFT** is studied further in [44].

Section 13 contains a list of defined notions. This Section may be used as a reference by the reader of the preceding Sections.

I thank the referees of earlier versions of this paper. Their thorough comments and criticisms were very useful and led to numerous improvements. The last referee deserves special mention. He wrote three very detailed reports and found many points where I expressed myself wrongly or confusingly.

I also thank U. Kohlenbach for providing some references and for making me repair a mistaken observation in Section 10.

2. The formal system BIM

2.1. The basic axioms. BIM, introduced in [43, Section 6], has two kinds of variables, numerical variables m, n, p, \ldots , whose intended range is the set ω of the natural numbers, and function variables $\alpha, \beta, \gamma, \ldots$, whose intended range is the set ω^{ω} of all infinite sequences of natural numbers. There is a numerical constant 0. There are five unary function constants: Id, a name for the identity function, $\underline{0}$, a name for the zero function, S, a name for the successor function, and K, L, names for the projection functions. There is one binary function symbol J, a name for the pairing function on ω . From these symbols numerical terms are formed in the usual way. The basic terms are the numerical variables and the numerical constant and other terms are obtained from earlier constructed terms by the use of a function symbol. The function constants Id, $\underline{0}$, S, K and L and the function variables are the only function terms.

BIM has two equality symbols, $=_0$ and $=_1$. The first symbol may be placed between numerical terms and the second one between function terms. When confusion seems improbable we simply write = and not $=_0$ or $=_1$. The usual axioms for equality are part of BIM. A basic formula is an equality between numerical terms or an equality between function terms. A basic formula in the strict sense is an equality between numerical terms. We obtain the formulas of the theory from the basic formulas by using the connectives, the numerical quantifiers and the function quantifiers.

The logic of the theory is intuitionistic logic.

Our first axiom is

Axiom 1 (Axiom of Extensionality).

$$\forall \alpha \forall \beta [\alpha =_1 \beta \leftrightarrow \forall n [\alpha(n) =_0 \beta(n)]]$$

The Axiom of Extensionality guarantees that every formula will be provably equivalent to a formula built up by means of connectives and quantifiers from basic formulas in the strict sense.

The second axiom is the axiom on the unary function constants Id, $\underline{0}$, S, K, L, and the binary function constant J.

Axiom 2.

$$\forall n[Id(n) = n] \land$$

$$\forall n[\neg(S(n) = 0)] \land \forall m \forall n[S(m) = S(n) \to m = n] \land$$

$$\forall n[\underline{0}(n) = 0] \land$$

$$\forall m \forall n[K(J(m, n)) = m \land L(J(m, n)) = n \land J(K(n), L(n)) = n]$$

Thanks to the presence of the pairing function we may treat binary, ternary and other non-unary operations on ω as unary functions. " $\alpha(m,n,p)$ ", for instance, will be an abbreviation of " $\alpha(J(J(m,n),p))$ ".

We also introduce the following notation: for each n, n' := K(n) and n'' := L(n), and, for all m, n, (m, n) := J(m, n). The last part of Axiom 2 now reads as follows: $\forall m \forall n [(m, n)' = m \land (m, n)'' = n \land (n', n'') = n]$.

The next axiom² asks for the closure of the set ω^{ω} under composition, pairing, primitive recursion and unbounded search.

Axiom 3.

$$\forall \alpha \forall \beta \exists \gamma \forall n [\gamma(n) = \alpha (\beta(n))] \land \forall \alpha \forall \beta \exists \gamma \forall n [\gamma(n) = (\alpha(n), \beta(n))] \land$$

²A referee made us see that this Axiom 3, as formulated in [43], is a little bit too weak.

$$\forall \alpha \forall \beta \exists \gamma \forall m \forall n [\gamma(m,0) = \alpha(m) \land \gamma(m,S(n)) = \beta(m,n,\gamma(m,n))] \land \\ \forall \alpha [\forall n \exists m [\alpha(n,m) = 0] \rightarrow \exists \gamma \forall n [\alpha(n,\gamma(n)) = 0]]$$

We also need the Axiom Scheme of Induction:

Axiom 4.

$$(P(0) \land \forall n[P(n) \rightarrow P(S(n))]) \rightarrow \forall n[P(n)]$$

Instances of this axiom scheme are obtained by substituting a formula $\phi = \phi(m_0, m_1, \dots, m_{k-1}, \alpha_0, \alpha_1, \dots, \alpha_{l-1}, n)$ for P and taking the universal closure of the resulting formula. The reader should understand the further axiom schemes we are to mention in this paper in the same way.

The axioms 1-4, together with the usual axioms for equality, define the system BIM.

Note BIM has the full induction scheme whereas RCA₀ has only Σ_1^0 -induction, see [26, Definition II.1.5], a fact that is indicated by the suffix ₀. We did not study the possibility of restricting induction likewise and we do not answer the question if our results might have been obtained in a system that probably should be called BIM₀.

We form a conservative extension of BIM by adding constants for all primitive recursive functions and relations and making their defining equations into axioms. Primitive recursive relations are present via their characteristic functions. 'x < y', for instance, will be short for: ' $\chi_{<}(x,y) \neq 0$ '. Somewhat loosely, we also denote this conservative extension of BIM by the acronym BIM although one might decide to use the acronym BIM⁺, see [30].

BIM may be compared to the system **H** introduced in [9] and to the system **EL** occurring in [29] and to the system **IRA**, proposed by J.R. Moschovakis and G. Vafeiadou, see [22]. A precise proof of the fact that BIM and these systems are essentially equivalent may be found in [30].

2.2. Possible further assumptions.

2.2.1. First Axiom Scheme of Countable Choice, $\mathbf{AC}_{\omega,\omega}(=\mathbf{AC}_{0,0})$:

$$\forall n \exists m [R(n,m)] \rightarrow \exists \gamma \forall n [R(n,\gamma(n))].$$

The intuitionistic mathematician accepts $\mathbf{AC}_{\omega,\omega}$. If $\forall n \exists m[R(n,m)]$, she builds the promised γ step by step, first choosing $\gamma(0)$ such that $R(0,\gamma(0))$, then choosing $\gamma(1)$ such that $R(1,\gamma(1))$, and so on. In her view, there is no need to give a finite description or algorithm that determines the infinitely many values of γ at one stroke.

2.2.2. Second Axiom Scheme of Countable Choice, $\mathbf{AC}_{\omega,\omega^{\omega}} = \mathbf{AC}_{0,1}$:

$$\forall n \exists \gamma [R(n,\gamma)] \rightarrow \exists \gamma \forall n [R(n,\gamma^{\upharpoonright n})].^3$$

The intuitionistic mathematician accepts $\mathbf{AC}_{\omega,\omega^{\omega}}$. If $\forall n \exists \gamma [R(n,\gamma)]$, she first starts a project for building $\gamma^{\uparrow 0}$ with the property $R(0,\gamma^{\uparrow 0})$ and determines $\gamma^{\uparrow 0}(0)$, she then starts a project for building $\gamma^{\uparrow 1}$ with the property $R(1,\gamma^{\uparrow 1})$ and determines $\gamma^{\uparrow 1}(0)$, and also, continuing the project started earlier, $\gamma^{\uparrow 0}(1)$, she then starts a project for building $\gamma^{\uparrow 2}$ with the property $R(2,\gamma^{\uparrow 2})$ and determines $\gamma^{\uparrow 2}(0)$ and also, continuing the projects started earlier, $\gamma^{\uparrow 1}(1)$ and $\gamma^{\uparrow 0}(2)$,

2.2.3. The Fan Theorem as an Axiom Scheme, FAN:

$$\forall \beta [(Fan(\beta) \land Bar_{\mathcal{F}_{\beta}}(B)) \rightarrow \exists a [D_a \subseteq B \land Bar_{\mathcal{F}_{\beta}}(D_a)]].$$

³For the notation $\gamma^{\uparrow n}$ see Subsection 13.4.

2.2.4. The (strict) Fan Theorem, **FT**:

$$\forall \alpha [Bar_{\mathcal{C}}(D_{\alpha}) \to \exists m [Bar_{\mathcal{C}}(D_{\overline{\alpha}m})]], \ or, \ equivalently,$$

$$\forall \beta [Explfan(\beta) \to \forall \alpha [Bar_{\mathcal{F}_{\beta}}(D_{\alpha}) \to \exists m [Bar_{\mathcal{F}_{\beta}}(D_{\overline{\alpha}m})]]] \ or, \ equivalently,$$
$$\forall \beta [Explfan(\beta) \to \forall \alpha [Bar_{\mathcal{F}_{\beta}}(D_{\alpha}) \to \exists m \forall \gamma \in \mathcal{F}_{\beta} \exists n \leq m [\overline{\gamma}n \in D_{\alpha}]]].$$

Theorem 2.1. $BIM \vdash FT \leftrightarrow$

 $\forall \alpha [(Thinbar_{\mathcal{C}}(D_{\alpha}) \land D_{\alpha} \subseteq Bin) \to \exists n \forall m > n[m \notin D_{\alpha}]].$

Proof. The proof is left to the reader.

2.2.5. The strengthened (strict) Fan Theorem, \mathbf{FT}^+ :

$$\forall \beta [Fan(\beta) \to \forall \alpha [Bar_{\mathcal{F}_{\beta}}(D_{\alpha}) \to \exists m [Bar_{\mathcal{F}_{\beta}}(D_{\overline{\alpha}m})]]] \text{ or, equivalently,}$$
$$\forall \beta [Fan(\beta) \to \forall \alpha [Bar_{\mathcal{F}_{\beta}}(D_{\alpha}) \to \exists m \forall \gamma \in \mathcal{F}_{\beta} \exists n \leq m [\overline{\gamma}n \in D_{\alpha}]]].$$

Theorem 2.2. BIM \vdash $\mathbf{FT}^+ \leftrightarrow$

$$\forall \beta [Fan(\beta) \to \forall \alpha [(Thinbar_{\mathcal{F}_{\beta}}(D_{\alpha}) \land \forall s \in D_{\alpha}[\beta(s) = 0]) \to \exists n \forall m > n[m \notin D_{\alpha}]]].$$

Proof. The proof is left to the reader.

Note: BIM + $\mathbf{FAN} \vdash \mathbf{FT}^+$.

In this paper, we restrict attention to \mathbf{FT} . For the equivalence of the three formulations of \mathbf{FT} , see [43, Section 2.3, Theorem 9.6(ii) and Theorem 7.7(v)], and also [28] and [29, vol. I, Chapter 4, Section 7.5]⁴.

In 4.5, we introduce Weak- Π_1^0 - $\mathbf{AC}_{\omega,\omega}$, a special case of $\mathbf{AC}_{\omega,\omega}$.

 $BIM+Weak-\Pi_1^0-AC_{\omega,\omega}\vdash FT\to FT^+$, see Theorem 4.2.

2.2.6. Brouwer's Thesis: Bar Induction as an Axiom Scheme, BARIND:

$$\left(Bar_{\omega^{\omega}}(B) \wedge \forall s[s \in B \to s \in C] \wedge \forall s[\forall n[s * \langle n \rangle \in C] \leftrightarrow s \in C]\right) \to \langle \rangle \in C.$$

Brouwer derived **FAN** from **BARIND**, see [43, Subsections 2.2 and 2.3]. We repeat the proof, in order to prepare the reader for Theorem 12.4.

Theorem 2.3. $BIM + BARIND \vdash FAN$.

Proof. Let β be given such that $Fan(\beta)$ and $\beta(\langle \rangle) = 0.5$ Assume $Bar_{\mathcal{F}_{\beta}}(B)$. Define $B' := B \cup \{s \mid \beta(s) \neq 0\}$.

We claim: $Bar_{\omega^{\omega}}(B')$. We prove this claim as follows. Let γ be given. Define γ^* such that, for each n, if $\beta(\overline{\gamma}(n+1)) = 0$, then $\gamma^*(n) = \gamma(n)$, and, if not, then $\gamma^*(n) = \mu p[\beta(\overline{\gamma}n * \langle p \rangle) = 0]$. Note: $\gamma^* \in \mathcal{F}_{\beta}$ and find n such that $\overline{\gamma^*}n \in B$. Either $\overline{\gamma}n = \overline{\gamma^*}n$ and $\overline{\gamma}n \in B$ or $\overline{\gamma}n \neq \overline{\gamma^*}n$ and $\beta(\overline{\gamma}n) \neq 0$. In both cases, $\overline{\gamma}n \in B'$.

We thus see: $\forall \gamma \exists n [\overline{\gamma} n \in B']$, i.e. $Bar_{\omega^{\omega}}(B')$.

Let C be the set of all s such that

either: $\beta(s) \neq 0$ or: $\beta(s) = 0$ and $\exists a [D_a \subseteq B \land Bar_{\mathcal{F}_{\beta} \cap s}(D_a)].$

For every s, if $\beta(s)=0$ and $s\in B$, define $a:=\overline{0}s*\langle 1\rangle$ and note: $\{s\}=D_a\subseteq B$ and $Bar_{\mathcal{F}_{\beta}\cap s}(D_a)$. Conclude: $B\subseteq C$.

Now let s be given such that $\forall m[s*\langle m\rangle \in C]$.

Find n such that $\forall m \geq n[\beta(s*\langle m \rangle) \neq 0]$. Find b in ω^n such that

 $\forall m < n[\beta(s * \langle m \rangle) = 0 \to (D_{b(m)} \subseteq B \land Bar_{\mathcal{F}_{\beta} \cap s * \langle m \rangle}(D_{b(m)}))] \text{ and }$

 $\forall m < n[\beta(s * \langle m \rangle) \neq 0 \rightarrow b(m) = \langle \ \rangle].$

Find a such that $p := length(a) = \max_{m < n} length(b(m))$ and

 $\forall t < p[a(t) \neq 0 \leftrightarrow \exists m < n[(b(m))(t) \neq 0]].$

Note: $D_a \subseteq B$ and $Bar_{\mathcal{F}_{\beta} \cap s}(D_a)$, and conclude: $s \in C$.

⁴At the latter two places, no attention is given to the rôle of Weak- Π_1^0 - $AC_{\omega,\omega}$, see 4.5.

⁵If $\beta(\langle \rangle) \neq 0$, then $\mathcal{F}_{\beta} = \emptyset$ and there is nothing to prove.

We thus see: $\forall s [\forall m [s * \langle m \rangle \in C] \rightarrow s \in C]$.

Obviously: $\forall s \forall m [s \in C \to s * \langle m \rangle \in C]$.

Using **BARIND**, we conclude: $\langle \rangle \in C$, i.e. $\exists a[D_a \subseteq B \land Bar_{\mathcal{F}_{\beta}}(D_a)].$

2.2.7. Church's Thesis, CT:

$$\exists \tau \exists \psi \forall \alpha \exists e \forall n \exists z [z = \mu i [\tau(e, n, i) \neq 0] \land \psi(z) = \alpha(n)].$$

Kleene has shown that \mathbf{CT} is true in the model of BIM given by the computable functions. He provided $\mathit{K\'almar-elementary}$ functions τ, ψ satisfying the above conditions.

Note that our formulation of **CT** is cautious and somewhat weaker than the usual one. We do not require that the set $\{(e,n,z) \mid \tau(e,n,z) \neq 0\}$ coincides with Kleene's set T, but only ask that the set behaves appropriately. A similar 'abstract' approach to Church's Thesis has been advocated by F. Richman, see [24] and [3, Chapter 3, Section 1].

2.2.8. Kleene's Alternative (to the Fan Theorem), ¬!**FT**:

$$\exists \alpha [Bar_{\mathcal{C}}(D_{\alpha}) \land \forall m [\neg Bar_{\mathcal{C}}(D_{\overline{\alpha}m})]].$$

The following result is due to Kleene.

Theorem 2.4. BIM + CT $\vdash \neg !FT$.

Proof. Let τ, ψ be as in CT. Define α such that, for all m, for all c in Bin_m ,

$$\alpha(c) = 0 \leftrightarrow \forall e < m \forall z < m[z = \mu i [\tau(e, e, i) \neq 0] \rightarrow c(e) = 1 - \psi(z)].$$

Let γ in \mathcal{C} be given. Find e such that, for all n, $\gamma(n) = \psi(\mu i [\tau(e,n,i) \neq 0])$. Define $z = \mu i [\tau(e,e,i) \neq 0]$. Note: $\gamma(e) = \psi(z)$. Define $m := \max(e,z) + 1$ and note: $\alpha(\overline{\gamma}m) \neq 0$. Conclude: $Bar_{\mathcal{C}}(D_{\alpha})$.

Let m be given. Find c in Bin_m such that

 $\forall e < m \forall z < m[z = \mu i[\tau(e,e,i) \neq 0] \rightarrow c(e) = 1 - \psi(z)].$ Note: $\forall n \leq m[\alpha(\overline{c}n) = 0].$ Define $\gamma := c * \underline{0}$ and note: $\forall n[c * \underline{\overline{0}}n > m]$ and conclude: $\neg \exists k[\overline{\gamma}k < m \land \alpha(\overline{\gamma}k) \neq 0]$ and: $\neg Bar_{\mathcal{C}}(D_{\overline{\alpha}m}).$

The above proof may be found in [29, vol. I, Chapter 4, Subsection 7.6]. In [43, Section 3] one finds two proofs.

We do not know the answer to the question if $BIM + \neg !FT \vdash CT$.

2.2.9. Brouwer's (unrestricted) Continuity Principle as an Axiom Scheme, BCP:

$$\forall \alpha \exists n [\alpha Rn] \to \forall \alpha \exists m \exists n \forall \beta [\overline{\alpha}m \sqsubseteq \beta \to \beta Rn].$$

Brouwer used this principle for the first time in 1918, see [4, Section 1, page 13]. The intuitionistic mathematician believes the axiom to be plausible.

She argues as follows.

If $\forall \alpha \exists n[\alpha Rn]$, I must be able, given any α , to find effectively an n as promised, also if the values of α are disclosed one by one and nothing is told about a law governing the development of α as a whole. I will decide upon n at some point of time and, at that point of time, only finitely many values of α , say

 $\alpha(0), \alpha(1), \ldots, \alpha(m-1)$, will be known to me. The number n will satisfy any infinite sequence that is a continuation of $\alpha(0), \alpha(1), \ldots, \alpha(m-1)$.

The Continuity Principle is revolutionary and changes one's mathematical perspective, see [35].

The classical mathematician may ask for a consistency proof for **BCP**. Kleene proved, using realizability methods, that his formal system **FIM** for intuitionistic analysis, actually an extension of BIM + FT + BCP, is (simply) consistent, see [12, Chapter II, Subsection 9.2]. Kleene's consistency proof should convince both

the classical mathematician and the constructive mathematician, should the latter accept **BARIND** but be plagued by doubts concerning **BCP**.

It is not difficult to see that BIM + CT + BCP is inconsistent, as it implies $\forall \alpha \exists m \forall \beta [\overline{\beta}m = \overline{\alpha}m \to \beta = \alpha]$, see [29, vol. I, Chapter 4, Theorem 6.7].

The next two axioms strengthen **BCP**. Kleene's consistency proof extends to these stronger forms of the Continuity Principle.

2.2.10. The First Axiom Scheme of Continuous Choice, $\mathbf{AC}_{\omega^{\omega},\omega}(=\mathbf{AC}_{1,0})$:

$$\forall \alpha \exists n[\alpha Rn] \to \exists \varphi : \omega^{\omega} \to \omega \forall \alpha [\alpha R\varphi(\alpha)].$$

2.2.11. The Second Axiom Scheme of Continuous Choice, $\mathbf{AC}_{\omega^{\omega},\omega^{\omega}}(=\mathbf{AC}_{1,1})$:

$$\forall \alpha \exists \beta [\alpha R \beta] \to \exists \varphi : \omega^\omega \to \omega^\omega \forall \alpha [\alpha R \varphi | \alpha].$$

2.2.12. Weak König's Lemma, WKL:

$$\forall \alpha [\forall n \exists a \in Bin_n \forall m \leq n [\alpha(\overline{a}n) = 0] \rightarrow \exists \gamma \in \mathcal{C} \forall n [\alpha(\overline{\gamma}n) = 0]],$$

or, equivalently,
$$\forall \alpha [\forall n [\neg Bar_{\mathcal{C}}(D_{\overline{\alpha}n})] \rightarrow \exists \gamma \in \mathcal{C} \forall n [\alpha(\overline{\gamma}n) = 0]].$$

WKL, as a classical theorem, dates from 1927, see [13]. It is a contraposition of **FT** and, from a classical point of view, the two are equivalent. The following result is due to U. Kohlenbach, see [14] and also [10].

Theorem 2.5. BIM \vdash WKL \rightarrow FT.

Proof. Assume WKL.

Let α be given such that $Bar_{\mathcal{C}}(D_{\alpha})$. We intend to prove: $\exists n[Bar_{\mathcal{C}}(D_{\overline{\alpha}n})]$.

Define α^* such that $\forall s \in Bin[\alpha^*(s) = 0 \leftrightarrow \forall t \sqsubseteq s[\alpha(t) = 0]].$

Define α^{**} such that, for all n, for all s in Bin_n , $\alpha^{**}(s) = 0$ if and only if either $\alpha^*(s) = 0$ or $\neg \exists t \in Bin_n[\alpha^*(t) = 0]$.

Clearly, for each n, there exists s in Bin_n such that $\alpha^{**}(s) = 0$.

Applying WKL, find γ in \mathcal{C} such that $\forall n[\alpha^{**}(\overline{\gamma}n) = 0]$.

Find n such that $\alpha(\overline{\gamma}n) \neq 0$.

Conclude: $\neg \exists t \in Bin_n[\alpha^*(t) = 0]$ and: $\forall \delta \in \mathcal{C} \exists j \leq n[\alpha(\overline{\delta}j) \neq 0]$ and: $\exists m[Bar_{\mathcal{C}}(D_{\overline{\alpha}m})].$

We thus see:
$$\forall \alpha [Bar_{\mathcal{C}}(D_{\alpha}) \to \exists m [Bar_{\mathcal{C}}(D_{\overline{\alpha}m})]]$$
, i.e. **FT**.

2.2.13. Weak König's Lemma with a uniqueness condition, WKL!:

$$\forall \alpha [(\forall m [\neg Bar_{\mathcal{C}}(D_{\overline{\alpha}m})] \land \forall \gamma \in \mathcal{C} \forall \delta \in \mathcal{C} [\gamma \perp \delta \rightarrow \exists m [\alpha(\overline{\gamma}m) \neq 0 \lor \alpha(\overline{\delta}m) \neq 0]]) \rightarrow \exists \gamma \forall n [\alpha(\overline{\gamma}n) = 0]].$$

The next two theorems, apart from being of interest in themselves, are useful for the discussion in Subsubsection 8.1.1.

Theorem 2.6. BIM \vdash WKL! \rightarrow FT.

Proof. ⁶ Assume **WKL**!.

Let α be given such that $Bar_{\mathcal{C}}(D_{\alpha})$. We intend to prove: $\exists m[Bar_{\mathcal{C}}(D_{\overline{\alpha}m})]$.

If $\alpha(0) = \alpha(\langle \rangle) \neq 0$, then $D_{\overline{\alpha}1} = \{\langle \rangle\}$ and $Bar_{\mathcal{C}}(D_{\overline{\alpha}1})$, and we are done.

We now assume $\alpha(\langle \rangle) = 0$, that is: $\langle \rangle \notin D_{\alpha}$.

Define α^* such that $\forall s \in Bin[\alpha^*(s) = 0 \leftrightarrow \forall t \subseteq s[\alpha(t) = 0]].$

Define α^{**} such that $\alpha^{**}(\langle \rangle) = 0$ and, for all n > 0, for all s in Bin_n , $\alpha^{**}(s) = 0$ if and only if either $\alpha^*(s) = 0$ and $\forall t \in Bin_n[\alpha^*(t) = 0 \to s \leq_{lex} t]$ or $\neg \exists t \in Bin_n[\alpha^*(t) = 0]$ and $\exists t \in Bin_{n-1}[\alpha^{**}(t) = 0 \land s = t * \langle 0 \rangle]$.

Using induction, one proves that, for each n, there is exactly one s in Bin_n such that $\alpha^{**}(s) = 0$.

⁶The equivalence of **FT** and **WKL**! is a result due to J. Berger and H. Ishihara, see [2]. Another proof has been given by H. Schwichtenberg, see [25].

Let γ, δ in \mathcal{C} be given such that $\gamma \perp \delta$.

Find n such that $\overline{\gamma}n \perp \overline{\delta}n$ and note: $\alpha^{**}(\overline{\gamma}n) \neq 0 \vee \alpha^{**}(\overline{\delta}n) \neq 0$.

We thus see: $\forall \gamma \in \mathcal{C} \forall \delta \in \mathcal{C} [\gamma \perp \delta \rightarrow \exists n [\alpha^{**}(\overline{\gamma}n) \neq 0 \lor \alpha^{**}(\overline{\delta}n) \neq 0]].$

Applying WKL!, find γ in \mathcal{C} such that $\forall n[\alpha^{**}(\overline{\gamma}n) = 0]$.

Find n such that $\alpha(\overline{\gamma}n) \neq 0$.

Conclude: $\neg \exists t \in Bin_n[\alpha^*(t) = 0]$ and: $\forall \delta \in \mathcal{C} \exists j \leq n[\alpha(\overline{\delta}j) \neq 0]$ and $\exists m[Bar_{\mathcal{C}}(D_{\overline{\alpha}m})].$

We thus see: $\forall \alpha [Bar_{\mathcal{C}}(D_{\alpha}) \to \exists m [Bar_{\mathcal{C}}(D_{\overline{\alpha}m})]]$, i.e. **FT**.

Theorem 2.7. BIM \vdash FT \rightarrow WKL!.

Proof. Assume **FT**. Let α be given such that $\forall n[\neg Bar_{\mathcal{C}}(D_{\overline{\alpha}n})]]$ and

 $\forall \gamma \in \mathcal{C} \forall \delta \in \mathcal{C}[\gamma \perp \delta \to \exists n[\alpha(\overline{\gamma}n) \neq 0 \lor \alpha(\overline{\delta}n) \neq 0]].$

We intend to prove: $\exists \gamma \in \mathcal{C} \forall n [\alpha(\overline{\gamma}n) = 0].$

Define α^* such that $\forall s \in Bin[\alpha^*(s) = 0 \leftrightarrow \forall t \sqsubseteq s[\alpha(t) = 0]].$

Let s in Bin be given.

Note: $\forall \gamma \in \mathcal{C} \forall \delta \in \mathcal{C} \exists k [\alpha(s * \langle 0 \rangle * \overline{\gamma} k) \neq 0 \lor \alpha(s * \langle 1 \rangle * \overline{\delta} k) \neq 0].$

Applying \mathbf{FT} , find m such that

 $\forall \gamma \in \mathcal{C} \forall \delta \in \mathcal{C} \exists k \leq m [\alpha(s * \langle 0 \rangle * \overline{\gamma} k) \neq 0 \lor \alpha(s * \langle 1 \rangle * \overline{\delta} k) \neq 0].$

Conclude: $\forall c \in Bin_m \forall d \in Bin_m [\alpha^*(s * \langle 0 \rangle * c) \neq 0 \lor \alpha^*(s * \langle 1 \rangle * d) \neq 0].$

Conclude: $\forall c \in Bin_m[\alpha^*(s*\langle 0 \rangle * c) \neq 0] \lor \forall d \in Bin_m[\alpha^*(s*\langle 1 \rangle * d) \neq 0].$

This last step is justified by the fact that BIM proves the following scheme⁷: $\forall n [\forall k < n \forall l < n[A(k) \lor B(l)] \rightarrow (\forall k < n[A(k)] \lor \forall l < n[B(l)])].$

Now define ζ such that, for every s in Bin,

 $\zeta(s) = \mu k [\forall c \in Bin_k [\alpha^*(s * \langle 0 \rangle * c) \neq 0] \lor \forall d \in Bin_k [\alpha^*(s * \langle 1 \rangle * d) \neq 0].$

Then define δ in \mathcal{C} such that, for every s in Bin,

 $\forall c \in Bin_{\zeta(s)}[\alpha^*(s*\langle 1-\delta(s)\rangle*c) \neq 0].$

Finally, define γ such that $\forall n[\gamma(n) = \delta(\overline{\gamma}n)].$

One may prove by induction, using $\forall n [\neg Bar_{\mathcal{C}}(D_{\overline{\alpha}n})]$:

for each $n, \forall m \geq n \exists s \in Bin_m [\overline{\gamma}n \sqsubseteq s \land \alpha^*(s) = 0].$

Conclude: $\forall n[\alpha^*(\overline{\gamma}n) = 0]$ and $\forall n[\alpha(\overline{\gamma}n) = 0]$.

We thus see: for each α , if $\forall n[\neg Bar_{\mathcal{C}}(D_{\overline{\alpha}n})]$ and

 $\forall \gamma \in \mathcal{C} \forall \delta \in \mathcal{C}[\gamma \perp \delta \to \exists n[\alpha(\overline{\gamma}n) \neq 0 \lor \alpha(\overline{\delta}n) \neq 0]], \text{ then } \exists \gamma \in \mathcal{C} \forall n[\alpha(\overline{\gamma}n) = 0],$ i.e. **WKL**!.

2.2.14. The Lesser Limited Principle of Omniscience, LLPO: 8

$$\forall \alpha \exists i < 2 \forall p[2p + i \neq \mu n[\alpha(n) \neq 0]].$$

Theorem 2.8. BIM \vdash BCP $\rightarrow \neg$ LLPO.

Proof. Assume LLPO.

Using **BCP**, find m,i such that i < 2 and $\forall \alpha [\overline{\underline{0}}m \sqsubset \alpha \rightarrow \forall p[2p+i \neq \mu n[\alpha(n) \neq 0]]]$. Define $\alpha := \overline{\underline{0}}(2m+i) * \langle 1 \rangle * \underline{0}$ and note: $\overline{\underline{0}}m \sqsubset \alpha$ and $2m+i = \mu n[\alpha(n) \neq 0]$. Contradiction.

Theorem 2.9. BIM \vdash WKL \rightarrow LLPO.

Proof. Assume **WKL**. Let α be given. Define β such that, for every s, $\beta(s)=0$ if and only if $\exists q \exists i < 2[s=\overline{i}q \land \forall p[2p+i < n \to 2p+i \neq \mu n[\alpha(n) \neq 0]]]$. Note: $\forall m \exists i < 2 \forall q \leq m[\beta(\overline{i}q)=0]$. Using **WKL**, find γ such that $\forall n[\beta(\overline{\gamma}n)=0]$. Define $i:=\gamma(0)$ and conclude: $\gamma=\underline{i}$ and $\forall p[2p+i \neq \mu n[\alpha(n) \neq 0]]$.

⁷This follows from our Lemma 9.1.

 $^{{}^8\}mu n[P(n)] = k \leftrightarrow (P(k) \land \forall n < k[\neg P(n)])$. Therefore: $\mu n[\alpha(n) \neq 0] \neq k \leftrightarrow (\alpha(k) = 0 \lor \exists i < k[\alpha(i) \neq 0])$.

Using a weak axiom of countable choice, one may also prove **LLPO** \rightarrow **WKL**, see Theorem 4.3.

2.2.15. The Limited Principle of Omniscience, LPO:

$$\forall \alpha [\exists n [\alpha(n) \neq 0] \lor \forall n [\alpha(n) = 0]].$$

LPO and **LLPO** were introduced by E. Bishop, see [3, Chapter 1, Section 1]. The following result is not difficult and well-known, see [20, Section 2.6] and [1, Theorem 3.1].

Theorem 2.10. BIM \vdash LPO \rightarrow LLPO.

Proof. Let α be given. Apply **LPO** and distinguish two cases.

Case (1). $\exists n[\alpha(n) \neq 0]$. Find q, k such that k < 2 and $2q + k = \mu n[\alpha(n) \neq 0]$. Conclude: $\forall p[2p + 1 - k \neq \mu n[\alpha(n) \neq 0]]$.

Case (2).
$$\forall n[\alpha(n) = 0]$$
. Then $\forall k < 2 \forall p[2p + k \neq \mu n[\alpha(n) \neq 0]]$.

The following Lemma shows that, in BIM + BCP, not every closed subset of ω^{ω} is a spread, see also [45, Theorem 2.10 (vi)].

Lemma 2.11. BIM $\vdash \forall \beta \exists \gamma [Spr(\gamma) \land \mathcal{F}_{\beta} = \mathcal{F}_{\gamma}] \rightarrow \mathbf{LPO}.$

Proof. Let α be given.

Define β such that $\beta(0) = 0$ and $\forall n \forall s [\beta(\langle n \rangle * s) = 0 \leftrightarrow \alpha(n) \neq 0]$.

Assume we find γ such that $Spr(\gamma)$ and $\mathcal{F}_{\beta} = \mathcal{F}_{\gamma}$.

If
$$\gamma(0) = 0$$
, then $\exists n[\alpha(n) \neq 0]$ and, if $\gamma(0) \neq 0$, then $\forall n[\alpha(n) = 0]$.

2.2.16. Markov's principle, MP:

$$\forall \alpha [\neg \neg \exists n [\alpha(n) \neq 0] \rightarrow \exists n [\alpha(n) \neq 0]].$$

For some discussion of this principle, see [29, Volume I, Chapter 4, Section 5]. In this paper, the principle figures only in Subsection 12.1.

We would like to make a philosophical observation. For a constructive mathematician, the assumptions **LPO**, **WKL**, **LLPO** and **MP** make no sense, as he does not know a situation in which these assumptions are true. Theorem 2.9: **WKL** \rightarrow **LLPO** concludes something which is never true from something which is never true. Nevertheless, the proof of Theorem 2.9 makes sense. It shows us how to find, given any α , a suitable β such that if β has the **WKL**-property, then α has the **LLPO**-property. This part of the argument does not use the assumption that every β has the **WKL**-property. Theorems 2.5 and 2.10 deserve a similar comment.

The reader may find more information on the axioms of intuitionistic analysis in [5], [6], [8], [12], [9], [29] and [46].

3. The Σ_1^0 -separation principle

3.1. In classical reverse mathematics, weak König's Lemma is equivalent to a principle called Σ_1^0 -separation, see [26, Lemma IV.4.4].

We call $X \subseteq \omega$ enumerable or Σ_1^0 if and only if $\exists \alpha [X = E_{\alpha}]$.

The following statement, formulated in the intuitionistic language of BIM, comes close to the just-mentioned classical principle.

 Σ_1^0 -separation principle, Σ_1^0 -Sep:

$$\forall \alpha [\neg \exists n \forall i < 2[(n,i) \in E_{\alpha}] \to \exists \gamma \forall n [(n,\gamma(n)) \notin E_{\alpha}]].$$

The next Theorem seems to confirm the just-mentioned classical result.

Theorem 3.1. BIM \vdash WKL $\leftrightarrow \Sigma_1^0$ -Sep.

Proof. (i) Assume **WKL**. We prove: Σ_1^0 -**Sep**.

Let α be given such that $\neg \exists n \forall i < 2[(n, i) \in E_{\alpha}].$

Define β such that $\forall n \forall a \in Bin_n[\beta(a) = 0 \leftrightarrow \forall m < n[(m, a(m)) \notin E_{\overline{\alpha}n}]].$

Let n be given. Note: $\forall m < n \exists i < 2[(m,i) \notin E_{\overline{\alpha}n}]$ and find a in Bin_n such that $\forall m < n[(m,a(m)) \notin E_{\overline{\alpha}n}]$.

Conclude: $\forall j \leq m[\beta(\overline{a}j) = 0].$

We thus see: $\forall n \exists a \in Bin_n \forall m \leq n[\beta(\overline{a}n) = 0].$

Using **WKL**, find γ in \mathcal{C} such that $\forall n[\beta(\overline{\gamma}n) = 0]$.

Conclude: $\forall n \forall m < n[(m, \gamma(m)) \notin E_{\overline{\alpha}n}]$ and: $\forall n[(n, \gamma(n)) \notin E_{\alpha}].$

We thus see: $\forall \alpha [\neg \exists n \forall i < 2[(n,i) \in E_{\alpha}] \to \exists \gamma \forall n [(n,\gamma(n)) \notin E_{\alpha}]$, i.e. Σ_1^0 -Sep.

(ii) Assume Σ_1^0 -Sep. We prove: WKL.

Let α be given such that $\forall m[\neg Bar_{\mathcal{C}}(D_{\overline{\alpha}m})].$

Define β such that, for both i < 2, for all n, for all s,

if $s \in Bin$ and $Bar_{\mathcal{C} \cap s*\langle i \rangle}(D_{\overline{\alpha}n})$ and $\neg Bar_{\mathcal{C} \cap s*\langle 1-i \rangle}(D_{\overline{\alpha}n})$, then $\beta(n,s) = (s,i)+1$, and, if not, then $\beta(n,s) = 0$.

Note: E_{β} is the set of all $s * \langle i \rangle$ such that $s \in Bin$ and i < 2 and

 $\exists n[Bar_{\mathcal{C}\cap s*\langle i\rangle}(D_{\overline{\alpha}n}) \land \neg Bar_{\mathcal{C}\cap s*\langle 1-i\rangle}(D_{\overline{\alpha}n})].$

Note: for all s in Bin, for all i < 2,

if $s * \langle i \rangle \notin E_{\beta}$, then $\forall n[Bar_{\mathcal{C} \cap s * \langle i \rangle}(D_{\overline{\alpha}n}) \to Bar_{\mathcal{C} \cap s * \langle 1-i \rangle}(D_{\overline{\alpha}n})].$

Note: $\neg \exists s \forall i < 2[(s,i) \in E_{\beta}].$

Using Σ_1^0 -Sep, find γ in \mathcal{C} such that $\forall s[(s, \gamma(s)) \notin E_\beta]$.

Define δ in \mathcal{C} by: $\forall n[\delta(n) = \gamma(\overline{\delta}n)].$

Suppose we find n such that $\alpha(\overline{\delta}n) \neq 0$.

Define $q := \overline{\delta}n + 1$ and note: $\overline{\delta}n \in D_{\overline{\alpha}q}$, that is: $Bar_{\mathcal{C} \cap \overline{\delta}n}(D_{\overline{\alpha}q})$.

We prove, using backwards induction: $\forall j \leq n[Bar_{\mathcal{C} \cap \overline{\delta}j}(D_{\overline{\alpha}q})].$

Our starting point is: $Bar_{\mathcal{C} \cap \overline{\delta}n}(D_{\overline{\alpha}q})$.

Now suppose $j+1 \leq n$ and $Bar_{\mathcal{C} \cap \overline{\delta}(j+1)}(D_{\overline{\alpha}q})$. Note: $\overline{\delta}(j+1) = \overline{\delta}j * \langle \gamma(\overline{\delta}j) \rangle$.

Also: $(\overline{\delta}j, \gamma(\overline{\delta}j)) \notin E_{\beta}$ and, therefore: $Bar_{\mathcal{C} \cap \overline{\delta}j*\langle 1-\gamma(\overline{\delta}(j)\rangle}(D_{\overline{\alpha}q})$ and: $Bar_{\mathcal{C} \cap \overline{\delta}j}(D_{\overline{\alpha}q})$.

This completes the proof of the induction step.

After n steps we reach the conclusion: $Bar_{\mathcal{C}}(D_{\overline{\alpha}q})$.

This contradicts the assumption: $\forall m[\neg Bar_{\mathcal{C}}(D_{\overline{\alpha}m})].$

Conclude: $\forall n[\alpha(\overline{\delta}n) = 0].$

We thus see:
$$\forall \alpha [\forall n [\neg Bar_{\mathcal{C}}(D_{\overline{\alpha}n})] \rightarrow \exists \delta \in \mathcal{C} \forall n [\alpha(\overline{\delta}n) = 0]]$$
, i.e. **WKL**.

Theorem 3.1 shows that Σ_1^0 -Sep, like WKL, is not constructive, see Theorem 2.9. It is not true in intuitionistic analysis and it also fails in the model of BIM given by the recursive functions.

4.
$$\mathbf{AC}_{\omega,\omega}$$
, SOME SPECIAL CASES

The following restricted version of $\mathbf{AC}_{\omega,\omega}$ is provable in BIM as it is a consequence of Axiom 3, see Section 2.

4.1. Minimal Axiom of Countable Choice, Δ_1^0 -AC_{ω,ω}:

$$\forall \alpha [\forall n \exists m [\alpha(n, m) = 0] \to \exists \gamma \forall n [\alpha(n, \gamma(n)) = 0]].$$

4.2. Axiom Scheme of Countable Unique Choice, $\mathbf{AC}_{\omega,\omega}! = \mathbf{AC}_{0,0}!$:

$$\forall n \exists ! m[R(n,m)] \rightarrow \exists \gamma \forall n[R(n,\gamma(n))].$$

where ' $\forall n \exists ! m[R(n,m)]$ ' abbreviates ' $\forall n \exists m[R(n,m) \land \forall p[R(n,p) \to m=p]]$ '.

 $\mathbf{AC}_{\omega,\omega}!$ is not a theorem of BIM, see Subsection 4.8 and [30].

4.3. Σ_1^0 -First Axiom of Countable Choice, Σ_1^0 - $\mathbf{AC}_{\omega,\omega}$:

$$\forall \alpha [\forall n \exists m [(n, m) \in E_{\alpha}] \to \exists \gamma \forall n [(n, \gamma(n)) \in E_{\alpha}]].$$

Theorem 4.1. BIM $\vdash \Sigma_1^0$ -AC_{ω,ω}.

