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# The large cardinals between supercompact and almost-huge

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Abstract I analyze the hierarchy of large cardinals between a supercompact cardinal and an almost-huge cardinal. Many of these cardinals are defined by modifying the definition of a high-jump cardinal. A high-jump cardinal is defined as the critical point of an elementary embedding  $j:V\to M$  such that M is closed under sequences of length  $\sup\{j(f)(\kappa)\mid f:\kappa\to\kappa\}$ . Some of the other cardinals analyzed include the super-high-jump cardinals, almost-high-jump cardinals, Shelah-for-supercompactness cardinals, Woodinfor-supercompactness cardinals, Vopěnka cardinals, hypercompact cardinals, and enhanced supercompact cardinals. I organize these cardinals in terms of consistency strength and implicational strength. I also analyze the superstrong cardinals, which are weaker than supercompact cardinals but are related to high-jump cardinals. Two of my most important results are as follows.

- Vopěnka cardinals are the same as Woodin-for-supercompactness cardinals.
- There are no excessively hypercompact cardinals.

Furthermore, I prove some results relating high-jump cardinals to forcing, as well as analyzing Laver functions for super-high-jump cardinals.

**Keywords** high-jump cardinals  $\cdot$  Vopěnka cardinals  $\cdot$  Woodin-for-supercompactness cardinals  $\cdot$  hypercompact cardinals  $\cdot$  forcing and large cardinals  $\cdot$  Laver functions

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#### 1 Introduction

The main purpose of this paper is to examine the consistency and implicational strengths of several large cardinals falling between supercompact and almost-huge cardinals. Many of these cardinals are variants of the high-jump cardinals, which are described in definition 2.2. I will also investigate superstrong cardinals, which are weaker than supercompact cardinals but are closely related to high-jump cardinals. Many of the cardinals that I will discuss have been used by Apter, Hamkins, and Sargsyan to prove several results about universal indestructibility in [3], [4], [1], [5] and [2]. This paper is adapted from the second chapter of my doctoral dissertation, [18].

Perhaps the most interesting result in this paper is the main result of section 5, that a Woodin-for-supercompactness cardinal is equivalent to a Vopěnka cardinal. Another noteworthy result is that there are no excessively hypercompact cardinals, which is proven in section 6.

Recall that an almost-huge cardinal  $\kappa$  is characterized by an elementary embedding  $j:V\to M$  with critical point  $\kappa$  such that M is closed under  $< j(\kappa)$ -sequences in  $V.^1$  Many of the large cardinals that I will discuss here are natural weakenings of an almost-huge cardinal, formed by reducing the level of closure of the target model. Indeed, in my study of these cardinals, a key methodology is to define new large cardinals by weakening, strengthening, or otherwise modifying existing large cardinal definitions. Often, the weaker large cardinals will still be sufficient for proving many of the same results as the stronger large cardinals. Eventually, by repeatedly weakening definitions, one hopes to obtain an equiconsistency, as is done in [5]. However, in this paper, I focus on the large cardinals themselves rather than their applications.

The chart at the end of the introduction summarizes the relationships between the large cardinals that I discuss in this paper. Most of the remaining sections will be dedicated to proving these relationships. The arrows on the chart represent relationships between the cardinals, as indicated in the key. A solid arrow from A to B indicates a direct implication: every cardinal with property A has property B. A dotted arrow means that a cardinal of type A is strictly stronger in consistency than a cardinal of type B. That is to say, if there is a cardinal of type A, then it is consistent with ZFC that there is a cardinal of type B. A double arrow indicates that both of these relationships hold. The arrows are labeled with theorem numbers referring to the theorems, propositions, and corollaries in which the corresponding results are proven. Dashed arrows are labeled with two numbers: one for a theorem demonstrating the consistency implication and one for a theorem demonstrating the failure of the direct implication.

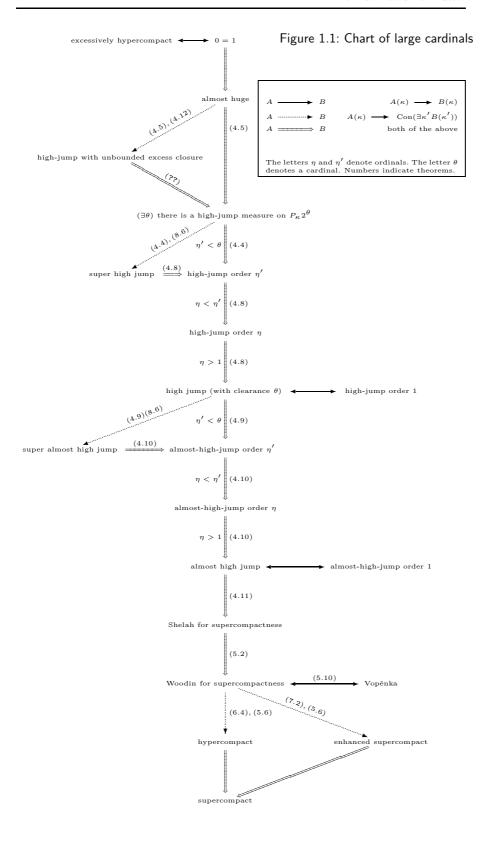
Throughout the paper, I will use the following seed theory notation, which has been popularized by Hamkins, to refer to factor embeddings and the related measures.