Proof. Assume $\forall n \exists m [(n,m) \in E_{\alpha}]$. Then $\forall n \exists m \exists p [\alpha(p) = (n,m)+1]$. Find δ such that $\forall n [\delta(n) = \mu q [\alpha(q') = (n,q'')+1]]$. Define γ such that $\forall n [\gamma(n) = \delta''(n)]$ and note: $\forall n [(n,\gamma(n)) \in E_{\alpha}]$.

We call $X \subseteq \omega$ co-enumerable or Π_1^0 if and only if there exists α such that $X = \omega \setminus E_{\alpha}$.

4.4. Π_1^0 -First Axiom of Countable Choice, Π_1^0 - $\mathbf{AC}_{\omega,\omega}$:

$$\forall \alpha [\forall n \exists m [(n, m) \notin E_{\alpha}] \to \exists \gamma \forall n [(n, \gamma(n)) \notin E_{\alpha}]].$$

 Π_1^0 - $\mathbf{AC}_{\omega,\omega}$ is unprovable in BIM, see Subsection 4.8. In [43, Section 6], we introduced the following special case of this axiom.

4.5. Weak Π_1^0 -First Axiom of Countable Choice, Weak- Π_1^0 - $\mathbf{AC}_{\omega,\omega}$:

$$\forall \alpha [\forall m \exists n \forall p \geq n [\alpha(m,p) \neq 0] \rightarrow \exists \gamma \forall m \forall p \geq \gamma(m) [\alpha(m,p) \neq 0]].$$

Weak- Π_1^0 - $\mathbf{AC}_{\omega,\omega}$ follows from $\mathbf{AC}_{\omega,\omega}! = \mathbf{AC}_{0,0}!$.

Weak- Π_1^0 - $\mathbf{AC}_{\omega,\omega}$ is a special case of the axiom scheme $AC_{0,0}^m$ introduced in [22, Subsection 3.1]. We suspect that, in BIM, Weak- Π_1^0 - $\mathbf{AC}_{\omega,\omega}$ does not imply Π_1^0 - $\mathbf{AC}_{\omega,\omega}$, but we have no proof.

Theorem 4.2. (i) BIM+ Weak- Π_1^0 -AC_{ω,ω} $\vdash \forall \beta [Fan(\beta) \to Explfan(\beta)]$. (ii) BIM+ Weak- Π_1^0 -AC_{ω,ω} \vdash FT \to FT⁺.

Proof. The proof is left to the reader.

One may also study statements one obtains from $\mathbf{AC}_{\omega,\omega}$ by limiting the number of alternatives one has at each choice.

4.6. Axiom Scheme of Countable Binary Choice, $AC_{\omega,2}$:

$$\forall n \exists m < 2[R(n,m)] \rightarrow \exists \gamma \in \mathcal{C} \forall n [R(n,\gamma(n))].$$

Here is a restricted version of $AC_{\omega,2}$:

4.7. Π_1^0 -Axiom of Countable Binary Choice, Π_1^0 -AC $_{\omega,2}$:

$$\forall \alpha [\forall n \exists m < 2[(n,m) \notin E_{\alpha}] \to \exists \gamma \in \mathcal{C} \forall n [(n,\gamma(n)) \notin E_{\alpha}]].$$

A result related to the following Theorem has been proven by U. Kohlenbach, see [15, Theorem 3]. A similar result is mentioned in [1, Subsection 2.2].

Theorem 4.3. BIM $\vdash (\Pi^0_1\text{-AC}_{\omega,2} \land \text{LLPO}) \leftrightarrow \text{WKL}.$

Proof. (i) Assume, in BIM, Π_1^0 -AC_{$\omega,2$} and LLPO. It suffices to prove: Σ_1^0 -Sep, as, according to Theorem 3.1, BIM $\vdash \Sigma_1^0$ -Sep \leftrightarrow WKL.

Let α be given such that $\forall n \neg \forall m < 2[(n,m) \in E_{\alpha}]$. Let n be given.

Define β such that $\forall q \forall i < 2[\beta(2q+i) \neq 0 \leftrightarrow q = \mu p[\alpha(p) = (n,i) + 1]].$

Apply **LLPO** and find i < 2 such that $\forall q[2q + i \neq \mu m[\beta(m) \neq 0]]$.

Assume $(n,i) \in E_{\alpha}$. Then $(n,1-i) \notin E_{\alpha}$ and $\neg \exists p[\alpha(p)=(n,1-i)+1]$ and $\forall q[\beta(2q+1-i)=0]$. Find $q:=\mu p[\alpha(p)=(n,i)+1]$. Note $\beta(2q+i)\neq 0$ and $\forall m<2q+i[\beta(m)=0]$, so $2q+i=\mu m[\beta(m)\neq 0]$. Contradiction. Therefore, $(n,i)\notin E_{\alpha}$.

Conclude: $\forall n \exists i < 2[(n, i) \notin E_{\alpha}].$

Apply Π_1^0 - $\mathbf{AC}_{\omega,2}$ and find γ in \mathcal{C} such that $\forall n[(n,\gamma(n)) \notin E_{\alpha}]$.

Conclude: $\forall \alpha [\forall n [\neg \forall m < 2[(n,m) \in E_{\alpha}] \rightarrow \exists \gamma \in \mathcal{C} \forall n [(n,\gamma(n))) \notin E_{\alpha}]]$, i.e.: Σ_1^0 -Sep.

(ii) Note that $\neg(P \land Q) \leftrightarrow \neg\neg(\neg P \lor \neg Q)$ is a valid scheme of intuitionistic logic, and, therefore:

BIM $\vdash \Sigma_1^0$ -Sep $\leftrightarrow (\forall \alpha [\forall n \neg \neg \exists m < 2[(n,m) \notin E_{\alpha}] \rightarrow \exists \gamma \in \mathcal{C} \forall n[(n,\gamma(n)) \notin E_{\alpha}]])$. Conclude: BIM $\vdash \Sigma_1^0$ -Sep $\rightarrow \Pi_1^0$ -AC_{$\omega,2$}.

The conclusion $BIM \vdash WKL \rightarrow LLPO$ has been drawn in Theorem 2.9.

From a constructive point of view, Σ_1^0 -**Sep**, or equivalently, **WKL**, is an axiom of countable choice that is formulated too strongly.

4.8. $AC_{\omega,2}!$ and Π_1^0 - $AC_{\omega,2}$ are unprovable in BIM.

Note that the theory $\mathsf{BIM} + \mathbf{CT}$ may be translated into intuitionistic arithmetic HA, by interpreting functions from ω to ω as indices of total computable functions. The negative translation due to Gödel and Gentzen, see [29, vol. I, Ch. 3, Subsection 3.4], shows that first order classical (Peano) arithmetic PA, the theory that results from HA by adding the axiom scheme $X \vee \neg X$, is consistent. It follows that also the theory $\mathsf{BIM} + \mathbf{CT}$ remains consistent upon adding the axiom scheme $X \vee \neg X$.

4.8.1. Note that the theory $\mathsf{BIM} + \mathbf{CT} + X \vee \neg X + \mathbf{AC}_{\omega,2}!$ is inconsistent. The argument is as follows.

Let τ, ψ be as in **CT**. Define $H := \{n \mid \exists z [\tau(n, n, z) \neq 0]\}.$

Using classical logic, conclude: $\forall n \exists ! i < 2[i = 0 \leftrightarrow n \in H].$

Using $\mathbf{AC}_{\omega,2}!$, find α such that $\forall n[\alpha(n) \neq 0 \leftrightarrow n \in H]$.

Define β such that, for each n, if $\alpha(n) \neq 0$, then $\beta(n) = \psi(\mu z[\tau(n, n, z) \neq 0]) + 1$.

Using **CT**, find n_0 such that $\forall n[\beta(n) = \psi(\mu z[\tau(n_0, n, z) \neq 0])].$

Note: $\alpha(n_0) \neq 0$ and: $\beta(n_0) = \beta(n_0) + 1$. Contradiction.

Conclude: If HA is consistent, then $AC_{\omega,2}$! is not derivable in BIM.

4.8.2. Theorem 4.3 implies: BIM + \neg !**FT** + **LLPO** + Π_1^0 -**AC**_{ω ,2} is not consistent, as \neg !**FT** contradicts **WKL**.

Conclude: BIM + \neg !**FT** + **LLPO** $\vdash \neg \Pi_1^0$ -**AC**_{$\omega, 2$}.

On the other hand, $\mathsf{BIM} + \neg ! \mathbf{FT} + \mathbf{LLPO}$ is consistent, as it is a subsystem of $\mathsf{BIM} + \mathbf{CT} + X \vee \neg X$. It follows that $\Pi^0_1 - \mathbf{AC}_{\omega,2}$ is not derivable in $\mathsf{BIM} + \neg ! \mathbf{FT} + \mathbf{LLPO}$ and, a fortiori, not derivable in BIM . The stronger axiom $\Pi^0_1 - \mathbf{AC}_{\omega,\omega}$ and the even stronger axiom $\Pi^0_1 - \mathbf{AC}_{\omega,\omega}$, to be introduced in Section 6, also are not derivable in BIM .

From Theorem 4.3, we also conclude BIM $+ \neg !\mathbf{FT} + \mathbf{\Pi}_1^0 - \mathbf{AC}_{\omega,2} \vdash \neg \mathbf{LLPO}$.

In the context of HA, *Church's Thesis* is sometimes introduced as an axiom scheme, CT₀, see [29, vol. I, Ch. 4, Sect. 3]:

$$\forall n \exists m [nAm] \rightarrow \exists e \forall n \exists z [T(\langle e, n, z \rangle) \land \forall i < z [\neg T(\langle e, n, i \rangle)] \land nA(U(z))].^{9}$$

It is not difficult to see that BIM + \mathbf{CT} + $\mathbf{AC}_{\omega,\omega}$ and also BIM + \mathbf{CT} + $\mathbf{AC}_{\omega,\omega}$ translate into HA+CT₀. There is no straightforward model for HA+CT₀ but, using realizability, one may show that, if HA is consistent, then so is HA+CT₀, see [29, vol. I, Ch. 4, Sect. 4].

It follows that, if HA is consistent, then BIM $+ \neg !\mathbf{FT} + \mathbf{\Pi}_{1}^{0} - \mathbf{AC}_{\omega,2}$ is consistent.

4.9. Axiom Scheme of Countable Finite Choice, $\mathbf{AC}_{\omega, <\omega}$:

$$\forall \beta [\forall m \exists n \leq \beta(m) [R(n,m)] \to \exists \gamma \forall n [\gamma(n) \leq \beta(n) \land nR\gamma(n)]]$$

 $^{{}^{9}}T$ is Kleene's T-predicate and U his result-extracting function.

4.10. Π_1^0 -Axiom of Countable Finite Choice, Π_1^0 -AC $_{\omega,<\omega}$:

$$\forall \alpha [\forall n \exists m \leq \alpha^{\dagger 0}(n) [(n,m) \notin E_{\alpha^{\dagger 1}}] \to \exists \gamma \forall n [\gamma(n) \leq \alpha^{\dagger 0}(n) \land (n,\gamma(n)) \notin E_{\alpha^{\dagger 1}}]].$$

We conjecture that Π_1^0 - $\mathbf{AC}_{\omega,<\omega}$ is not provable in $\mathsf{BIM} + \Pi_1^0$ - $\mathbf{AC}_{\omega,2}$, but we have no proof of this conjecture. There might be many statements intermediate in strength like Π_1^0 - $\mathbf{AC}_{\omega,3}$, the Π_1^0 -Axiom of Countable Ternary Choice.

5. Contrapositions of some special cases of $\mathbf{AC}_{\omega,\omega}$

The following statement is a contraposition of $\mathbf{AC}_{\omega,\omega}$:

5.1. First Axiom Scheme of Reverse Countable Choice, $\overleftarrow{\mathbf{AC}_{\omega,\omega}}$:

$$\forall \gamma \exists n [R(n, \gamma(n))] \rightarrow \exists n \forall m [R(n, m)].$$

A special case is:

5.2. Minimal Axiom of Reverse Countable Choice, $\Delta_1^0 - \overleftarrow{\mathbf{AC}_{\omega,\omega}}$:

$$\forall \alpha [\forall \gamma \exists n [(n, \gamma(n)) \in D_{\alpha}] \to \exists n \forall m [(n, m) \in D_{\alpha}]].$$

The following result may be found in [32, Section 2].

Theorem 5.1. BIM $+\Delta_1^0 - \overleftarrow{\mathbf{AC}_{\omega,\omega}} \vdash \mathbf{LPO}$.

Proof. Let β be given.

Assuming $\Delta_1^0 - \overline{\mathbf{AC}}_{\omega,\omega}$, we prove: $\exists n[\beta(n) \neq 0] \lor \forall n[\beta(n) = 0]$.

To this end, define α such that $\forall n \forall m [\alpha(n,m) = 0 \leftrightarrow (\overline{\beta}n = \overline{0}n \land \overline{\beta}m \neq \overline{0}m)].$

For every γ , either: $\overline{\beta}(\gamma(0)) = \overline{\underline{0}}(\gamma(0))$ and, therefore, $\alpha(0, \gamma(0)) \neq 0$, that is $(0, \gamma(0)) \in D_{\alpha}$, or: $\overline{\beta}(\gamma(0)) \neq \overline{\underline{0}}(\gamma(0))$, and, therefore, $\alpha(\gamma(0), \gamma(\gamma(0))) \neq 0$, that is: $(\gamma(0), \gamma(\gamma(0))) \in D_{\alpha}$.

Conclude: $\forall \gamma \exists n [(n, \gamma(n)) \in D_{\alpha}].$

Applying $\Delta_1^0 - \overleftarrow{\mathbf{AC}_{\omega,\omega}}$, find n such that $\forall m[(n,m) \in D_{\alpha}]$.

Either: $\overline{\beta}n \neq \overline{\underline{0}}n$ and $\exists j[\beta(j) \neq 0]$, or: $\overline{\beta}n = \overline{\underline{0}}n$. In the latter case, for each m, $\overline{\beta}m = \overline{\underline{0}}m$ and $\forall j[\beta(j) = 0]$.

5.3. Axiom Scheme of Reverse Countable Binary Choice, $\overleftarrow{\mathbf{AC}_{\omega,2}}$:

$$\forall \gamma \in \mathcal{C} \exists n [R(n, \gamma(n))] \rightarrow \exists n \forall i < 2 [R(i, m)].$$

In [32, Section 4], $\overleftarrow{\mathbf{AC}_{\omega,2}}$ has been shown to be a consequence of $\mathbf{FT} + \mathbf{AC}_{\omega,\omega^{\omega}}$. We introduce a restricted version:

 Δ_1^0 -Axiom of Reverse Countable Binary Choice, Δ_1^0 - $\overrightarrow{AC}_{\omega,2}$:

$$\forall \alpha [\forall \gamma \in \mathcal{C} \exists n [(n, \gamma(n)) \in D_{\alpha}] \to \exists n \forall i < 2 [(n, i) \in D_{\alpha}]].$$

Theorem 5.2. BIM $\vdash \Delta_1^0 - \overleftarrow{AC_{\omega,2}}$.

Proof. Let α be given such that $\forall \gamma \in \mathcal{C} \exists n [(n, \gamma(n)) \in D_{\alpha}]$. Define γ in \mathcal{C} such that $\forall n [(n, 0) \in D_{\alpha} \leftrightarrow \gamma(n) = 1]$. Find n such that $(n, \gamma(n)) \in D_{\alpha}$. Note: $\gamma(n) = 1$ and $\forall i < 2 [(n, i) \in D_{\alpha}]$.

We now introduce a less restricted version:

5.4. Σ_1^0 -Axiom of Reverse Countable Binary Choice, Σ_1^0 - $\overrightarrow{AC}_{\omega,2}$:

$$\forall \alpha [\forall \gamma \in \mathcal{C} \exists n [(n, \gamma(n)) \in E_{\alpha}] \to \exists n \forall i < 2 [(n, i) \in E_{\alpha}]].$$

We define a formula that we want to call its *strong negation*:

5.5.
$$\neg ! (\mathbf{\Sigma}_1^0 - \overleftarrow{\mathbf{AC}_{\omega,2}}) :$$

$$\exists \alpha [\forall \gamma \in \mathcal{C} \exists n [(n, \gamma(n)) \in E_{\alpha}] \land \neg \exists n \forall i < 2 [(n, i) \in E_{\alpha}]].$$

Note: BIM proves:

$$\forall \alpha [\neg \exists n \forall i < 2[(n,i) \in E_{\alpha}] \leftrightarrow \forall n \forall p \forall q [\alpha(p) = (n,0) + 1 \to \alpha(q) \neq (n,1) + 1]].$$

Lemma 5.3. BIM proves the following:

(i)
$$\mathbf{FT} \to \mathbf{\Sigma}_{1}^{0} \cdot \overleftarrow{\mathbf{AC}_{\omega,2}} \ and \ \neg ! (\mathbf{\Sigma}_{1}^{0} - \overleftarrow{\mathbf{AC}_{\omega,2}}) \to \neg ! \mathbf{FT}.$$

(ii) $\mathbf{\Sigma}_{1}^{0} \cdot \overleftarrow{\mathbf{AC}_{\omega,2}} \to \mathbf{FT} \ and \ \neg ! \mathbf{FT} \to \neg ! (\mathbf{\Sigma}_{1}^{0} \cdot \overleftarrow{\mathbf{AC}_{\omega,2}}).$

(ii)
$$\Sigma_1^0 - \overline{AC_{\omega,2}} \to FT$$
 and $\neg !FT \to \neg !(\Sigma_1^0 - \overline{AC_{\omega,2}})$.

Proof. (i) We prove, in BIM: for each α , there exists β such that

(1)
$$\forall \gamma \in \mathcal{C} \exists n [(n, \gamma(n)) \in E_{\alpha}] \to Bar_{\mathcal{C}}(D_{\beta})]$$
 and

$$(2) \ \exists m[Bar_{\mathcal{C}}(D_{\overline{\beta}m})] \to \exists n \forall i < 2[(n,i) \in E_{\alpha}].$$

The two promised conclusions then follow easily.

Let α be given. Define β such that

$$\forall m \forall a \in Bin_m[\beta(a) \neq 0 \leftrightarrow \exists n < m[(n, a(n)) \in E_{\overline{\alpha}m}]].$$

- (1) Assume: $\forall \gamma \in \mathcal{C} \exists n[(n, \gamma(n)) \in E_{\alpha}]$. Let γ in \mathcal{C} be given. Find n, p such that $(n, \gamma(n)) \in E_{\overline{\alpha}p}$ and n < p. Note: $\beta(\overline{\gamma}p) \neq 0$. We thus see: $\forall \gamma \in \mathcal{C} \exists p [\beta(\overline{\gamma}p) \neq 0]$, i.e. $Bar_{\mathcal{C}}(D_{\beta})$.
 - (2) Let m be given such that $Bar_{\mathcal{C}}(D_{\overline{\beta}m})$.

Note: $\forall a \in Bin_m[m < a]$ and: $\forall a \in Bin_m \exists n \leq m[\beta(\overline{a}n) \neq 0]$.

Assume $\forall n < m \exists i < 2[(n,i) \notin E_{\overline{\alpha}m}].$

Define a in Bin_m such that $\forall n < m[(n,0) \notin E_{\overline{\alpha}m} \leftrightarrow a(n) = 0]$.

Then $\forall n < m[(n, a(n)) \notin E_{\overline{\alpha}m}]$ and $\forall n \leq m[\beta(\overline{a}n) = 0]$. Contradiction.

Conclude: $\exists n < m \forall i < 2[(n,i) \in E_{\overline{\alpha}m} \subseteq E_{\alpha}].$

- (ii) We prove, in BIM: for each α , there exists β such that
 - (1) $Bar_{\mathcal{C}}(D_{\alpha}) \to \forall \gamma \in \mathcal{C} \exists n [(n, \gamma(n)) \in E_{\beta}]$ and

$$(2) \ \exists n \forall i < 2[(n,i) \in E_{\beta}] \to \exists m [Bar_{\mathcal{C}}(D_{\overline{\alpha}m})].$$

The two promised conclusions then follow easily.

Let α be given. Define β such that, for all n, for every s in Bin, for all i < 2, if either: (a) $Bar_{\mathcal{C}}(D_{\overline{\alpha}n})$, or: (b) $Bar_{\mathcal{C}\cap s*\langle i\rangle}(D_{\overline{\alpha}n})$ and not $Bar_{\mathcal{C}\cap s*\langle 1-i\rangle}(D_{\overline{\alpha}n})$, then $\beta(n, s*\langle i \rangle) = (s, i) + 1$, and, (c) if both (a) and (b) fail, then $\beta(n, s*\langle i \rangle) = 0$. Furthermore, for all n, for all s, if $s \notin Bin \setminus \{\langle \rangle \}$, then $\beta(n,s) = 0$.

Note: E_{β} is the set of all pairs (s,i) such that $s \in Bin$ and i < 2 and either $\exists n[Bar_{\mathcal{C}}(D_{\overline{\alpha}n})] \text{ or } \exists n[Bar_{\mathcal{C}\cap s*\langle i\rangle}(D_{\overline{\alpha}n}) \land \neg Bar_{\mathcal{C}\cap s*\langle 1-i\rangle}(D_{\overline{\alpha}n})].$

Note that, for all s in Bin, if $\forall i < 2[(s,i) \in E_{\beta}]$, then $\exists n[Bar_{\mathcal{C}}(D_{\overline{\alpha}n})].$

(1) Assume $Bar_{\mathcal{C}}(D_{\alpha})$. We shall prove: $\forall \gamma \in \mathcal{C} \exists n[(n, \gamma(n)) \in E_{\beta}]$.

Let γ in \mathcal{C} be given. Define δ in \mathcal{C} such that $\forall n[\delta(n) = \gamma(\overline{\delta}n)]$. Find n such that $\alpha(\overline{\delta}n) \neq 0$. Define $q := \overline{\delta}n + 1$ and note: $\overline{\delta}n \in D_{\overline{\alpha}q}$.

We claim: for all $j \leq n$, either $\exists i \leq n[(\overline{\delta}i, \gamma(\overline{\delta}i)) \in E_{\beta}]$, or $Bar_{\mathcal{C} \cap \overline{\delta}j}(D_{\overline{\alpha}q})$. We prove this claim by backwards induction, starting from j = n. Note: $Bar_{\mathcal{C} \cap \overline{\delta}n}(D_{\overline{\alpha}q})$. Now assume j < n and $Bar_{\mathcal{C} \cap \overline{\delta}(j+1)}(D_{\overline{\alpha}q})$, that is: $Bar_{\mathcal{C} \cap \overline{\delta}(j)*\langle \delta(j) \rangle}(D_{\overline{\alpha}p})$. Find out if also $Bar_{\mathcal{C} \cap \overline{\delta}(j)*\langle 1-\delta(j)\rangle}(D_{\overline{\alpha}q})$.

If so, then $Bar_{\mathcal{C}\cap\overline{\delta}(j)}(D_{\overline{\alpha}q})$, and, if not, then $\beta(q,\overline{\delta}(j+1)) = (\overline{\delta}j,\delta(j)) + 1$, and $(\overline{\delta}j,\delta(j)) = (\overline{\delta}j,\gamma(\overline{\delta}(j)) \in E_{\beta}.$

¹⁰The reader should compare this proof to the proof of Theorem 3.1(ii).

We may conclude: either $\exists j \leq n[(\overline{\delta}j, \delta(j)) \in E_{\beta}]$, or $Bar_{\mathcal{C}}(D_{\overline{\alpha}q})$. Note that, if $Bar_{\mathcal{C}}(D_{\overline{\alpha}q})$, then $\forall s \in Bin \forall i < 2[\beta(q, s * \langle i \rangle) = (s, i) + 1]$ and: $\forall s \in Bin[(s, \gamma(s)) \in E_{\beta}].$

We thus see: $\forall \gamma \in \mathcal{C} \exists s [(s, \gamma(s)) \in E_{\beta}].$

(2) Now assume: $\exists n \forall i < 2[(n,i) \in E_{\beta}]$. Conclude, using the observation we made just after the definition of β : $\exists n[Bar_{\mathcal{C}}(D_{\overline{\alpha}n})].$

Theorem 5.4. BIM proves: $\mathbf{FT} \leftrightarrow \Sigma_1^0 - \overleftarrow{\mathbf{AC}_{\omega,2}}$ and: $\neg ! \mathbf{FT} \leftrightarrow \neg ! (\Sigma_1^0 - \overleftarrow{\mathbf{AC}_{\omega,2}})$.

Proof. Use Lemma 5.3.
$$\Box$$

5.6. Axiom Scheme of Reverse Countable Finite Choice, $\overleftarrow{\mathbf{AC}_{\omega < \omega}}$:

$$\forall \beta [\forall \gamma \exists n [\gamma(n) \leq \beta(n) \to R(n, \gamma(n))] \to \exists m \forall n \leq \beta(m) [R(n, m)]].$$

 $\overline{\mathbf{AC}_{\omega,<\omega}}$ may be concluded from $\mathbf{FT} + \mathbf{AC}_{\omega^{\omega},\omega}$, by a slight extension of the argument given in [32, Section 4].

The following is a restricted version:

$$\Sigma_1^0$$
- $\overline{\mathbf{AC}_{\omega,<\omega}}$:

$$\forall \alpha [\forall \gamma \exists n [\gamma(n) < \alpha^{\uparrow 0}(n) \to (n, \gamma(n)) \in E_{\alpha^{\uparrow 1}}] \to \exists n \forall i < \alpha^{\uparrow 0}(n) [(n, i) \in E_{\alpha^{\uparrow 1}}]].$$

We introduce a strong negation of this restricted version:

$$\neg ! (\mathbf{\Sigma}_{1}^{0} - \overleftarrow{\mathbf{AC}_{\omega, <\omega}}) :$$

$$\exists \alpha [\forall \gamma \exists n [\gamma(n) \leq \alpha^{\restriction 0}(n) \to (n, \gamma(n)) \in E_{\alpha^{\restriction 1}}] \land \neg \exists n \forall i \leq \alpha^{\restriction 0}(n) [(n, i) \in E_{\alpha^{\restriction 1}}]].$$

Note: BIM proves:

$$\forall \alpha [\neg \exists n \forall i \leq \alpha^{\dagger 0}(n)[(n,i) \in E_{\alpha}] \leftrightarrow \\ \forall n \forall t \in \omega^{\alpha^{\dagger 0}(n)+1} \exists i \leq \alpha^{\dagger 0}(n)[\alpha(t(i)) \neq (n,i)+1]].$$

Lemma 5.5. BIM proves:

(i)
$$\Sigma_{1}^{0} - \overrightarrow{\mathbf{AC}}_{\omega,<\omega} \to \Sigma_{1}^{0} - \overrightarrow{\mathbf{AC}}_{\omega,2}$$
 and $\neg ! (\Sigma_{1}^{0} - \overleftarrow{\mathbf{AC}}_{\omega,2}) \to \neg ! (\Sigma_{1}^{0} - \overleftarrow{\mathbf{AC}}_{\omega,<\omega})$
(ii) $\mathbf{FT} \to \Sigma_{1}^{0} - \overleftarrow{\mathbf{AC}}_{\omega,<\omega}$ and $\neg ! (\Sigma_{1}^{0} - \overleftarrow{\mathbf{AC}}_{\omega,<\omega}) \to \neg ! \mathbf{FT}$.

(ii)
$$\mathbf{FT} \to \mathbf{\Sigma}_1^0 - \overleftarrow{\mathbf{AC}_{\omega,<\omega}} \ and \ \neg ! (\mathbf{\Sigma}_1^0 - \overleftarrow{\mathbf{AC}_{\omega,<\omega}}) \to \neg ! \mathbf{FT}.$$

Proof. (i) We prove, in BIM: for each α , there exists β such that

$$(1) \ \forall \gamma \in \mathcal{C} \exists n [(n, \gamma(n)) \in E_{\alpha}] \to \forall \gamma \exists n [\gamma(n) \leq \beta^{\uparrow 0}(n) \to (n, \gamma(n)) \in E_{\beta^{\uparrow 1}}] \text{ and }$$

$$(2) \ \exists n \forall i \le \beta^{\uparrow 0}(n) [(n,i) \in E_{\beta^{\uparrow 1}}] \to \exists n \forall i < 2 [(n,i) \in E_{\alpha}].$$

The two promised conclusions then follow easily.

Let α be given. Define β such that $\beta^{\dagger 0} = \underline{1}$ and $\beta^{\dagger 1} = \alpha$.

- (1) Assume $\forall \gamma \in \mathcal{C} \exists n[(n,\gamma(n)) \in E_{\alpha}]$. Let γ be given. Define γ^* such that $\forall n[\gamma^*(n) = \min(1, \gamma(n))].$ Note: $\gamma^* \in \mathcal{C}$ and find n such that $(n, \gamma^*(n)) \in E_{\alpha}$. If $\gamma^*(n) \neq \gamma(n)$, then $\gamma(n) > 1 = \beta^{\uparrow 0}(n)$, and if $\gamma^*(n) = \gamma(n)$, then $(n, \gamma(n)) \in E_{\alpha}$. Conclude: $\forall \gamma \exists n [\gamma(n) \leq \beta^{\uparrow 0}(n) \to (n, \gamma(n)) \in E_{\beta^{\uparrow 1}}].$
- (2) Let n be given such that $\forall i \leq \beta^{\uparrow 0}(n)[(n,i) \in E_{\beta^{\uparrow 1}}].$ Clearly, then, $\forall i < 2[(n, i) \in E_{\alpha}].$
 - (ii)¹¹ We prove, in BIM: for each α , there exists β such that

(1)
$$\forall \gamma \exists n [\gamma(n) \leq \alpha^{\dagger 0}(n) \to (n, \gamma(n)) \in E_{\alpha^{\dagger 1}}] \to Bar_{\mathcal{C}}(D_{\beta})$$
 and

$$(2) \ \exists m[Bar_{\mathcal{C}}(D_{\overline{\beta}m})] \to \exists n \forall m \le \alpha^{\upharpoonright 0}(n)[(n,m) \in E_{\alpha^{\upharpoonright 1}}].$$

The two promised conclusions then follow easily.

¹¹The argument extends the argument for Lemma 5.3(i).

First, define $Cod_2: \omega \to Bin$ such that

 $Cod_2(\langle \rangle) = \langle \rangle$ and $\forall s \forall n [Cod_2(s * \langle n \rangle) = Cod_2(s) * \overline{\underline{0}}n * \langle 1 \rangle].$

Note: $\forall t \in Bin \exists s \exists i [t = Cod_2(s) * \overline{\underline{0}}i].$

Now, let α be given. Define β such that, for all $s, i, \beta(Cod_2(s)*\overline{0}i) \neq 0$ if and only $\text{if } \exists n < length(s)[s(n) > \alpha^{\restriction 0}(n) \ \lor \ \left(n, s(n)\right) \in E_{\overline{\alpha^{\restriction 1}}length(s)} \ \lor \ \alpha^{\restriction 0}\left(length(s)\right) < i].$

(1) Assume: $\forall \gamma \exists n [\gamma(n) \leq \alpha^{\uparrow 0}(n) \to (n, \gamma(n)) \in E_{\alpha^{\uparrow 1}}].$

Assume: $\delta \in \mathcal{C}$. Define γ , by induction, such that, for each n,

if $\exists i \leq \alpha^{\uparrow 0}(n)[Cod_2(\overline{\gamma}n * \langle i \rangle) \subset \delta]$, then $Cod_2(\overline{\gamma}(n+1)) \subset \delta$, and, if not, then $\gamma(n) = 0$.

Note: $\forall n [\gamma(n) \leq \alpha^{\uparrow 0}(n)].$

Find n, p such that $(n, \gamma(n)) \in E_{\overline{\alpha}^{+1}p}$. Define $q := \max(n, p)$.

Note: $\beta(Cod_2(\overline{\gamma}(q+1))) \neq 0$ and distinguish two cases.

Case (a). $c := Cod_2(\overline{\gamma}(q+1)) \sqsubset \delta \text{ and } \beta(c) \neq 0.$

Case (b). $\exists m \forall i \leq \alpha^{\uparrow 0}(m) [Cod_2(\overline{\gamma}m * \langle i \rangle) \perp \delta].$ Find $m_0 := \mu m \forall i \leq \alpha^{\uparrow 0}(m) [Cod_2(\overline{\gamma}m * \langle i \rangle) \perp \delta]$ and note:

 $d := Cod_2(\overline{\gamma}m_0) * \overline{\underline{0}}(\alpha^{\uparrow 0}(m_0) + 1) \sqsubset \delta \text{ and } \beta(d) \neq 0.$

In both cases $\exists p[\beta(\overline{\delta}p) \neq 0].$

We thus see: $\forall \delta \in \mathcal{C} \exists p [\beta(\overline{\delta}p) \neq 0]$, i.e. $Bar_{\mathcal{C}}(D_{\beta})$.

(2) Let m be given such that $Bar_{\mathcal{C}}(D_{\overline{\beta}m})$.

Suppose: $\forall i < m \exists j \leq \alpha^{\uparrow 0}(i)[(i,j) \notin E_{\overline{\alpha^1}m}]].$

Find s such that length(s) = m and $\forall i < m[s(i) \le \alpha^{\uparrow 0}(i) \land (i, s(i)) \notin E_{\overline{\alpha^{\uparrow 1}}m}]$.

Note: $Cod_2(s) > m$ and $\forall t \subseteq Cod_2(s)[\beta(t) = 0]$, so $\neg Bar_{\mathcal{C}}(D_{\overline{\beta}m})$.

Contradiction.

Conclude: $\exists i < m \forall j \leq \alpha^{\uparrow 0}(i)[(i,j) \in E_{\overline{\alpha^{\uparrow 1}}m} \subseteq E_{\alpha^{\uparrow 1}}].$

Theorem 5.6. BIM proves: $\mathbf{FT} \leftrightarrow \mathbf{\Sigma}_{1}^{0} - \overleftarrow{\mathbf{AC}_{\omega,<\omega}}$ and $\neg ! \mathbf{FT} \leftrightarrow \neg ! (\mathbf{\Sigma}_{1}^{0} - \overleftarrow{\mathbf{AC}_{\omega,<\omega}})$.

Proof. These statements follow from Lemma 5.5 and Theorem 5.4.

5.7. No Double Negation Shift.

Assume $\neg ! \mathbf{FT}$. Using $\neg ! (\mathbf{\Sigma}_{1}^{0} - \overline{\mathbf{AC}_{\omega,2}})$, find α such that $\forall \gamma \in \mathcal{C} \exists n [(n, \gamma(n)) \in E_{\alpha}]$ and: $\neg \exists n \forall m < 2[(n,m) \in E_{\alpha}].$

Then, for each $n, \neg \forall m < 2[(n,m) \in E_{\alpha}]$ and: $\neg \forall m < 2[\neg \neg ((n,m) \in E_{\alpha})]$ and: $\neg\neg\exists m < 2[(n,m) \notin E_{\alpha}].$

We thus see: $\forall n \neg \neg \exists m < 2[(n, m) \notin E_{\alpha}].$

Also: $\neg \exists \gamma \in \mathcal{C} \forall n [(n, \gamma(n)) \notin E_{\alpha}].$

Using Π_1^0 - $\mathbf{AC}_{\omega,2}$, we conclude: $\neg \forall n \exists m < 2[(n,m) \notin E_{\alpha}].$

We thus see that if we assume both \neg !**FT** and Π_1^0 -**AC**_{ω ,2} we can find Π_1^0 -subsets $P = \{n \mid (n,0) \notin E_{\alpha}\}$ and $Q = \{n \mid (n,1) \notin E_{\alpha}\}$ of ω such that $\forall n [\neg \neg (P(n) \lor Q(n))] \text{ and } \neg \forall n [P(n) \lor Q(n)].$

S. Kuroda's scheme of Double Negation Shift

$$\forall n[\neg \neg T(n)] \rightarrow \neg \neg \forall n[T(n)]$$

(see [18, page 45] and [8, page 105]) thus is refuted.

In [29, vol. I, Chapter 4, Proposition 3.4, Corollary 1], the same conclusion is obtained in HA from CT_0 .

6.
$$\mathbf{AC}_{\omega,\omega^{\omega}}$$
, some special cases

6.1. Σ_1^0 -Second Axiom of Countable Choice, Σ_1^0 - $\mathbf{AC}_{\omega,\omega^{\omega}}$:

$$\forall \alpha [\forall n \exists \gamma [\gamma \in \mathcal{G}_{\alpha^{\restriction n}}] \to \exists \gamma \forall n [\gamma^{\restriction n} \in \mathcal{G}_{\alpha^{\restriction n}}]].$$

Theorem 6.1. BIM $\vdash \Sigma_1^0$ -AC_{ω,ω^{ω}}.

Proof. Assume $\forall n \exists \gamma [\gamma \in \mathcal{G}_{\alpha^{\dagger n}}]$. Then $\forall n \exists s [\alpha^{\dagger n}(s) \neq 0]$. Find δ such that $\forall n [\delta(n) = \mu s [\alpha^{\dagger n}(s) \neq 0]]$. Find γ such that $\forall n [\gamma^{\dagger n} = \delta(n) * \underline{0}]$. Note: $\forall n [\gamma^{\dagger n} \in \mathcal{G}_{\alpha^{\dagger n}}]$.

6.2. Π_1^0 -Second Axiom of Countable Choice, Π_1^0 - $\mathbf{AC}_{\omega,\omega^{\omega}}$:

$$\forall \alpha [\forall n \exists \gamma [\gamma \notin \mathcal{G}_{\alpha^{\restriction n}}] \to \exists \gamma \forall n [\gamma^{\restriction n} \notin \mathcal{G}_{\alpha^{\restriction n}}]].$$

Theorem 6.2. BIM $\vdash \Pi_1^0 \text{-}AC_{\omega,\omega^{\omega}} \to \Pi_1^0 \text{-}AC_{\omega,\omega}$.

Proof. Let α be given such that $\forall n \exists m [(n,m) \notin E_{\alpha}]$. Define β such that $\forall n \forall a [\beta^{\uparrow n}(a) \neq 0 \leftrightarrow \exists m \exists b \exists p \leq a [\alpha(p) = (n,m) + 1 \land a = \langle m \rangle * b]]$.

Note: $\forall n \forall m [(n, m) \in E_{\alpha} \leftrightarrow \forall \gamma [\langle m \rangle * \gamma \in \mathcal{G}_{\beta^{\restriction n}}]].$

Conclude: $\forall n \exists \gamma [\gamma \notin \mathcal{G}_{\beta^{\dagger n}}]$. Using Π_1^0 - $\mathbf{AC}_{\omega,\omega^{\omega}}$, find γ such that $\forall n [\gamma^{\dagger n} \notin \mathcal{G}_{\beta^{\dagger n}}]$. Define δ such that $\forall n [\delta(n) = \gamma^{\dagger n}(0)]$ and note: $\forall n [(n, \delta(n)) \notin E_{\alpha}]$.

One may conclude that Π_1^0 - $\mathbf{AC}_{\omega,\omega^{\omega}}$ is unprovable in BIM, see Subsection 4.8. Not every Π_1^0 subset of ω^{ω} is a spread, see Lemma 2.11. For spreads, which are a special kind of Π_1^0 sets, countable choice is easier:

Theorem 6.3. BIM $\vdash \forall \alpha [(\forall n [Spr(\alpha^{\restriction n})] \land \forall n \exists \gamma [\gamma \notin \mathcal{G}_{\alpha^{\restriction n}}]) \rightarrow \exists \gamma \forall n [\gamma^{\restriction n} \notin \mathcal{G}_{\alpha^{\restriction n}}]].$

Proof. Let α be given such that $\forall n[Spr(\alpha^{\uparrow n})]$ and $\forall n\exists \gamma[\gamma\notin\mathcal{G}_{\alpha^{\uparrow n}}]$. Define γ such that, for each n, for each m, $\gamma^{\uparrow n}(m) = \mu k[\alpha^{\uparrow n}((\overline{\gamma^{\uparrow n}}m)*\langle k\rangle) = 0]$. \square

6.3. Axiom Scheme of Countable Compact Choice, $AC_{\omega,C}$:

$$\forall n \exists \gamma \in \mathcal{C}[R(n,\gamma)] \to \exists \gamma \in \mathcal{C} \forall n [R(n,\gamma^{\restriction n})].$$

Here is a restricted version of $\mathbf{AC}_{\omega,\mathcal{C}}$:

6.4. Π_1^0 -Axiom of Countable Compact Choice, Π_1^0 -AC $_{\omega,\mathcal{C}}$:

$$\forall \alpha [\forall n \exists \gamma \in \mathcal{C}[\gamma \notin \mathcal{G}_{\alpha^{\restriction n}}] \to \exists \gamma \in \mathcal{C} \forall n [\gamma^{\restriction n} \notin \mathcal{G}_{\alpha^{\restriction n}}]]$$

Theorem 6.4. BIM $\vdash \Pi_1^0 \text{-}AC_{\omega,\mathcal{C}} \to \Pi_1^0 \text{-}AC_{\omega,2}$.

Proof. The proof is almost the same as the proof of Theorem 6.2 and is left to the reader. \Box

We may conclude: Π_1^0 - $\mathbf{AC}_{\omega,\mathcal{C}}$ is unprovable in BIM, see Subsection 4.8.

The treatment of real numbers in BIM is sketched in Subsection 13.7.

6.5. Π_1^0 -**AC**_{$\omega,[0,1]$}:

$$\forall \alpha [\forall n \exists \delta \in [0,1] [\delta \notin \mathcal{H}_{\alpha^{\dagger n}}] \to \exists \delta \in [0,1]^{\omega} \forall n [\delta^{\dagger n} \notin \mathcal{H}_{\alpha^{\dagger n}}]].$$

Theorem 6.5. BIM $\vdash \Pi_1^0$ -AC_{ω,C} $\leftrightarrow \Pi_1^0$ -AC_{$\omega,[0,1]$}.

Proof. First assume Π_1^0 - $\mathbf{AC}_{\omega,\mathcal{C}}$.

Using Lemma 13.3, find $\sigma: \mathcal{C} \to [0,1]$ and $\psi: \omega^{\omega} \to \omega^{\omega}$ such that

- (1) $\forall \delta \in [0,1] \exists \gamma \in \mathcal{C}[\delta =_{\mathcal{R}} \sigma | \gamma]$ and
- (2) $\forall \alpha \forall \gamma \in \mathcal{C}[\gamma \in \mathcal{G}_{\psi|\alpha} \leftrightarrow \sigma | \gamma \in \mathcal{H}_{\alpha}].$

Let α be given such that $\forall n \exists \delta \in [0,1][\delta \notin \mathcal{H}_{\alpha^{\dagger n}}]$. Then $\forall n \exists \gamma \in \mathcal{C}[\sigma | \gamma \notin \mathcal{H}_{\alpha^{\dagger n}}]$ and $\forall n \exists \gamma \in \mathcal{C}[\gamma \notin \mathcal{G}_{\psi | (\alpha^{\dagger n})}]$. Find γ in \mathcal{C} such that $\forall n [\gamma^{\dagger n} \notin \mathcal{G}_{\psi | (\alpha^{\dagger n})}]$. Conclude: $\forall n [\sigma | \gamma^{\dagger n} \notin \mathcal{H}_{\alpha^{\dagger n}}]$ and $\exists \delta \forall n [\delta^{\dagger n} \notin \mathcal{H}_{\alpha^{\dagger n}}]$.

Now assume Π_1^0 - $\mathbf{AC}_{\omega,[0,1]}$.

Using Lemma 13.4, find $\tau: \mathcal{C} \to [0,1]$ and $\chi: \omega^{\omega} \to \omega^{\omega}$ such that

- (1) $\forall \gamma \in \mathcal{C} \forall \delta \in \mathcal{C} [\gamma \# \delta \to \tau | \gamma \#_{\mathcal{R}} \tau | \delta]$, and
- (2) $\forall \alpha \forall \gamma \in \mathcal{C}[\gamma \in \mathcal{G}_{\alpha} \leftrightarrow \tau | \gamma \in \mathcal{H}_{\chi | \alpha}]$. and
- $(3) \ \forall \alpha \forall \delta \in [0,1]^{\omega} \exists \gamma \in \mathcal{C} \forall n [\delta^{\uparrow n} \#_{\mathcal{R}} \tau | (\gamma^{\uparrow n}) \to \delta^{\uparrow n} \in \mathcal{H}_{\chi | (\alpha^{\uparrow n})}].$

Let α be given such that $\forall n \exists \gamma \in \mathcal{C}[\gamma \notin \mathcal{G}_{\alpha^{\upharpoonright n}}].$ Conclude: $\forall n \exists \gamma \in \mathcal{C}[\tau | \gamma \notin \mathcal{H}_{\chi|(\alpha^{\dagger n})}]$ and: $\forall n \exists \delta \in [0, 1] [\delta \notin \mathcal{H}_{\chi|(\alpha^{\dagger n})}].$ Using Π_1^0 - $\mathbf{AC}_{\omega,[0,1]}$, find δ in $[0,1]^{\omega}$ such that $\forall n[\delta^{\uparrow n} \notin \mathcal{H}_{\chi|(\alpha^{\uparrow n})}]$. Using (3), find γ in \mathcal{C} such that $\forall n[\delta^{\uparrow n} \#_{\mathcal{R}} \tau | (\gamma^{\uparrow n}) \to \delta^{\uparrow n} \in \mathcal{H}_{\gamma | (\alpha^{\uparrow n})}].$ Conclude: $\forall n[\delta^{\uparrow n} =_{\mathcal{R}} \tau | (\gamma^{\uparrow n})]]$, and, using (2): $\forall n[\gamma^{\uparrow n} \notin \mathcal{G}_{\alpha^{\uparrow n}}]$. Clearly, $\exists \gamma \in \mathcal{C} \forall n[\gamma^{\uparrow n} \notin \mathcal{G}_{\alpha^{\uparrow n}}]$.

7. Contrapositions of some special cases of $\mathbf{AC}_{\omega,\omega^{\omega}}$

The axiom $\Delta_1^0 - \overleftarrow{\mathbf{AC}_{\omega,\omega^{\omega}}}$ implies $\Delta_1^0 - \overleftarrow{\mathbf{AC}_{\omega,\omega}}$ and therefore **LPO**. Let us consider an Axiom Scheme of Reverse Countable Compact Choice:

7.1.
$$\overleftarrow{\mathbf{AC}_{\omega,\mathcal{C}}}$$
: $\forall \gamma \in \mathcal{C} \exists n [R(n,\gamma^{\upharpoonright n})] \to \exists n \forall \gamma \in \mathcal{C}[R(n,\gamma)].$

In [32] it is shown that $\overline{\mathbf{AC}_{\omega,\mathcal{C}}}$ is a consequence of the First Axiom of Continuous Choice $\mathbf{AC}_{\omega^{\omega},\omega}$ and \mathbf{FAN} .

We now require the relation R to be Σ_1^0 and obtain the Σ_1^0 -Axiom of Reverse Countable Compact Choice:

7.2.
$$\Sigma_1^0 - \overline{\mathbf{AC}_{\omega,\mathcal{C}}}$$
: $\forall \alpha [\forall \gamma \in \mathcal{C} \exists n [\gamma^{\uparrow n} \in \mathcal{G}_{\alpha^{\uparrow n}}] \to \exists n [\mathcal{C} \subseteq \mathcal{G}_{\alpha^{\uparrow n}}]]$. We also introduce a strong negation:

7.3.
$$\neg!(\mathbf{\Sigma}_{1}^{0} - \overleftarrow{\mathbf{AC}_{\omega,\mathcal{C}}}): \exists \alpha [\forall \gamma \in \mathcal{C} \exists n [\gamma^{\upharpoonright n} \in \mathcal{G}_{\alpha^{\upharpoonright n}}] \land \neg \exists n [\mathcal{C} \subseteq \mathcal{G}_{\alpha^{\upharpoonright n}}]].$$
 We introduce a 'real' version:

7.4.
$$\Sigma_1^0 - \overleftarrow{\mathbf{AC}_{\omega,[0,1]}} : \forall \alpha [\forall \delta \in [0,1]^{\omega} \exists n [\delta^{\uparrow n} \in \mathcal{H}_{\alpha^{\dagger n}}] \to \exists n [[0,1] \subseteq \mathcal{H}_{\alpha^{\dagger n}}]].$$
 and a strong negation:

7.5.
$$\neg ! (\mathbf{\Sigma}_{1}^{0} - \overleftarrow{\mathbf{AC}_{\omega,[0,1]}}) : \exists \alpha [\forall \delta \in [0,1]^{\omega} \exists n [\delta^{\uparrow n} \in \mathcal{H}_{\alpha^{\uparrow n}}] \land \neg \exists n [[0,1] \subseteq \mathcal{H}_{\alpha^{\uparrow n}}]].$$

The treatment of real numbers in BIM is sketched in Subsection 13.7.

Lemma 7.1. BIM proves:

(i)
$$\mathbf{FT} \to \mathbf{\Sigma}_1^0 - \overleftarrow{\mathbf{AC}_{\omega,\mathcal{C}}}$$
 and $\neg ! (\mathbf{\Sigma}_1^0 - \overleftarrow{\mathbf{AC}_{\omega,\mathcal{C}}}) \to \neg ! \mathbf{FT}$.

(ii)
$$\Sigma_1^0 - \overrightarrow{\mathbf{AC}}_{\omega,\mathcal{C}} \to \Sigma_1^0 - \overrightarrow{\mathbf{AC}}_{\omega,[0,1]}$$
 and $\neg ! (\Sigma_1^0 - \overleftarrow{\mathbf{AC}}_{\omega,[0,1]}) \to \neg ! (\Sigma_1^0 - \overleftarrow{\mathbf{AC}}_{\omega,\mathcal{C}})$

$$\begin{array}{lll} \text{(i)} & \mathbf{FT} \xrightarrow{} \mathbf{\Sigma}_{1}^{0} \cdot \overleftarrow{\mathbf{AC}_{\omega,\mathcal{C}}} & and \ \neg ! (\mathbf{\Sigma}_{1}^{0} \cdot \overleftarrow{\mathbf{AC}_{\omega,\mathcal{C}}}) \xrightarrow{} \neg ! \mathbf{FT}. \\ \text{(ii)} & \mathbf{\Sigma}_{1}^{0} \cdot \overleftarrow{\mathbf{AC}_{\omega,\mathcal{C}}} \xrightarrow{} \mathbf{\Sigma}_{1}^{0} \cdot \overleftarrow{\mathbf{AC}_{\omega,[0,1]}} & and \ \neg ! (\mathbf{\Sigma}_{1}^{0} \cdot \overleftarrow{\mathbf{AC}_{\omega,[0,1]}}) \xrightarrow{} \neg ! (\mathbf{\Sigma}_{1}^{0} \cdot \overleftarrow{\mathbf{AC}_{\omega,\mathcal{C}}}) \\ \text{(iii)} & \mathbf{\Sigma}_{1}^{0} \cdot \overleftarrow{\mathbf{AC}_{\omega,[0,1]}} \xrightarrow{} \mathbf{\Sigma}_{1}^{0} \cdot \overleftarrow{\mathbf{AC}_{\omega,2}} & and \ \neg ! (\mathbf{\Sigma}_{1}^{0} \cdot \overleftarrow{\mathbf{AC}_{\omega,2}}) \xrightarrow{} \neg ! (\mathbf{\Sigma}_{1}^{0} \cdot \overleftarrow{\mathbf{AC}_{\omega,[0,1]}}) \end{array}$$

Proof. (i)¹² We prove, in BIM: for each α , there exists β such that

$$\forall \gamma \in \mathcal{C} \exists n[\gamma^{\uparrow n} \in \mathcal{G}_{\alpha^{\uparrow n}}] \to Bar_{\mathcal{C}}(D_{\beta}) \text{ and } \exists m[Bar_{\mathcal{C}}(D_{\overline{\beta}m})] \to \exists n[\mathcal{C} \subseteq \mathcal{G}_{\alpha^{\uparrow n}}].$$

The two promised statements then follow easily.

Let α be given. Define β such that, for every s,

$$\beta(s) \neq 0 \leftrightarrow \left(s \in Bin \ \land \ \exists n < length(s) \exists p \leq length(s^{\upharpoonright n}) [\alpha^{\upharpoonright n}(\overline{s^{\upharpoonright n}}p) \neq 0]\right).$$

Assume: $\forall \gamma \in \mathcal{C} \exists n \exists p [\alpha^{\upharpoonright n}(\overline{\gamma^{\upharpoonright n}}p) \neq 0].$

Clearly $\forall \gamma \in \mathcal{C} \exists n [\beta(\overline{\gamma}n) \neq 0]$, i.e. $Bar_{\mathcal{C}}(D_{\beta})$.

Let m be given such that $Bar_{\mathcal{C}}(D_{\overline{\beta}m})$. Suppose there is no n < m such that $Bar_{\mathcal{C}}(D_{\overline{\alpha^{n}}m})$. For each n < m, there exists u in Bin such that length(u) = m and u does not meet D_{α^n} . Let s be an element of Bin such that length(s) = m and, for each n < m, s^n does not meet $D_{\alpha \uparrow n}$. Note: s does not meet D_{β} . Contradiction. Thus we see there must exist n < m such that every s in Bin_m meets $D_{\alpha^{\dagger n}}$. Conclude: $\exists n [\mathcal{C} \subseteq \mathcal{G}_{\alpha^{\upharpoonright n}}].$

 $^{^{12}}$ The argument may be compared to the arguments for Lemma 5.3(i) and for Lemma 5.5(ii).

(ii) We prove, in BIM: for each α , there exists β such that

$$\forall \delta \in [0,1]^{\omega} \exists n [\delta^{\uparrow n} \in \mathcal{H}_{\alpha^{\uparrow n}}] \to \forall \gamma \in \mathcal{C} \exists n [\gamma^{\uparrow n} \in \mathcal{G}_{\beta^{\uparrow n}}] \text{ and}$$
$$\exists n [\mathcal{C} \subseteq \mathcal{G}_{\beta^{\uparrow n}}] \to \exists n [[0,1] \subseteq \mathcal{H}_{\alpha^{\uparrow n}}].$$

Using Lemma 13.3, find $\sigma: \mathcal{C} \to [0,1]$ and $\psi: \omega^{\omega} \to \omega^{\omega}$ such that $\forall \delta \in [0,1] \exists \gamma \in \mathcal{C}[\sigma | \gamma =_{\mathcal{R}} \delta] \text{ and } \forall \alpha \forall \gamma \in \mathcal{C}[\gamma \in \mathcal{G}_{\psi | \alpha} \leftrightarrow \sigma | \gamma \in \mathcal{H}_{\alpha}].$

Let α be given. Define β such that, for every n, $\beta^{\uparrow n} = \psi | (\alpha^{\uparrow n})$.

Assume $\forall \delta \in [0,1]^{\omega} \exists n[\delta^{\uparrow n} \in \mathcal{H}_{\alpha^{\uparrow n}}]$. Then $\forall \gamma \in \mathcal{C} \exists n[\sigma|(\gamma^{\uparrow n}) \in \mathcal{H}_{\alpha^{\uparrow n}}]$ and $\forall \gamma \in \mathcal{C} \exists n [\gamma^{\restriction n} \in \mathcal{G}_{\psi \mid (\alpha^{\restriction n})}] \text{ and: } \forall \gamma \in \mathcal{C} \exists n [\gamma^{\restriction n} \in \mathcal{G}_{\beta^{\restriction n}}].$

Let n be given such that such that $C \subseteq \mathcal{G}_{\beta^{\uparrow n}} = \mathcal{G}_{\psi \mid (\alpha^{\uparrow n})}$.

Note: $\forall \gamma \in \mathcal{C}[\sigma | \gamma \in \mathcal{H}_{\alpha^{\dagger n}}]$ and, therefore: $[0,1] \subseteq \mathcal{H}_{\alpha^{\dagger n}}$.

(iii) We prove, in BIM: for each α , there exists β such that

$$\forall \gamma \in \mathcal{C} \exists n [(n, \gamma(n)) \in E_{\alpha}] \to \forall \delta \in [0, 1]^{\omega} \exists n [\delta^{\dagger n} \in \mathcal{H}_{\beta^{\dagger n}}] \text{ and}$$
$$\exists n [[0, 1] \subseteq \mathcal{H}_{\beta^{\dagger n}}] \to \exists n \forall i < 2 [(n, i) \in E_{\alpha}].$$

Let α be given. Define β such that $\forall n \forall s \in \mathbb{S}[\beta^{n}(s) \neq 0 \leftrightarrow$

$$\exists i < s[\left(\alpha(i) = (n,0) + 1 \ \land \ s'' <_{\mathbb{Q}} 1_{\mathbb{Q}}\right) \ \lor \ \left(\alpha(i) = (n,1) + 1 \ \land \ 0_{\mathbb{Q}} <_{\mathbb{Q}} s'\right)]].$$

Note $\forall n[((n,0) \in E_{\alpha} \leftrightarrow [0,1) \subseteq \mathcal{H}_{\beta \uparrow n}]) \land ((n,1) \in E_{\alpha} \leftrightarrow (0,1] \subseteq \mathcal{H}_{\beta \uparrow n})].$

Assume: $\forall \gamma \in \mathcal{C} \exists n[(n, \gamma(n)) \in E_{\alpha}], \text{ and } \delta \in [0, 1]^{\omega}.$

Define ε such that $\forall n[\varepsilon(n) = \mu m[0_{\mathbb{Q}} <_{\mathbb{Q}} (\delta^n(m))' \vee (\delta^n(m))'' <_{\mathbb{Q}} 1_{\mathbb{Q}}].$

Define γ in \mathcal{C} such that $\forall n [\gamma(n) = 0 \leftrightarrow (\delta^n(\varepsilon(n)))'' <_{\mathbb{Q}} 1_{\mathbb{Q}}].$

Note: $\forall n [\gamma(n) = 1 \to 0_{\mathcal{R}} <_{\mathcal{R}} \delta^n].$

Find n such that $(n, \gamma(n)) \in E_{\alpha}$ and conclude:

either: $\gamma(n) = 0$ and $\delta^{\uparrow n} <_{\mathcal{R}} 1_{\mathcal{R}}$ and $[0,1) \subseteq \mathcal{H}_{\beta^{\uparrow n}}$, so $\delta^{\uparrow n} \in \mathcal{H}_{\beta^{\uparrow n}}$,

or: $\gamma(n) = 1$ and $0_{\mathcal{R}} <_{\mathcal{R}} \delta^{\uparrow n}$ and $(0,1] \subseteq \mathcal{H}_{\beta^{\uparrow n}}$, so, again, $\delta^{\uparrow n} \in \mathcal{H}_{\beta^{\uparrow n}}$.

Conclude: $\forall \delta \in [0,1]^{\omega} \exists n [\delta^{\uparrow n} \in \mathcal{H}_{\beta^{\uparrow n}}].$

Let n be given such that $[0,1] \subseteq \mathcal{H}_{\beta \upharpoonright n}$. Conclude: $\forall i < 2[(n,i) \in E_{\alpha}]$.

$$\begin{array}{ll} \textbf{Theorem 7.2.} & (i) \ \ \mathsf{BIM} \vdash \mathbf{FT} \leftrightarrow \boldsymbol{\Sigma}^0_1 \text{-} \overleftarrow{\mathbf{AC}_{\omega,\mathcal{C}}} \leftrightarrow \boldsymbol{\Sigma}^0_1 \text{-} \overleftarrow{\mathbf{AC}_{\omega,[0,1]}}. \\ \\ (ii) \ \ \mathsf{BIM} \vdash \neg ! \mathbf{FT} \leftrightarrow \neg ! (\boldsymbol{\Sigma}^0_1 \text{-} \overleftarrow{\mathbf{AC}_{\omega,\mathcal{C}}}) \leftrightarrow \neg ! (\boldsymbol{\Sigma}^0_1 \text{-} \overleftarrow{\mathbf{AC}_{\omega,[0,1]}}). \end{array}$$

Proof. These statements follow from Lemmas 7.1 and 5.3.

8. On the Contraposition of Twofold Compact Choice

We introduce a limited version of $\Sigma_1^0 - \overleftarrow{\mathbf{AC}_{\omega,C}}$:

8.1.
$$\Sigma_1^0 - \overline{\mathbf{AC}}_{2,\mathcal{C}}$$
: $\forall \alpha [\forall \gamma \in \mathcal{C} \exists i < 2[\gamma^{\restriction i} \in \mathcal{G}_{\alpha^i}] \to \exists i < 2[\mathcal{C} \subseteq \mathcal{G}_{\alpha^{\restriction i}}]]$. This statement should be called the Σ_1^0 -Axiom of Reverse Twofold Compact

Choice. It is a contraposition of a special case of the following scheme:

$$\forall i < 2\exists \gamma \in \mathcal{C}[R(i,\gamma)] \to \exists \gamma \in \mathcal{C} \forall i < 2[R(i,\gamma^{\dagger i})].$$

and the latter scheme is provable in BIM.

For each α , we define the following statement, called **LLPO** $^{\alpha}$:

$$\forall \varepsilon [\forall p [2p = \mu m[\varepsilon(m) \neq 0] \to Bar_{\mathcal{C}}(D_{\overline{\alpha}p})] \lor \forall p [2p + 1 = \mu m[\varepsilon(m) \neq 0] \to Bar_{\mathcal{C}}(D_{\overline{\alpha}p})]].$$

Lemma 8.1. (i) BIM \vdash LLPO $\leftrightarrow \forall \alpha [$ LLPO $^{\alpha}]$.

(ii) BIM
$$+ \Sigma_1^0 - AC_{2,C} \vdash \forall \alpha [Bar_C(D_\alpha) \to \mathbf{LLPO}^\alpha]$$

(ii) BIM
$$+ \Sigma_1^0 - \overrightarrow{AC}_{2,C} \vdash \forall \alpha [Bar_{\mathcal{C}}(D_{\alpha}) \to \mathbf{LLPO}^{\alpha}].$$

(iii) BIM $+ \Pi_1^0 - \mathbf{AC}_{\omega,2} \vdash \forall \alpha [Bar_{\mathcal{C}}(D_{\alpha}) \to \mathbf{LLPO}^{\alpha}] \to \mathbf{FT}.$

(iv) BIM $\vdash \mathbf{FT} \to \Sigma_1^0 - \overleftarrow{\mathbf{AC}}_{2,\mathcal{C}}$.

¹³The argument may be compared to the argument for the first half of Theorem 6.5.

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Proof. (i) Assume LLPO and let \alpha, \varepsilon be given.
 Either \forall p[2p \neq \mu n[\varepsilon(n) \neq 0]] and, therefore, \forall \varepsilon [\forall p[2p = \mu m[\varepsilon(m) \neq 0] \rightarrow Bar_{\mathcal{C}}(D_{\overline{\alpha}p})],
 or \forall p[2p+1 \neq \mu n[\varepsilon(n) \neq 0]] and: \forall \varepsilon [\forall p[2p+1 = \mu m[\varepsilon(m) \neq 0] \rightarrow Bar_{\mathcal{C}}(D_{\overline{\alpha}p})].
We thus see: LLPO^{\alpha}.
      For the converse, note: \mathbf{LLPO} \leftrightarrow \mathbf{LLPO}^{\underline{0}}.
      (ii) Let \alpha be given such that Bar_{\mathcal{C}}(D_{\alpha}).
Using \Sigma_0^1-\mathbf{AC}_{2,\mathcal{C}}, we now prove: \mathbf{LLPO}^{\alpha}.
      Let \varepsilon be given. Define \eta such that, for each p,
        (1) if \overline{\underline{0}}(2p+2) \subset \varepsilon, then \eta^{\uparrow 0}(p) = \eta^{\uparrow 1}(p) = \alpha(p), and,
        (2) if 2p = \mu m[\varepsilon(m) \neq 0], then \forall m \geq p[\eta^{\uparrow 0}(m) = 0 \land \eta^{\uparrow 1}(m) = \alpha(m)], and
        (3) if 2p+1=\mu m[\varepsilon(m)\neq 0], then \forall m\geq p[\eta^{\uparrow 1}(m)=0 \land \eta^{\uparrow 0}(m)=\alpha(m)].
      Note: if \eta^{\uparrow 0} \# \alpha, then \eta^{\uparrow 1} = \alpha.
      Let \gamma in \mathcal{C} be given. Find n such that \alpha(\overline{\gamma^{\lceil 0}}n) \neq 0.
Either: \eta^{\uparrow 0}(\overline{\gamma^{\uparrow 0}}n) = \alpha(\overline{\gamma^{\uparrow 0}}n) \neq 0,
or: \eta^{\upharpoonright 0} \# \alpha and \eta^{\upharpoonright 1} = \alpha and \exists m [\eta^{\upharpoonright 1}(\overline{\gamma^{\upharpoonright 1}}m) = \alpha(\overline{\gamma^{\upharpoonright 1}}m) \neq 0].
We thus see: \forall \gamma \in \mathcal{C}[\gamma^{\upharpoonright 0} \in \mathcal{G}_{\eta^{\upharpoonright 0}} \vee \gamma^{\upharpoonright 1} \in \mathcal{G}_{\eta^{\upharpoonright 1}}].
      Use \Sigma_0^1-\overline{\mathbf{AC}}_{2,\mathcal{C}} and find i < 2 such that \mathcal{C} \subseteq \mathcal{G}_{\eta \upharpoonright i}.
      Assume: C \subseteq \mathcal{G}_{\eta^{\uparrow i}}, i.e. Bar_{\mathcal{C}}(D_{\eta^{\uparrow i}}). Assume: 2p + i = \mu m[\varepsilon(m) \neq 0].
Note: \forall m \geq p[\eta^{\dagger i}(m) = 0]. Conclude: Bar_{\mathcal{C}}(D_{\overline{\eta^{\dagger i}}p}) and: Bar_{\mathcal{C}}(D_{\overline{\alpha}p}).
      We thus see: \forall p[2p+i=\mu m[\varepsilon(m)\neq 0]\to Bar_{\mathcal{C}}(D_{\overline{\alpha}p})].
      Conclude: \exists i < 2 \forall p[2p+i = \mu m[\varepsilon(m) \neq 0] \rightarrow Bar_{\mathcal{C}}(D_{\overline{\alpha}p})], i.e. LLPO<sup>\alpha</sup>.
      (iii) Assume \forall \alpha [Bar_{\mathcal{C}}(D_{\alpha}) \to \mathbf{LLPO}^{\alpha}].
Using \Pi_1^0-AC<sub>\omega,2</sub>, we now prove: FT.
     Let \alpha be given such that Bar_{\mathcal{C}}(D_{\alpha}) and, therefore, LLPO^{\alpha}.
      Assume: s \in Bin. Define \varepsilon such that, \forall i < 2 \forall n [\varepsilon(2n+i) \neq 0 \leftrightarrow Bar_{\mathcal{C} \cap s*\langle i \rangle}(D_{\overline{\alpha}n})].
Using LLPO<sup>\alpha</sup>, find i < 2 such that \forall p[2p + i = \mu m[\varepsilon(m) \neq 0] \rightarrow Bar_{\mathcal{C}}(D_{\overline{\alpha}p})].
      Assume we find n such that Bar_{\mathcal{C} \cap s*\langle i \rangle}(D_{\overline{\alpha}n}). Then \varepsilon(2n+i) \neq 0.
Find p := \mu j[\varepsilon(j) \neq 0]. Find q \leq n such that p = 2q or p = 2q + 1.
Either p = 2q + i and Bar_{\mathcal{C}}(D_{\overline{\alpha}q}), or p = 2q + 1 - i and Bar_{\mathcal{C} \cap s*(1-i)}(D_{\overline{\alpha}q}).
In both cases: Bar_{\mathcal{C} \cap s*\langle 1-i\rangle}(D_{\overline{\alpha}n}).
We thus see: \forall n[Bar_{\mathcal{C} \cap s*\langle i\rangle}(D_{\overline{\alpha}n}) \to Bar_{\mathcal{C} \cap s*\langle 1-i\rangle}(D_{\overline{\alpha}n})].
      Conclude: \forall s \in Bin \exists i < 2 \forall n [Bar_{\mathcal{C} \cap s*\langle i \rangle}(D_{\overline{\alpha}n}) \to Bar_{\mathcal{C} \cap s*\langle 1-i \rangle}(D_{\overline{\alpha}n})].
      Now use \Pi_1^0-AC<sub>\omega,2</sub> and find \gamma in \mathcal{C} such that
      \forall s \in Bin \forall n [Bar_{\mathcal{C} \cap s*\langle \gamma(s) \rangle}(D_{\overline{\alpha}n}) \to Bar_{\mathcal{C} \cap s*\langle 1-\gamma(s) \rangle}(D_{\overline{\alpha}n})].
      Observe that, for each s in Bin, for all n, if Bar_{\mathcal{C}\cap s*\langle\gamma(s)\rangle}(D_{\overline{\alpha}n}), then also
Bar_{\mathcal{C}\cap s*\langle 1-\gamma(s)\rangle}(D_{\overline{\alpha}n}), and, therefore, Bar_{\mathcal{C}\cap s}(D_{\overline{\alpha}n}).
      Define \delta in \mathcal{C} such that, for each n, \delta(n) = \gamma(\overline{\delta}n). Find p such that \alpha(\overline{\delta}p) \neq 0 and
define n := \overline{\delta}p + 1. Note: Bar_{\mathcal{C} \cap \overline{\delta}p}(D_{\overline{\alpha}n}). One may prove, by backwards induction:
for each j \leq p, Bar_{\mathcal{C} \cap \overline{\delta}j}(D_{\overline{\alpha}n}). For assume j+1 \leq n and Bar_{\mathcal{C} \cap \overline{\delta}(j+1)}(D_{\overline{\alpha}n}). As
\overline{\delta}(j+1) = \overline{\delta}j * \langle \gamma(\overline{\delta}j) \rangle, one may conclude: Bar_{\mathcal{C} \cap \overline{\delta}(j)}(D_{\overline{\alpha}n}). After n steps we obtain
the conclusion: Bar_{\mathcal{C}}(D_{\overline{\alpha}n}).
      Conclude: \forall \alpha [Bar_{\mathcal{C}}(D_{\alpha}) \to \exists n [Bar_{\mathcal{C}}(D_{\overline{\alpha}n})]], i.e. FT.
      (iv) Assume FT. Use Theorem 7.2 and conclude: \Sigma_1^0 + \overleftarrow{\mathbf{AC}_{\omega,\mathcal{C}}} and its corollary:
\Sigma_1^0-\overline{\mathbf{AC}}_{2,\mathcal{C}}.
```

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Theorem 8.2. $BIM + \Pi_1^0 - AC_{\omega,2} \vdash \Sigma_1^0 - \overleftarrow{AC_{2,\mathcal{C}}} \leftrightarrow FT$.

Proof. Use Lemma 8.1.

8.1.1. Mark Bickford called my attention to the fact that $\Sigma_1^0 - \overleftarrow{\mathbf{AC}}_{2,\mathcal{C}}$ occurs in [21, §2] and is called there the separation principle SP.

After having proven $\mathbf{FT} \to \Sigma_1^0 - \overline{\mathbf{AC}_{2,\mathcal{C}}}$, see our Lemma 8.1(iv), the author of [21] gives a proof of $FT \to WKL!$. She does so as follows.

Assume FT.

Let α be given such that $\forall n[\neg Bar_{\mathcal{C}}(D_{\overline{\alpha}n})]$ and

 $\forall \gamma \in \mathcal{C}[\gamma^{\uparrow 0} \perp \gamma^{\uparrow 1} \to \exists i < 2 \exists n [\alpha(\overline{\gamma^{\uparrow i}}n) \neq 0]].$

We intend to prove: $\exists \gamma \in \mathcal{C} \forall n [\alpha(\overline{\gamma}n) = 0].$

Define α^* such that $\forall s \in Bin[\alpha^*(s) = 0 \leftrightarrow \forall t \sqsubseteq s[\alpha(t) = 0]].$

Using induction, we prove:

 $(*): \forall n \exists ! s \in Bin_n \forall m > n \exists t \in Bin_m [s \sqsubset t \land \alpha^*(t) = 0]].$

If n = 0, then $s = \langle \rangle$ satisfies the requirements.

Let n be given and let s be the unique element of Bin_n such that

 $\forall m \ge n \exists t \in Bin_m[s \sqsubseteq t \land \alpha^*(t) = 0].$

Observe: $\forall \gamma \in \mathcal{C} \exists i < 2 \exists n [\alpha(s * \langle i \rangle * \overline{\gamma^{\dagger i}}n) \neq 0].$

Using $\Sigma_1^0 - \overline{AC}_{2,\mathcal{C}}$, find j < 2 such that $\forall \gamma \in \mathcal{C} \exists n [\alpha(s * \langle j \rangle * \overline{\gamma} n) \neq 0]$.

Using **FT**, find m such that $\forall \gamma \in \mathcal{C} \exists n \leq m [\alpha(s * \langle j \rangle * \overline{\gamma} n) \neq 0].$

Define p := n + 1 + m and note:

 $\forall q > p \forall t \in Bin_q[s * \langle j \rangle \sqsubseteq t \to \alpha^*(t) \neq 0]$ and:

 $s*\langle 1-j\rangle$ is the unique element u of Bin_{n+1} such that

 $\forall q \ge n + 1 \exists t \in Bin_q[u \sqsubseteq t \land \alpha^*(t) = 0].$

This completes our proof of (*).

Using (*) and Π_1^0 - $\mathbf{AC}_{\omega,\omega}$!, find δ such that

 $\forall n[\delta(n) \in Bin_n \land \forall m \ge n \exists t \in Bin_m[\delta(n) \sqsubseteq t \land \alpha^*(t) = 0]].$

Note: $\forall n[\delta(n) \sqsubset \delta(n+1) \land \alpha^*(\delta(n)) = 0]$].

Define γ such that $\forall n[\delta(n) \sqsubseteq \gamma]$ and note: $\gamma \in \mathcal{C}$ and $\forall n[\alpha(\overline{\gamma}n) = 0]$.

The author of [21] quotes the result $\mathbf{WKL}! \to \mathbf{FT}$, see our Theorem 2.6.

She does not prove **FT**, or equivalently, **WKL**!, from $\Sigma_1^0 - \overline{\mathbf{AC}}_{2,\mathcal{C}}$ and $\Pi_1^0 - \mathbf{AC}_{\omega,\omega}!$.

Our proof of Theorem 2.7 shows that no choice is needed for a proof of $\mathbf{FT} \to \mathbf{WKL!}.$

We introduce a 'real' version of Σ_1^0 - $\overleftarrow{\mathbf{AC}}_{2,\mathcal{C}}$:

8.2. $\Sigma_1^0 - \overline{AC_{2,[0,1]}}$:

$$\forall \alpha [\forall \delta \in [0,1]^2 \exists i < 2[\delta^{\uparrow i} \in \mathcal{H}_{\alpha^{\uparrow i}}] \to \exists i < 2[[0,1] \subseteq \mathcal{H}_{\alpha^{\uparrow i}}]].$$

Theorem 8.3. BIM
$$\vdash \Sigma_1^0 - \overleftarrow{\mathbf{AC}}_{2,\mathcal{C}} \leftrightarrow \Sigma_1^0 - \overleftarrow{\mathbf{AC}}_{2,[0,1]}$$
.

Proof. Using Lemma 13.3, find $\sigma: \mathcal{C} \to [0,1]$ and $\psi: \omega^{\omega} \to \omega^{\omega}$ such that $\forall \delta \in [0, 1] \exists \gamma [\delta =_{\mathcal{R}} \sigma | \gamma] \text{ and } \forall \alpha \forall \gamma \in \mathcal{C} [\gamma \in \mathcal{G}_{\psi | \alpha} \leftrightarrow \sigma | \gamma \in \mathcal{H}_{\alpha}].$

First assume: $\Sigma_1^0 - \overleftarrow{\mathbf{AC}_{2,\mathcal{C}}}$.

Let α be given such that $\forall \delta \in [0,1]^2 \exists i < 2[\delta^{\dagger i} \in \mathcal{H}_{\alpha^{\dagger i}}]$. Define β such that, for both i < 2, $\beta^{\dagger i} = \psi | (\alpha^{\dagger i})$. Then $\forall \gamma \in \mathcal{C} \exists i < 2[\sigma | \gamma^{\dagger i} \in \mathcal{H}_{\alpha^{\dagger i}}]$ and, therefore: $\forall \gamma \in \mathcal{C} \exists i < 2[\gamma^{\dagger i} \in \mathcal{G}_{\beta^{\dagger i}}]$. Find i < 2 such that $\mathcal{C} \subseteq \mathcal{G}_{\beta^{\dagger i}}$. Conclude: $[0,1] \subseteq \mathcal{H}_{\alpha^{\dagger i}}$.

We nay conclude: $\Sigma_1^0 - \overleftarrow{\mathbf{AC}}_{2,[0,1]}$.

Now assume: $\Sigma_1^0 - \overline{\mathbf{AC}_{2,[0,1]}}$.

Let α be given such that

Using Lemma 13.4, find $\tau: \mathcal{C} \to [0,1]$ and $\chi: \omega^{\omega} \to \omega^{\omega}$ such that

- (1) $\forall \gamma \in \mathcal{C} \forall \delta \in \mathcal{C}[\gamma \# \delta \to \tau | \gamma \#_{\mathcal{R}} \tau | \delta]$, and
- (2) $\forall \alpha \forall \gamma \in \mathcal{C}[\gamma \in \mathcal{G}_{\alpha} \leftrightarrow \tau | \gamma \in \mathcal{H}_{\chi|\alpha}]$ and
- (3) $\forall \delta \in [0, 1] \exists \gamma \in \mathcal{C}[\delta \#_{\mathcal{R}} \tau | \gamma \to \forall \alpha [\delta \in \mathcal{H}_{\chi | \alpha}]].$

Let α be given such that $\forall \gamma \in \mathcal{C} \exists i < 2[\gamma^{\uparrow i} \in \mathcal{G}_{\alpha^{\uparrow i}}].$

Let δ in $[0,1]^2$ be given.

Find γ in \mathcal{C} such that $\forall i < 2[\delta^{\uparrow i} \#_{\mathcal{R}} \tau | (\gamma^{\uparrow i}) \to \delta^{\uparrow i} \in \mathcal{H}_{\chi | (\alpha^{\uparrow i})}]$.

Find i < 2 such that $\gamma^{\uparrow i} \in \mathcal{G}_{\alpha^{\uparrow i}}$.

Conclude: $\tau|(\gamma^{\uparrow i}) \in \mathcal{H}_{\chi|\alpha^{\uparrow i}}$. Find s, n such that $(\chi|\alpha^{\uparrow i})(s) \neq 0$ and $(\tau|(\gamma^{\uparrow i}))(n) \sqsubseteq_{\mathbb{S}} s$.

Using Lemma 13.2, find p such that either: $\delta^{\uparrow i}(p) \sqsubseteq_{\mathbb{S}} s$, and, therefore: $\delta^{\uparrow i} \in \mathcal{H}_{\chi|(\alpha^i)}$,

or: $\delta^{\uparrow i}(p) \#_{\mathbb{S}} (\tau | (\gamma^{\uparrow i}))(n)$, and, therefore, again: $\delta^{\uparrow i} \in \mathcal{H}_{\chi | (\alpha^{i})}$.

We thus see: $\forall \delta \in [0,1]^2 \exists i < 2[\delta^{\uparrow i} \in \mathcal{H}_{\chi|(\alpha^{\uparrow i})}].$

Applying $\Sigma_1^0 - \overleftarrow{\mathbf{AC}}_{2,[0,1]}$, find i < 2 such that $[0,1] \subseteq \mathcal{H}_{\chi|(\alpha^{\dagger i})}$.

Conclude: $\forall \gamma \in \mathcal{C}[\tau | \gamma \in \mathcal{H}_{\chi | (\alpha^{\dagger i})}]$ and: $\mathcal{C} \subseteq \mathcal{G}_{\alpha^{\dagger i}}$.

We may conclude: $\Sigma_1^0 - \overleftarrow{\mathbf{AC}}_{2,\mathcal{C}}$.

Corollary 8.4. $\mathsf{BIM} + \Pi^0_1 \text{-}\mathbf{AC}_{\omega,2} \vdash \mathbf{FT} \leftrightarrow \mathbf{\Sigma}^0_1 \text{-}\overleftarrow{\mathbf{AC}_{2,[0,1]}}.$

Proof. Use Theorems 8.2 and 8.3.

8.3. Some logical consequences.

In this Subsection, we want to formulate the result of Theorem 8.2 in modeltheoretic terms and draw an even sharper conclusion.

For every δ , we define a proposition Pr_{δ} , as follows: $Pr_{\delta} := \exists n[\delta(n) \neq 0]$.

Tarski's truth definition makes sense intuitionistically as well as classically.

For every structure $\mathfrak{A}=(A,\ldots)$, for every sentence φ in the elementary language of the structure \mathfrak{A} , we write:

$$\mathfrak{A} \models \varphi$$

if the sentence φ is true in the structure \mathfrak{A} .

More generally, for every structure $\mathfrak{A} = (A, ...)$, for every formula

 $\varphi=\varphi(x_0,x_1,\ldots,x_{n-1})$ in the elementary language of the structure $\mathfrak{A},$

for all $a_0, a_1, \ldots, a_{n-1}$ in A, we write:

$$\mathfrak{A} \models \varphi[a_0, a_1, \dots, a_{n-1}]$$

if the formula φ is true in the structure \mathfrak{A} , provided we interpret the individual variables $x_0, x_1, \ldots, x_{n-1}$ by $a_0, a_1, \ldots, a_{n-1}$, respectively.

Theorem 8.5. The following statements are equivalent in BIM:

- (i) $\forall \alpha[(\mathcal{C}, \mathcal{G}_{\alpha^{\upharpoonright 0}}, Pr_{\alpha^{\upharpoonright 1}}) \models \forall x[P(x) \lor A] \to (\forall x[P(x)] \lor A).$
- (ii) $\forall \alpha [(\mathcal{C}, \mathcal{G}_{\alpha^{\uparrow 0}}, \mathcal{G}_{\alpha^{\uparrow 1}}) \models \forall x [P(x) \lor Q(x)] \to (\forall x [P(x)] \lor \exists x [Q(x)]).$

Proof. (i) \Rightarrow (ii). Let α be given such that $(\mathcal{C}, \mathcal{G}_{\alpha^{\dagger 0}}, \mathcal{G}_{\alpha^{\dagger 1}}) \models \forall x [P(x) \vee Q(x)].$

Note: $(C, \mathcal{G}_{\alpha^{\uparrow 0}}, \mathcal{G}_{\alpha^{\uparrow 1}}) \models \forall x [P(x) \lor \exists y [Q(y)]].$

Define β such that $\beta^{\dagger 0} = \alpha^{\dagger 0}$ and $\forall n [\beta^{\dagger 1}(n) \neq 0 \leftrightarrow (\alpha^{\dagger 1}(n) \neq 0 \land n \in Bin)].$

Note: $(C, \mathcal{G}_{\beta^{\uparrow 0}}, Pr_{\beta^{\uparrow 1}}) \models \forall x [P(x) \lor A]$, and thus, according to (i):

 $(\mathcal{C}, \mathcal{G}_{\beta^{\dagger 0}}, Pr_{\beta^{\dagger 1}}) \models \forall x[P(x)] \lor A, \text{ and, therefore: } (\mathcal{C}, \mathcal{G}_{\alpha^{\dagger 0}}, \mathcal{G}_{\alpha^{\dagger 1}}) \models \forall x[P(x)] \lor \exists x[Q(x)].$

(ii) \Rightarrow (i). Let α be given such that $(\mathcal{C}, \mathcal{G}_{\alpha^{\dagger 0}}, Pr_{\alpha^{\dagger 1}}) \models \forall x [P(x) \vee A].$

Define β such that $\beta^{\uparrow 0} = \alpha^{\uparrow 0}$ and $\forall n [\beta^{\uparrow 1}(n) \neq 0 \leftrightarrow (\exists i < n [\alpha^{\dagger 1}(i) \neq 0] \land n \in Bin)].$

Note that, for each γ in \mathcal{C} , $\gamma \in \mathcal{G}_{\beta^{\dagger 1}}$ if and only if $Pr_{\alpha^{\dagger 1}}$.

Therefore, $(\mathcal{C}, \mathcal{G}_{\beta^{\uparrow 0}}, \mathcal{G}_{\beta^{\uparrow 1}}) \models \forall x[P(x) \lor Q(x)]$ and, according to (ii): $(\mathcal{C}, \mathcal{G}_{\beta^{\uparrow 0}}, \mathcal{G}_{\beta^{\uparrow 1}}) \models \forall x[P(x)] \lor \exists x[Q(x)]$, and thus: $(\mathcal{C}, \mathcal{G}_{\alpha^{\uparrow 0}}, Pr_{\alpha^{\uparrow 1}}) \models \forall x[P(x)] \lor A$. \square

For each α , we define the following statement:

$$\mathbf{LPO}^{\alpha}: \ \forall \varepsilon [\forall p [\overline{\underline{0}}(2p+2) \perp \varepsilon \rightarrow Bar_{\mathcal{C}}(D_{\overline{\alpha}p})] \ \lor \ \exists n [\varepsilon(n) \neq 0]].$$

Lemma 8.6. (i) BIM \vdash **LPO** $\rightarrow \forall \alpha [\mathbf{LPO}^{\alpha}]$.