<sup>&</sup>lt;sup>1</sup> When I speak of an elementary embedding, I always intend to denote an elementary embedding with a critical point between transitive proper class models of ZFC, unless otherwise stated

**Definition 1.1** Let  $j:V\to M$  be an elementary embedding with critical point  $\kappa$ , and let  $\lambda$  be a cardinal greater than  $\kappa$ . Let U be a normal fine measure on  $P_{\kappa}\lambda$  given by  $A\in U\iff j"\lambda\in j(A)$ . Then U is the normal fine measure on  $P_{\kappa}\lambda$  induced via j by the seed  $j"\lambda$ , and the ultrapower embedding generated by U is the  $\lambda$ -supercompactness factor embedding of j induced by the seed  $j"\lambda$ .

The organization of the paper is as follows. The sections after section 2 can mostly be read out of order. I have noted the most important dependencies between the sections below. In section 2, I define the clearance of an elementary embedding and use this property to define the high-jump cardinals. I also define and analyze the related notions of almost-high-jump cardinals, Shelahfor-supercompactness cardinals, and high-jump functions. In section 3, which depends on section 2, I analyze properties of the clearance of an embedding and prove theorems tying together the ideas of the clearance of an embedding, the almost-high-jump cardinals, and the superstrong cardinals. The next few sections are arranged mostly by decreasing strength of the large cardinal notions studied. In section 4, which depends on section 2 and on lemma 3.4, I define and analyze several large cardinals above a Vopěnka cardinal and below an almost-huge cardinal. In section 5, I define the Vopěnka and Woodin-forsupercompactness cardinals and prove that they are equivalent. In section 6, I define the hypercompact cardinals and the excessively hypercompact cardinals, and I show that the existence of an excessively hypercompact cardinal is inconsistent with ZFC. In section 7, I define the enhanced supercompact cardinals and analyze their place in the large cardinal hierarchy. In section 8, which depends on section 2, I consider the relationship between high-jump cardinals and forcing. In section 9, which depends on section 2 and on lemma 3.4, I develop analogues of Laver functions for high-jump cardinals and related cardinals. In section 10, I review open problems and directions for further re-

I use the label *theorem* to denote very important results. The results labeled as *propositions* vary in their mathematical depth. Some of them might more appropriately be considered as examples or observations.



# 2 High-jump cardinals, almost-high-jump cardinals, and Shelah-for-supercompactness cardinals

In this section, I define high-jump cardinals, almost-high-jump cardinals, and Shelah-for-supercompactness cardinals.<sup>2</sup> I also give characterizations for these large cardinals in terms of ultrafilters and prove a lemma about factor embeddings that will be very useful for the rest of the paper.

The clearance of an elementary embedding, defined below in definition 2.1, is a key concept for defining several large cardinals. The motivation for defining the clearance is for use as a weaker substitute for  $j(\kappa)$  in large cardinal definitions.

**Definition 2.1** Let  $j: M \to N$  be an elementary embedding with critical point  $\kappa$ . The **clearance** of j denotes the ordinal

$$\sup\{j(f)(\kappa) \mid f : \kappa \to \kappa\}.$$

The notation clearance is borrowed from the sport of pole vaulting, where the clearance is the height of the bar that the pole vaulter must clear. A high-jump embedding is like a pole vaulter: for a cardinal to be high jump, the closure of the embedding must successfully clear the clearance, as is described precisely in the following definition. The term high-jump cardinal comes from [3]. However, these cardinals were previously defined in [19, p.111], where they were given the designation  $A_4$ .

**Definition 2.2** The cardinal  $\kappa$  is a **high jump cardinal** if and only if there exists a cardinal  $\theta$  and an elementary embedding  $j:V\to M$  with critical point  $\kappa$  and clearance  $\theta$  such that  $M^{\theta}\subseteq M$ .