```
(ii) BIM \vdash \forall \alpha [\mathbf{LPO}^{\alpha} \to \mathbf{LLPO}^{\alpha}].
Proof. (i) Given any \varepsilon, by LPO.
 either \varepsilon = \underline{0}, and, therefore: \forall p[\underline{\overline{0}}(2p+2) \perp \varepsilon \rightarrow Bar_{\mathcal{C}}(D_{\overline{\alpha}p})], \text{ or } \exists n[\varepsilon(n) \neq 0].
       (ii) Let \alpha be given such that LPO^{\alpha}. We have to prove, for all \varepsilon, either
\forall p[2p = \mu m[\varepsilon(m) \neq 0] \rightarrow Bar_{\mathcal{C}}(D_{\overline{\alpha}p})] \text{ or } \forall p[2p + 1 = \mu m[\varepsilon(m) \neq 0] \rightarrow Bar_{\mathcal{C}}(D_{\overline{\alpha}p})].
      Let \varepsilon be given. Use LPO<sup>\alpha</sup> and distinguish two cases.
       Case (1). \forall p[\overline{\underline{0}}(2p+2) \perp \varepsilon \rightarrow Bar_{\mathcal{C}}(D_{\overline{\alpha}p})].
Then \forall p[2p = \mu m[\varepsilon(m) \neq 0] \rightarrow Bar_{\mathcal{C}}(D_{\overline{\alpha}p})], and also
\forall p[2p+1 = \mu p[\varepsilon(m) \neq 0] \to Bar_{\mathcal{C}}(D_{\overline{\alpha}p})].
       Case (2). \exists n[\varepsilon(n) \neq 0]. Define m := \mu n[\varepsilon(n) \neq 0].
Find q such that m = 2q or m = 2q + 1.
If m = 2q + 1, then \forall p[2p = \mu m[\varepsilon(m) \neq 0] \to Bar_{\mathcal{C}}(D_{\overline{\alpha}p})].
If m = 2q, then \forall p[2p + 1 = \mu m[\varepsilon(m) \neq 0] \rightarrow Bar_{\mathcal{C}}(D_{\overline{\alpha}p})].
      Conclude: LLPO^{\alpha}.
                                                                                                                                                                               Theorem 8.7. The following statements are equivalent in BIM + \Pi_1^0-AC<sub>\omega,2</sub>:
   (ii) \ \forall \alpha [(\mathcal{C}, \mathcal{G}_{\alpha^{\dagger 0}}, \mathcal{G}_{\alpha^{\dagger 1}}) \models \forall x \forall y [P(x) \lor Q(y)] \to (\forall x [P(x)] \lor \forall y [Q(y)])].
 (iii) \forall \alpha [(\mathcal{C}, \mathcal{G}_{\alpha^{\dagger 0}}, Pr_{\alpha^{\dagger 1}}) \models \forall x [P(x) \lor A] \to (\forall x [P(x)] \lor A)].
 (iv) \forall \alpha [Bar_{\mathcal{C}}(D_{\alpha}) \to \mathbf{LPO}^{\alpha}].
Proof. (i) \Rightarrow (ii). Note that, in BIM + \Pi_1^0-AC<sub>\omega,2</sub>, FT implies \Sigma_1^0-\overleftarrow{AC_{2,\mathcal{C}}}:
\forall \alpha [\forall \gamma \in \mathcal{C} \exists i < 2[\gamma^{\restriction i} \in \mathcal{G}_{\alpha^{\restriction i}}] \to \exists i < 2[\mathcal{C} \subseteq \mathcal{G}_{\alpha^{\restriction i}}]], \text{ see Theorem 8.2.}
      (ii) \Rightarrow (iii). Let \alpha be given such that (\mathcal{C}, \mathcal{G}_{\alpha^{\uparrow 0}}, Pr_{\alpha^{\uparrow 1}}) \models \forall x [P(x) \vee A], i.e.
\forall \gamma \in \mathcal{C} \exists n \exists i < 2 [\alpha^{\restriction 0}(\overline{\gamma}n) \neq 0 \ \lor \ \alpha^{\restriction 1}(n) \neq 0].
Define \beta such that \beta^{\uparrow 0} = \alpha^{\uparrow 0} and \forall n \forall s \in Bin_n[\beta^{\uparrow 1}(s) = \alpha^{\uparrow 1}(n)].
Note (C, \mathcal{G}_{\beta^{\dagger 0}}, \mathcal{G}_{\beta^{\dagger 1}}) \models \forall x \forall y [P(x) \lor Q(y)] and conclude, using (i),
(\mathcal{C},\mathcal{G}_{\beta^{\dagger 0}},\mathcal{G}_{\beta^{\dagger 1}}) \models \forall x[\mathsf{P}(\mathsf{x})] \vee \forall y[\mathsf{Q}(\mathsf{y})] \text{ and } (\mathcal{C},\mathcal{G}_{\alpha^{\dagger 0}},\mathit{Pr}_{\alpha^{\dagger 1}}) \models \forall x[\mathsf{P}(\mathsf{x})] \vee \mathsf{A}.
      (iii) \Rightarrow (iv). Assume (iii).
      Let \alpha be given such that Bar_{\mathcal{C}}(D_{\alpha}). We have to prove: LPO^{\alpha}.
Let \varepsilon be given.
Define \eta in \mathcal{C} such that \eta^{\uparrow 1} = \varepsilon, and, for each p, if \overline{\underline{0}}(2p+2) \sqsubset \varepsilon, then \eta^{\uparrow 0}(p) = \alpha(p),
and, if \overline{0}(2p+2) \perp \varepsilon, then \eta^{\uparrow 0}(p) = 0. Note: if \eta^{\uparrow 0} \# \alpha, then \exists n [\varepsilon(n) = \eta^{\uparrow 1}(n) \neq 0].
      Let \gamma in \mathcal{C} be given. Find n such that \alpha(\overline{\gamma}n) \neq 0.
Either: \eta^0(\overline{\gamma}n) = \alpha(\overline{\gamma}n) \neq 0 or: \eta^{\uparrow 0} \# \alpha and \exists m[\eta \upharpoonright 1(m) \neq 0].
We thus see: \forall \gamma \in \mathcal{C}[(\exists n[\eta^{\uparrow 0}(\overline{\gamma}n) \neq 0]) \lor (\exists m[\eta^{\uparrow 1}(m) \neq 0])].
Use (iii) and conclude: \forall \gamma \in \mathcal{C} \exists n [\eta^{\uparrow 0}(\overline{\gamma}n) = 1] \lor \exists m [\eta^{\uparrow 1}(m) \neq 0].
      Assume: \forall \gamma \in \mathcal{C} \exists n [\eta^{\uparrow 0}(\overline{\gamma}n) \neq 0], that is: Bar_{\mathcal{C}}(D_{\eta^{\uparrow 0}}). Let p be given such that
\overline{0}(2p+2) \perp \varepsilon. Note \forall m \geq p[\eta^{\uparrow 0}(m) = 0]. Conclude: Bar_{\mathcal{C}}(D_{\overline{n}|0}) and: Bar_{\mathcal{C}}(D_{\overline{\alpha p}}).
We thus see: \forall p[\overline{\underline{0}}(2p+2) \perp \varepsilon \to Bar_{\mathcal{C}}(D_{\overline{\alpha}p})].
      Assume: \exists n[\eta^{\uparrow 1}(n) \neq 0], Then, of course: \exists n[\varepsilon(n) \neq 0].
      We thus see: \forall \varepsilon [\overline{0}(2p+2) \perp \varepsilon \to Bar_{\mathcal{C}}(D_{\overline{\alpha}p})] \vee \exists n[\varepsilon(n) \neq 0]], i.e. LPO<sup>\alpha</sup>.
      (iv) \Rightarrow (i). Use Lemma 8.6(ii) and Theorem 8.2(iii).
                                                                                                                                                                               The sentences mentioned in Theorems 8.5 and 8.7 are true in every structure
```

The sentences mentioned in Theorems 8.5 and 8.7 are true in every structure $(\{0,1,\ldots,n\},P,Q,A)$ where n is a natural number, P,Q are arbitrary subsets of $\{0,1,\ldots,n\}$ and A is an arbitrary proposition, that is, these sentences hold in every finite structure. They sometimes fail to be true in countable structures, as appears from the next two theorems.

Theorem 8.8. The following statements are equivalent in BIM.

(i) LLPO.

```
(ii) \forall \alpha [(\omega, D_{\alpha^{\dagger 0}}, D_{\alpha^{\dagger 1}}) \models \forall x \forall y [P(x) \lor Q(y)] \rightarrow (\forall x [P(x)] \lor \forall y [Q(y)]).
Proof. (i) \Rightarrow (ii). Let \alpha be given.
Define \beta such that \forall p \forall i < 2[\beta(2p+i) = 0 \leftrightarrow \alpha^{\lceil i \rceil}(p) \neq 0].
Assume: (\omega, D_{\alpha^{\dagger 0}}, D_{\alpha^{\dagger 1}}) \models \forall x \forall y [P(x) \lor Q(y)]. Conclude: \forall p \forall q [\beta(2p) = 0 \lor \beta(2p)]
\beta(2q+1)=0]. Apply LLPO and distinguish two cases.
      Case (1). \forall p[2p+1 \neq \mu m[\beta(m) \neq 0]]. Assume we find p such that \beta(2p+1) \neq 0.
Determine q such that q \leq p and \beta(2q) \neq 0. Contradiction.
     Conclude: \forall p[\beta(2p+1)=0] and: D_{\alpha^{\dagger 1}}=\omega.
      Case (2). \forall p[2p \neq \mu m[\beta(m) \neq 0]]. Then, for similar reasons: D_{\alpha^{\uparrow 0}} = \omega.
     In both cases: (\omega, D_{\alpha^{\uparrow 0}}, D_{\alpha^{\uparrow 1}}) \models \forall x [P(x)] \vee \forall y [Q(y)].
      (ii) \Rightarrow (i). Let \alpha be given. Define \beta such that, for each p, for both i < 2, \beta^{\dagger i}(p) =
0 if and only if \overline{\alpha}(2p+i) = \overline{\underline{0}}(2p+i) and \alpha(2p+i) \neq 0. Note: (\omega, D_{\beta^{\dagger 0}}, D_{\beta^{\dagger 1}}) \models
\forall x \forall y [P(x) \lor Q(y)]. Conclude: (\omega, D_{\beta \uparrow 0}, D_{\beta \uparrow 1}) \models \forall x [P(x)] \lor \forall y [Q(y)].
      Either: (\omega, D_{\beta \uparrow 0}, D_{\beta \uparrow 1}) \models \forall x [P(x)] \text{ and: } \forall p [\overline{\alpha}(2p) = \overline{0}(2p) \to \alpha(2p) = 0],
or: (\omega, D_{\beta^{\dagger 0}}, D_{\beta^{\dagger 1}}) \models \forall y[Q(y)] \text{ and: } \forall p[\overline{\alpha}(2p+1) = \overline{0}(2p+1) \to \alpha(2p+1) = 0].
We thus see: LLPO.
Theorem 8.9. The following statements are equivalent in BIM.
    (i) LPO.
  (ii) \forall \alpha [(\omega, D_{\alpha^{\uparrow 0}}, D_{\alpha^{\uparrow 1}}) \models \forall x [P(x) \lor Q(x)] \to (\forall x [P(x)] \lor \exists x [Q(x)]).
 (iii) \forall \alpha [(\omega, D_{\alpha^{\dagger 0}}, Pr_{\alpha^{\dagger 1}}) \models \forall x [P(x) \lor A] \to (\forall x [P(x)] \lor A).
Proof. (i) \Rightarrow (ii). Let \alpha be given such that (\omega, D_{\alpha^{\uparrow 0}}, D_{\alpha^{\uparrow 1}}) \models \forall x [P(x) \lor Q(x)], i.e.
\forall n[\alpha^{\uparrow 0}(n) \neq 0 \lor \alpha^{\uparrow 1}(n) \neq 0]. Using LPO, distinguish two cases.
Either \forall n [\alpha^{\dagger 0}(n) \neq 0] and (\omega, D_{\alpha^{\dagger 0}}, D_{\alpha^{\dagger 1}}) \models \forall x [P(x)],
or \exists n[\alpha^{\restriction 0}(n)=0] and \exists n[\alpha^{\restriction 1}(n)\neq 0] and (\omega,D_{\alpha^{\restriction 0}},D_{\alpha^{\restriction 1}})\models \exists x[Q(x)].
     (ii) \Rightarrow (i). Let \alpha be given. Define \beta such that \forall n [\beta^{\dagger 0}(n) = 0 \leftrightarrow \alpha(n) \neq 0] and
\beta^{\uparrow 1} = \alpha. Note: \forall n [\beta^{\uparrow 0}(n) \neq 0 \lor \beta^{1}(n) \neq 0], i.e.
(\omega, D_{\beta^{\dagger 0}}, D_{\beta^{\dagger 1}}) \models \forall x [P(x) \lor Q(x)]. Use (ii) and distinguish two cases.
Either (\omega, D_{\beta \upharpoonright 0}, D_{\beta \upharpoonright 1}) \models \forall x [P(x)] \text{ and } \forall n [\alpha(n) = 0],
or (\omega, D_{\beta \uparrow 0}, D_{\beta \uparrow 1}) \models \exists x [Q(x)] \text{ and } \exists n [\alpha(n) \neq 0].
We thus see: LPO.
      (i) \Rightarrow (iii). Let \alpha be given such that (\omega, D_{\alpha^{\uparrow 0}}, Pr_{\alpha^{\uparrow 1}}) \models \forall x [P(x) \vee A], i.e.
\forall n[\alpha^{\uparrow 0}(n) \neq 0 \lor \exists m[\alpha^{\uparrow 1}(m) \neq 0]]. Using LPO, distinguish two cases.
Either \forall n[\alpha^{\uparrow 0}(n) \neq 0] and (\omega, D_{\alpha^{\uparrow 0}}, Pr_{\alpha^{\uparrow 1}}) \models \forall x[P(x)],
or \exists n[\alpha^{\restriction 0}(n)=0] and \exists m[\alpha^{\restriction 1}(m)\neq 0] and (\omega,D_{\alpha^{\restriction 0}},Pr_{\alpha^{\restriction 1}})\models A.
      (iii) \Rightarrow (i). Let \alpha be given. Define \beta such that \forall n [\beta(n) = 0 \leftrightarrow \alpha(n) \neq 0].
Note: \forall n[\beta(n) \neq 0 \lor \alpha(n) \neq 0], that is, (\omega, D_{\beta}, Pr_{\alpha}) \models \forall x[P(x) \lor A].
Use (iii) and distinguish two cases.
Either (\omega, D_{\beta}, Pr_{\alpha}) \models \forall x [P(x)] \text{ and } \forall n [\beta(n) \neq 0] \text{ and } \forall n [\alpha(n) = 0],
or (\omega, D_{\beta}, Pr_{\alpha}) \models A and \exists n [\alpha(n) \neq 0].
```

8.4. **A note.** According to Theorem 8.7(iii), BIM + Π_1^0 -AC_{ω ,2} proves that **FT** is equivalent to:

```
For all \alpha, if \forall \gamma \in \mathcal{C} \exists n [\alpha^{\upharpoonright 0}(\overline{\gamma}n) \neq 0 \lor \alpha^{\upharpoonright 1}(n) \neq 0], then either \forall \gamma \in \mathcal{C} \exists n [\alpha^{\upharpoonright 0}(\overline{\gamma}n) \neq 0] or \exists n [\alpha^{\upharpoonright 1}(n) \neq 0].
```

One might ask if $\mathsf{BIM} + \Pi_1^0 - \mathbf{AC}_{\omega,2}$ proves that $\neg ! \mathbf{FT}$ is equivalent to the following statement:

```
There exists \alpha such that \forall \gamma \in \mathcal{C} \exists n[\alpha^{\uparrow 0}(\overline{\gamma}n) \neq 0 \lor \alpha^{\uparrow 1}(n) \neq 0] and \neg \forall \gamma \in \mathcal{C} \exists n[\alpha^{\uparrow 0}(\overline{\gamma}n) \neq 0] and \alpha^{\uparrow 1} = \underline{0},
```

i.e.

We thus see: **LPO**.

(*): There exists α such that

 $\forall \gamma \in \mathcal{C} \exists n [\alpha(\overline{\gamma}n) \neq 0] \ and \ \neg \forall \gamma \in \mathcal{C} \exists n [\alpha(\overline{\gamma}n) \neq 0].$

The formula (*) is an outright contradiction!

9. The determinacy of finite and infinite games

We consider games of perfect information for players I, II. First finite games, then games with finitely many moves where the players may choose out of infinitely many alternatives, and then games of infinite length. In the last Subsection we prove: in BIM, **FT** is an equivalent of the Intuitionistic Determinacy Theorem: every subset of $(\omega \times 2)^{\omega}$ is weakly determinate.

9.1. Finite games.

9.1.1. Finite Choice and a contraposition of Finite Choice.

Lemma 9.1. BIM proves the following scheme:

$$\forall m [\forall n < m[A(n) \lor B] \to (\forall n < m[A(n)] \lor B)]$$

П

Proof. The proof is straightforward, by induction.

Lemma 9.2. BIM proves the following schemes:

- (i) $\forall k [\forall n < k \exists m [R(n,m)] \rightarrow \exists s \forall n < k [R(n,s(n))]].$
- (ii) $\forall k \forall l [\forall s : k \to l \exists n < k [R(n, s(n))] \to \exists n < k \forall m < l[R(n, m)]].$

Proof. (i) The proof is by induction on k and left to the reader.

(ii) The proof is by induction on k and uses Lemma 9.1. Note that there is nothing to prove if k = 0. Now assume the statement holds for a certain k.

Assume $\forall s : (k+1) \to l \exists n < k+1 [R(n,s(n))]$. Note:

$$\forall j < l \forall s : k \to l [\exists n < k [R(n, s(n))] \lor R(k, j)],$$

and, therefore, by Lemma 9.1:

$$\forall j < l[R(k,j) \lor \forall s : k \to l \exists n < k[R(n,s(n))]].$$

Using the induction hypothesis, we conclude:

$$\forall j < l[R(k,j)] \lor \exists n < k \forall m < l[R(n,m)],$$

that is: $\exists n < k + 1 \forall m < l[R(n, m)].$

9.1.2. Finite games. We want to study finite and infinite games for players I and II of perfect information. We first consider finite games: there are finitely many moves, and for each move there are only finitely many alternatives.

Assume $X \subseteq \omega$. Let n, l be given such that l > 0.

Players I and II play the I-game for X in Seq(n,l) and the II-game for X in Seq(n,l) in the same way, as follows. First, player I chooses $i_0 < l$, then player II chooses $i_1 < l$, and they continue until a finite sequence $\langle i_0, i_1, \ldots, i_{n-1} \rangle$ of length n has been formed. Player I wins the play $\langle i_0, i_1, \ldots, i_{n-1} \rangle$ in the I-game for X if and only if $\langle i_0, i_1, \ldots, i_{n-1} \rangle \in X$. Player II wins the play $\langle i_0, i_1, \ldots, i_{n-1} \rangle$ in the II-game for X if and only if $\langle i_0, i_1, \ldots, i_{n-1} \rangle \in X$.

We define φ such that, for all n, for all l > 0,

$$\varphi(n,l) := \mu a \forall i < n \forall c \in Seq(i,l)[c < a].$$

Every number coding a position in Seq(n, l) that is not a final position is smaller than $\varphi(n, l)$.

We define ψ such that, for all n, for all l > 0,

$$\psi(n, l) := \mu a \forall s \in Seq(\varphi(n, l), l)[s < a].$$

When studying games in Seq(n, l) it suffices to consider strategies s, t for player I, II, respectively, such that $s, t < \psi(n, l)$.

The reader should consult Subsubsection 13.10 for some of the notations we are going to use, like ' \in_I ' and ' \in_{II} '.

We define: $X \subseteq \omega$ is *I*-determinate in Seq(n,l), $Det_{Seq(n,l)}^{I}(X)$,

if and only if $\forall t < \psi(n, l) \exists c[c \in_{II} t \land c \in X] \rightarrow \exists s \forall c \in Seq(n, l)[c \in_{I} s \rightarrow c \in X],$ and: $X \subseteq \omega$ is II-determinate in Seq(n, l), $Det_{Seq(n, l)}^{II}(X)$,

if and only if $\forall s < \psi(n, l) \exists c[c \in_I s \land c \in X] \rightarrow \exists t \forall c \in Seq(n, l)[c \in_{II} t \rightarrow c \in X].$

Theorem 9.3 (Determinacy of finite games). Let $X \subseteq \omega$ be given.

For every l > 0, for every n, $Det^{I}_{Seq(n,l)}(X)$ and $Det^{II}_{Seq(n,l)}(X)$.

Proof. Let $X \subseteq \omega$ and l > 0 be given.

For every u, we define: $X_u := \{c \in \omega \mid u * c \in X\}.$

We intend to prove a statement seemingly stronger than the statement of the theorem:

(*): for every n, for every u, $Det_{Seq(n,l)}^{I}(X_u)$ and $Det_{Seq(n,l)}^{II}(X_u)$.

The proof is by induction on n. If n = 0, then, for every u, the statements: X_u is I-determinate in Seq(0, l) and: X_u is II-determinate in Seq(0, l) both assert: if $\langle \ \rangle \in X_u$, then $\langle \ \rangle \in X_u$, and thus are obviously true.

Now assume the statement (*) has been established for a certain n. We prove that (*) is also true for n+1.

Let u be given.

First, assume: $\forall t < \psi(n+1,l) \exists c[c \in_{II} t \land c \in X_u]$, or, equivalently ¹⁴

 $\forall t < \psi(n+1,l) \exists k < l \exists c [c \in_I t^{\uparrow k} \land \langle k \rangle * c \in X_u].$

Note: $\forall k < l \forall c [\langle k \rangle * c \in X_u \leftrightarrow c \in X_{u*\langle k \rangle}].$

Conclude: $\forall t: l \to \psi(n, l) \exists k < l \exists c [c \in_I t(k) \land c \in X_{u*\langle k \rangle}].$

Use Lemma 9.2(ii) and find k < l such that $\forall s < \psi(n, l) \exists c [c \in_I s \land c \in X_{u*(k)}].$

Use the induction hypothesis and find $t < \psi(n, l)$ such that

 $\forall c \in Seq(n, l)[c \in_{II} t \to c \in X_{u*\langle k \rangle}].$

Define $s < \psi(n+1, l)$ such that $s(\langle \rangle) = k$ and $s^{\uparrow k} = t$.

Note: $\forall c \in Seq(n+1,l)[c \in_I s \to c \in X_u].$

We thus see that X_u is *I*-determinate.

Next, assume: $\forall s < \psi(n+1,l) \exists c[c \in_I s \land c \in X_u]$, or, equivalently:

 $\forall s < \psi(n+1,l) \exists c [c \in_{II} s^{\upharpoonright s(\langle \rangle)} \land \langle s(\langle \rangle) \rangle * c \in X_u].$

Note: $\forall k < l \forall t < \psi(n, l) \exists s < \psi(n+1, l) [s(\langle \rangle) = k \land s^{\uparrow k} = t].$

We thus see: $\forall k < l \forall t < \psi(n, l) \exists c[c \in_{II} t \land \langle k \rangle * c \in X_u].$

Note: $\forall k < l \forall c [\langle k \rangle * c \in X_u \leftrightarrow c \in X_{u*\langle k \rangle}].$

Conclude: $\forall k < l \forall t < \psi(n, l) \exists c [c \in_{II} t \land c \in X_{u*\langle k \rangle}].$

Use the induction hypothesis and conclude:

 $\forall k < l \exists s < \psi(n, l) \forall c \in Seq(n, l) [c \in_I s \to s \in X_{u*\langle k \rangle}].$

Use Lemma 9.2(i) and find $t < \psi(n+1,l)$ such that

 $\forall k < l \forall c \in Seq(n, l)[c \in_I t^{\restriction k} \to c \in X_{u*\langle k \rangle}].$

Note: $\forall c \in Seq(n+1,l)[c \in_{II} t \to c \in X_u].$

We thus see that X_u is II-determinate.

9.1.3. Comparison with the classical theorem. Note that, in classical mathematics, the *I*-determinacy of finite games is stated as follows:

For every $X \subseteq \omega$, for every l > 0, for every n,

either:
$$\exists t < \psi(n, l) \forall c \in Seq(n, l) [c \in_{II} t \to c \notin X],$$

or: $\exists s < \psi(n, l) \forall c \in Seq(n, l) [c \in_{I} s \to c \in X],$

¹⁴For the notation $t^{\uparrow k}$, see Subsection 13.4.

that is: either player II has a strategy ensuring that the result of the game will not be in X, or player I has a strategy ensuring that it does.

Taken constructively, this statement fails to be true already in the case n=0, because it then implies: for every subset X of $\{\langle \, \rangle \}$, either $\langle \, \rangle \notin X$ or $\langle \, \rangle \in X$, and therefore, for any proposition $P, \neg P \lor P$, the principle of the excluded third.

9.2. Infinitely many alternatives.

9.2.1. Infinitely many alternatives for player II.

Let $X \subseteq 2 \times \omega$ be given. Players I and II play the I-game for X in $2 \times \omega$ in the following way. First, player I chooses i < 2, then player II chooses n and the play is finished. Player I wins the play $\langle i, n \rangle$ if and only if $\langle i, n \rangle \in X$.

We define: X is I-determinate in $2 \times \omega$, $Det_{2 \times \omega}^{I}(X)$, if and only if

$$\forall t \exists c \in 2 \times \omega [c \in II \ t \ \land \ c \in X] \to \exists s < 2 \forall n [\langle s, n \rangle \in X].$$

Theorem 9.4. BIM $\vdash \forall \alpha [Det^I_{2 \times \omega}(D_{\alpha})] \leftrightarrow \mathbf{LLPO}.$

Proof. This Theorem is a reformulation of Theorem 8.8.

In order to see this, make two observations:

(i) Note that, for each α , there exists β such that

 $Det_{2\times\omega}^{I}(D_{\alpha}) \leftrightarrow (\omega, D_{\beta^{\dagger 0}}, D_{\beta^{\dagger 1}}) \models \forall \mathsf{x} \forall \mathsf{y} [\mathsf{P}(\mathsf{x}) \vee \mathsf{Q}(\mathsf{y})] \rightarrow (\forall \mathsf{x} [\mathsf{P}(\mathsf{x})] \vee \forall \mathsf{y} [\mathsf{Q}(\mathsf{y})]).$ Given α , define β such that, for each n, $\beta^{\dagger 0}(n) = \alpha(\langle 0, n \rangle)$ and $\beta^{\dagger 1}(n) = \alpha(\langle 1, n \rangle).$

(ii) Note that, for each α , there exists β such that

 $((\omega, D_{\alpha^{\uparrow 0}}, D_{\alpha^{\uparrow 1}}) \models \forall \mathsf{x} \forall \mathsf{y} [\mathsf{P}(\mathsf{x}) \vee \mathsf{Q}(\mathsf{y})] \to (\forall \mathsf{x} [\mathsf{P}(\mathsf{x})] \vee \forall \mathsf{y} [\mathsf{Q}(\mathsf{y})])) \leftrightarrow Det_{2 \times \omega}^{I}(D_{\beta}).$ Given α , define β such that, for each n, $\beta(\langle 0, n \rangle) = \alpha^{0}(n)$ and $\beta(\langle 1, n \rangle) = \alpha^{1}(n)$. \square

9.2.2. Infinitely many alternatives for player I.

Let $X \subseteq \omega \times 2$ be given. Players I and II play the I-game for X in $\omega \times 2$ in the following way. First, player I chooses a natural number n, then player II chooses a number i from $\{0,1\}$. Player I wins the play $\langle n,i\rangle$ if and only if $\langle n,i\rangle \in X$.

Note that a strategy for player I in such a two-move-game coincides with his first move and thus is a natural number. A strategy for player II, on the other hand, is an infinite sequence τ in \mathcal{C} that expresses player II's intention to play $\tau(\langle n \rangle)$ once player I has brought them to the position $\langle n \rangle$.

We define: $X \subseteq \omega \times 2$ is *I-determinate in* $\omega \times 2$, $Det_{\omega \times 2}^{I}(X)$, if and only if:

$$\forall \tau \in \mathcal{C} \exists c \in \omega \times 2[c \in_{II} \tau \land c \in X] \to \exists s \forall i < 2[\langle s, i \rangle \in X].$$

Theorem 9.5. BIM $\vdash \forall \alpha [Det^I_{\omega \times 2}(D_\alpha)].$

Proof. Let α be given. Assume: $\forall \tau \in \mathcal{C} \exists n [\langle n, \tau(\langle n \rangle) \rangle \in D_{\alpha}]$. Find τ in \mathcal{C} such that $\forall n [\tau(\langle n \rangle) = 1 \leftrightarrow \langle n, 0 \rangle \in D_{\alpha}]$. Find n such that $\langle n, \tau(\langle n \rangle) \rangle \in D_{\alpha}$. Note: $\tau(\langle n \rangle) = 1$ and $\forall i < 2 [\langle n, i \rangle \in D_{\alpha}]$.

We define: $X \subseteq \omega \times \omega$ is II-determinate in $\omega \times \omega$, $Det_{\omega \times \omega}^{II}(X)$, if and only if:

$$\forall m \exists n [\langle m, n \rangle \in X] \to \exists \tau \forall m [\langle m, \tau(\langle m \rangle) \rangle \in X].$$

Theorem 9.6. BIM $\vdash \forall \alpha [Det^{II}_{\omega \times \omega}(E_{\alpha})].$

Proof. Let α be given such that $\forall m \exists n [\langle m, n \rangle \in E_{\alpha}]$, that is $\forall m \exists n \exists p [\alpha(p) = \langle m, n \rangle + 1]$. Find γ such that $\forall m [\alpha(\gamma'(m)) = \langle m, \gamma''(m) \rangle + 1]$. Define τ such that $\forall m [\tau(\langle m \rangle) = \gamma''(m)]$.

We define: $X \subseteq 2 \times \omega$ is II-determinate in $2 \times \omega$, $Det_{2 \times \omega}^{II}(X)$, if and only if:

$$\forall m < 2\exists n[\langle m, n \rangle \in X] \to \exists t \forall m < 2[\langle m, t(\langle m \rangle) \rangle \in X].$$

Note: BIM proves the scheme $Det_{2\times\omega}^{II}(X)$.

9.2.3. Longer games. We also consider games in which players I, II make more than one move. Which of those games are determinate from the viewpoint of Player I? Because of Theorem 9.4, we restrict ourselves to games in which player I has, for each one of his moves, countably many alternatives, whereas player II always has to choose one of two possibilities.

For every n, for every $X \subseteq (\omega \times 2)^n$, we define:

X is I-determinate in $(\omega \times 2)^n$, $Det_{(\omega \times 2)^n}(X)$, if and only if

$$\forall \tau \in \mathcal{C} \exists c [c \in_{II} \tau \land c \in X] \to \exists s \forall c \in (\omega \times 2)^n [c \in_I s \to c \in X].$$

This definition extends in the obvious way to subsets X of $(\omega \times 2)^n \times \omega$.

9.3. Infinitely many moves. We also want to consider games of infinite length. We imagine players I, II to build together an infinite sequence γ in ω^{ω} , as follows. First, player I chooses $\gamma(0)$, then player II chooses $\gamma(1)$, then player I chooses $\gamma(2)$, and so on.

We define a number of notions of determinacy.

$$\mathcal{X} \subseteq (\omega \times 2)^{\omega}$$
 is I -determinate in $(\omega \times 2)^{\omega}$, $Det_{(\omega \times 2)^{\omega}}^{I}(\mathcal{X})$, if and only if $\forall \tau \in \mathcal{C} \exists \gamma [\gamma \in_{II} \tau \land \gamma \in \mathcal{X}] \to \exists \sigma \forall \gamma \in (\omega \times 2)^{\omega} [\gamma \in_{I} \sigma \to \gamma \in \mathcal{X}].$

$$\mathcal{X} \subseteq (\omega \times 2)^{\omega}$$
 is finitely *I*-determinate in $(\omega \times 2)^{\omega}$, $^*Det^I_{(\omega \times 2)^{\omega}}(\mathcal{X})$ if and only if $\forall \tau \in \mathcal{C} \exists \gamma [\gamma \in_{II} \tau \land \gamma \in \mathcal{X}] \rightarrow \exists s \forall \gamma \in (\omega \times 2)^{\omega} [\gamma \in_{I} s \rightarrow \gamma \in \mathcal{X}].$

$$\mathcal{X} \subseteq \mathcal{C}$$
 is *I-determinate in* \mathcal{C} , $Det_{\mathcal{C}}^{I}(\mathcal{X})$, if and only if
$$\forall \tau \in \mathcal{C} \exists \gamma [\gamma \in_{II} \tau \land \gamma \in \mathcal{X}] \rightarrow \exists \sigma \in \mathcal{C} \forall \gamma \in \mathcal{C} [\gamma \in_{I} \sigma \rightarrow \gamma \in \mathcal{X}].$$

$$\mathcal{X} \subseteq \mathcal{C}$$
 is finitely I-determinate in \mathcal{C} , $^*Det_{\mathcal{C}}^I(\mathcal{X})$, if and only if $\forall \tau \in \mathcal{C} \exists \gamma [\gamma \in_I I \tau \land \gamma \in \mathcal{X}] \rightarrow \exists s \in Bin \forall \gamma \in \mathcal{C} [\gamma \in_I s \rightarrow \gamma \in \mathcal{X}].$

$$\mathcal{X} \subseteq \mathcal{C}$$
 is II -determinate in \mathcal{C} , $Det_{\mathcal{C}}^{II}(\mathcal{X})$, if and only if $\forall \sigma \in \mathcal{C} \exists \gamma [\gamma \in_I \sigma \land \gamma \in \mathcal{X}] \rightarrow \exists \tau \in \mathcal{C} \forall \gamma \in \mathcal{C} [\gamma \in_I I \tau \rightarrow \gamma \in \mathcal{X}].$

$$\mathcal{X} \subseteq \mathcal{C}$$
 is finitely II-determinate in \mathcal{C} , * $Det_{\mathcal{C}}^{II}(\mathcal{X})$, if and only if $\forall \sigma \in \mathcal{C} \exists \gamma [\gamma \in_I \sigma \land \gamma \in \mathcal{X}] \rightarrow \exists t \in Bin \forall \gamma \in \mathcal{C} [\gamma \in_{II} t \rightarrow \gamma \in \mathcal{X}].$

We are going to study the following statements:

$$\Sigma_1^0$$
- $Det_{\omega \times 2}^I$: $\forall \alpha [Det^I(E_\alpha)]$.

$$\mathbf{\Delta}_{1}^{0}$$
- $Det_{\omega \times 2 \times \omega}^{I}$: $\forall \alpha [Det_{\omega \times 2 \times \omega}^{I}(D_{\alpha})]$.

$$\mathbf{\Delta}_{1}^{0}\text{-}Det_{(\omega\times2)^{m}}^{I}\colon\forall\alpha[Det_{(\omega\times2)^{m}}^{I}(D_{\alpha})].$$

$$\Sigma_1^0$$
- $Det^I_{(\omega \times 2)^\omega}$: $\forall \alpha [Det^I_{(\omega \times 2)^\omega}(\mathcal{G}_\alpha)].$

$$\Sigma_1^0$$
 * $Det^I_{(\omega \times 2)^\omega}$: $\forall \alpha [*Det^I_{(\omega \times 2)^\omega}(\mathcal{G}_\alpha)].$

$$\Sigma_1^0$$
- $Det_{\mathcal{C}}^I$: $\forall \alpha [Det_{\mathcal{C}}^I(\mathcal{G}_{\alpha})].$

$$\Sigma_1^0$$
- * $Det_{\mathcal{C}}^I$: $\forall \alpha [*Det_{\mathcal{C}}^I(\mathcal{G}_{\alpha})].$

$$\Sigma_1^0$$
- $Det_{\mathcal{C}}^I$: $\forall \alpha [Det_{\mathcal{C}}^{II}(\mathcal{G}_{\alpha})].$

$$\Sigma_1^0$$
- * $Det_{\mathcal{C}}^{II}$: $\forall \alpha [*Det_{\mathcal{C}}^{II}(\mathcal{G}_{\alpha})].$

Each of the above formulas X has the form: $\forall \alpha [P(\alpha) \to Q(\alpha)]$. For each of these nine formulas X, we define the statement $\neg !X$, the *strong negation* of X, as follows:

$$\neg!X := \neg! \big(\forall \alpha [P(\alpha) \to Q(\alpha)] \big) := \exists \alpha [P(\alpha) \land \neg Q(\alpha)].$$

Note that these strong negations contain the negation symbol \neg , a possibility we mentioned in Subsection 1.4.

Note that the symbol $\neg!$ is used as a metamathematical notation, not as part of the language of BIM. One should also not consider $\neg!$ as the name of a syntactical operation on formulas.

9.4. Simulating a game in $(\omega \times 2)^{\omega}$ by a game in \mathcal{C} . From the point of view of player I, a game in $(\omega \times 2)^{\omega}$ may be simulated by a game in Cantor space \mathcal{C} . Where player I would play n in $(\omega \times 2)^{\omega}$, he will play n times 0 and one time 1 in \mathcal{C} . So he plays the finite sequence $\overline{0}n * \langle 1 \rangle$. Every time he plays 0, he makes what we call a postponing move. Player II has to react, in \mathcal{C} , to these postponing moves of player I, but these reactions do not matter. As soon as player I plays 1 and completes $\overline{0}n * \langle 1 \rangle$, player II gives, in the play in \mathcal{C} , the answer he would give to player I's move n in $(\omega \times 2)^{\omega}$. The reader should keep this in mind when reading the following definitions.

Define $Halfbin := (\omega \times 2)^{<\omega} \cup ((\omega \times 2)^{<\omega} \times \omega) = \bigcup_n \{\overline{\gamma}n \mid \gamma \in (\omega \times 2)^{\omega}\}.$ Define $\boldsymbol{\pi_{bin}}$ such that

- 1. $\pi_{bin}(\langle \rangle) = \langle \rangle$, and,
- 2. for each c, if length(c) is even, then, for each n, $\pi_{bin}(c*\langle n \rangle) = \pi_{bin}(c)*\overline{\underline{0}}2n*\langle 1 \rangle$, and, for both i < 2, $\pi_{bin}(c*\langle n, i \rangle) = \pi_{bin}(c*\langle n \rangle)*\langle i \rangle$.

The function π_{bin} associates to every position in Halfbin a position in Bin. Note that, for each c, $length(\pi_{bin}(c)) \ge length(c)$.

Define ρ_{bin} in ω^{ω} such that

- 1. $\rho_{bin}(\langle \rangle) = \langle \rangle$, and,
- 2. for each d in Bin, if length(d) is even, then

 $\rho_{bin}(d*\langle 0 \rangle) = \rho_{bin}(d*\langle 0, 0 \rangle) = \rho_{bin}(d*\langle 0, 1 \rangle) = \rho_{bin}(d)$, and

3. for each d in Bin, if length(d) is even, then $\rho_{bin}(d * \langle 1 \rangle) = \rho_{bin}(d) * \langle n \rangle$, and, for both i < 2, $\rho_{bin}(d * \langle 1, i \rangle) = \rho_{bin}(d) * \langle n, i \rangle$, where n satisfies: either: 2n = length(d) and $\forall i < n[d(2i) = 0]$, or:

for some k > 0, length(d) = 2k + 2n and d(2k-2) = 1 and $\forall i < n[d(2k+2i) = 0]$.

The function ρ_{bin} associates to every position in Bin a position in Halfbin.

Note that, for every c in Halfbin, $\rho_{bin} \circ \pi_{bin}(c) = c$.

Note that, for each c in Halfbin, length(c) is even if and only if $length(\pi_{bin}(c))$ is even.

Lemma 9.7. The following is provable in BIM.

For each α , there exists β such that

```
\forall \tau \in \mathcal{C} \exists \gamma \in (\omega \times 2)^{\omega} [\gamma \in_{II} \tau \land \gamma \in \mathcal{G}_{\alpha}] \to \forall \tau \in \mathcal{C} \exists \delta \in \mathcal{C} [\delta \in_{II} \tau \land \delta \in \mathcal{G}_{\beta}] \text{ and}
\exists \sigma \forall \delta \in \mathcal{C} [\delta \in_{I} \sigma \to \delta \in \mathcal{G}_{\beta}] \to \exists \sigma \forall \gamma \in (\omega \times 2)^{\omega} [\gamma \in_{I} \sigma \to \gamma \in \mathcal{G}_{\alpha}] \text{ and}
\exists s \forall \delta \in \mathcal{C} [\delta \in_{I} s \to \delta \in \mathcal{G}_{\beta}] \to \exists s \forall \gamma \in (\omega \times 2)^{\omega} [\gamma \in_{I} s \to \gamma \in \mathcal{G}_{\alpha}].
```

Proof. Let α be given. Define $\beta := \alpha \circ \rho_{bin}$.

Assume $\forall \tau \in \mathcal{C} \exists \gamma \in (\omega \times 2)^{\omega} [\gamma \in_{II} \tau \land \gamma \in \mathcal{G}_{\alpha}].$

Let τ be given as a strategy for player II in C.

We want to prove: $\exists \delta \in \mathcal{C}[\delta \in_{II} \tau \land \delta \in \mathcal{G}_{\beta}].$

To this end, we define τ^{\dagger} as a strategy for player II in $(\omega \times 2)^{\omega}$. We define τ^{\dagger} on all positions in Halfbin of odd length, by induction on the length of the position. It suffices to define τ^{\dagger} on positions c satisfying the condition: $c \in II$ τ^{\dagger} .

We shall take care that, for each c in Halfbin, if $c \in_{II} \tau^{\dagger}$, then there exists d in Bin such that $\rho_{bin}(d) = c$ and $d \in_{II} \tau$.

We first define τ^{\dagger} on positions of length 1.

Let n be given. We have to define $\tau^{\dagger}(\langle n \rangle)$.

Find d in Bin such that length(d) = 2n + 1 and $d \in_{II} \tau$ and d(2n) = 1 and $\forall i < n[d(2i) = 0]$. Define $\tau^{\dagger}(\langle n \rangle) := \tau(d)$.

Note: $\rho_{bin}(d) = \langle n \rangle$ and $\rho_{bin}(d * \langle \tau(d) \rangle) = \langle n, \tau^{\dagger}(\langle n \rangle) \rangle$.

Now assume k > 0. Let c in $(\omega \times 2)^k$ be given such that $c \in II$ τ^{\dagger} .

Let n be given. We have to define $\tau^{\dagger}(c * \langle n \rangle)$.

First find d in Bin such that $d \in_{II} \tau$ and $\rho_{bin}(d) = c$. Find l := length(d). Find e in Bin such that $d \sqsubseteq e$ and length(e) = l + 2n + 1 and $e \in_{II} \tau$ and e(l + 2n) = 1 and $\forall i < n[e(l + 2i) = 0]$. Note: $\rho_{bin}(e) = c * \langle n \rangle$. Define $\tau^{\dagger}(c * \langle n \rangle) := \tau(e)$.

Note: $\rho_{bin}(e) = c * \langle n \rangle$ and $\rho_{bin}(e * \langle \tau(e) \rangle) = c * \langle \tau^{\dagger}(c) \rangle$.

This completes the definition of τ^{\dagger} .

Now find γ in $(\omega \times 2)^{\omega}$ such that $\gamma \in II \tau^{\dagger} \wedge \gamma \in \mathcal{G}_{\alpha}$.

Note: $\forall n \exists d \in Bin[d \in_{II} \tau \land \rho_{bin}(d) = \overline{\gamma}n].$

Note: $\forall d \in Bin \forall e \in Bin[(d \in_{II} \tau \land e \in_{II} \tau) \rightarrow (d \sqsubset e \leftrightarrow \rho_{bin}(d) \sqsubset \rho_{bin}(e))].$

Using this fact, construct δ in \mathcal{C} such that $\delta \in_{II} \tau$ and $\forall n \exists m [\overline{\gamma}n = \rho_{bin}(\overline{\delta}m)]$. Find n such that $\overline{\gamma}n \in D_{\alpha}$.

Find m such that $\overline{\gamma}n = \rho_{bin}(\overline{\delta}m)$.

Note: $\overline{\gamma}n = \rho_{bin}(\overline{\delta}m) \in D_{\alpha}$ and: $\overline{\delta}m \in \mathcal{G}_{\beta}$ and: $\delta \in \mathcal{G}_{\beta}$.

We thus see: $\forall \tau \in \mathcal{C} \exists \delta \in \mathcal{C} [\delta \in_{II} \tau \land \delta \in \mathcal{G}_{\beta}].$

Now assume $\exists \sigma \forall \delta \in \mathcal{C}[\delta \in_I \sigma \to \delta \in \mathcal{G}_{\beta}].$

Find σ such that $\forall \delta \in \mathcal{C}[\delta \in_I \sigma \to \delta \in \mathcal{G}_{\beta}].$

We define σ^* as a strategy for player I in $(\omega \times 2)^{\omega}$ such that, for each c in Halfbin, if $c \in I$ σ^* , then $either \pi_{bin}(c) \in I$ σ or $\exists e \sqsubset c[e \in D_{\alpha}]$.

We first define $\sigma^*(\langle \rangle)$.

Define δ in \mathcal{C} such that $\delta \in_I \sigma$ and $\forall i [\delta(2i+1) = 0]$. Find m such that $\beta(\overline{\delta}m) \neq 0$ and distinguish two cases.

Case (a). $\exists n[2n < m \land \delta(2n) = 1]$. Define: $n_0 := \mu n[\delta(2n) = 1]$ and $\sigma^*(\langle \rangle) := n_0$. Note $\langle n_0 \rangle \in_I \sigma^*$ and $\pi_{bin}(\langle n_0 \rangle) = \overline{0}(2n_0) *\langle 1 \rangle = \overline{\delta}(2n_0 + 1) \in_I \sigma$.

Case (b). $\forall n[2n < m \to \delta(2n) = 0]$. Conclude: $\rho_{bin}(\overline{\delta}m) = \langle \rangle$ and $\beta(\langle \rangle) \neq 0$ and also $\alpha(\langle \rangle) \neq 0$. Define: $\sigma^*(\langle \rangle) := 0$. Note: $\langle 0 \rangle \in_I \sigma^*$ and $\exists e \sqsubset \langle 0 \rangle [e \in D_{\alpha}]$.

Now assume k > 0. Let c in $(\omega \times 2)^k$ be given such that $c \in I$ σ^* . We have to define $\sigma^*(c)$.

We distinguish two cases.

Case 1. $\exists e \sqsubset c[e \in D_{\alpha}]$. We then define $\sigma^*(c) := 0$. Note: $\exists e \sqsubset c * \langle 0 \rangle [e \in D_{\alpha}]$. Case 2. $\neg \exists e \sqsubset c[e \in D_{\alpha}]$. Then $\pi_{bin}(c) \in I$ σ .

Note: $length(\pi_{bin}(c))$ is even and find l such that $2l := length(\pi_{bin}(c))$.

Define δ in \mathcal{C} such that $\delta \in_I \sigma$ and $\pi_{bin}(c) \sqsubset \delta$ and $\forall i[2i+1 > 2l \to \delta(2i+1) = 0]$. Find m such that $\beta(\overline{\delta}m) \neq 0$ and distinguish two cases.

Case (2a). $\exists n[2l \le 2n < m \land \delta(2n) = 1].$

Define: $n_0 := \mu n[2l \le 2n < m \land \delta(2n) = 1]$ and $\sigma^*(c) := n_0 - l$.

Note $c * \langle n_0 - l \rangle \in_I \sigma^*$ and $\boldsymbol{\pi_{bin}}(c * \langle n_0 - l \rangle) = \boldsymbol{\pi_{bin}}(c) * \overline{\underline{0}}(2n_0 - 2l) * \langle 1 \rangle = \overline{\delta}(2n_0 + 1) \in_I \sigma$.

Case (2b). $\forall n[2n < m \to \delta(2n) = 0]$. Conclude: $\rho_{bin}(\overline{\delta}m) = c$ and $\beta(\overline{\delta}m) \neq 0$ and $\alpha(c) = \beta(\overline{\delta}m) \neq 0$. Define: $\sigma^*(\langle \rangle) := 0$. Note: $c*\langle 0 \rangle \in_I \sigma^*$ and $\exists e \sqsubset c*\langle 0 \rangle [e \in D_{\alpha}]$.

This completes the definition of σ^* .

Now assume $\gamma \in (\omega \times 2)^{\omega}$ and $\gamma \in_I \sigma^*$. Find $\delta \in \mathcal{C}$ such that $\forall n[\pi_{bin}(\overline{\gamma}n) \sqsubseteq \delta]$. Find ε in \mathcal{C} such that such that $\varepsilon \in_I \sigma$ and, for each n, if $\overline{\delta}n \in_I \sigma$, then $\overline{\varepsilon}n = \overline{\delta}n$. Find m such that $\overline{\varepsilon}m \in D_{\beta}$ and distinguish two cases.

Case (*). $\overline{\delta}m = \overline{\varepsilon}m$. Conclude: $\overline{\delta}m \in D_{\beta}$ and $\rho_{bin}(\overline{\delta}m) \in D_{\alpha}$ and $\gamma \in \mathcal{G}_{\alpha}$.

Case (**). $\overline{\delta}m \neq \overline{\varepsilon}m$. Then $\overline{\delta}m \notin_I \sigma$ and $\pi_{bin}(\overline{\gamma}m) \notin_I \sigma$ and $\exists e \sqsubset \overline{\gamma}m[e \in D_\alpha]$. Conclude: $\gamma \in \mathcal{G}_\alpha$.

We thus see: $\forall \gamma \in (\omega \times 2)^{\omega} [\gamma \in_I \sigma^* \to \gamma \in \mathcal{G}_{\alpha}].$

Assume $\exists s \forall \delta \in \mathcal{C}[\delta \in_I s \to \delta \in \mathcal{G}_{\beta}].$

Find s such that $\forall \delta \in \mathcal{C}[\delta \in_I s \to \delta \in \mathcal{G}_{\beta}]$. Consider p := length(s). Find q such that, for all c in Halfbin, if $c \geq q$, then $\pi_{bin}(c) \geq p$. Define s^* such that $length(s^*) = q$, inductively. For each c < q in $\bigcup_k (\omega \times 2)^k$ such that $c \in I$ s^* , $s^*(c)$ is defined just as, in the previous paragraph, where we were given $\sigma \in \mathcal{C}$, $\sigma^*(c)$ was defined, for each c in $\bigcup (\omega \times 2)^k$ such that $c \in I$ σ^* .

One then may prove: $\forall \gamma \in (\omega \times 2)^{\omega} [\gamma \in s^* \to \gamma \in \mathcal{G}_{\alpha}].$

Lemma 9.8. One may prove the following statements in BIM.

- (i) Σ_1^0 - $Det_{\omega \times 2}^I \to \Sigma_1^0$ - $\overleftarrow{AC_{\omega,2}}$ and $\neg ! (\Sigma_1^0 \overleftarrow{AC_{\omega,2}}) \to \neg ! (\Sigma_1^0 Det_{\omega \times 2}^I)$.
- (ii) Δ_1^0 - $Det_{\omega \times 2 \times \omega}^I \to \Sigma_1^0$ - $Det_{\omega \times 2}^I$ and $\neg! (\Sigma_1^0$ - $Det_{\omega \times 2}^I) \to \neg! (\Delta_1^0$ - $Det_{\omega \times 2 \times \omega}^I)$.
- (iii) $\forall m[\mathbf{\Delta}_1^0 Det^I_{(\omega \times 2)^m}] \to \mathbf{\Delta}_1^0 Det^I_{\omega \times 2 \times \omega}$ and $\neg ! (\boldsymbol{\Delta}^0_1 \text{-} Det^I_{\omega \times 2 \times \omega}) \to \exists m [\neg ! (\boldsymbol{\Delta}^0_1 \text{-} Det^I_{(\omega \times 2)^m})].$
- (iv) $\forall m[\mathbf{\Sigma}_1^0 Det^I_{(\omega \times 2)^\omega} \to \mathbf{\Delta}_1^0 Det^I_{(\omega \times 2)^m} \land$ $\neg !(\boldsymbol{\Delta}_{1}^{0} - Det_{(\omega \times 2)^{m}}^{I}) \rightarrow \neg !(\boldsymbol{\Sigma}_{1}^{0} - Det_{(\omega \times 2)^{\omega}}^{I})].$
- $(v) \quad \Sigma_{1}^{0}-Det_{\mathcal{C}}^{I} \to \Sigma_{1}^{0}-Det_{(\omega \times 2)^{\omega}}^{I} \quad and \quad \neg!(\Sigma_{1}^{0}-Det_{(\omega \times 2)^{\omega}}^{I}) \to \neg!(\Sigma_{1}^{0}-Det_{\mathcal{C}}^{I}), \quad and$ $\Sigma_{1}^{0}-*Det_{\mathcal{C}}^{I} \to \Sigma_{1}^{0}-*Det_{(\omega \times 2)^{\omega}}^{I} \quad and \quad \neg!(\Sigma_{1}^{0}-*Det_{(\omega \times 2)^{\omega}}^{I}) \to \neg!(\Sigma_{1}^{0}-*Det_{\mathcal{C}}^{I}), \quad and$ $\Sigma_{1}^{0}-*Det_{\mathcal{C}}^{II} \to \Sigma_{1}^{0}-Det_{\mathcal{C}}^{I} \quad and \quad \neg!(\Sigma_{1}^{0}-Det_{\mathcal{C}}^{I}) \to \neg!(\Sigma_{1}^{0}-Det_{\mathcal{C}}^{II}), \quad and$ $\Sigma_{1}^{0}-*Det_{\mathcal{C}}^{II} \to \Sigma_{1}^{0}-*Det_{\mathcal{C}}^{I} \quad and \quad \neg!(\Sigma_{1}^{0}-*Det_{\mathcal{C}}^{I}) \to \neg!(\Sigma_{1}^{0}-*Det_{\mathcal{C}}^{II}), \quad and$ $\Sigma_{1}^{0}-*Det_{\mathcal{C}}^{II} \to \Sigma_{1}^{0}-*Det_{\mathcal{C}}^{I} \quad and \quad \neg!(\Sigma_{1}^{0}-*Det_{\mathcal{C}}^{I}) \to \neg!(\Sigma_{1}^{0}-*Det_{\mathcal{C}}^{II}).$ $(\text{vii)} \quad \mathbf{FT} \to \Sigma_{1}^{0}-*Det_{\mathcal{C}}^{II} \quad and \quad \neg!(\Sigma_{1}^{0}-*Det_{\mathcal{C}}^{I}) \to \neg!\mathbf{FT}.$

Proof. (i) We prove: given any α , one may construct β such that

$$\forall \gamma \in \mathcal{C} \exists n [(n, \gamma(n)) \in E_{\alpha}] \rightarrow \forall \tau \in \mathcal{C} \exists n [(n, \tau(n)) \in E_{\beta}] \text{ and}$$

$$\exists n[\langle n,0\rangle\in E_{\beta}\ \wedge\ \langle n,1\rangle\in E_{\beta}]\rightarrow \exists n[(n,0)\in E_{\alpha}\ \wedge\ (n,1)\in E_{\alpha}].$$

The two promised conclusions then follow easily.

Given α , define β such that $\forall n \forall i < 2[\alpha(p) = (n, i) + 1 \leftrightarrow \beta(p) = \langle n, i \rangle + 1]$ and $\forall p[\neg \exists n \exists i < 2[\alpha(p) = (n, i) + 1] \to \beta(p) = 0].$

Clearly, β satisfies the requirements.

(ii) We prove: given any α , one may construct β such that

$$\forall \tau \in \mathcal{C} \exists n [\langle n, \gamma(n) \rangle \in E_{\alpha}] \to \forall \tau \in \mathcal{C} \exists n \exists p [\langle n, \tau(n), p \rangle \in D_{\beta}] \text{ and}$$

$$\exists n \forall i < 2 \exists p [\langle n, i, p \rangle \in D_{\beta}] \to \exists n \forall i < 2 [\langle n, i \rangle \in E_{\alpha}].$$

The two promised conclusions then follow easily.

Given α , define β such that $\forall n \forall i < 2 \forall p [\beta(\langle n, i, p \rangle) \neq 0 \leftrightarrow \alpha(p) = \langle n, i \rangle + 1].$ Note: $\forall n \forall i < 2[\langle n, i \rangle \in E_{\alpha} \leftrightarrow \exists p[\langle n, i, p \rangle \in D_{\beta}]].$

Clearly, β satisfies the requirements.

(iii) Note:
$$\Delta_1^0 - Det^I_{(\omega \times 2)^2} \to \Delta_1^0 - Det^I_{\omega \times 2 \times \omega}$$
 and $\neg ! (\Delta_1^0 - Det^I_{\omega \times 2 \times \omega}) \to \neg ! (\Delta_1^0 - Det^I_{(\omega \times 2)^2}).$

(iv) Let m be given. We prove: given any α one may construct β such that

$$\forall \tau \in \mathcal{C} \exists c \in (\omega \times 2)^m [c \in_H \tau \land c \in D_\alpha] \to \forall \tau \in \mathcal{C} \exists \gamma \in (\omega \times 2)^\omega [\gamma \in_H \tau \land \gamma \in \mathcal{G}_\beta] \text{ and}$$

$$\exists \sigma \forall \gamma \in (\omega \times 2)^{\omega} [\gamma \in I \ \sigma \to \gamma \in \mathcal{G}_{\beta}] \to \exists s \forall c \in (\omega \times 2)^m [c \in I \ s \to c \in D_{\alpha}].$$

The two promised conclusions then follow easily.

Given α , define β such that $\forall s[\beta(s) \neq 0 \leftrightarrow (s \in (\omega \times 2)^m \land \alpha(s) \neq 0)].$

Observe that, if $\forall \tau \in \mathcal{C} \exists c \in (\omega \times 2)^m [c \in_{II} \tau \land c \in D_\alpha]$, then

$$\forall \tau \in \mathcal{C} \exists \gamma \in \mathcal{C} [\gamma \in_{II} \tau \land \beta(\overline{\gamma}(2m)) \neq 0], \text{ that is: } \forall \tau \in \mathcal{C} \exists \gamma \in \mathcal{C} [\gamma \in_{II} \tau \land \gamma \in \mathcal{G}_{\beta}].$$

Let σ in \mathcal{C} be given such that $\forall \gamma \in (\omega \times 2)^{\omega} [\gamma \in_I \sigma \to \exists n [\beta(\overline{\gamma}n) \neq 0]].$ Conclude: $\forall \gamma \in (\omega \times 2)^{\omega} [\gamma \in_I \sigma \to \alpha(\overline{\delta}(2m)) \neq 0].$

Find N such that $\forall c \in \bigcup_{n \leq 2m} \omega^n [c \in_I \gamma \to c < N]$ and define $s := \overline{\sigma}N$. Conclude: $\forall c \in (\omega \times 2)^m [c \in_I s \to c \in D_\alpha]$.

We thus see that β satisfies the requirements.

(v) For each α , one may construct β such that

$$\forall \tau \in \mathcal{C} \exists \gamma \in (\omega \times 2)^{\omega} [\gamma \in_{II} \tau \land \gamma \in \mathcal{G}_{\alpha}] \to \forall \tau \in \mathcal{C} \exists \delta \in \mathcal{C} [\delta \in_{II} \tau \land \delta \in \mathcal{G}_{\beta}] \text{ and}$$

$$\exists \sigma \forall \delta \in \mathcal{C} [\delta \in_{I} \sigma \to \delta \in \mathcal{G}_{\beta}] \to \exists \sigma \forall \gamma \in (\omega \times 2)^{\omega} [\gamma \in_{I} \sigma \to \gamma \in \mathcal{G}_{\alpha}] \text{ and}$$

$$\exists s \forall \delta \in \mathcal{C} [\delta \in_{I} s \to \delta \in \mathcal{G}_{\beta}] \to \exists s \forall \gamma \in (\omega \times 2)^{\omega} [\gamma \in_{I} s \to \gamma \in \mathcal{G}_{\alpha}].$$

The proof has been given in Lemma 9.7.

The promised conclusions follow easily.

(vi) We prove: given any α , one may construct β such that

$$\forall \sigma \in \mathcal{C} \exists c \in Bin[c \in_I \sigma \land \alpha(c) \neq 0] \rightarrow \forall \tau \in \mathcal{C} \exists d \in Bin[d \in_{II} \tau \land \beta(d) \neq 0] \text{ and}$$
$$\exists \tau \in \mathcal{C} \forall \delta \in \mathcal{C} [\delta \in_{II} \tau \rightarrow \exists n[\beta(\overline{\delta}n) \neq 0]] \rightarrow \exists \sigma \in \mathcal{C} \forall \delta \in \mathcal{C} [\delta \in_I \sigma \rightarrow \exists n[\alpha(\overline{\delta}n) \neq 0]]$$

and
$$\exists t \forall \delta \in \mathcal{C}[\delta \in_H t \to \exists n[\beta(\overline{\delta}n) \neq 0]] \to \exists t \forall \delta \in \mathcal{C}[\delta \in_H t \to \exists n[\alpha(\overline{\delta}n) \neq 0]].$$

The promised conclusions then follow easily.

Given α , define β such that $\beta(0) = 0$ and $\forall c \in Bin[\beta(\langle 0 \rangle * c) = \beta(\langle 1 \rangle * c) = \alpha(c)]$.

Assume: $\forall \sigma \in \mathcal{C} \exists c \in Bin[c \in_I \sigma \land \alpha(c) \neq 0].$

Let τ in \mathcal{C} be given.

Define σ such that $\forall c \in Bin[\sigma(c) = \tau(\langle 0 \rangle * c)].$

Find c in Bin such that $c \in I \sigma$ and $\alpha(c) \neq 0$.

Define $d := \langle 0 \rangle * c$ and note $d \in_{II} \tau$ and $\beta(d) \neq 0$.

We thus see: $\forall \tau \in \mathcal{C} \exists d \in Bin[d \in_{II} \tau \land \beta(d) \neq 0].$

Let τ in \mathcal{C} be given such that $\forall \delta \in \mathcal{C}[\delta \in_{II} \tau \to \exists n[\beta(\overline{\delta}n) \neq 0]].$

Define σ such that $\forall c \in Bin \forall i < 2[\sigma(\langle i \rangle * c) = \tau(c)].$

Note: $\forall \delta \in \mathcal{C}[\delta \in_I \sigma \to \langle 0 \rangle * \delta \in_{II} \tau]$, so $\forall \delta \in \mathcal{C}[\delta \in_I \sigma \to \exists n[\beta(\overline{\langle 0 \rangle} * \delta n) \neq 0]]$ and: $\forall \delta \in \mathcal{C}[\delta \in_I \sigma \to \exists n[\alpha(\overline{\delta}n) \neq 0]]$.

Let t be given such that $\forall \delta \in \mathcal{C}[\delta \in_{II} t \to \exists n[\beta(\overline{\delta}n) \neq 0]]$. Define s such that, $\forall c \in Bin[\langle 0 \rangle * c < length(t) \to s(c) = t(\langle 0 \rangle * c)]$.

Conclude, as above: $\forall \delta \in \mathcal{C}[\delta \in_I s \to \exists n[\alpha(\overline{\delta}n) \neq 0]].$

We thus see that β satisfies the requirements.

(vii) We prove: for each α , there exists β such that

$$\forall \gamma \in \mathcal{C} \exists s [s \in_I \gamma \land \alpha(s) \neq 0] \rightarrow Bar_{\mathcal{C}}(D_{\beta}) \text{ and}$$

$$\exists m[Bar_{\mathcal{C}}(D_{\overline{\beta}m})] \to \exists c \forall \delta \in \mathcal{C}[\delta \in_{II} c \to \exists n[\alpha(\overline{\delta}n) \neq 0]].$$

The two promised conclusions then follow easily.

Given α , define β such that $\forall c \in Bin[\beta(c) \neq 0 \leftrightarrow \exists s < c[s \in_I c \land \alpha(s) \neq 0]].$

Assume $\forall \gamma \in \mathcal{C} \exists s [s \in_I \gamma \land \alpha(s) \neq 0].$

Clearly, $\forall \gamma \in \mathcal{C} \exists n [\beta(\overline{\gamma}n) \neq 0]$, that is: $Bar_{\mathcal{C}}(D_{\beta})$.

Let m be given such that $Bar_{\mathcal{C}}(D_{\overline{\beta}m})$.

Define $X:=\{s\in Bin_m\mid \exists n\leq m[\alpha(\overline{s}n)\neq 0]\}$. Note: $\forall b\exists s[s\in_I b \land s\in X]$. According to Theorem 9.3, $Det_{Bin_m}^{II}(X)$.

Find c such that $\forall s \in Bin_m[s \in_{II} c \to s \in X]$.

Conclude: $\forall \delta \in \mathcal{C}[\delta \in_{II} c \to \overline{\delta}m \in X]$ and: $\forall \delta \in \mathcal{C}[\delta \in_{II} c \to \exists n[\alpha(\overline{\delta}n) \neq 0]].$

We thus see that β satisfies the requirements.

$$\begin{array}{c} \textbf{Theorem 9.9.} \quad \text{(i)} \ \ \mathsf{BIM} \vdash \boldsymbol{\Sigma}^0_1 \text{-} \overleftarrow{\mathbf{A}} \mathbf{C}_{\omega,2} \leftrightarrow \boldsymbol{\Sigma}^0_1 \text{-} Det^I_{\omega \times 2} \leftrightarrow \boldsymbol{\Delta}^0_1 \text{-} Det^I_{\omega \times 2 \times \omega} \leftrightarrow \\ \forall m [\boldsymbol{\Delta}^0_1 \text{-} Det^I_{(\omega \times 2)^m}] \leftrightarrow \boldsymbol{\Sigma}^0_1 \text{-} Det^I_{(\omega \times 2)^\omega} \leftrightarrow \boldsymbol{\Sigma}^0_1 \text{-} * Det^I_{(\omega \times 2)^\omega} \leftrightarrow \boldsymbol{\Sigma}^0_1 \text{-} Det^I_{\mathcal{C}} \leftrightarrow \\ \boldsymbol{\Sigma}^0_1 \text{-} * Det^I_{\mathcal{C}} \leftrightarrow \boldsymbol{\Sigma}^0_1 \text{-} Det^{II}_{\mathcal{C}} \leftrightarrow \boldsymbol{\Sigma}^0_1 \text{-} * Det^{II}_{\mathcal{C}} \leftrightarrow \mathbf{F} \mathbf{T}. \end{array}$$

$$\begin{array}{l} \text{(ii)} \ \ \mathsf{BIM} \vdash \neg ! (\mathbf{\Sigma}_{1}^{0} \text{-} \overline{\mathbf{A}} \overline{\mathbf{C}_{\omega,2}}) \leftrightarrow \neg ! (\mathbf{\Sigma}_{1}^{0} \text{-} Det_{\omega \times 2}^{I}) \leftrightarrow \neg ! (\mathbf{\Delta}_{1}^{0} \text{-} Det_{\omega \times 2 \times \omega}^{I}) \leftrightarrow \\ \exists m [\neg ! (\mathbf{\Delta}_{1}^{0} \text{-} Det_{(\omega \times 2)^{m}}^{I})] \leftrightarrow \neg ! (\mathbf{\Sigma}_{1}^{0} \text{-} Det_{(\omega \times 2)^{\omega}}^{I}) \leftrightarrow \neg ! (\mathbf{\Sigma}_{1}^{0} \text{-} * Det_{(\omega \times 2)^{\omega}}^{I}) \leftrightarrow \neg ! (\mathbf{\Sigma}_{1}^{0} \text{-} * Det_{\mathcal{C}}^{I}) \leftrightarrow \neg ! (\mathbf{\Sigma}_{1}^{0} \text{-} * Det_{\mathcal{C}}^{I}) \leftrightarrow \neg ! \mathbf{F} \mathbf{T}. \end{array}$$

Proof. Use Lemmas 5.3 and 9.8.

9.5. The Intuitionistic Determinacy Theorem. Recall that $\mathcal{X} \subseteq (\omega \times 2)^{\omega}$ is I-determinate in $(\omega \times 2)^{\omega}$ if and only if

$$\forall \tau \in \mathcal{C} \exists \gamma [\gamma \in_{II} \tau \land \gamma \in \mathcal{X}] \to \exists \sigma \forall \gamma \in (\omega \times 2)^{\omega} [\gamma \in_{I} \sigma \to \gamma \in \mathcal{X}].$$

We now define: $\mathcal{X} \subseteq (\omega \times 2)^{\omega}$ is weakly *I*-determinate in $(\omega \times 2)^{\omega}$ if and only if $\forall \varphi : \mathcal{C} \to (\omega \times 2)^{\omega} [\forall \tau \in \mathcal{C}[\varphi | \tau \in_{II} \tau \land \varphi | \tau \in \mathcal{X}] \to \exists \sigma \forall \gamma \in (\omega \times 2)^{\omega} [\gamma \in_{I} \sigma \to \gamma \in \mathcal{X}]].$ A (continuous) function $\varphi : \mathcal{C} \to (\omega \times 2)^{\omega}$ such that $\forall \tau \in \mathcal{C}[\varphi | \tau \in_{II} \tau]$ will be called an anti-strategy for player *I* in $(\omega \times 2)^{\omega}$.

Note that the Second Axiom of Continuous Choice, $\mathbf{AC}_{\omega^{\omega},\omega^{\omega}} = \mathbf{AC}_{1,1}$, implies, for every subset $\mathcal{X} \subseteq (\omega \times 2)^{\omega}$: if $\forall \tau \in \mathcal{C} \exists \gamma \in (\omega \times 2)^{\omega} [\gamma \in_{II} \tau \land \gamma \in \mathcal{X}]$, then $\exists \varphi : \mathcal{C} \to (\omega \times 2)^{\omega} \forall \tau \in \mathcal{C} [\varphi | \tau \in_{II} \tau \land \varphi | \tau \in \mathcal{X}]$.

 $\mathbf{AC}_{\omega^{\omega},\omega^{\omega}}$ thus implies: if $\mathcal{X} \subseteq (\omega \times 2)^{\omega}$ is weakly *I*-determinate in $(\omega \times 2)^{\omega}$, then \mathcal{X} is *I*-determinate in $(\omega \times 2)^{\omega}$.

Earlier versions of the next Theorem may be found in [31, Chapter 16] and [41].

Theorem 9.10 (Intuitionistic Determinacy Theorem). The following statements are equivalent in BIM:

- (i) **FT**.
- (ii) For every anti-strategy φ for player I in $(\omega \times 2)^{\omega}$ there exists a strategy σ for player I in $(\omega \times 2)^{\omega}$ such that $\forall \gamma \in (\omega \times 2)^{\omega} [\gamma \in I \ \sigma \to \exists \tau \in \mathcal{C}[\gamma = \varphi | \tau]]$.

Proof. (i)
$$\Rightarrow$$
 (ii). Let $\varphi : \mathcal{C} \to (\omega \times 2)^{\omega}$ satisfy: $\forall \tau \in \mathcal{C}[\varphi | \tau \in_{II} \tau]$.

We intend to develop a strategy σ for player I in $(\omega \times 2)^{\omega}$ such that Player I, whenever he uses σ and develops, together with player II, γ in $(\omega \times 2)^{\omega}$, will be able to construct, simultaneously, a strategy τ for player II in $(\omega \times 2)^{\omega}$ such that $\gamma = \varphi | \tau$. The infinite sequence γ must be the answer given by the anti-strategy φ to player II's strategy τ . So, while playing γ , player I conjectures a strategy τ that player II may be assumed to follow during this very play.

Let c, t be given such that $c \in \bigcup_k (\omega \times 2)^k$ and $t \in Bin$. We define: with respect to the given anti-strategy φ , t is, at the position c, a safe (partial) conjecture by player I about the strategy followed by player II, notation: Safe(c,t), if and only if I

$$\forall \rho \in \mathcal{C} \exists u \in Bin[t \sqsubseteq u \land {}^{c}u \sqsubseteq \rho \land c \sqsubseteq \varphi|u].$$

Note¹⁶:
$$Safe(c, t) \leftrightarrow \forall \rho \in \mathcal{C} \exists \tau \in \mathcal{C}[t \sqsubset \tau \land {}^{c}\tau = \rho \land c \sqsubset \varphi | \tau].$$

If Safe(c,t), then player I, at the position c, may extend every strategy ρ of player II 'above c', that is, on positions d in $(\omega \times 2)^{<\omega} \times \omega$ such that $c \sqsubseteq d$, to a strategy τ of player II on the whole of $(\omega \times 2)^{<\omega} \times \omega$ such that $t \sqsubseteq \tau$ and $c \sqsubseteq \varphi | \tau$.

We now prove the following important observation:

$$\forall c \in \bigcup_k (\omega \times 2)^k \forall t \in Bin[Safe(c,t) \vee \neg Safe(c,t)].$$

Let c in $\bigcup_k (\omega \times 2)^k$ and t in Bin be given.

Define β such that $\forall u [\beta(u) \neq 0 \leftrightarrow (u \in Bin \land (c \sqsubseteq \varphi | u \lor c \perp \varphi | u))].$

Note: $Bar_{\mathcal{C}}(D_{\beta})$, and: $\forall u[u \in D_{\beta} \to \forall i < 2[u * \langle i \rangle \in D_{\beta}]]$.

Using **FT**, find n > length(t) such that $Bin_n \subseteq D_{\beta}$.

¹⁵For the notation ^{c}u , see Subsection 13.4.

¹⁶For the notation $^{c}\tau$, see Subsection 13.4.

Note: $\forall u \in Bin_n[length(^cu) \leq length(u) \leq n]$ and $Safe(c,t) \leftrightarrow \forall r \in Bin_n \exists u \in Bin_n [t \sqsubseteq u \land {}^c u \sqsubseteq r \land c \sqsubseteq \varphi | u].$

We thus see that one may prove: $Safe(c,t) \vee \neg Safe(c,t)$.

Going one step further, we want to define σ such that

$$\forall c \in \bigcup_{k} (\omega \times 2)^{k} \forall t \in Bin[\sigma(c,t) \neq 0 \leftrightarrow Safe(c,t)].$$

To this end, we first define ψ such that, for all c in $\bigcup_k (\omega \times 2)^k$, for all t in Bin,

$$\psi(c,t) := \mu n[n > length(t) \land \forall u \in Bin_n[c \sqsubseteq \varphi | u \lor c \bot \varphi | u]],$$

and then define σ such that, for all c in $\bigcup_k (\omega \times 2)^k$, for all t in Bin,

$$\sigma(c,t) \neq 0 \leftrightarrow \forall r \in Bin_{\psi(c,t)} \exists u \in Bin_{\psi(c,t)} [t \sqsubseteq u \land {}^{c}u \sqsubseteq r \land c \sqsubseteq \varphi | u].$$
We also define a graph that for all similar (a. v. 2)k for all t in Bin

We also define χ such that, for all c in $\bigcup_k (\omega \times 2)^k$, for all t in Bin, if $\sigma(c,t)=0$, then

 $\chi(c,t) = \mu r[r \in Bin_{\psi(c,t)} \land \neg \exists u \in Bin_{\psi(c,t)}[t \sqsubseteq u \land {}^{c}u \sqsubseteq r \land c \sqsubseteq \varphi[u]].$ Note that, if $\sigma(c,t)=0$, then

 $\forall \rho \in \mathcal{C}[\chi(c,t) \sqsubset \rho \rightarrow \neg \exists u \in Bin[t \sqsubseteq u \land {}^{c}u \sqsubset \rho \land c \sqsubseteq \varphi|u]].$

Let n, c, t be given such that $c \in (\omega \times 2)^n$ and t in Bin and Safe(c, t). We prove:

$$\forall \rho \in \mathcal{C} \exists d \in \bigcup_{k>0} (\omega \times 2)^k [d \in_{II} \rho \land \exists i < 2[Safe(c*d, t*\langle i \rangle)]].$$

Let ρ in \mathcal{C} be given. Find u in Bin such that $t \sqsubset u \land {}^c u \sqsubset \rho \land c \sqsubseteq \varphi | u$.

Find i < 2 such that $t * \langle i \rangle \sqsubseteq u$.

Find p such that, for all d in $\bigcup_k (\omega \times 2)^k \times \omega$,

if $d < length(^cu)$, then length(d) < 2p.

Define $D := \{ d \in (\omega \times 2)^p \mid d \in_{II} \rho \land \exists \tau \in \mathcal{C}[c * d \sqsubset \varphi | \tau] \}.$

It follows from **FT** that $E := \{ \overline{\phi | \tau}(2n+2p) \mid \tau \in \mathcal{C} \}$ is a finite¹⁷ subset of ω . Note that, for all d in $(\omega \times 2)^p$, $d \in D$ if and only if $\exists b \in E[c*d=b]$ and conclude: also D is a finite subset of ω .

Assume $\forall d \in D[\neg Safe(c * d, t * \langle i \rangle)].$

Define ρ^* in \mathcal{C} such that $^cu \sqsubseteq \rho^*$ and $\forall d \in D[d \in_{II} \rho^*]$ and $\forall d \in D[\chi(c*d, t*\langle i \rangle) \sqsubset {}^d(\rho^*)].$

Find τ in \mathcal{C} such that $u \sqsubseteq \tau$ and $c\tau = \rho^*$ and consider $\varphi | \tau$.

Note: $c \sqsubseteq \varphi | u \sqsubset \varphi | \tau$.

Find d in D such that $c * d \sqsubset \varphi | \tau$.

Note: ${}^{c*d}\tau = {}^d(\rho^*)$ and $\chi(c*d,t*\langle i\rangle) \sqsubset {}^d(\rho^*)$. Conclude: $\neg \exists v \in Bin[t*\langle i\rangle \sqsubseteq v \land {}^{c*d}v \sqsubseteq {}^d(\rho^*) \land c*d \sqsubseteq \varphi|v]$.

On the other hand, $t * \langle i \rangle \sqsubseteq u \sqsubset \tau$ and $\exists m[c * d \sqsubseteq \varphi | \overline{\tau}m]$.

Note that ${}^c\tau = \rho^*$ and, therefore, ${}^{c*d}\overline{\tau}m \sqsubseteq {}^d(\rho^*)$.

Conclude: $\exists m[t*\langle i\rangle \sqsubseteq \overline{\tau}m \land c*d\overline{\tau}m \sqsubseteq d(\rho^*) \land c*d \sqsubseteq \varphi|\overline{\tau}m].$

Contradiction.

We thus see: $\neg \forall d \in D[\neg Safe(c * d, t * \langle i \rangle)].$

As D is finite, conclude: $\exists d \in D[Safe(c * d, t * \langle i \rangle)].$

We thus see: $\forall \rho \in \mathcal{C} \exists d \in \bigcup_{k>0} (\omega \times 2)^k [d \in_{II} \rho \land Safe(c * d, t * \langle i \rangle)].$

Using Σ_1^0 - $Det^I_{(\omega \times 2)^\omega}$, an equivalent of **FT**, see Theorem 9.9(i), find σ such that $\forall \delta[\delta \in_I \sigma \to \exists k[Safe(c * \overline{\delta}(2k+2), t * \langle i \rangle)]].$

Note that the set $\{\delta \in (\omega \times 2)^{\omega} \mid \delta \in_I \sigma\}$ is an explicit fan.

 $^{^{17}}D \subseteq \omega$ is finite if and only if there exists a such that $D = D_a = \{n < length(a) \mid a(n) \neq 0\},\$ see Subsection 13.2.

```
Note: for all d in (\omega \times 2)^{<\omega}, if d > length(s) and d \in_I s, then
\exists n[0 < 2n \leq length(d) \land Safe(c * \overline{d}(2n), t * \langle i \rangle)].
     Note that, for all s, the set of all d in (\omega \times 2)^{<\omega} such that d > length(s) and
\forall n[2n < length(d) \rightarrow \overline{d}(2n) < length(s)] \text{ and } d \in_I s \text{ is a finite subset of } \omega.
     Define \alpha such that for all c in (\omega \times 2)^{<\omega}, for all t in Bin, for all s,
\alpha(c,t,s)\neq 0 if and only if, for all d in (\omega\times 2)^{<\omega},
if d > length(s) and \forall n[2n < length(d) \rightarrow \overline{d}(2n) < length(s)] and d \in I s, then
\exists n[0 < 2n \leq length(d) \land \exists i < 2[Safe(c * \overline{d}(2n), t * \langle i \rangle)]].
     We have proven: \forall c \in (\omega \times 2)^{<\omega} \forall t \in Bin[Safe(c,t) \to \exists s[\alpha(c,t,s) \neq 0]].
     Define \eta such that
                  \forall c \in (\omega \times 2)^{<\omega} \forall t \in Bin[Safe(c,t) \to \eta(c,t) = \mu s[\alpha(c,t,s) \neq 0]].
     Define \lambda such that \lambda(\langle \rangle) = \langle \rangle, and, for all c in (\omega \times 2)^{<\omega}, for all \langle p, i \rangle in \omega \times 2, if
there exists j < 2 such that Safe(c * \langle p, i \rangle, \lambda(c) * \langle j \rangle), then \lambda(c * \langle p, i \rangle) = \lambda(c) * \langle j_0 \rangle,
where j_0 is the least such j, and, if not, then \lambda(c * \langle p, i \rangle) = \lambda(c).
     Note: \forall c \in (\omega \times 2)^{<\omega}[Safe(c, \lambda(c))].
     Note: \forall c \in (\omega \times 2)^{<\omega} \forall d \in (\omega \times 2)^{<\omega} [c \sqsubseteq d \to \lambda(c) \sqsubseteq \lambda(d)].
     Define \sigma as a strategy for player I in (\omega \times 2)^{\omega} as follows.
     Let c in (\omega \times 2)^{<\omega} be given.
     Find n_0 := \mu n[Safe(\overline{c}(2n), \lambda(c))].
     Find d such that c = \overline{c}(2n_0) * d.
     Define \sigma(c) := (\eta(\overline{c}(2n_0), \lambda(c)))(d).
     Let \gamma be an element of (\omega \times 2)^{\omega} such that \gamma \in I \sigma.
     We claim: \forall n \exists m \geq n [\lambda(\overline{\gamma}(2m)) \sqsubset \lambda(\overline{\gamma}(2m+2))].
     We prove this claim as follows.
     Let n be given. Find n_0 := \mu k[Safe(\overline{\gamma}(2k), \lambda(\overline{\gamma}(2n))].
     Define c := \overline{\gamma}(2n_0) and t := \lambda(c).
     Consider s := \eta(c, t).
     Find \delta such that \delta \in I s and \forall k[\delta(2k+1) = \gamma(2n_0 + 2k + 1)].
     Find k_0 := \mu k [\exists i < 2[Safe(c * \overline{\delta}(2k+2), t * \langle i \rangle)].
     Note: c * \overline{\delta}(2k_0 + 2) \sqsubset \gamma.
     Note: \lambda(c) = \lambda(\overline{\gamma}(2n)) = \lambda(\overline{\gamma}(2n_0 + 2k_0)) \subset \lambda(\overline{\gamma}(2n_0 + 2k_0 + 2)).
     Defining m := n_0 + k_0, we see: m \ge n and \lambda(\overline{\gamma}(2m)) \sqsubset \lambda(\overline{\gamma}(2m+2)).
     This ends the proof of our claim.
     Find \tau in \mathcal{C} such that \forall n[\lambda(\overline{\gamma}(2n)) \sqsubset \tau].
     Note: \forall n \exists \zeta \in \mathcal{C}[\overline{\tau}n \sqsubseteq \zeta \land \overline{\gamma}(2n) \sqsubseteq \varphi|\zeta]. Conclude: \varphi|\tau = \gamma.
     We thus see: \forall \gamma \in (\omega \times 2)^{\omega} [\gamma \in_I \sigma \to \exists \tau \in \mathcal{C}[\varphi | \tau = \gamma]].
     (ii) \Rightarrow (i). Assume (ii). We prove: \Sigma_1^0-Det_{\omega \times 2}^I.
Using Theorem 9.9(i), we then may conclude: FT.
     Let \alpha be given such that \forall \tau \in \mathcal{C} \exists n [\langle n, \tau(\langle n \rangle) \rangle \in E_{\alpha}], i.e.
\forall \tau \in \mathcal{C} \exists n \exists p [\alpha(p) = \langle n, \tau(\langle n \rangle) \rangle + 1], \text{ i.e. } \forall \tau \in \mathcal{C} \exists p [\alpha(p') = \langle p'', \tau(\langle p'' \rangle) \rangle + 1].
Define \psi: \mathcal{C} \to \omega such that \forall \tau \in \mathcal{C}[\psi(\tau) = \mu p[\alpha(p') = \langle p'', \tau(\langle p'' \rangle) \rangle + 1].
Define \varphi: \mathcal{C} \to (\omega \times 2)^{\omega} such that \forall \tau \in \mathcal{C}[\varphi | \tau = \langle \psi''(\tau), \tau(\langle \psi''(\tau) \rangle) \rangle * \underline{0}].
Find \sigma such that \forall \gamma \in (\omega \times 2)^{\omega} [\gamma \in I \ \sigma \to \exists \tau \in \mathcal{C} [\gamma = \varphi | \tau]], and, therefore,
\forall \gamma \in (\omega \times 2)^{\omega} [\gamma \in_I \sigma \to \exists q [\alpha(q) = \overline{\gamma}2 + 1]] \text{ and } \forall \gamma \in (\omega \times 2)^{\omega} [\gamma \in_I \sigma \to \overline{\gamma}2 \in E_{\alpha}].
Define n := \sigma(\langle \rangle) and note: \forall i < 2[\langle n, i \rangle \in I \sigma] and, therefore: \forall i < 2[\langle n, i \rangle \in E_{\alpha}].
     We thus see: \forall \alpha [\forall \tau \in \mathcal{C} \exists n [\langle n, \tau(\langle n \rangle) \rangle \in E_{\alpha}] \rightarrow \exists n \forall i < 2 [\langle n, i \rangle \in E_{\alpha}]], i.e.
\Sigma_1^0-Det_{\omega\times 2}^I.
                                                                             37
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Using **FT**, find m such that $\forall \delta[\delta \in_I \sigma \to \exists k \leq m[Safe(c * \overline{\delta}(2k+2), t * \langle i \rangle)]].$

Find p such that $\forall \delta [\delta \in_I \sigma \to \overline{\delta}(2m+2) < p]$ and define $s := \overline{\sigma}p$.

Corollary 9.11. BIM + FT proves the following scheme:

Every $\mathcal{X} \subseteq (\omega \times 2)^{\omega}$ is weakly I-determinate.

The theorem and its corollary may be generalized. One may replace $(\omega \times 2)^{\omega}$ by any spread \mathcal{F} satisfying the condition: $\exists \zeta \forall \alpha \in \mathcal{F} \forall n [\alpha(2n+1) \leq \zeta(n)].$

Let $\varphi: \mathcal{C} \to (\omega \times 2)^{\omega}$ be an anti-strategy for player I in $(\omega \times 2)^{\omega}$. We define: φ fails to translate into a strategy for player I if and only if

$$\neg \exists \sigma \forall \gamma \in (\omega \times 2)^{\omega} [\gamma \in_I \sigma \to \exists \tau \in \mathcal{C} [\gamma = \varphi | \tau]].$$

¬!FT implies the existence of an anti-strategy for player I in $(\omega \times 2)^{\omega}$ that fails to translate into a strategy for player I in $(\omega \times 2)^{\omega}$: using \neg !**FT** and Theorem 9.9(ii), find α such that $\forall \tau \in \mathcal{C} \exists n [\langle n, \tau(\langle n \rangle) \rangle \in E_{\alpha}]$ and $\neg \exists n \forall i < 2 [\langle n, i \rangle \in E_{\alpha}]$. As in the proof of Theorem 9.10(ii) \Rightarrow (i), find an anti-strategy φ for player I in $(\omega \times 2)^{\omega}$ such that $\forall \tau \in \mathcal{C}[(\varphi|\tau)2 \in E_{\alpha}]$. Assume σ is a strategy for player I in $(\omega \times 2)^{\omega}$ such that $\forall \gamma \in (\omega \times 2)^{\omega} [\gamma \in I \ \sigma \to \exists \tau \in \mathcal{C}[\gamma = \varphi | \tau]]$. Consider $n := \sigma(\langle \rangle)$ and conclude: $\forall i < 2[\langle n, i \rangle \in E_{\alpha}]$. Contradiction.

We did not find an argument proving ¬!FT from the assumption of the existence of an anti-strategy for player I in $(\omega \times 2)^{\omega}$ that fails to translate into a strategy for player I in $(\omega \times 2)^{\omega}$.

10. The (uniform) intermediate value theorem

10.1. The Intermediate Value Theorem, IVT^{18}

For all φ in $\mathbb{R}^{[0,1]}$,

if
$$\exists \gamma \in [0,1]^2 [\varphi^{\mathcal{R}}(\gamma^{\uparrow 0}) \leq_{\mathcal{R}} 0_{\mathcal{R}} \leq_{\mathcal{R}} \varphi^{\mathcal{R}}(\gamma^{\uparrow 1})]$$
, then $\exists \gamma \in [0,1] [\varphi^{\mathcal{R}}(\gamma) =_{\mathcal{R}} 0_{\mathcal{R}}]$.

IVT fails constructively. The next two theorems are similar to [3, Chapter 3, Theorem 2.4] and [27, Theorem 6(iv) and (iii)].

Theorem 10.1. BIM \vdash IVT \rightarrow LLPO.

Proof. Assume **IVT**. Let β be given.

Define δ in \mathcal{R} such that, for each n,

if $\overline{\underline{0}}n \sqsubset \beta$, then $\delta(n) = (-\frac{1}{2^n}, \frac{1}{2^n})$, and, if $\overline{\underline{0}}n \perp \beta$ and $p_0 := \mu p[\beta(p) \neq 0]$, then $\delta(n) = ((-1)^{p_0} \frac{1}{2^{p_0+1}} - \frac{1}{2^{n+3}}, (-1)^{p_0} \frac{1}{2^{p_0+1}} + \frac{1}{2^{n+3}})$. Note: $\delta >_{\mathcal{R}} 0_{\mathcal{R}} \leftrightarrow \exists p[2p = \mu n[\beta(n) \neq 0]]$ and $\delta <_{\mathcal{R}} 0_{\mathcal{R}} \leftrightarrow \exists p[2p + 1 = \mu n[\beta(n) \neq 0]]$. Find φ in $\mathcal{R}^{[0,1]}$ such that $\varphi'^{\mathcal{R}}(0_{\mathcal{R}}) =_{\mathcal{R}} (-1)_{\mathcal{R}}$, and $\varphi'^{\mathcal{R}}(\frac{1}{3}) =_{\mathcal{R}} \varphi'^{\mathcal{R}}(\frac{2}{3}) =_{\mathcal{R}} \delta$ and $\varphi'^{\mathcal{R}}(1_{\mathcal{R}}) =_{\mathcal{R}} 1_{\mathcal{R}}$ and φ is linear on $[0, \frac{1}{3}]$, on $[\frac{1}{3}, \frac{2}{3}]$ and on $[\frac{2}{3}, 1]$.

Note: $\varphi^{\mathcal{R}}(0_{\mathcal{R}}) \leq_{\mathcal{R}} 0_{\mathcal{R}} \leq_{\mathcal{R}} \varphi^{\mathcal{R}}(1_{\mathcal{R}}).$

Using **IVT**, find γ in [0,1] such that $\varphi'^{\mathcal{R}}(\gamma) =_{\mathcal{R}} 0_{\mathcal{R}}$.

Either $\gamma >_{\mathcal{R}} \frac{1}{3}$ or $\gamma <_{\mathcal{R}} \frac{2}{3}$.

If $\gamma >_{\mathcal{R}} \frac{1}{3}$, then $\neg(\delta >_{\mathcal{R}} 0)$ and: $\forall p[2p \neq \mu n[\beta(n) \neq 0]]$, and,

if $\gamma <_{\mathcal{R}} \frac{2}{3}$, then $\neg (\delta <_{\mathcal{R}} 0)$ and: $\forall p[2p+1 \neq \mu n[\beta(n) \neq 0]]$.

We thus see: $\forall \beta [\forall p[2p \neq \mu n[\beta(n) \neq 0]] \lor \forall p[2p+1 \neq \mu n[\beta(n) \neq 0]], i.e.$ LLPO.

Theorem 10.2. $BIM + \Pi_1^0 - AC_{\omega,2} \vdash LLPO \rightarrow IVT$.

Proof. Let φ in $\mathcal{R}^{[0,1]}$ and γ in $[0,1]^2$ be given such that $\varphi'^{\mathcal{R}}(\gamma^{\uparrow 0}) \leq_{\mathcal{R}} 0_{\mathcal{R}} \leq_{\mathcal{R}} \varphi'^{\mathcal{R}}(\gamma^{\uparrow 1})$. Define β such that, for all n, $\beta(2n) \neq 0 \leftrightarrow \gamma^{\uparrow 0}(n) <_{\mathbb{S}} \gamma^{\uparrow 1}(n)$ and: $\beta(2n+1) \neq 0 \leftrightarrow \gamma^{\uparrow 1}(n) <_{\mathbb{S}} \gamma^{\uparrow 0}(n)$. By LLPO,

either: $\forall p[2p+1 \neq \mu n[\beta(n) \neq 0]]$ and $\forall n[\gamma^{\uparrow 0}(n) \leq_{\mathbb{S}} \gamma^{\uparrow 1}(n)]$ and $\gamma^{\uparrow 0} \leq_{\mathcal{R}} \gamma^{\uparrow 1}$, or: $\forall p[2p \neq \mu n[\beta(n) \neq 0]]$ and $\forall n[\gamma^{\uparrow 1}(n) \leq_{\mathbb{S}} \gamma^{\uparrow 0}(n)]$ and $\gamma^{\uparrow 1} \leq_{\mathcal{R}} \gamma^{\uparrow 0}$.

¹⁸For some notation used in this Section, see Subsection 13.9.

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Assume \gamma^{\restriction 0} \leq_{\mathcal{R}} \gamma^{\restriction 1}. Using LLPO like we used it just now, conclude: for all n, for all m \leq 2^n, either: \varphi^{\cdot \mathcal{R}}(\frac{2^n - m}{2^n} \cdot_{\mathcal{R}} \gamma^{\restriction 0} +_{\mathcal{R}} \frac{m}{2^n} \cdot_{\mathcal{R}} \gamma^{\restriction 1}) \leq_{\mathcal{R}} 0_{\mathcal{R}},
or: 0_{\mathcal{R}} \leq_{\mathcal{R}} \varphi^{\mathcal{R}}(\frac{2^{n}-m}{2^{n}} \cdot_{\mathcal{R}} \gamma^{\uparrow 0} +_{\mathcal{R}} \frac{m}{2^{n}} \cdot_{\mathcal{R}} \gamma^{\uparrow 1}).
Using \mathbf{\Pi}_{1}^{0}-\mathbf{AC}_{\omega,2}, find \beta such that \beta^{0}(0) = 0 and \beta^{0}(1) \neq 0 and \forall n \forall m \leq 2^{n} [\beta^{n}(m) = 0 \rightarrow \varphi^{\mathcal{R}}(\frac{2^{n}-m}{2^{n}} \cdot_{\mathcal{R}} \gamma^{\uparrow 0} +_{\mathcal{R}} \frac{m}{2^{n}} \cdot_{\mathcal{R}} \gamma^{\uparrow 1}) \leq_{\mathcal{R}} 0_{\mathcal{R}}], and \forall n \forall m \leq 2^{n} [\beta^{n}(m) \neq 0 \rightarrow 0_{\mathcal{R}} \leq_{\mathcal{R}} \varphi^{\mathcal{R}}(\frac{2^{n}-m}{2^{n}} \cdot_{\mathcal{R}} \gamma^{\uparrow 0} +_{\mathcal{R}} \frac{m}{2^{n}} \cdot_{\mathcal{R}} \gamma^{\uparrow 1})].
       Now define \delta such that \delta(0) = 0 and
\forall n[\beta^{n+1}(2\delta(n)+1)\neq 0 \rightarrow \delta(n+1)=2\delta(n)], \text{ and}
\forall n[\beta^{n+1}(2\delta(n)+1) = 0 \to \delta(n+1) = 2\delta(n)+1].
       Note: \forall n[\delta(n) < 2^n \land \beta^n(\delta(n)) = 0 \land \beta^n(\delta(n) + 1) \neq 0] and
\forall n[\varphi'^{\mathcal{R}}(\frac{\delta(n)}{2^n}) \leq_{\mathcal{R}} 0_{\mathcal{R}} \leq_{\mathcal{R}} \varphi'^{\mathcal{R}}(\frac{\delta(n)+1}{2^n})].
Define \varepsilon such that, for each n, \varepsilon(n) = (\frac{\delta(n)}{2^n}, \frac{\delta(n)+1}{2^n}) and define \rho such that, for each n, \rho(n) = double_{\mathbb{S}}(\varepsilon(n)). Note: \rho \in [0, 1].
        Assume \varphi^{\mathcal{R}}(\rho) >_{\mathcal{R}} 0_{\mathcal{R}}.
      Find n, p such that p \in E_{\varphi} and \rho(n) \sqsubseteq_{\mathbb{S}} p' and (p'')' >_{\mathbb{Q}} 0_{\mathbb{Q}}.

Note \rho'(n) <_{\mathbb{Q}} \frac{\delta(n)}{2^n} <_{\mathbb{Q}} \rho''(n) and \varphi'^{\mathcal{R}}(\frac{\delta(n)}{2^n}) \leq_{\mathbb{R}} 0_{\mathbb{R}}. Contradiction.

Conclude: \varphi'^{\mathcal{R}}(\rho) \leq_{\mathcal{R}} 0_{\mathcal{R}}.
       For a similar reason, 0_{\mathcal{R}} \leq_{\mathcal{R}} \varphi^{\mathcal{R}}(\rho) and, therefore, \varphi^{\mathcal{R}}(\rho) =_{\mathcal{R}} 0_{\mathcal{R}}.
       The case \gamma^{\uparrow 1} \geq_{\mathcal{R}} \gamma^{\uparrow 0} is treated similarly.
                                                                                                                                                                                                                         Corollary 10.3. BIM + \Pi_1^0-AC<sub>\omega,2</sub> \vdash IVT \leftrightarrow LLPO \leftrightarrow WKL.
 Proof. Use Theorems 4.3, 10.1 and 10.2.
                                                                                                                                                                                                                         10.2. A contraposition of the Intermediate Value Theorem, \overline{\text{IVT}}:
          For each \varphi in \mathbb{R}^{[0,1]}, if \forall \gamma \in [0,1][\varphi^{\mathcal{R}}(\gamma) \#_{\mathcal{R}} 0_{\mathcal{R}}], then
        either \forall \gamma \in [0,1][\varphi'^{\mathcal{R}}(\gamma) >_{\mathcal{R}} 0_{\mathcal{R}}] or \forall \gamma \in [0,1][\varphi'^{\mathcal{R}}(\gamma) <_{\mathcal{R}} 0_{\mathcal{R}}].
Theorem 10.4. BIM \vdash \overrightarrow{IVT}.
 Proof. Assume \varphi \in \mathcal{R}^{[0.1]} and \forall \gamma \in [0, 1] [\varphi'^{\mathcal{R}}(\gamma) \#_{\mathcal{R}} 0_{\mathcal{R}}].
Assume: \varphi^{\mathcal{R}}(0_{\mathcal{R}}) <_{\mathcal{R}} 0_{\mathcal{R}}.
Suppose: \gamma \in [0,1] and \varphi^{\mathcal{R}}(\gamma) >_{\mathcal{R}} 0_{\mathcal{R}}.
We will find a contradiction by the method of successive bisection.
Find q in \mathbb{Q} such that \varphi^{\mathcal{R}}(q_{\mathcal{R}}) >_{\mathcal{R}} 0_{\mathbb{Q}}. Define \delta as follows, by induction.
We define \delta(0) := (0, q). Now, let n, s be given such that \delta(n) = s.
Note: \varphi'^{\mathcal{R}}((\frac{s'+\mathbb{Q}s''}{2})_{\mathcal{R}}) \# 0_{\mathcal{R}}. Find (r,t) in E_{\varphi} such that r' <_{\mathbb{Q}} \frac{1}{2}(s'+\mathbb{Q}s'') <_{\mathbb{Q}} r''
and either: 0_{\mathbb{Q}} <_{\mathbb{Q}} t' or: t'' <_{\mathbb{Q}} 0_{\mathbb{Q}}. If 0_{\mathbb{Q}} <_{\mathbb{Q}} t', define \delta(n+1) = (s', \frac{s' +_{\mathbb{Q}} s''}{2}), and
if t'' <_{\mathbb{Q}} 0_{\mathbb{Q}}, define \delta(n+1) = (\frac{s'+s''}{2}, s'').
Note: for each n, \delta(n+1) \sqsubseteq_{\mathbb{S}} \delta(n), and \varphi'^{\mathcal{R}}((\delta'(n))_{\mathcal{R}}) <_{\mathcal{R}} 0_{\mathcal{R}} <_{\mathcal{R}} \varphi'^{\mathcal{R}}((\delta''(n))_{\mathcal{R}}).
Note: \delta \in [0,1] and \varphi^{\mathcal{R}}(\delta) \#_{\mathcal{R}} 0_{\mathcal{R}}. Determine (r,s) in E_{\varphi} and n in \omega such that
\delta(n) \sqsubseteq_{\mathbb{S}} r and either s'' <_{\mathbb{Q}} 0 or 0 <_{\mathbb{Q}} s', that is, either \varphi'^{\mathcal{T}}((\delta''(n))_{\mathcal{R}}) <_{\mathcal{R}} 0_{\mathcal{R}} or
0_{\mathcal{R}} <_{\mathcal{R}} \varphi^{'\mathcal{R}} ((\delta'(n))_{\mathcal{R}}). Contradiction. Clearly, then, \neg (\varphi^{'\mathcal{R}}(\gamma) >_{\mathcal{R}} 0_{\mathcal{R}}).
        From \neg(\varphi^{\mathcal{R}}(\gamma) >_{\mathcal{R}} 0_{\mathcal{R}}) and \varphi^{\mathcal{R}}(\gamma) \#_{\mathcal{R}} 0_{\mathcal{R}}, we conclude: \varphi^{\mathcal{R}}(\gamma) <_{\mathcal{R}} 0_{\mathcal{R}}.
       We thus see: if \varphi'^{\mathcal{R}}(0_{\mathcal{R}}) <_{\mathcal{R}} 0_{\mathcal{R}}, then \forall \gamma \in [0,1][\varphi'^{\mathcal{R}}(\gamma) <_{\mathcal{R}} 0_{\mathcal{R}}].
       In the same way, one proves: if \varphi'^{\mathcal{R}}(0_{\mathcal{R}}) >_{\mathcal{R}} 0_{\mathcal{R}}, then \forall \gamma \in [0,1][\varphi'^{\mathcal{R}}(\gamma) >_{\mathcal{R}} 0_{\mathcal{R}}].
       We thus see: either \forall \gamma \in [0,1] [\varphi'^{\mathcal{R}}(\gamma) >_{\mathcal{R}} 0_{\mathcal{R}}] or \forall \gamma \in [0,1] [\varphi'^{\mathcal{R}}(\gamma) <_{\mathcal{R}} 0_{\mathcal{R}}].
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10.3. **FT** is unprovable in BIM+IVT. As we observed in Subsection 4.8, BIM+ $\mathbf{CT}+X\vee\neg X$ is consistent. According to Theorem 10.4, BIM $\vdash \widehat{\mathbf{IVT}}$, and, therefore: BIM+ $X\vee\neg X\vdash \mathbf{IVT}$. Assume BIM $\vdash \mathbf{IVT}\to \mathbf{FT}$. Then BIM+ $X\vee\neg X\vdash \mathbf{FT}$. As we know from Theorem 2.4, BIM+ $\mathbf{CT}\vdash\neg !\mathbf{FT}$, and, therefore,

 $BIM + CT \vdash \neg FT$. Conclude: $BIM + IVT \nvdash FT$, and also, in view of Theorem 2.5, $BIM + IVT \nvdash WKL$.

Note that, in view of Corollary 10.3, this gives another proof of BIM $\not\vdash \Pi_1^0$ -AC_{ω ,2}, a fact established in Subsection 4.8.

One may even conclude: BIM + IVT $\nvdash \Pi_1^0$ -AC_{ω ,2}.

One may ask if $BIM + LLPO \vdash IVT$, i.e. if the proof of Theorem 10.2 can be given without recourse to Π_1^0 -AC_{ω , 2}, but we do not know the answer to this question.

10.4. The Uniform Intermediate Value Theorem, **UIVT**:

For each φ such that $\forall n[\varphi^n \in \mathcal{R}^{[0,1]}],$ $if \, \forall n \exists \gamma \in [0,1]^2 [(\varphi^n)^{\cdot \mathcal{R}} (\gamma^0) \leq_{\mathcal{R}} 0_{\mathcal{R}} \leq_{\mathcal{R}} (\varphi^n)^{\cdot \mathcal{R}} (\gamma^1)],$ $then \, \exists \gamma \in [0,1]^{\omega} \forall n [(\varphi^n)^{\cdot \mathcal{R}} (\gamma^n) =_{\mathcal{R}} 0_{\mathcal{R}}].$

In [26, Exercise IV.2.12, page 137], the reader is asked to prove that, in the classical system RCA_0 , UIVT is an equivalent of WKL. As $RCA_0 \vdash IVT$, RCA_0 proves the equivalence of **UIVT** and:

10.5. **Uzero**:

For all φ such that $\forall n [\varphi^n \in \mathcal{R}^{[0,1]}],$ if $\forall n \exists \gamma \in [0,1] [(\varphi^n)^{\prime \mathcal{R}}(\gamma) =_{\mathcal{R}} 0_{\mathcal{R}}]$, then $\exists \gamma \in [0,1]^{\omega} \forall n [(\varphi^n)^{\prime \mathcal{R}}(\gamma^n) =_{\mathcal{R}} 0_{\mathcal{R}}]$.

We want to study **Uzero** in BIM. We need the following Lemma.

Lemma 10.5. BIM proves:

- (i) $\exists \psi : \omega^{\omega} \to \omega^{\omega} \forall \varphi \in \mathcal{R}^{[0,1]}[\mathcal{H}_{\psi|\varphi} = \{ \gamma \in [0,1] \mid \varphi^{\mathcal{R}}(\gamma) \#_{\mathcal{R}} 0_{\mathcal{R}} \}], \text{ and}$ (ii) $\exists \tau : \omega^{\omega} \to \omega^{\omega} \forall \alpha [\tau | \alpha \in \mathcal{R}^{[0,1]} \land \mathcal{H}_{\alpha} = \{ \gamma \in [0,1] \mid (\tau | \alpha)^{\mathcal{R}}(\gamma) \#_{\mathcal{R}} 0_{\mathcal{R}} \}].$

Proof. Define $\psi:\omega^{\omega}\to\omega^{\omega}$ such that, for each φ , for each s,

$$(\psi|\varphi)(s) \neq 0 \leftrightarrow \exists p \in E_{\overline{\varphi}s}[s \sqsubseteq_{\mathbb{S}} p' \land (0_{\mathbb{Q}} <_{\mathbb{Q}} (p'')' \lor (p'')'' <_{\mathbb{Q}} 0_{\mathbb{Q}})].$$

Note: $\forall \varphi \in \mathcal{R}^{[0,1]} \forall \gamma \in [0,1] [\varphi^{\mathcal{R}}(\gamma) \#_{\mathcal{R}} 0_{\mathcal{R}} \leftrightarrow \gamma \in \mathcal{H}_{\psi|\varphi}].$

Now define ρ such that, for each s in $\mathbb{S}, \, \rho^s \in \mathcal{R}^{[0,1]}$ and, if not $(-1)_{\mathbb{Q}} \leq_{\mathbb{Q}} s' \leq_{\mathbb{Q}} s'' \leq_{\mathbb{Q}} (2)_{\mathbb{Q}}$, then, for all γ in [0,1], $(\rho^s)^{i\mathcal{R}}(\gamma) =_{\mathcal{R}} 0_{\mathcal{R}}$, and, if $(-1)_{\mathbb{Q}} \leq_{\mathbb{Q}} s' \leq_{\mathbb{Q}} s'' \leq_{\mathbb{Q}} (2)_{\mathbb{Q}}$, then, for all γ in [0,1],

- (1) if $\gamma \leq_{\mathcal{R}} (s')_{\mathcal{R}}$ or $(s'')_{\mathcal{R}} \leq_{\mathcal{R}} \gamma$, then $\rho^s(\gamma) =_{\mathcal{R}} 0_{\mathcal{R}}$, and,
- (2) if $(s')_{\mathcal{R}} \leq_{\mathcal{R}} \gamma \leq_{\mathcal{R}} (s'')_{\mathcal{R}}$, then $\rho^s(\gamma) =_{\mathcal{R}} \inf(\gamma -_{\mathcal{R}} (s')_{\mathcal{R}}, (s'')_{\mathcal{R}} -_{\mathcal{R}} \gamma)$.

(Note that ρ^s codes the restriction to [0,1] of the 'tent' function from \mathcal{R} tot \mathcal{R} that is zero outside of [s',s''], linear on both $[s',\frac{s'+s''}{2}]$ and $[\frac{s'+s''}{2},s'']$ and takes the values $0_{\mathcal{R}}, \frac{s''-s'}{2}, 0_{\mathcal{R}}$ at $s', \frac{s'+s''}{2}, s''$, respectively. Note that, for all s in \mathbb{S} , for all γ in $[0,1], (\rho^s)^{i_{\mathcal{R}}}(\gamma) \leq_{\mathcal{R}} (\frac{3}{2})_{\mathcal{R}}$.

Define $\tau:\omega^{\omega}\to\omega^{\omega}$ such that, for all $\alpha,\,\tau|\alpha\in\mathcal{R}^{[0,1]}$ and, for each γ in [0,1], $(\tau | \alpha)^{\prime \mathcal{R}}(\gamma) =_{\mathcal{R}} \sum_{s, \alpha(s) \neq 0} (\frac{1}{2^{s}})_{\mathcal{R}} \cdot_{\mathcal{R}} (\rho^{s})^{\prime \mathcal{R}}(\gamma).$

Note:
$$\forall \gamma \in [0,1] [\gamma \in \mathcal{H}_{\alpha} \leftrightarrow (\tau | \alpha)^{\prime \mathcal{R}} (\gamma) \#_{\mathcal{R}} 0_{\mathcal{R}}], \text{ and:}$$
 $\forall \gamma \in [0,1] [\gamma \notin \mathcal{H}_{\alpha} \leftrightarrow (\tau | \alpha)^{\prime \mathcal{R}} (\gamma) =_{\mathcal{R}} 0_{\mathcal{R}}].$

Theorem 10.6. BIM $\vdash \Pi_1^0$ -AC_{$\omega,[0,1]$} \leftrightarrow Uzero.

Proof. First, assume Π_1^0 - $\mathbf{AC}_{\omega,[0,1]}$.

Let φ be given such that $\forall n[\varphi^n \in \mathcal{R}^{[0,1]} \land \exists \gamma \in [0,1][(\varphi^n)^{\mathcal{R}}(\gamma) =_{\mathcal{R}} 0_{\mathcal{R}}]].$ Using Lemma 10.5(i), find α such that, $\forall n[\mathcal{H}_{\alpha^n} = \{ \gamma \in [0,1] \mid (\varphi^n)^{\mathcal{R}}(\gamma) \#_{\mathcal{R}} 0_{\mathcal{R}} \}].$ Conclude: $\forall n \exists \gamma \in [0,1] [\gamma \notin \mathcal{H}_{\alpha^n}]$ and, by Π_1^0 - $\mathbf{AC}_{\omega,[0,1]}$: $\exists \gamma \in [0,1]^\omega \forall n [\gamma^n \notin \mathcal{H}_{\alpha^n}]$ and: $\exists \gamma \in [0,1]^{\omega} \forall n [(\varphi^n)^{\mathcal{R}}(\gamma^n)] =_{\mathcal{R}} 0_{\mathcal{R}}$].

Conclude: Uzero.

Secondly, assume **Uzero**.

Let α be given such that $\forall n \exists \gamma \in [0,1] [\gamma \notin \mathcal{H}_{\alpha^n}]$. Using Lemma 10.5(ii), find φ such that $\forall n[\varphi^n \in \mathcal{R}^{[0,1]}]$ and $\forall n[\mathcal{H}_{\alpha^n} = \{\gamma \in [0,1] \mid (\varphi^n)^{\mathcal{R}}(\gamma) \#_{\mathcal{R}} 0_{\mathcal{R}}\}].$ Conclude: $\forall n \exists \gamma \in [0, 1] [(\varphi^n)^{\mathcal{R}}(\gamma) =_{\mathcal{R}} 0_{\mathcal{R}}], \text{ and } \exists \gamma \in [0, 1]^{\omega} \forall n [(\varphi^n)^{\mathcal{R}}(\gamma^n) =_{\mathcal{R}} 0_{\mathcal{R}}],$ and $\exists \gamma \in [0,1]^{\omega} \forall n [\gamma^n \notin \mathcal{H}_{\alpha^n}].$

Conclude: Π_1^0 - $\mathbf{AC}_{\omega,[0,1]}$. П

10.6. A uniform contrapositive Intermediate Value Theorem $\mathbf{\dot{U}IVT}$: For each φ such that $\forall n[\varphi^n \in \mathcal{R}^{[0,1]}]$, if $\forall \gamma \in [0,1]^{\omega} \exists n[(\varphi^n)^{\mathcal{R}}(\gamma^n) \#_{\mathcal{R}} 0_{\mathcal{R}}]$, then $\exists n[\forall \gamma \in [0,1][(\varphi^n)^{\mathcal{R}}(\gamma) >_{\mathcal{R}} 0_{\mathcal{R}}] \lor \forall \gamma \in [0,1][(\varphi^n)^{\mathcal{R}}(\gamma) <_{\mathcal{R}} 0_{\mathcal{R}}]]$.

As $BIM \vdash \overleftarrow{IVT}$, BIM proves the equivalence of \overleftarrow{UIVT} and:

10.7. **Uzero**:

For all φ such that $\forall n [\varphi^n \in \mathcal{R}^{[0,1]}],$ if $\forall \gamma \in [0,1]^{\omega} \exists n [(\varphi^n)^{\mathcal{R}}(\gamma^n) \#_{\mathcal{R}} 0_{\mathcal{R}}], \text{ then } \exists n \forall \gamma \in [0,1] [(\varphi^n)^{\mathcal{R}}(\gamma) \#_{\mathcal{R}} 0_{\mathcal{R}}].$

We define a strong negation of this statement. This strong negation itself contains a negation sign, a possibility mentioned in Subsection 1.4.

10.8. ¬!**Uzero**:

There exists φ such that $\forall n[\varphi^n \in \mathcal{R}^{[0,1]}]$ and $\forall \gamma \in [0,1]^{\omega} \exists n[(\varphi^n)^{\mathcal{R}}(\gamma^n) \#_{\mathcal{R}} 0_{\mathcal{R}}]$ and $\neg \exists n \forall \gamma \in [0,1][(\varphi^n)^{\mathcal{R}}(\gamma) \#_{\mathcal{R}} 0_{\mathcal{R}}].$

Lemma 10.7. One may prove in BIM:

(i)
$$\Sigma_1^0 - \overleftarrow{\mathbf{AC}}_{\omega,[0,1]} \to \overleftarrow{\mathbf{Uzero}}$$
 and $\neg ! \overleftarrow{\mathbf{Uzero}} \to \neg ! \Sigma_1^0 - \overleftarrow{\mathbf{AC}}_{\omega,[0,1]}$.
(ii) $\overleftarrow{\mathbf{Uzero}} \to \Sigma_1^0 - \overleftarrow{\mathbf{AC}}_{\omega,[0,1]}$ and $\neg ! \Sigma_1^0 - \overleftarrow{\mathbf{AC}}_{\omega,[0,1]} \to \neg ! \overleftarrow{\mathbf{Uzero}}$

Proof. (i) We prove, in BIM: for all φ such that $\forall n[\varphi^n \in \mathcal{R}^{[0,1]}]$, there exists β such that

$$\forall \gamma \in [0,1]^{\omega} \exists n [(\varphi^n)^{\hat{}\mathcal{R}}(\gamma^n) \#_{\mathcal{R}} 0_{\mathcal{R}}] \to \forall \gamma \in [0,1]^{\omega} \exists n [\gamma^n \in \mathcal{H}_{\beta^n}] \text{ and}$$
$$\exists n [[0,1] \subseteq \mathcal{H}_{\beta^n}] \to \exists n \forall \gamma \in [0,1] [(\varphi^n)^{\hat{}\mathcal{R}}(\gamma) \#_{\mathcal{R}} 0_{\mathcal{R}}].$$

The two promised conclusions then follow easily.

Let φ be given such that $\forall n[\varphi^n \in \mathcal{R}^{[0,1]}]$. Using Lemma 10.5(i) find β such that $\forall n[\mathcal{H}_{\beta^n} = \{ \gamma \in [0,1] \mid (\varphi^n)^{\mathcal{R}}(\gamma) \#_{\mathcal{R}} 0_{\mathcal{R}} \}].$

Assume $\forall \gamma \in [0,1]^{\omega} \exists n[(\varphi^n)^{\mathcal{R}}(\gamma^n) \#_{\mathcal{R}} 0_{\mathcal{R}}]].$

Conclude: $\forall \gamma \in [0,1]^{\omega} \exists n [\gamma^n \in \mathcal{H}_{\beta^n}].$

Now let n be given such that $[0,1] \subseteq \mathcal{H}_{\beta^n}$. Clearly, $\forall \gamma \in [0,1][(\varphi^n)^{\prime \mathcal{R}}(\gamma) \#_{\mathcal{R}} 0_{\mathcal{R}}]$.

(ii) We prove, in BIM: for each α , there exists φ such that $\forall n [\varphi^n \in \mathcal{R}^{[0,1]}]$ and

$$\forall \gamma \in [0,1]^{\omega} \exists n [\gamma^n \in \mathcal{H}_{\alpha^n}] \to \forall \gamma \in [0,1]^{\omega} \exists n [(\varphi^n)^{{}^{\iota}\mathcal{R}}(\gamma^n) \ \#_{\mathcal{R}} \ 0_{\mathcal{R}}] \text{ and }$$

$$\exists n \forall \gamma \in [0,1][(\varphi^n)^{\mathcal{R}}(\gamma) \#_{\mathcal{R}} 0_{\mathcal{R}}]) \to \exists n[[0,1] \subseteq \mathcal{H}_{\alpha^n}].$$

The two promised conclusions then follow easily.

Let α be given. Using Lemma 10.5(ii), find φ such that

 $\forall n [\varphi^n \in \mathcal{R}^{[0,1]} \land \mathcal{H}_{\alpha^n} = \{ \gamma \in [0,1] \mid (\varphi^n)^{\cdot \mathcal{R}}(\gamma) \not\#_{\mathcal{R}} 0_{\mathcal{R}} \} \}.$ Assume: $\forall \gamma \in [0,1]^{\omega} \exists n [\gamma^n \in \mathcal{H}_{\alpha^n}]. \text{ Conclude: } \forall \gamma \in [0,1]^{\omega} \exists n [(\varphi^n)^{\cdot \mathcal{R}}(\gamma^n) \not\#_{\mathcal{R}} 0_{\mathcal{R}}].$ Now assume: $\exists n \forall \gamma \in [0,1] [(\varphi^n)^{\cdot \mathcal{R}}(\gamma) \not\#_{\mathcal{R}} 0_{\mathcal{R}}]. \text{ Conclude: } \exists n [[0,1] \subseteq \mathcal{H}_{\alpha^n}]. \quad \Box$

Theorem 10.8. BIM proves $\overleftarrow{\mathbf{Uzero}} \leftrightarrow \overleftarrow{\mathbf{UIVT}} \leftrightarrow \mathbf{FT}$ and $\neg !\dot{\mathbf{U}}\mathbf{zero} \leftrightarrow \neg !\dot{\mathbf{U}}\mathbf{I}\mathbf{V}\mathbf{T} \leftrightarrow \neg !\mathbf{F}\mathbf{T}.$

Proof. Use Lemma 10.7 and Theorem 7.2.

11. The compactness of classical propositional logic

In this Section, we prove that \mathbf{FT} is equivalent to a contraposition of a restricted version of the compactness theorem for classical propositional logic. We prove also the corresponding result for $\neg !\mathbf{FT}$.

We introduce the symbols \neg , \bigwedge and \bigvee as natural numbers: $\neg := 1$, $\bigwedge := 2$ and $\bigvee := 3$. We define (the characteristic function of) $Form \subseteq \omega$, as follows, by recursion: for each $n, n \in Form$ if and only if

$$n' = 0 \lor (n' = \neg \land n'' \in Form) \lor$$

$$\big((n' = \bigwedge \vee n' = \bigvee) \wedge \forall i < length(n'')[n''(i) \in Form]\big).$$

We define $\top := (\bigwedge, \langle \rangle)$ and $\bot := (\bigvee, \langle \rangle)$.

Assume $\gamma \in \mathcal{C}$. We define $\tilde{\gamma}$ in \mathcal{C} such that, for every n in Form,

- (i) if $n \notin Form$, then $\tilde{\gamma}(n) = 0$, and,
- (ii) if $n \in Form$ and n' = 0, then $\tilde{\gamma}(n) = \gamma(n'')$, and,
- (iii) if $n \in Form$ and $n' = \neg$, then $\tilde{\gamma}(n) = 1 \tilde{\gamma}(n'')$, and,
- (iv) if $n \in Form$ and $n' = \bigwedge$, then $\tilde{\gamma}(n) = \min\{\tilde{\gamma}(n''(i))|i < length(n'')\}$, and
- (v) if $n \in Form$ and $n' = \bigvee$, then $\tilde{\gamma}(n) = \max{\{\tilde{\gamma}(n''(i)) | i < length(n'')\}}$.

Note: $0 = \langle \rangle$. We define $\min(\emptyset) = 1$ and $\max(\emptyset) = 0$.

Note: $\tilde{\gamma}(\top) = \tilde{\gamma}((\wedge, 0)) = 1$ and $\tilde{\gamma}(\bot) = \tilde{\gamma}((\vee, 0)) = 0$.

For all m, n in Form, we define: $m \equiv n$ if and only if $\forall \gamma \in \mathcal{C}[\tilde{\gamma}(m) = \tilde{\gamma}(n)]$.

Assume $c \in Bin$. We define \tilde{c} in Bin such that $length(c) = length(\tilde{c})$, as follows. First, define $\gamma = c * \underline{0}$. Then define, for all m < length(c), $\tilde{c}(m) := \tilde{\gamma}(m)$.

 $X \subseteq \omega$ is realizable, Real(X), if and only if $\exists \gamma \in \mathcal{C} \forall n \in X[\tilde{\gamma}(n) = 1]$, and positively unrealizable, Unreal(X), if and only if $\forall \gamma \in \mathcal{C} \exists n \in X[\tilde{\gamma}(n) = 0]$.

We define a mapping Fm from Bin to Form, as follows. Assume $a \in Bin$. Find s such that length(s) = length(a), and, for all i < length(a), if a(i) = 0, then $s(i) = (\neg, (0, i))$, and, if a(i) = 1, then s(i) = (0, i). Define $Fm(a) = (\bigwedge, s)$.

Lemma 11.1.

- (i) $\forall a \in Bin \forall \gamma \in \mathcal{C}[\tilde{\gamma}(Fm(a)) = 1 \leftrightarrow a \sqsubset \gamma].$
- (ii) There exists β such that

 $\forall m \in Form \forall p [\beta(m, p)) > p \land \beta((m, p)) \in Form \land \beta((m, p)) \equiv m].$

(iii) ¹⁹ For all α in \mathcal{C} , there exists δ in $[\omega]^{\omega}$ such that $\forall m[\delta(m) \in Form \land \forall \gamma \in \mathcal{C}[\tilde{\gamma}(\delta(m)) = 1 \leftrightarrow \forall n \leq m[\alpha(\overline{\gamma}n) = 0]]].$

Proof. (i) We prove, by induction:

for each $n, \forall a \in Bin_n \forall \gamma \in \mathcal{C}[\tilde{\gamma}(Fm(a)) = 1 \leftrightarrow a \sqsubset \gamma].$

First, note $Fm(\langle \rangle) = \top$ and: $\forall \gamma \in \mathcal{C}[\tilde{\gamma}(\top) = 1]$ and $\forall \gamma \in \mathcal{C}[\langle \rangle \sqsubset \gamma]$.

Then, let a, n be given such that $a \in Bin_n$ and $\forall \gamma \in \mathcal{C}[\tilde{\gamma}(Fm(a)) = 1 \leftrightarrow a \sqsubseteq \gamma]$. Note that for each γ in \mathcal{C} , for each i < 2,

 $\tilde{\gamma}\big(Fm(a*\langle i\rangle)\big) = 1 \leftrightarrow \big(\tilde{\gamma}\big(Fm(a)\big) = 1 \ \land \ \gamma(n) = i\big) \leftrightarrow a*\langle i\rangle \sqsubset \gamma.$

- (ii) The proof is an exercise in calculating codes of formulas. Given m in Form and p, one might first find $q := \max(m, p)$ and then s in ω^{q+1} such that s(0) = m and $\forall j < q[s(j) = \top]$ and then define $\beta((m, p)) := (\bigwedge, s)$.
 - (iii) Let α be given. We define the promised α by induction.

If $\alpha(\langle \rangle) = 0$, define $\delta(0) := \top$, and, if $\alpha(\langle \rangle) \neq 0$, define $\delta(0) := \bot$.

Note that $\delta(0)$ satisfies the requirements.

Let m be given such that $\delta(m)$ has been defined and satisfies the requirements. Find t such that $\{t(i) \mid i < length(t)\} = \{a \in Bin_{m+1} \mid \forall n \leq m+1[\alpha(\overline{a}n)=0]\}$. Then find s such that length(s) = length(t) and $\forall i < length(s)[s(i) = Fm(t(i))]$.

 $^{^{19}[\}omega]^{\omega}=\{\zeta\mid \forall n[\zeta(n)<\zeta(n+1)]\},$ see Subsection 13.1.

Note that, for each γ in \mathcal{C} , $\tilde{\gamma}((\bigvee, s)) = 1 \leftrightarrow$

 $\exists a \in Bin_{m+1} [\forall n \leq m[\alpha(\overline{a}n) = 0] \land \tilde{\gamma}(Fm(a)) = 1] \leftrightarrow$

 $\exists a \in Bin_{m+1} [\forall n \le m[\alpha(\overline{a}n) = 0] \land a \sqsubset \gamma] \leftrightarrow \forall n \le m[\alpha(\overline{\gamma}n) = 0].$

Define $\delta(m+1) = \beta((\bigvee, s), \delta(m))$, where β is the function we found in (ii). Note that $\delta(m+1)$ satisfies the requirements.

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Lemma 11.2. The following statements are provable in BIM.

- (i) $\mathbf{FT} \to \forall \alpha [Unreal(E_{\alpha}) \to \exists n [Unreal(E_{\overline{\alpha}n})]].$
- (ii) $\exists \alpha [Unreal(E_{\alpha}) \land \forall n[Real(E_{\overline{\alpha}n})]] \rightarrow \neg !\mathbf{FT}.$
- (iii) $\mathbf{WKL} \to \forall \alpha [\forall n [Real(E_{\overline{\alpha}n})] \to Real(E_{\alpha})]].$
- (iv) $\forall \alpha[Unreal(D_{\alpha}) \rightarrow \exists n[Unreal(D_{\overline{\alpha}n})]] \rightarrow \mathbf{FT}.$
- (v) $\neg ! \mathbf{FT} \to \exists \alpha [Unreal(D_{\alpha}) \land \forall n [Real(D_{\overline{\alpha}n})]].$
- (vi) $\forall \alpha [\forall n [Real(D_{\overline{\alpha}n})] \rightarrow Real(D_{\alpha})]] \rightarrow \mathbf{WKL}.$

Proof. (i), (ii) and (iii). We argue in BIM.

Let α be given. Define β such that, for all m, for all c in Bin_m ,

 $\beta(c) \neq 0 \leftrightarrow \exists n < m[n \in E_{\overline{\alpha}m} \land \tilde{c}(n) = 0]].$ We shall prove:

$$Unreal(E_{\alpha}) \to Bar_{\mathcal{C}}(D_{\beta})$$
 and

$$\exists m[Bar_{\mathcal{C}}(D_{\overline{\beta}m})] \rightarrow \exists n[Unreal(E_{\overline{\alpha}n})].$$

Assume: $Unreal(E_{\alpha})$. Let γ in \mathcal{C} be given. Find n, p such that $n \in E_{\overline{\alpha}p}$ and $\tilde{\gamma}(n) = 0$. Define $m := \max\{n, p\} + 1$ and note: $\beta(\overline{\gamma}m) \neq 0$. Conclude: $Bar_{\mathcal{C}}(D_{\beta})$.

Let m be given such that $Bar_{\mathcal{C}}(D_{\overline{\beta}m})$. For all c in Bin_m , $\exists n \leq m[\beta(\overline{c}n) \neq 0]$, and, therefore: $\exists n < length(c)[n \in E_{\overline{\alpha}m} \land \tilde{c}(n) = 0]$. Conclude: $Unreal(E_{\overline{\alpha}m})$ and $\exists n[Unreal(E_{\overline{\alpha}n})]$.

Note: if $Unreal(E_{\alpha})$, then $Bar_{\mathcal{C}}(D_{\beta})$, and by **FT**, there exist m such that $Bar_{\mathcal{C}}(D_{\overline{\beta}m})$ and n such that $Unreal(E_{\overline{\alpha}n})$. This establishes (i).

Note: if $Unreal(E_{\alpha}) \wedge \forall n[Real(E_{\overline{\alpha}n}], \text{ then } Bar_{\mathcal{C}}(D_{\beta}) \text{ and } \forall m[\neg Bar(D_{\overline{\beta}m})],$ i.e. $\neg !\mathbf{FT}$. This establishes (ii).

Note: if $\forall n[Real(E_{\overline{\alpha}n})]$, then $\forall m[\neg Bar(D_{\overline{\beta}m})]$, and, by **WKL**, there exists γ such that $\forall n[\beta(\overline{\gamma}n)=0]$. Conclude: $\forall m \forall n < m[n \in E_{\overline{\alpha}m} \to \tilde{\gamma}(n)=1]$, i.e. γ realizes E_{α} and $Real(E_{\alpha})$. This establishes (iii).

(iv), (v) and (vi). We argue in BIM.

Let α be given. Using Lemma 11.1(iii), find δ in $[\omega]^{\omega}$ such that

 $\forall m[\delta(m) \in Form \land \forall \gamma \in \mathcal{C}[\tilde{\gamma}(\delta(m)) = 1] \leftrightarrow \forall n \leq m[\alpha(\overline{\gamma}n) = 0]]].$

Note: δ is strictly increasing and, therefore,

 $\forall m[\exists n[m = \delta(n)] \leftrightarrow \exists n \le m[m = \delta(n)]].$

Define β such that $\forall m[\beta(m) \neq 0 \leftrightarrow \exists n[m = \delta(n)]]$. We shall prove:

$$Bar_{\mathcal{C}}(D_{\alpha}) \to Unreal(D_{\beta})$$
 and

$$\exists n[Unreal(D_{\overline{\alpha}n})] \rightarrow \exists m[Bar_{\mathcal{C}}(D_{\overline{\alpha}m})].$$

Assume: $Bar_{\mathcal{C}}(D_{\alpha})$. Given any $\gamma \in \mathcal{C}$, find m such that $\alpha(\overline{\gamma}m) \neq 0$ and, therefore, $\tilde{\gamma}(\delta(m)) = 0$ and: $\exists n \in D_{\beta}[\tilde{\gamma}(n) \neq 1]$. Conclude: $Unreal(D_{\beta})$.

Let n be given such that $Unreal(D_{\overline{\beta}n})$. Let m_0 be the largest m such that $\delta(m) < n$. Note: $\neg \exists \gamma \in \mathcal{C}[\tilde{\gamma}(\delta(m_0)) = 1]$, and, therefore,

 $\forall a \in Bin_{m_0+1} \exists n \leq m_0[\alpha(\overline{a}n) \neq 0].$ Find k such that $\forall a \in Bin_{m_0+1}[a \leq k]$ and conclude: $Bar_{\mathcal{C}}(D_{\overline{\alpha}k})$ and $\exists m[Bar_{\mathcal{C}}(D_{\overline{\alpha}m})].$

Note: if $Bar_{\mathcal{C}}(D_{\alpha})$ and $Unreal(D_{\beta}) \to \exists n[Unreal(D_{\overline{\beta}n})]$, then $\exists m[Bar_{\mathcal{C}}(D_{\overline{\alpha}m})]$. This establishes (iv).

Note: if $Bar_{\mathcal{C}}(D_{\alpha})$ and $\neg \exists n[Bar_{\mathcal{C}}(D_{\overline{\alpha}n}], \text{ then } Unreal(D_{\beta}) \text{ and } \forall n[Real(D_{\overline{\beta}n})].$ This establishes (v).

Note: if $\forall n[\neg Bar_{\mathcal{C}}(D_{\overline{\alpha}n}] \text{ and } \forall n[Real(D_{\overline{\beta}n})] \rightarrow Real(D_{\beta}), \text{ then there exists } \gamma \text{ in }$ \mathcal{C} realizing D_{β} , so $\forall m[\tilde{\gamma}(\delta(m)) = 1]$ and $\forall m \forall n \leq m[\alpha(\overline{\gamma}n) = 0]$, i.e. $\forall n[\alpha(\overline{\gamma}n) = 0]$. This establishes (vi).

Theorem 11.3. (i) BIM \vdash **FT** $\leftrightarrow \forall \alpha [Unreal(E_{\alpha}) \rightarrow \exists n [Unreal(E_{\overline{\alpha}n})]] \leftrightarrow$ $\forall \alpha [Unreal(D_{\alpha}) \rightarrow \exists n [Unreal(D_{\overline{\alpha}n})]].$

- (ii) BIM $\vdash \neg ! \mathbf{FT} \leftrightarrow \exists \alpha [Unreal(E_{\alpha}) \land \forall n [Real(E_{\overline{\alpha}n})]] \leftrightarrow$ $\exists \alpha [Unreal(D_{\alpha}) \land \forall n [Real(D_{\overline{\alpha}n})]].$
- (iii) BIM \vdash WKL $\leftrightarrow \forall \alpha [\forall n[Real(E_{\overline{\alpha}n})] \rightarrow Real(E_{\alpha})] \leftrightarrow$ $\forall \alpha [\forall n [Real(D_{\overline{\alpha}n})] \to Real(D_{\alpha})].$

Proof. Use Lemma 11.2 and: $\forall \alpha \exists \beta [D_{\alpha} = E_{\beta}].$

Theorem 11.3(i) also is a consequence of [19, Theorem 6.5].

V.N. Krivtsov has shown, among other things, that **FT** is an equivalent of an intuitionistic (generalized) completeness theorem for intuitionistic first-order predicate logic, see [17].

12. STRONGER 'FAN THEOREMS'?

In this Section, we indicate what, on our opinion, should be the subject of the next chapter in intuitionistic reverse mathematics. The Fan Theorem may be seen as a replacement, for the intuitionistic mathematician, of that enviable tool of the classical mathematician: (Weak) König's Lemma. We hope to make clear that the greater subtlety of the language of the intuitionistic mathematician allows for many other possible replacements.

12.1. Notions of finiteness. Let α be given.

We consider the set $D_{\alpha} := \{n \mid \alpha(n) \neq 0\}$, the subset of ω decided by α .

We define: D_{α} is finite if and only if $\exists n \forall m \geq n [\alpha(m) = 0]$.

We define: D_{α} is bounded-in-number if and only if

 $\exists n \forall t \in [\omega]^{n+1} \exists i < n+1 [\alpha \circ t(i) = 0].$

We define: D_{α} is almost-finite if and only if $\forall \zeta \in [\omega]^{\omega} \exists n [\alpha \circ \zeta(n) = 0]$.

We define: D_{α} is not-not-finite if and only if $\neg \neg \exists n \forall m \geq n [\alpha(m) = 0]$.

We define: D_{α} is $not\text{-}infinite^{20}$ if and only if $\neg \forall n \exists m \geq n[\alpha(m) \neq 0]$. Note that D_{α} is infinite if and only if $\exists \zeta \in [\omega]^{\omega} \forall n[\alpha \circ \zeta(n) \neq 0]$, and that

 D_{α} is not-infinite if and only if $\neg \exists \zeta \in [\omega]^{\omega} \forall n [\alpha \circ \zeta(n) \neq 0]$.

Decidable subsets of ω that are bounded-in-number are introduced and discussed in [33].

Almost-finite decidable subsets of ω were introduced in [33] and [34], and are also studied in [38].

Lemma 12.1. (i) BIM $\vdash \forall \alpha [D_{\alpha} \text{ is finite } \rightarrow D_{\alpha} \text{ is bounded-in-number}].$

- (ii) BIM $\vdash \forall \alpha [D_{\alpha} \ is \ bounded\text{-}in\text{-}number \rightarrow D_{\alpha} \ is \ finite] \rightarrow \mathbf{LPO}.$
- (iii) BIM $\vdash \forall \alpha [D_{\alpha} \ is \ bounded\ -in\ -number \rightarrow D_{\alpha} \ is \ almost\ -finite].$
- (iv) BIM $\vdash \forall \alpha [D_{\alpha} \ is \ almost \ finite \rightarrow D_{\alpha} \ is \ bounded \ in \ -number] \rightarrow \mathbf{LPO}$.
- $\text{(v) BIM} \vdash \forall \alpha [D_\alpha \ is \ almost\mbox{-}finite \ \rightarrow \ D_\alpha \ is \ not\mbox{-}infinite].$
- (vi) $\mathsf{BIM} + \mathbf{BARIND}^{21} \vdash \forall \alpha [D_{\alpha} \ is \ almost-finite \rightarrow D_{\alpha} \ is \ not-not-finite].$

Proof. (i) Let α, n be given such that $\forall m \geq n[\alpha(m) = 0]$.

Note: $\forall t \in [\omega]^{n+1}[t(n) \ge n]$ and conclude: $\forall t \in [\omega]^{n+1}[\alpha \circ t(n) = 0]$.

(ii) Let α be given. Define α^* such that $\forall n[\alpha^*(n) \neq 0 \leftrightarrow n = \mu m[\alpha(m) \neq 0]]$.

 $^{^{20}\}mathrm{A}$ referee suggested to consider this notion too.

²¹See 2.2.6. We do not know if the use of this priniple here is unavoidable.

Note: $\forall t \in [\omega]^2 \exists i < 2[\alpha^* \circ t(i) = 0]$, so D_{α^*} is bounded-in-number.

Assuming that D_{α^*} is finite, find n such that $\forall m \geq n[\alpha^*(m) = 0]$.

Either $\exists m < n[\alpha^*(m) \neq 0]$ or $\forall m[\alpha^*(m) = 0]$, and, therefore,

either $\exists m[\alpha(m) \neq 0]$ or $\forall m[\alpha(m) = 0]$.

We thus see how, starting from the assumption:

 $\forall \alpha[D_{\alpha} \ is \ bounded\text{-}in\text{-}number \rightarrow D_{\alpha} \ is \ finite], \text{ one obtains the conclusion:} \ \forall \alpha[\exists m[\alpha(m) \neq 0] \ \lor \ \forall m[\alpha(m) = 0]], \text{ i.e. } \mathbf{LPO}.$

(iii) Let α, n be given such that $\forall t \in [\omega]^{n+1} \exists i < n+1 [\alpha \circ t(i)=0].$

Conclude: $\forall \zeta \in [\omega]^{\omega} \exists i < n+1 [\alpha \circ \zeta(i)=0].$

(iv) Let α be given. Define α^* such that

 $\forall n[\alpha^*(n) \neq 0 \leftrightarrow \mu m[\alpha(m) \neq 0] \leq n < 2 \cdot \mu m[\alpha(m) \neq 0]].$

Observe: for all k, if $k = \mu[\alpha(m) \neq 0]$, then $\forall n[k \leq n < 2 \cdot k \leftrightarrow \alpha^*(n) \neq 0]$ and $\exists t \in [\omega]^k \forall i < k[\alpha^* \circ t(i) \neq 0]$ and $\forall t \in [\omega]^{k+1} \exists i < k+1[\alpha^* \circ t(i) = 0]$ Let ζ in $[\omega]^{\omega}$ be given.

We want to prove; $\exists n[\alpha^* \circ \zeta(n) = 0]$ and and distinguish two cases.

Case (a). $\alpha^* \circ \zeta(0) = 0$. Then we are done.

Case (b). $\alpha^* \circ \zeta(0) \neq 0$. Then $\exists m[\alpha(m) \neq 0]$. Define $k := \mu m[\alpha(m) \neq 0]$ and note: $\forall m \geq 2 \cdot k[\alpha^*(m) = 0]$, and, in particular, $\alpha^* \circ \zeta(2 \cdot k) = 0$.

We thus see: $\forall \zeta \in [\omega]^{\omega} \exists n [\alpha^* \circ \zeta(n) = 0]$, i.e. D_{α^*} is almost-finite.

Assuming that D_{α^*} is bounded-in-number, find n such that

 $\forall t \in [\omega]^{n+1} \exists i < n+1 [\alpha^* \circ t(i) = 0].$

Conclude: for all k, if $k = \mu m[\alpha(m) \neq 0]$, then k < n + 1.

Either $\exists k < n+1 [\alpha \neq 0]$ or $\forall k < n+1 [\alpha(k)=0]$, and, therefore,

either $\exists k[\alpha(k) \neq 0]$ or $\forall k[\alpha(k) = 0]$.

We thus see how, starting from the assumption:

 $\forall \alpha [D_{\alpha} \ is \ almost\mbox{-}finite \rightarrow D_{\alpha} \ is \ bounded\mbox{-}in\mbox{-}number], one obtains the conclusion:$ $<math>\forall \alpha [\exists k [\alpha(k) \neq 0] \ \lor \ \forall k [\alpha(k) = 0]], \ i.e. \ \mathbf{LPO}.$

(v) Let α be given such that $\forall \zeta \in [\omega]^{\omega} \exists n [\alpha \circ \zeta(n) = 0]$.

Define $B := \bigcup_n \{ s \in \omega^n \mid s \notin [\omega]^n \lor \exists i < n[\alpha \circ s(i) = 0] \}.$

We prove: $Bar_{\omega^{\omega}}(B)$.

Let γ be given. Define γ^* such that $\gamma^*(0) = \gamma(0)$, and, for each n, if $\overline{\gamma}(n+2) \in [\omega]^{<\omega}$, then $\gamma^*(n+1) = \gamma(n+1)$, and, if not, then $\gamma^*(n+1) = \gamma^*(n) + 1$. Note: $\gamma^* \in [\omega]^{\omega}$ and find n such that $\overline{\gamma^*n} \in B$. Either $\overline{\gamma}n = \overline{\gamma^*n}$ or $\overline{\gamma}n \notin [\omega]^n$.

In both cases; $\overline{\gamma}n \in B$.

Define $C := \bigcup_n \{ s \in \omega^n \mid s \notin [\omega]^n \lor \exists i < n[\alpha \circ s(i) = 0] \lor D_\alpha \text{ is not-not-finite}] \}$. Note: $B \subseteq C$.

Let s, n be given such that $s \in \omega^n$ and $\forall m[s * \langle m \rangle \in C]$.

We intend to prove: $s \in C$.

We may assume: $s \in [\omega]^n \land \neg \exists i < n[\alpha \circ s(i) = 0]$, and distinguish two cases.

Case(a). $\exists m[s*\langle m\rangle \in [\omega]^{n+1} \land \alpha(m) \neq 0]$. Finding such m, we consider $s*\langle m\rangle$ and conclude: $s*\langle m\rangle \in C$ and, therefore: D_{α} is not-not-finite.

Case (b). $\neg \exists m[s * \langle m \rangle \in [\omega]^{n+1} \land \alpha(m) \neq 0]$, and, therefore,

 $\forall m[s*\langle m\rangle \in [\omega]^{n+1} \to \alpha(m) = 0].$ Conclude: D_{α} is finite.

Defining $P := \exists m [s * \langle m \rangle \in [\omega]^{n+1} \land \alpha(m) \neq 0]$, we thus see:

 $(P \vee \neg P) \rightarrow D_{\alpha}$ is not-not-finite.

By intuitionistic logic²², we conclude: D_{α} is not-not-finite and: $s \in C$.

We thus see: $\forall s [\forall m [s * \langle m \rangle \in C] \rightarrow s \in C]$.

Obviously: $\forall s \forall m [s \in C \rightarrow s * \langle m \rangle \in C]$.

Using **BARIND**, we conclude: $\langle \rangle \in C$, and: D_{α} is not-not-finite.

 $^{^{22}\}neg\neg(P\vee\neg P)$ is provable, and from $A\to B$ one may conclude $\neg B\to\neg A$ and also $\neg\neg A\to\neg\neg B$. Furthermore, $\neg\neg\neg C$ is equivalent to $\neg C$ and $\neg\neg\neg\neg C$ is equivalent to $\neg C$.

We thus see:

 $\mathsf{BIM} + \mathbf{BARIND} \vdash \forall \alpha [D_{\alpha} \ is \ almost\ -finite \ \rightarrow \ D_{\alpha} \ is \ not\ -not\ -finite]. \ \Box$

Lemma 12.2. $BIM + MP^{23}$ proves

 $\forall \alpha [D_{\alpha} \ is \ not\text{-}infinite \ \leftrightarrow \ D_{\alpha} \ is \ not\text{-}not\text{-}finite \ \leftrightarrow D_{\alpha} \ is \ almost\text{-}finite].$

Proof. First, let α be given such that D_{α} is not-finite. Then $\neg \exists n \forall m \geq n [\alpha(m) = 0]$. Conclude $\forall n \neg \neg \exists m \geq n [\alpha(m) \neq 0]$.

Using MP, conclude: $\forall n \exists m \geq n [\alpha(m) \neq 0]$, i.e. D_{α} is infinite.

We thus see, using **MP**: for all α , if D_{α} is not-finite, then D_{α} is infinite, and may conclude: for all α , if D_{α} is not- infinite, then D_{α} is not-not-finite.

Secondly, let α be given such that $\neg\neg(D_{\alpha} \text{ is almost-finite})$, i.e. $\neg\neg\forall\zeta\in[\omega]^{\omega}\exists n[\alpha\circ\zeta(n)=0]$. Conclude: $\forall\zeta\in[\omega]^{\omega}\neg\neg\exists n[\alpha\circ\zeta(n)=0]$, and, by $\mathbf{MP},\,\forall\zeta\in[\omega]^{\omega}\exists n[\alpha\circ\zeta(n)=0]$, i.e. D_{α} is almost-finite.

We thus see: for all α , if $\neg\neg(D_{\alpha}$ is almost-finite), then D_{α} is almost-finite.

As we also know, from Lemma 12.1: for all α , if D_{α} is finite, then D_{α} is almost-finite, we conclude: for all α , if D_{α} is not-not-finite, then D_{α} is almost-finite.

Finally, it is clear that, for all α , if D_{α} is almost-finite, then D_{α} is not-infinite. \square

We now extend the notion 'almost-finite' from decidable subsets of ω to enumerable subsets of ω .

Let α be given. We consider the set

 $E_{\alpha} := \{n \mid \exists m[\alpha(m) = n+1]\}, \text{ the subset of } \omega \text{ enumerated by } \alpha. \text{ We define:}$ $E_{\alpha} \text{ is almost-finite if and only if } \forall \zeta \in [\omega]^{\omega} \exists m \exists n[m < n \land \alpha \circ \zeta(m) = \alpha \circ \zeta(n)].$

The first item of the next Lemma show that this definition is consistent with the definition given earlier for decidable subsets of ω .

The second item shows that the definition is a good definition indeed as it does not depend on the enumeration α of E_{α} .

The fifth item shows that an almost-finite union of almost-finite enumerable subsets of ω is almost-finite and enumerable.

Lemma 12.3. BIM proves the following.

- (i) $\forall \alpha \forall \beta [D_{\alpha} = E_{\beta} \rightarrow (\forall \zeta \in [\omega]^{\omega} \exists n [\alpha \circ \zeta(n) = 0] \leftrightarrow \forall \zeta \in [\omega]^{\omega} \exists m \exists n [m < n \land \beta(m) = \beta(n)]).$
- (ii) $\forall \alpha \forall \beta [(E_{\alpha} = E_{\beta} \land \forall \zeta \in [\omega]^{\omega} \exists m \exists n [m < n \land \alpha \circ \zeta(m) = \alpha \circ \zeta(n)]) \rightarrow \forall \zeta \in [\omega]^{\omega} \exists m \exists n [m < n \land \beta \circ \zeta(m) = \beta \circ \zeta(n)]].$
- (iii) $\forall \alpha [\forall i < 2[E_{\alpha^i} \ is \ almost\mbox{-}finite] \rightarrow \bigcup_{i < 2} E_{\alpha^i} \ is \ almost\mbox{-}finite].$
- (iv) $\forall \alpha \forall n [\forall i < n[E_{\alpha^i} \ is \ almost-finite] \rightarrow \bigcup_{i < n} E_{\alpha^i} \ is \ almost-finite].$
- (v) $\forall \alpha[(\forall n[E_{\alpha^n} \ is \ almost-finite] \land \forall \zeta \in [\omega]^{\omega} \exists n[\alpha^{\zeta(n)} = \underline{0}]) \rightarrow \bigcup_n E_{\alpha^n} \ is \ almost-finite].$

Proof. (i) The proof is left to the reader.

- (ii) The proof is left to the reader.
- (iii) Let α be given such that E_{α^0} , E_{α^1} are almost-finite. Define α^* such that, for each n, $\alpha^*(2n) = \alpha^0(n)$ and $\alpha^*(2n+1) = \alpha^1(n)$. Note: $E_{\alpha^*} = E_{\alpha^0} \cup E_{\alpha^1}$.

Let ζ in $[\omega]^{\omega}$ be given. Define $QED := \exists m \exists n [m < n \land \alpha^* \circ \zeta(m) = \alpha^* \circ \zeta(n)]$. We prove: $\forall k \exists l > k [\exists p [\zeta(l) = 2p + 1] \lor QED]$.

Let k be given. Define ζ^* such that, for each i, if $\forall j \leq i \exists p [\zeta(k+1+j)=2p]$, then $\zeta^*(i)=\zeta(k+1+i)$, and, if not, then $\zeta^*(i)=2\cdot\zeta(k+1+i)$. Note: $\forall i \exists p [\zeta^*(i)=2p]$. Define ζ^{**} such that $\forall i [\zeta^*(i)=2\cdot\zeta^{**}(i)]$ and note: $\forall i [\alpha^*\circ\zeta^*(i)=\alpha^0\circ\zeta^{**}(i)]$. Find m,n such that m< n and $\alpha^0\circ\zeta^{**}(m)=\alpha^0\circ\zeta^{**}(n)$ and note:

²³For Markov's Principle **MP**, see Subsubsection 2.2.16.

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either \zeta^*(m) = \zeta(k+1+m) and \zeta^*(n) = \zeta(k+1+n) and: QED
or \exists j \leq n \exists p [\zeta(k+1+j) = 2p+1].
     Using Theorem 4.3, find \delta such that \forall k[\delta(k) > k \land (\exists p[\zeta \circ \delta(k) = 2p+1] \lor QED)].
    Define \zeta^{\dagger} such that \zeta^{\dagger}(0) = \delta(0) and, for each k, \zeta^{\dagger}(k+1) = \delta(\zeta^{\dagger}(k)).
Note: for each k, if \exists p[\zeta^{\dagger}(k) = 2p], then QED.
    Define \zeta^{\dagger *} such that, for each i, if \forall j \leq i \exists p[\zeta^{\dagger}(j) = 2p + 1], then \zeta^{\dagger *}(i) = \zeta^{\dagger}(i),
and, if not, then \zeta^{\dagger *}(i) = 2 \cdot \zeta^{\dagger}(i) + 1. Note: \forall i \exists p [\zeta^{\dagger *}(i) = 2p + 1]. Define \zeta^{\dagger **} such
that \forall i [\zeta^{\dagger *}(i) = 2 \cdot \zeta^{\dagger **}(i) + 1] and note: \forall i [\alpha^* \circ \zeta^{\dagger *}(i) = \alpha^1 \circ \zeta^{\dagger **}(i)]. Find m, n
such that m < n and \alpha^1 \circ \zeta^{\dagger **}(m) = \alpha^1 \circ \zeta^{\dagger **}(n) and note:
either \zeta^{\dagger *}(m) = \zeta^{\dagger}(m) and \zeta^{\dagger *}(n) = \zeta^{\dagger}(n) and: QED
or \exists j \exists p [\zeta^{\dagger}(j) = 2p] and QED, so in any case QED.
     We thus see: \forall \zeta \in [\omega]^{\omega} \exists m \exists n [m < n \land \alpha^* \circ \zeta(m) = \alpha^* \circ \zeta(n)], i.e.
E_{\alpha^*} = E_{\alpha^0} \cup E_{\alpha^1} is almost-finite.
     (iv) Use (iii) and induction.
     (v) Let \alpha be given such that, for all n, E_{\alpha^n} is almost-finite, and
\forall \zeta \in [\omega]^{\omega} \exists n[\alpha^{\zeta(n)} = \underline{0}]. We are going to prove: \bigcup_n E_{\alpha^n} is almost-finite.
Define \alpha^* such that, for all n, m, \alpha^*(n, m) = \alpha^n(m) and note: E_{\alpha^*} = \bigcup_n E_{\alpha^n}.
Let \zeta in [\omega]^{\omega} be given. Define QED := \exists m \exists n [m < n \ \land \ \alpha^* \circ \zeta(m) = \alpha^* \circ \zeta(n)].
     We first prove: \forall k \forall n \exists l [l > k \land (QED \lor (\zeta(l))' > n)].
    Let k, n be given.
If (\zeta(k+1))' > n, there is nothing to prove, so we assume (\zeta(k+1))' \leq n.
Define \zeta^* such that \zeta^*(0) = \zeta(k+1) and, for all i, if \forall j \leq i+1 [(\zeta(k+1+j))' \leq n],
then \zeta^*(i+1) = \zeta(k+2+i), and, if not, then \zeta^*(i+1) = \mu p[p] > \zeta^*(i) \land p' = n.
Note: \forall i [(\zeta^*(i))' \leq n].
Using (iii), find p, q such that p < q and \alpha^* \circ \zeta^*(p) = \alpha^* \circ \zeta^*(q) and note:
either \zeta^*(p) = \zeta(k+1+p) and \zeta^*(q) = \zeta(k+1+q) and QED,
or \exists j \leq q[(\zeta(k+1+j))' > n].
    We thus see: \forall k \forall n \exists l [l > k \land (QED \lor (\zeta(l))' > n)].
     Using Theorem 4.3, find \delta such that
\forall k \forall n [\delta(k,n) > k \land (QED \lor (\zeta \circ \delta(k,n))' > n)].
    Now define \eta such that \eta(0) = \delta(0,0), and, for each n, \eta(n+1) = \delta(\eta(n),n).
    Note: \eta \in [\omega]^{\omega} and \forall n[(\zeta \circ \eta(n))' > n \vee QED].
    Find \rho in [\omega]^{\omega} such that \forall n[((\zeta \circ \eta \circ \rho(n+1))' > ((\zeta \circ \eta \circ \rho(n))' \vee QED].
    Find p such that \alpha^{(\zeta \circ \eta \circ \rho(p))'} = 0.
    Conclude: either QED, or
\alpha^* \circ \zeta \circ \eta \circ \rho(\langle p, 0 \rangle) = \alpha^* \circ \zeta \circ \eta \circ \rho(\langle p, 1 \rangle) = 0, and again: QED.
     We thus see: \forall \zeta \in [\omega]^{\omega} \exists m \exists n [m < n \land \alpha^* \circ \zeta(m) = \alpha^* \circ \zeta(n)], i.e.
E_{\alpha^*} = \bigcup_n E_{\alpha^n} is almost-finite.
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12.2. Fans, approximate fans and almost-fans.

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Let \beta be given. We define: \mathcal{F}_{\beta} := \{ \alpha \mid \forall n [\beta(\overline{\alpha}n) = 0] \}.
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We define: β is a *spread-law*, $Spr(\beta)$, if and only if $\forall s[\beta(s) = 0 \leftrightarrow \exists n[\beta(s * \langle n \rangle) = 0]]$.

If β is a spread-law, then \mathcal{F}_{β} is a spread.

We define: β is a fan-law, $Fan(\beta)$, if and only if $Spr(\beta)$ and $\forall s \exists n \forall m [\beta(s * \langle m \rangle) = 0 \to m \le n]$.

If β is a fan-law, then \mathcal{F}_{β} is a fan.

We define: β is an *explicit fan-law* if and only if

 $Fan(\beta)$ and $\exists \gamma \forall s \forall m [\beta(s * \langle m \rangle) = 0 \rightarrow m \leq \gamma(s)].$

If β is an explicit fan-law, then \mathcal{F}_{β} is an explicit fan.

One may prove in BIM, see Lemma 13.1:

 $Explfan(\beta) \leftrightarrow (Spr(\beta) \land \exists \gamma \forall n \forall s \in [\omega]^n [\beta(s) = 0 \rightarrow s \leq \gamma(n)]).$

We define: β is an approximate-fan-law, $Appfan(\beta)$, if and only if

 $Spr(\beta)$ and $\forall n \exists k \forall t \in [\omega]^{k+1} \exists i < k+1[t(i) \notin \omega^n \lor \beta(t(i)) \neq 0].$

If β is an approximate-fan-law, then \mathcal{F}_{β} is an approximate fan.

We define: β is an almost-fan-law, $Almfan(\beta)$, if and only if $Spr(\beta)$ and $\forall s \forall \zeta \in [\omega]^{\omega} \exists m [\beta(s * \langle \zeta(m) \rangle) \neq 0]$.

If β is an almost-fan-law, then \mathcal{F}_{β} is an almost-fan.

12.3. The Almost-fan Theorem as a Scheme, ALMFAN:

 $\forall \beta [(Alm fan(\beta) \land Bar_{\mathcal{F}_{\beta}}(B)) \rightarrow$

 $\exists \alpha [E_{\alpha} \subseteq B \land E_{\alpha} \text{ is almost-finite } \land Bar_{\mathcal{F}_{\beta}}(E_{\alpha})]].$

The following theorem may be compared to Theorem 2.3.

Theorem 12.4. BIM + BARIND + $AC_{\omega,\omega^{\omega}} \vdash ALMFAN$. ²⁴

Proof. Let β be given such that $Almfan(\beta)$ and $\beta(\langle \rangle) = 0.25$ Assume $Bar_{\mathcal{F}_{\beta}}(B)$. Define $B' := B \cup \{s \mid \beta(s) \neq 0\}$.

We claim: $Bar_{\omega^{\omega}}(B')$. In the proof of Theorem 2.3 we have seen how to prove this claim.

Let C be the set of all s such that either: $\beta(s) \neq 0$ or:

 $\beta(s) = 0$ and $\exists \alpha [E_{\alpha} \subseteq B \land E_{\alpha} \text{ is almost-finite } \land Bar_{\mathcal{F}_{\beta} \cap s}(E_{\alpha})].$

One easily sees: $B \subseteq C$: for every s, if $\beta(s) = 0$ and $s \in B$, define α such that $\forall n[\alpha(n) = s + 1]$ and note: $\{s\} = E_{\alpha} \subseteq B$ and E_{α} is finite and $Bar_{\mathcal{F}_{\beta} \cap s}(E_{\alpha})$.

Now let s be given such that $\forall m[s*\langle m\rangle \in C]$. Using $\mathbf{AC}_{\omega,\omega^{\omega}}$, find α such that, for all m, if $\beta(s*\langle m\rangle) = 0$, then $E_{\alpha^m} \subseteq B$ and E_{α^m} is almost-finite and $Bar_{\mathcal{F}_{\beta} \cap s*\langle m\rangle}(E_{\alpha^m})$, and, if $\beta(s*\langle m\rangle) \neq 0$, then $\alpha^m = \underline{0}$ and $E_{\alpha^m} = \emptyset$.

Note: $Alm fan(\beta)$ and, therefore, $\forall \zeta \in [\omega]^{\omega} \exists m [\alpha^{\zeta(m)} = \underline{0}].$

Use Lemma 12.3(iv) and conclude: $\bigcup_m E_{\alpha^m}$ is almost-finite.

Also: $Bar_{\mathcal{F}_{\beta} \cap s}(\bigcup_{n} E_{\alpha^{n}})$. Conclude: $s \in C$.

We thus see: $\forall s [\forall m [s * \langle m \rangle \in C] \rightarrow s \in C]$.

Obviously: $\forall s \forall m [s \in C \to s * \langle m \rangle \in C].$

Using **BARIND**, we conclude: $\langle \ \rangle \in C$, i.e.

 $\exists \alpha [E_{\alpha} \subseteq B \land E_{\alpha} \text{ is almost-finite } \land Bar_{\mathcal{F}_{\beta}}(E_{\alpha})].$

12.4. The Almost-fan Theorem, **AlmFT**:

$$\forall \beta [Alm fan(\beta) \to \forall \alpha [(Thinbar_{\mathcal{F}_{\beta}}(D_{\alpha}) \land \forall s \in D_{\alpha}[\beta(s) = 0]) \to \\ \forall \zeta \in [\omega]^{\omega} \exists n [\zeta(n) \notin D_{\alpha}]]],$$

i.e.: every decidable subset of ω that is a thin bar in some almost-fan is almost-finite.

Theorem 12.5. $BIM + ALMFAN \vdash AlmFT$.

Proof. Let β, α be given such that $Appfan(\beta)$ and $Thinbar_{\mathcal{F}_{\beta}}(D_{\alpha})$ and $\forall s \in D_{\alpha}[\beta(s) = 0]$. Applying Theorem 12.4, find γ such that $E_{\gamma} \subseteq D_{\alpha}$ and $Bar_{\mathcal{F}_{\beta}}(E_{\gamma})$ and E_{γ} is almost-finite. As $\forall s \in D_{\alpha} \forall t \in D_{\alpha}[s \sqsubseteq t \to s = t]$, conclude: $E_{\gamma} = D_{\alpha}$ and, by Lemma 12.1(i), $\forall \zeta \in [\omega]^{\omega} \exists n[\zeta(n) \notin D_{\alpha}]$.

The Almost-fan Theorem occurs in [36] and [37].

 $^{^{24}\}text{We}$ do not know id the use of the Axiom $\mathbf{AC}_{\omega,\omega^\omega}$ is avoidable.

²⁵If $\beta(\langle \rangle) \neq 0$, then $\mathcal{F}_{\beta} = \emptyset$ and there is nothing to prove.

12.5. The Approximate-fan Theorem, AppFT:

$$\forall \beta \forall \alpha [(Appfan(\beta) \land Thinbar_{\mathcal{F}_{\beta}}(D_{\alpha}) \land \forall s \in D_{\alpha}[\beta(s) = 0]) \rightarrow \\ \forall \zeta \in [\omega]^{\omega} \exists n [\zeta(n) \notin D_{\alpha}]],$$

i.e.: every decidable subst of ω that is a thin bar in some approximate fan is almost-finite.

Theorem 12.6. $BIM + AlmFT \vdash AppFT$.

Proof. Obvious, as every approximate fan is an almost-fan.

I do not know if BIM proves $AppFT \rightarrow AlmFT$.

Theorem 12.7. $BIM + AppFT \vdash FT$.

Proof. Assume AppFT. We want to prove FT and use Theorem 2.1.

Let α be given such that $Thinbar_{\mathcal{C}}(D_{\alpha})$.

Using **AppFT**, conclude: D_{α} is almost-finite.

Define ζ in $[\omega]^{\omega}$ such that, for each n,

if $\neg \forall \gamma \in \mathcal{C} \exists i < n[\zeta(i) \sqsubset \gamma]$, then $\zeta(n) = \mu p[p \in D_{\alpha} \land \forall i < n[p \neq \zeta(i)]]$.

Find p such that $\zeta(p) \notin D_{\alpha}$.

Conclude: $\forall n > \zeta(p)[n \notin D_{\alpha}].$

We thus see: $\forall \alpha [Thinbar_{\mathcal{C}}(D_{\alpha}) \to \exists m \forall n > m[n \notin D_{\alpha}]].$

Using Theorem 2.1, we conclude: \mathbf{FT} .

BIM does not prove $\mathbf{FT} \to \mathbf{AppFT}$, see [44, Corollary 10.6].

In BIM, **AppFT** has a number of important equivalents, see [44]. One of them is the following principle of Open Induction:

$$\mathbf{OI}([0,1]): \forall \alpha [\forall \delta \in [0,1][[0,\delta) \subseteq \mathcal{H}_{\alpha} \to \delta \in \mathcal{H}_{\alpha}] \to [0,1] \subseteq \mathcal{H}_{\alpha}].$$

The relation between \mathbf{FT} and \mathbf{AppFT} in BIM may be compared to the relation between \mathbf{WKL} and \mathbf{KL} in RCA_0 .

In the classical context of RCA_0 , one studies two extensions of WKL,

Bounded König's Lemma **BKL**, that (for a classical reader) would coincide with (a contraposition of) the second formulation of **FT** 2.2.4 and König's Lemma **KL**, that, similarly, would coincide with **FT** $^+$ 2.2.5.

BKL is, in RCA₀, equivalent to **WKL**, see [26, Lemma IV.1.4], just as, in BIM, the first and second formulation of **FT** are equivalent.

 \mathbf{KL} , on the other hand, is definitely stronger than \mathbf{WKL} , as $\mathsf{RCA}_0 + \mathbf{KL}$ is equivalent to ACA_0 .

As we observed before, see Theorem 4.2, $\mathsf{BIM} + Weak \cdot \Pi_1^0 \cdot \mathbf{AC}_{\omega,\omega} \vdash \mathbf{FT} \leftrightarrow \mathbf{FT}^+$. ²⁶ From a constructive point of view, the axiom $Weak \cdot \Pi_1^0 \cdot \mathbf{AC}_{\omega,\omega}$ is weak indeed, as

From a constructive point of view, the axiom Weak- Π_1^0 - $\mathbf{AC}_{\omega,\omega}$ is weak indeed, as the antecedent is read constructively. Also note: $\mathsf{BIM} + \mathbf{AC}_{\omega,\omega}! \vdash Weak$ - Π_1^0 - $\mathbf{AC}_{\omega,\omega}$.

In a classical context, the rôle of a countable axiom of choice is very different from the rôle of such an axiom in a constructive context, see [9, Appendix 1].

It seems to us that \mathbf{FT}^+ is still too close to \mathbf{FT} for being a good candidate for playing, in the intuitionistic context, a rôle like the rôle played by \mathbf{KL} in the classical context.

Note that our 'axiom' **AppFT** is a possibly better candidate, as, for a classical reader, **AppFT** is still indistinguishable from **KL**.

Some authors have called our **FT** the *Weak Fan Theorem* **WFT**, see for instance [19], but we decided not to follow them.

 $^{^{26}}$ One may learn from [16, Theorem 9.20] that, using countable choice, in fact only Weak- Π_1^0 -AC $_{\omega,\omega}$, one may constructively derive KL from WKL. On the problem of treating non-constructive assumptions in a constructive context, see the end of Section 2.

There is a constructive version of Weak Weak König's Lemma WWKL, see [26, Definition X.1.7], that is called WWFT, see [23].

13. NOTATION AND CONVENTIONS

In this Section we explain how, in BIM, some useful notation is introduced and some elementary results are proven.

13.1. Finite and infinite sequences of natural numbers.

BIM contains a constant p denoting the function enumerating the prime numbers: $p(0) = 2, p(1) = 3, \dots$

We code finite sequences of natural numbers by natural numbers:

 $\langle \rangle := 0$ and, for each k > 0, for all $m_0, m_1, \dots m_{k-1}$,

$$\langle m_0, \dots, m_{k-1} \rangle = p(k-1) \cdot \prod_{i < k} p(i)^{m_i} - 1$$

length(0) := 0 and,

for each s > 0, length(s) := 1 + the largest k such that <math>p(k) divides s + 1.

If i < length(s) - 1, then $s(i) := the largest m such that <math>p(i)^m$ divides s + 1, and, if i = length(s) - 1, then $s(i) := the largest m such that <math>p(i)^{m+1}$ divides s + 1, and, if $i \ge length(s)$ then s(i) := 0.

Note: if length(s) = k, then $s = \langle s(0), s(1), \dots, s(k-1) \rangle$.

Also note: $\forall s[s \geq length(s)].$

a*b is the number s satisfying: length(s) = length(a) + length(b) and,

for each n, if n < length(a), then s(n) = a(n) and,

if $length(a) \le n < length(s)$, then s(n) = b(n - length(a)).

 $a * \alpha$ is the element β of ω^{ω} satisfying: for each n, if n < length(a),

then $\beta(n) = a(n)$, and, if $length(a) \le n$, then $\beta(n) = \alpha(n - length(a))$.

For $n \leq length(a)$, $\overline{a}(n) := \langle a(0), \dots, a(n-1) \rangle$.

If confusion seems unlikely, we sometimes write: " $\overline{a}n$ " and not: " $\overline{a}(n)$ ".

 $a \sqsubseteq b \leftrightarrow \exists n \leq length(b)[a = \overline{b}n], \text{ and: } a \sqsubseteq b \leftrightarrow (a \sqsubseteq b \land a \neq b).$

 $s \leq_{lex} t \leftrightarrow \forall m < length(s)[\overline{s}m = \overline{t}m \to s(m) \leq t(m)].$

 $a \perp b \leftrightarrow \neg (a \sqsubseteq b \lor b \sqsubseteq a).$

 $\overline{\alpha}(n) := \overline{\alpha}n := \langle \alpha(0), \dots \alpha(n-1) \rangle.$

 $a \sqsubset \alpha \leftrightarrow \exists n [\overline{\alpha}n = a].$

 $a \perp \alpha \leftrightarrow \alpha \perp a \leftrightarrow \neg (a \sqsubset \alpha).$

 $\alpha \# \beta \leftrightarrow \alpha \perp \beta \leftrightarrow \exists n [\alpha(n) \neq \beta(n)].$

For each p, p is the element of ω^{ω} satisfying $\forall n[p(n) = p]$.

$$\forall \alpha \forall n [\alpha'(n) = (\alpha(n))' \land \alpha''(n) = (\alpha(n))''].$$

We extend the language of BIM by introducing \in and terms denoting subsets of ω . This is not a real extension of the language of BIM. Formulas in which the new symbols occur are abbreviations of formulas in which they do not occur.

Given a formula $\varphi = \varphi(n)$ we may introduce a 'set term' T_{φ} . The basic formula ' $t \in X_{\varphi}$ ' is an abbreviation of ' $\varphi(t)$ '.

Here is a first example.

 $Bin := \{ a \mid \forall n < length(a) [a(n) = 0 \lor a(n) = 1] \}.$

' $a \in Bin$ ' is an abbreviation of ' $\forall n < length(a)[a(n) = 0 \lor a(n) = 1]$ '.

 $Bin_m := \{ s \in Bin \mid length(s) = m \} \}.$

$$\omega^m := \{ s \mid length(s) = m \}.$$

Given terms A, B denoting subsets of ω ,

 $A \subseteq B$ is an abbreviation of $\forall n [n \in A \to n \in B]$.

We also introduce terms denoting subsets of ω^{ω} , for instance:

```
\mathcal{C} := \{ \gamma \mid \forall n [\gamma(n) = 0 \lor \gamma(n) = 1] \}.
```

' $\alpha \in \mathcal{C}$ ' is ean abbreviation of ' $\forall n < length(a)[\alpha(n) = 0 \lor \alpha(n) = 1]$ '.

$$[\omega]^{\omega} := \{ \zeta \mid \forall n [\zeta(n) < \zeta(n+1)] \}.$$

Given terms \mathcal{X}, \mathcal{Y} denoting subsets of ω^{ω} ,

 $\mathcal{X} \subseteq \mathcal{Y}'$ abbreviates $\forall \alpha [\alpha \in \mathcal{X} \to \alpha \in \mathcal{Y}]'$.

Given a term \mathcal{X} denoting a subset of ω^{ω} , we introduce, for all s,

$$\mathcal{X} \cap s := \{ \alpha \in \mathcal{X} \mid s \sqsubset \alpha \}.$$

Given a term \mathcal{X} denoting a subset of ω^{ω} , and a term B denoting a subset of ω , we let ' $Bar_{\mathcal{X}}(B)$ ' be an abbreviation of ' $\forall \alpha \in \mathcal{X} \exists n [\overline{\alpha}n \in B]$ '.

Note that $Bar_{\mathcal{X}}(B)$ is a *formula scheme*, that becomes a formula if one substitutes formulas defining \mathcal{X} , B, respectively.

From now on, we will express ourselves more informally, as in the following example:

For each $\mathcal{X} \subseteq \omega^{\omega}$, for each $B \subseteq \omega$,

 $Thinbar_{\mathcal{X}}(B) \leftrightarrow (Bar_{\mathcal{X}}(B) \land \forall s \in B \forall t \in B[s \sqsubseteq t \rightarrow s = t]).$

Note that we are not extending the language of BIM by second-order variables.

13.2. Decidable and enumerable subsets of ω .

 $D_{\alpha} := \{i \mid \alpha(i) \neq 0\}.$ D_{α} is the subset of ω decided by α .

The expression $i \in D_{\alpha}$ is an abbreviation of $\alpha(i) \neq 0$.

 $X \subseteq \omega$ is decidable or Δ_1^0 if and only if $\exists \alpha [X = D_{\alpha}]$.

$$D_a := \{i \mid i < length(a) \mid a(i) \neq 0\}.$$

 $X \subseteq \omega$ is *finite* if and only if $\exists a[X = D_a]$.

Note: for each α , $D_{\alpha} = \bigcup_{n \in \omega} D_{\overline{\alpha}n}$.

 $E_{\alpha} := \{n \mid \exists p[\alpha(p) = n+1]\}.$ E_{α} is the subset of ω enumerated by α .

 $X \subseteq \omega$ is enumerable or Σ_1^0 if and only if $\exists \alpha [X = E_{\alpha}]$.

$$E_a := \{ n \mid \exists p < length(a)[a(p) = n + 1] \}.$$

Note: for each α , $E_{\alpha} = \bigcup_{n \in \omega} E_{\overline{\alpha}n}$.

 $X \subseteq \omega$ is co-enumerable or Π_1^0 if and only if

$$\exists \alpha [X = \omega \setminus E_{\alpha} = \{n \mid \forall p [\alpha(p) \neq n+1]\}].$$

Given any α , define β such that

 $\forall n[\alpha(n) = \beta(n) = 0 \lor (\alpha(n) \neq 0 \land \beta(n) = n+1)], \text{ and note: } D_{\alpha} = E_{\beta}.$

We thus see: BIM $\vdash \forall \alpha \exists \beta [D_{\alpha} = E_{\beta}].$

13.3. Open and closed subsets of ω^{ω} , spreads and fans.

 $\mathcal{G}_{\beta} := \{ \gamma \mid \exists n [\beta(\overline{\gamma}n) \neq 0] \}.$

 $\mathcal{G} \subseteq \omega^{\omega}$ is open or Σ_1^0 if and only if $\exists \beta [\mathcal{G} = \mathcal{G}_{\beta}]$.

 $\mathcal{F}_{\beta} := \omega^{\omega} \setminus \mathcal{G}_{\beta} = \{ \gamma \mid \forall n [\beta(\overline{\gamma}n) = 0] \}.$

 $\mathcal{F} \subseteq \omega^{\omega}$ is *closed* or Π_1^0 if and only if $\exists \beta [\mathcal{F} = \mathcal{F}_{\beta}]$.

 $Spr(\beta) \leftrightarrow \forall s [\beta(s) = 0 \leftrightarrow \exists n [\beta(s * \langle n \rangle) = 0]].$

 $\mathcal{X} \subseteq \omega^{\omega}$ is a spread if and only if $\exists \beta [Spr(\beta) \land \mathcal{X} = \mathcal{F}_{\beta}].$

In intuitionistic mathematics, not every closed subset of ω^{ω} is a spread, see Lemma 2.11.

$$Fan(\beta) \leftrightarrow (Spr(\beta) \land \forall s \exists n \forall m [\beta(s * \langle m \rangle) = 0 \rightarrow m \leq n])$$
 and

$$Explfan(\beta) \leftrightarrow \big(Spr(\beta) \ \land \ \exists \gamma \forall s \forall m [\beta(s * \langle m \rangle) = 0 \rightarrow m \leq \gamma(s)]\big).$$

 $\mathcal{X} \subseteq \omega^{\omega}$ is a fan if and only if $\exists \beta [Fan(\beta) \land \mathcal{X} = \mathcal{F}_{\beta}].$

 $\mathcal{X} \subseteq \omega^{\omega}$ is an explicit fan if and only if $\exists \beta [Explfan(\beta) \land \mathcal{X} = \mathcal{F}_{\beta}].$

Lemma 13.1. One may prove in BIM:

$$\forall \beta [Explfan(\beta) \leftrightarrow (Spr(\beta) \land \exists \gamma \forall n \forall s \in \omega^n [\beta(s) = 0 \rightarrow s \leq \gamma(n)])].$$

```
Proof. Let \beta be given such that Explfan(\beta).
Find \gamma such that \forall s \forall m [\beta(s * \langle m \rangle) = 0 \rightarrow m \leq \gamma(s)].
Define \delta such that \delta(0) = 0 = \langle \rangle, and, for each n,
\delta(n+1) = \max(\{s * \langle m \rangle \mid \beta(\{s * \langle m \rangle) = 0 \land s \leq \delta(n) \land m \leq \gamma(s)\}).
One proves by induction: \forall n \forall s \in \omega^n [\beta(s) = 0 \to s \leq \delta(n)].
     Conversely, let \beta, \gamma be given such that \forall n \forall s \in \omega^n [\beta(s) = 0 \to s \le \gamma(n)].
Define \delta such that, for each n, for each s in \omega^n such that \beta(s) = 0,
\delta(s) = \max(\{m \mid \beta(s * \langle m \rangle) = 0 \land s * \langle m \rangle \le \gamma(n+1)\}).
Note: \forall s \forall m [\beta(s * \langle m \rangle) = 0 \rightarrow m \leq \delta(s)] and conclude: Explfan(\beta).
                                                                                                                                                                    13.4. Subsequences.
     \forall n \forall m [\alpha^{\restriction n}(m) := \alpha(\langle n \rangle * m)].
     \alpha^{\upharpoonright n} is called the n-th subsequence of the infinite sequence \alpha.
     length(s^{\upharpoonright n}) := \mu p \leq s[\langle n \rangle * p \geq length(s)] and
\forall m < length(s^{\upharpoonright n})[s^{\upharpoonright n}(m) = s(\langle n \rangle * m)].
     Note: \forall \alpha \forall m \forall n [(\overline{\alpha}m)^{\restriction n} \sqsubset \alpha^{\restriction n}].
     \forall m[^a\alpha(m) := \alpha(a*m)].
     Note: \forall n [\langle n \rangle \alpha = \alpha^{\upharpoonright n}].
     length(^a s) := \mu p \le s[a * p \ge length(s)] and \forall m < length(^a s)[^a s(m) = s(a * m)].
     Note: \forall \alpha \forall m \forall a [^a(\overline{\alpha}m) \sqsubseteq {}^a\alpha].
13.5. Partial continuous functions from \omega^{\omega} to \omega and from \omega^{\omega} to \omega^{\omega}.
     Fun_0(\varphi) \leftrightarrow \forall a \in E_{\varphi} \forall b \in E_{\varphi}[a' \sqsubseteq b' \to a'' = b''].
     Dom_0(\varphi) := \{ \alpha \mid \forall \alpha \exists a \in E_{\varphi}[a' \sqsubset \alpha] \}.
     Assume: Fun_0(\varphi) and \alpha \in Dom_0(\varphi).
     Then \varphi(\alpha) := the number c such that \exists n(\overline{\alpha}n, c) \in E_{\varphi}].
     For every \mathcal{X} \subseteq \omega^{\omega}, for every \varphi, \varphi : \mathcal{X} \to \omega \leftrightarrow (Fun_0(\varphi) \land \mathcal{X} \subseteq Dom_0(\varphi)).
     Fun_1(\varphi) \leftrightarrow \forall a \in E_{\varphi} \forall b \in E_{\varphi}[a' \sqsubseteq b' \to a'' \sqsubseteq b''].
     Dom_1(\varphi) := \{ \alpha \mid \forall n \exists a \in E_{\varphi}[a' \sqsubset \alpha \land length(a'') \ge n] \}.
     \varphi|a:=\max(\{t\mid \exists b\in E_{\overline{\varphi}a}[b'\sqsubseteq a\ \wedge b''=t]\}).
     \varphi: \alpha \mapsto \gamma \leftrightarrow \forall n \exists m [\overline{\gamma} n \sqsubseteq \varphi | \overline{\alpha} m]]
     Assume: Fun_1(\varphi) and \alpha \in Dom_1(\varphi).
     Then \varphi | \alpha := \text{ the element } \gamma \text{ of } \omega^{\omega} \text{ such that } \varphi : \alpha \mapsto \gamma.
     For every \mathcal{X} \subseteq \omega^{\omega}, for every \varphi, \varphi : \mathcal{X} \to \omega^{\omega} \leftrightarrow (Fun_1(\varphi) \land \mathcal{X} \subseteq Dom_1(\varphi)).
13.6. Integers and rationals.
     m =_{\mathbb{Z}} n \leftrightarrow m' + n'' = m'' + n'.
     m <_{\mathbb{Z}} n \leftrightarrow m' + n'' < m'' + n'.
     0_{\mathbb{Z}} := (0,0)
     m +_{\mathbb{Z}} n = (m' + n', m'' + n'').
     m - \mathbb{Z} n = (m' + n'', m'' + n').
     m \cdot_{\mathbb{Z}} n := (m' \cdot n' + m'' \cdot n'', m' \cdot n'' + m'' \cdot n').
     \mathbb{Q} := \{ n \mid n'' >_{\mathbb{Z}} 0_{\mathbb{Z}} \}.
     m =_{\mathbb{O}} n \leftrightarrow m' \cdot_{\mathbb{Z}} n'' =_{\mathbb{Z}} m'' \cdot_{\mathbb{Z}} n'.
     m <_{\mathbb{Q}} n \leftrightarrow m' \cdot_{\mathbb{Z}} n'' <_{\mathbb{Z}} m'' \cdot_{\mathbb{Z}} n'.
     m \leq_{\mathbb{Q}} n \leftrightarrow m' \cdot_{\mathbb{Z}} n'' \leq_{\mathbb{Z}} m'' \cdot_{\mathbb{Z}} n'.
     m \leq_{\mathbb{Q}} n \leftrightarrow \max_{\mathbb{Q}}(m, n) =_{\mathbb{Q}} n \leftrightarrow \min_{\mathbb{Q}}(m, n) =_{\mathbb{Q}} m.
     m +_{\mathbb{Q}} n = (m' \cdot_{\mathbb{Z}} n'' +_{\mathbb{Z}} m'' \cdot_{\mathbb{Z}} n', m'' \cdot_{\mathbb{Z}} n'').
     m - \mathbb{Q} n = (m' \cdot \mathbb{Z} n'' - \mathbb{Z} m'' \cdot \mathbb{Z} n', m'' \cdot \mathbb{Z} n'').
     m \cdot_{\mathbb{Q}} n = (m' \cdot_{\mathbb{Z}} n', m'' \cdot_{\mathbb{Z}} n'').
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²⁷See [43, Subsections 7.2 and 7.5]

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\mathbb{S} := \{ s \mid s' \in \mathbb{Q} \ \land \ s'' \in \mathbb{Q} \ \land \ s' <_{\mathbb{Q}} s'' \}.
       s \sqsubseteq_{\mathbb{S}} t \leftrightarrow (t' <_{\mathbb{Q}} s' \land s'' <_{\mathbb{Q}} t'').
       s \sqsubseteq_{\mathbb{S}} t \leftrightarrow (t' \leq_{\mathbb{Q}} s' \land s'' \leq_{\mathbb{Q}} t'').
       s <_{\mathbb{S}} t \leftrightarrow s'' <_{\mathbb{Q}} t'.
       s \leq_{\mathbb{S}} t \leftrightarrow s' \leq_{\mathbb{Q}} t''.
       s \#_{\mathbb{S}} t \leftrightarrow (s <_{\mathbb{S}} t \lor t <_{\mathbb{S}} s).
       For each n, we define n_{\mathbb{Q}} in \mathbb{Q} by: n_{\mathbb{Q}} = ((n,0),(1,0)).
       For all s in \mathbb{S}, double_{\mathbb{S}}(s) is the element u of \mathbb{S} satisfying:
       u' +_{\mathbb{Q}} u'' =_{\mathbb{Q}} s' +_{\mathbb{Q}} s'' and u'' -_{\mathbb{Q}} u' =_{\mathbb{Q}} 2_{\mathbb{Q}} \cdot_{\mathbb{Q}} (s'' -_{\mathbb{Q}} s').
       Note: \forall s \in \mathbb{S} \forall t \in \mathbb{S}[s \sqsubseteq_{\mathbb{S}} t \to double_{\mathbb{S}}(s) \sqsubseteq_{\mathbb{S}} double_{\mathbb{S}}(t)].
      s +_{\mathbb{S}} t := (s' +_{\mathbb{O}} t', s'' +_{\mathbb{O}} t'').
       s \cdot \mathbf{s} t :=
\left(\min_{\mathbb{Q}}(s'\cdot_{\mathbb{Q}}t',s''\cdot_{\mathbb{Q}}t',s'\cdot_{\mathbb{Q}}t'',s''\cdot_{\mathbb{Q}}t''),\max_{\mathbb{Q}}(s'\cdot_{\mathbb{Q}}t',s''\cdot_{\mathbb{Q}}t',s''\cdot_{\mathbb{Q}}t'',s''\cdot_{\mathbb{Q}}t'')\right).
13.7. Real numbers.
       \mathcal{R} := \{ \alpha \mid \forall n [\alpha(n) \in \mathbb{S} \ \land \ \alpha(n+1) \sqsubseteq_{\mathbb{S}} \alpha(n)] \ \land \ \forall m \exists n [\alpha''(n) -_{\mathbb{Q}} \alpha'(n) <_{\mathbb{Q}} \frac{1}{2^m}] \}.^{28}
       \alpha <_{\mathcal{R}} \beta \leftrightarrow \exists n [\alpha(n) <_{\mathbb{S}} \beta(n)].
       \alpha \leq_{\mathcal{R}} \beta \leftrightarrow \forall n[\alpha(n) \leq_{\mathbb{S}} \beta(n)].
       \forall n[\inf(\alpha,\beta)(n) := (\min_{\mathbb{Q}}(\alpha'(n),\beta'(n)), \min_{\mathbb{Q}}(\alpha''(n),\beta''(n)))].
      \forall n[\sup(\alpha,\beta)(n) := (\max_{\mathbb{Q}}(\alpha'(n),\beta'(n)), \max_{\mathbb{Q}}(\alpha''(n),\beta''(n)))].
       \alpha \#_{\mathcal{R}} \beta \leftrightarrow (\alpha <_{\mathcal{R}} \beta \lor \beta <_{\mathcal{R}} \alpha).
       \alpha =_{\mathcal{R}} \beta \leftrightarrow (\alpha \leq_{\mathcal{R}} \beta \land \beta \leq_{\mathcal{R}} \alpha).
       \forall n[(\alpha +_{\mathcal{R}} \beta)(n) := \alpha(n) +_{\mathbb{S}} \beta(n)].
       \forall n [(\alpha \cdot_{\mathcal{R}} \beta)(n) := \alpha(n) \cdot_{\mathbb{S}} \beta(n)].
       For each q in \mathbb{Q}, we define q_{\mathcal{R}} in \mathcal{R} by: for each n, q_{\mathcal{R}}(n) = (q - \mathbb{Q} \frac{1}{2^n}, q + \mathbb{Q} \frac{1}{2^n}).
      0_{\mathcal{R}} := (0_{\mathbb{Q}})_{\mathcal{R}} and 1_{\mathcal{R}} := (1_{\mathbb{Q}})_{\mathcal{R}}.
Lemma 13.2. One may prove in BIM:
\forall s \in \mathbb{S} \forall t \in \mathbb{S}[s \sqsubseteq_{\mathbb{S}} t \to \forall \alpha \in \mathcal{R} \exists n[s \#_{\mathbb{S}} \alpha(n) \lor \alpha(n) \sqsubseteq_{\mathbb{S}} t]].
```

Proof. The proof is left to the reader.

13.8. [0,1] and C.

 $[0,1] := \{ \alpha \in \mathcal{R} \mid 0_{\mathcal{R}} \leq_{\mathcal{R}} \alpha \leq_{\mathcal{R}} 1_{\mathcal{R}} \}.$

 $(0,1] := \{ \alpha \in \mathcal{R} \mid 0_{\mathcal{R}} <_{\mathcal{R}} \alpha \leq_{\mathcal{R}} 1_{\mathcal{R}} \}, \text{ and } [0,1) := \{ \alpha \in \mathcal{R} \mid 0_{\mathcal{R}} \leq_{\mathcal{R}} \alpha <_{\mathcal{R}} 1_{\mathcal{R}} \}.$

For all α, β in \mathcal{R} , $[\alpha, \beta) := \{ \gamma \in \mathcal{R} \mid \alpha \leq_{\mathcal{R}} \gamma <_{\mathcal{R}} \beta \}.$

 $[0,1]^2 := \{ \gamma \mid \forall i < 2[\gamma^{\uparrow i} \in [0,1]] \} \text{ and } [0,1]^{\omega} := \{ \gamma \mid \forall n[\gamma^{\uparrow n} \in [0,1]] \}.$

 $\mathcal{H}_{\alpha} := \{ \gamma \in [0, 1] \mid \exists n \exists s \in \mathbb{S}[\alpha(s) \neq 0 \land \gamma(n) \sqsubset_{\mathbb{S}} s] \}.$

 $\mathcal{H} \subseteq \mathcal{R}$ is open if and only if $\exists \alpha [\mathcal{H} = \mathcal{H}_{\alpha}].$

Lemma 13.3. One may prove in BIM:

There exist σ, ψ such that

- (i) $\sigma: \mathcal{C} \to [0,1]$ and $\forall \delta \in [0,1] \exists \gamma \in \mathcal{C}[\sigma | \gamma =_{\mathcal{R}} \delta]$.
- (ii) $\psi: \omega^{\omega} \to \omega^{\omega}$ and $\forall \alpha \forall \gamma \in \mathcal{C}[\gamma \in \mathcal{G}_{\psi|\alpha} \leftrightarrow \sigma | \gamma \in \mathcal{H}_{\alpha}].$

Proof. (i) Define λ and ρ such that, for each s in \mathbb{S} , $\lambda(s) = (s', \frac{1}{3}s' + \mathbb{O}(\frac{2}{3}s''))$ and $\rho(s) = (\frac{2}{3}s' + \mathbb{Q} \frac{1}{3}s'', s'').$

For each s in S, $\lambda(s)$ is the *left-two-third part of s* and $\rho(s)$ is the *right-two-third* part of s.

Define ν such that $\nu(\langle \rangle) = (0_{\mathbb{Q}}, 1_{\mathbb{Q}})$ and, for all s in Bin, $\nu(s * \langle 0 \rangle) = \lambda(\nu(s))$ and $\nu(s * \langle 1 \rangle) = \rho(\nu(s))$.

²⁸This is not the same definition as in [43, Subsubsection 8.1.4]. We replaced ' $\sqsubset_{\mathbb{S}}$ ' by ' $\sqsubset_{\mathbb{S}}$ '.

```
Define \sigma: \mathcal{C} \to [0,1] such that \forall \gamma \in \mathcal{C}[(\sigma|\gamma)(n) = \nu(\overline{\gamma}n)].
One may prove: \forall \delta \in [0, 1] \exists \gamma \in \mathcal{C}[\sigma | \gamma =_{\mathcal{R}} \delta].
       (ii) Define \psi:\omega^{\omega}\to\omega^{\omega} such that
\forall \alpha \forall s [(\psi | \alpha)(s) \neq 0 \leftrightarrow \big(s \in Bin \ \land \ \exists t < s [\nu(s) \sqsubseteq_{\mathbb{S}} t \ \land \ \alpha(t) \neq 0]\big)].
One may prove: \forall \alpha \forall \gamma \in \mathcal{C}[\gamma \in \mathcal{G}_{\psi|\alpha} \leftrightarrow \sigma | \gamma \in \mathcal{H}_{\alpha}].
                                                                                                                                                                                         Lemma 13.4. One may prove in BIM:
There exist \tau, \chi such that
    (i) \tau: \mathcal{C} \to [0,1] and \forall \gamma \in \mathcal{C} \forall \delta \in \mathcal{C} [\gamma \# \delta \to \tau | \gamma \#_{\mathcal{R}} \tau | \delta].
  (ii) \chi: \omega^{\omega} \to \omega^{\omega} and \forall \alpha \forall \gamma \in \mathcal{C}[\gamma \in \mathcal{G}_{\alpha} \leftrightarrow \tau | \gamma \in \mathcal{H}_{\chi|\alpha}].
 (iii) \forall \delta \in [0, 1] \exists \gamma \in \mathcal{C}[\tau | \gamma \#_{\mathcal{R}} \delta \to \forall \alpha [\delta \in \mathcal{H}_{\chi | \alpha}]].
 (iv) \forall \delta \in [0,1]^{\omega} \exists \gamma \in \mathcal{C} \forall n [\tau | \gamma^n \#_{\mathcal{R}} \delta^n \to \forall \alpha [\delta^n \in \mathcal{H}_{\chi | \alpha}]].
Proof. (i) Define \pi_0, \pi_1, \pi_2, \pi_3 and \pi_4 such that, for each s in \mathbb{S}, for each i < 5, \pi_i(s) := (\frac{5-i}{5}s' + \mathbb{Q}\frac{i}{5}s'', \frac{5-i-1}{5}s' + \mathbb{Q}\frac{i+1}{5}s'').
      For each s in S, for each i < 5, \pi_i(s) is the i-th fifth part of s.
      Define \varepsilon such that \varepsilon(\langle \rangle) = (0_{\mathbb{Q}}, 1_{\mathbb{Q}}) and, for all a in Bin,
\varepsilon(a * \langle 0 \rangle) = \pi_1(\varepsilon(a)) \text{ and } \varepsilon(s * \langle 1 \rangle) = \pi_3(\varepsilon(a)).
      Define \tau: \mathcal{C} \to [0,1] such that \forall \gamma \in \mathcal{C}[(\tau|\gamma)(n) = \varepsilon(\overline{\gamma}n)].
One may prove: \forall \gamma \in \mathcal{C} \forall \delta \in \mathcal{C} [\gamma \# \delta \to \tau | \gamma \#_{\mathcal{R}} \tau | \delta].
       (ii) Define \chi:\omega^{\omega}\to\omega^{\omega} such that, for all \alpha, for all s, (\chi|\alpha)(s)\neq 0 if and only if
\exists t < s[t \in Bin \ \land \ s \sqsubseteq_{\mathbb{S}} \varepsilon(t) \ \land \ \alpha(t) \neq 0] \ \lor \ \exists n < s \forall t \in Bin_n[s \#_{\mathbb{S}} \varepsilon(t)].
We now prove: \forall \alpha \forall \gamma \in \mathcal{C}[\gamma \in \mathcal{G}_{\alpha} \leftrightarrow \tau | \gamma \in \mathcal{H}_{\gamma | \alpha}]
      Let \alpha be given and assume: \gamma \in \mathcal{C}.
      Assume: \gamma \in \mathcal{G}_{\alpha}. Find n such that \alpha(\overline{\gamma}n) \neq 0.
Find k > n such that \varepsilon(\overline{\gamma}k) > \overline{\gamma}n.
Note \varepsilon(\overline{\gamma}k) \sqsubseteq_{\mathbb{S}} \varepsilon(\overline{\gamma}n) and conclude: (\chi|\alpha)(\varepsilon(\overline{\gamma}k)) \neq 0.
As (\tau|\gamma)(k+1) \sqsubset_{\mathbb{S}} (\tau|\gamma)(k) = \varepsilon(\overline{\gamma}k), conclude: \tau|\gamma \in \mathcal{H}_{\chi|\alpha}.
      Conversely, assume \tau | \gamma \in \mathcal{H}_{\chi | \alpha}. Note: \forall n [\varepsilon(\overline{\gamma}n) \sqsubset \tau | \gamma].
Conclude: \neg \exists s [s \sqsubset \tau | \gamma \land \exists n \forall t \in Bin_n [s \#_{\mathbb{S}} \varepsilon(t)]].
Find n, s such that (\tau | \gamma)(n) \sqsubset_{\mathbb{S}} s and (\chi | \alpha)(s) \neq 0.
Find t in Bin such that s \sqsubset \varepsilon(t) and \alpha(t) \neq 0.
Note: t \sqsubset \gamma and conclude: \gamma \in \mathcal{G}_{\alpha}.
      We thus see: \forall \gamma \in \mathcal{C}[\gamma \in \mathcal{G}_{\alpha} \leftrightarrow \tau | \gamma \in \mathcal{H}_{\chi|\alpha}].
      (iii) Assume \delta \in [0, 1].
      Note: \forall a \in Bin[\varepsilon''(a * \langle 0 \rangle) <_{\mathbb{Q}} \varepsilon'(a * \langle 1 \rangle)].
      Define \gamma in \mathcal{C} such that, for all m, p,
if p = \mu q[\varepsilon''(\overline{\gamma}m * \langle 0 \rangle) <_{\mathbb{Q}} \delta'(q) \lor \delta''(q) <_{\mathbb{Q}} \varepsilon'(\overline{\gamma}m * \langle 1 \rangle)],
then \gamma(m) = 0 \leftrightarrow \delta''(p) <_{\mathbb{Q}} \varepsilon'(\overline{\gamma}m * \langle 1 \rangle).
      One may prove, by induction on n:
for each n, there exists p such that \forall t \in Bin_n[t \perp \overline{\gamma}n \to \delta(p) \#_{\mathbb{S}} \varepsilon(t)].
      Assume: \delta \#_{\mathcal{R}} \tau | \gamma. Find n such that \delta(n) \#_{\mathbb{S}} (\tau | \gamma)(n) = \varepsilon(\overline{\gamma}(n)).
      Find p > n such that \forall t \in Bin_n[t \perp \overline{\gamma}n \to \delta(p) \#_{\mathbb{S}} \varepsilon(t)].
      Conclude: \forall t \in Bin_n[\delta(p) \#_{\mathbb{S}} \varepsilon(t)] and, for each \alpha, \delta \in \mathcal{H}_{\chi|\alpha}.
      We thus see: if \delta \#_{\mathcal{R}} \tau | \gamma, then \forall \alpha [\delta \in \mathcal{H}_{\chi | \alpha}].
      (iv) Assume: \delta \in [0,1]^{\omega}.
      Define \gamma in \mathcal{C} such that, for all n, m, p,
if p = \mu q[\varepsilon''(\overline{\gamma^n}m * \langle 0 \rangle) <_{\mathbb{Q}} (\delta^n)'(q) \lor (\delta^n)''(q) <_{\mathbb{Q}} \varepsilon'(\overline{\gamma^n}m * \langle 1 \rangle)],
then \gamma^n(m) = 0 \leftrightarrow \delta''(p) <_{\mathbb{Q}} \varepsilon'((\gamma^n)m * \langle 1 \rangle).
      Conclude, following the argument for (iii): \forall n[\delta^n \#_{\mathcal{R}} \tau | (\gamma^n) \to \delta^n \in \mathcal{H}_{\alpha^n}].
13.9. Real functions from [0,1] to \mathcal{R}.
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²⁹The definition slightly deviates from the one used in [43, Subsection 8.4].

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\varphi: [0,1] \to_{\mathcal{R}} \mathcal{R} if and only if
       (1) \forall n \in E_{\varphi}[n' \in \mathbb{S} \land n'' \in \mathbb{S}], and
       (2) \forall m \in E_{\varphi} \forall n \in E_{\varphi}[m' \sqsubseteq_{\mathbb{S}} n' \to m'' \sqsubseteq_{\mathbb{S}} n''], and
       (3) \forall \alpha \in [0,1] \forall n \exists m \exists s [(\alpha(m),s) \in E_{\varphi} \land s'' -_{\mathbb{Q}} s' <_{\mathbb{Q}} \frac{1}{2n}].
     \mathcal{R}^{[0,1]} := \{ \varphi \mid \varphi : [0,1] \to_{\mathcal{R}} \mathcal{R} \}.
     Assume: \varphi: [0,1] \to_{\mathcal{R}} \mathcal{R}.
     We define, for each \alpha in [0,1], for each \beta in \mathcal{R},
\varphi: \alpha \mapsto_{\mathcal{R}} \beta \text{ if and only if } \forall n \exists m \exists p \in E_{\varphi}[|\alpha(m) \sqsubseteq_{\mathbb{S}} p' \land p'' \sqsubseteq_{\mathbb{S}} \beta(n)].
     For each \alpha in [0,1], we let \varphi'^{\mathcal{R}}(\alpha) be the element \beta of \mathcal{R} such that, for each n,
\beta(n) = double_{\mathbb{S}}(s''), where s is the least t such that t \in \mathbb{S} and t'' - \mathbb{O} t' = \mathbb{O} \frac{1}{2^n} and
\exists p \le t \exists r \le t \exists m \le t [\varphi(r) = p + 1 \land \alpha(m) \sqsubseteq_{\mathbb{S}} p' \land p'' \sqsubseteq_{\mathbb{S}} t].
Note: \varphi : \alpha \mapsto_{\mathcal{R}} \varphi^{\mathcal{R}}(\alpha).
13.10. Game-theoretic terminology.
     s: n \to k \leftrightarrow (length(s) = n \land \forall j < n[s(j) < k]).
     Seq(n, l) := \{s \mid s : n \to l\}. Note: Bin_n = Seq(n, 2).
    c \in I s \leftrightarrow \forall i [2i < length(c) \rightarrow (\overline{c}(2i) < length(s) \land c(2i) = s(\overline{c}(2i)))].
     c \in H t \leftrightarrow \forall i[2i+1 < length(c) \rightarrow
(\overline{c}(2i+1) < length(t) \land c(2i+1) = t(\overline{c}(2i+1)))].
     (The numbers s, t should be thought of as strategies for player I, II, respectively.)
     c \in_I \sigma \leftrightarrow \forall i[2i < length(c) \rightarrow c(2i) = \sigma(\overline{c}(2i))].
     c \in II \ \tau \leftrightarrow \forall i[2i+1 < length(c) \rightarrow c(2i+1) = \tau(\overline{c}(2i+1))].
    \gamma \in_I \sigma \leftrightarrow \forall i [\gamma(2i) = \sigma(\overline{\gamma}(2i))].
     \gamma \in_{II} \tau \leftrightarrow \forall i [\gamma(2i+1) = \tau(\overline{\gamma}(2i+1))].
     \gamma \in_I s \leftrightarrow \forall i [\overline{\gamma}(2i) < length(s) \rightarrow \gamma(2i) = s(\overline{\gamma}(2i))].
     \gamma \in II \ t \leftrightarrow \forall i [\overline{\gamma}(2i+1) < length(t) \rightarrow \gamma(2i+1) = t(\overline{\gamma}(2i+1))].
    \omega \times \omega := \{ s \mid length(s) = 2 \}.
    2 \times \omega := \{s \mid length(s) = 2 \land s(0) < 2\}.
    \omega \times 2 := \{ s \mid length(s) = 2 \land s(1) < 2 \}.
     For each n > 0, (\omega \times 2)^n := \{s \mid length(s) = 2n \land \forall i < n[s(2i+1) < 2]\} and
(\omega \times 2)^n \times \omega := \{ s \mid length(s) = 2n+1 \land \forall i < n[s(2i+1) < 2] \}.
    (\omega \times 2)^{<\omega} := \bigcup_n (\omega \times 2)^n.
     (\omega \times 2)^{\omega} := \{ \delta \mid \forall n [\delta(2n+1) < 2] \}.
     Halfbin := (\omega \times 2)^{<\omega} \cup ((\omega \times 2)^{<\omega} \times \omega) = \bigcup_{n} \{ \overline{\gamma} n \mid \gamma \in (\omega \times 2)^{\omega} \}.
     For every \delta, \delta_I, \delta_{II} satisfy: \forall n[\delta_I(n) = \delta(2n)] and \forall n[\delta_{II}(n) = \delta(2n+1)].
     For every s, length(s_I) = \mu n[length(s) \le 2n] and \forall n < length(s_I)[s_I(n) = s(2n)],
and length(s_{II}) = \mu n[length(s) \le 2n+1] and \forall n < length(s_{II})[s_{II}(n) = s(2n+1)].
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