An embedding witnessing that  $\kappa$  is high jump is called a **high-jump embedding** for  $\kappa$ . A normal fine measure on some  $P_{\kappa}\lambda$  generating an ultrapower embedding that is a high-jump embedding is called a **high-jump measure**.

The clearance of an embedding has strong properties, as I will show in the next section. In particular, I will show in corollary 3.5 that if  $\theta$  is the clearance of any elementary embedding  $j: V \to N$  with critical point  $\kappa$ , then  $N_{\theta} \prec N_{j(\kappa)}$ .

The following lemma provides a criterion for showing that a factor embedding of a high-jump embeddings is a high-jump embedding. It will be used many times throughout the paper.

**Lemma 2.3** Let  $j: V \to M$  be a high-jump embedding for  $\kappa$  with clearance  $\theta$ . Let  $\lambda \geq \theta$  be a cardinal such that j "  $\lambda \in M$ . Let  $j_0: V \to M_0$  be the factor embedding induced via j by the seed j "  $\lambda$ . Then  $j_0$  is a high-jump embedding, and the clearance of  $j_0$  is  $\theta$ .

 $<sup>^2</sup>$  In many cases, the English usage rule for the punctuation of compound adjectives is to hyphenate compound adjectives coming before a noun, but not compound adjectives coming after a noun. Hence, I will write that  $\kappa$  is a high-jump cardinal, but also that the cardinal  $\kappa$  is high jump.

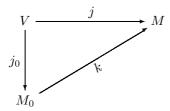


Figure 2.1: Factor embeddings of a high-jump embedding

*Proof* Let  $f: \kappa \to \kappa$  be any function. Referring to the diagram above, note that the critical point of k is greater than  $\lambda$ . It follows that

$$j(f)(\kappa) = (k \circ h)(f)(\kappa) = (k(h(f)))(k(\kappa)) = k(h(f)(\kappa)).$$

The ordinal  $j(f)(\kappa)$  is less than  $\theta$ , since  $\theta$  is the clearance of j. Therefore, again since  $\operatorname{crit}(k) > \lambda$ , it must be the case that  $h(f)(\kappa) = j(f)(\kappa)$ . Since f was arbitrary, the embedding  $j_0$  must have the same clearance as the embedding j, namely  $\theta$ . Since  $\lambda \geq \theta$ , it follows that  $j_0$  is a high-jump embedding.  $\square$ 

Next, I provide a combinatorial characterization of high-jump measures.

**Lemma 2.4** Given an ordered pair of cardinals  $(\kappa, \theta)$ , the following are equivalent.

- 1. There exists a high-jump embedding  $j: V \to M$  with critical point  $\kappa$  such that  $M^{\theta} \subseteq M$  and the clearance of j is at most  $\theta$ .
- 2. There exists a normal fine measure U on  $P_{\kappa}\theta$  such that for every function  $f: \kappa \to \kappa$ , the set  $\{A \in P_{\kappa}\theta \mid f(\operatorname{ot}(A \cap \kappa)) < \operatorname{ot}(A)\}$  is a member of U. (The operator of denotes order type.)

*Proof* The proof consists of a straightforward argument using the Łoś Theorem and lemma 2.3. For the details, see [18, lemma 55].

Next, I define the almost-high-jump cardinals by a slight weakening of the closure property used for defining high-jump cardinals. An almost-high-jump cardinal is to a high-jump cardinal as an almost-huge cardinal is to a huge cardinal.

**Definition 2.5** A cardinal  $\kappa$  is **almost high jump** if and only if there exists an elementary embedding  $j:V\to M$  with critical point  $\kappa$  and clearance  $\theta$  such that  $M^{<\theta}\subseteq M$ . Such an embedding is called an almost-high-jump embedding for  $\kappa$ .

Another way to look at the definition of an almost-high-jump cardinal is as follows. The cardinal  $\kappa$  is almost high jump if and only if there exists an elementary embedding  $j:V\to M$  with critical point  $\kappa$  such that for every function  $f:\kappa\to\kappa$ , the closure property  $M^{j(f)(\kappa)}\subseteq M$  holds. The almost-high-jump cardinals have a combinatorial characterization in terms of coherent sequences of normal fine measures. See [18, lemma 57] for details.

Weakening the definition of an almost-high-jump cardinal to allow for distinct embeddings to witness closure with respect to distinct functions  $f:\kappa\to\kappa$  produces the definition of a Shelah-for-supercompactness cardinal. The analogue of this definition for strongness (in place of supercompactness) was originally formulated by Shelah.

**Definition 2.6** A cardinal  $\kappa$  is **Shelah for supercompactness** if and only if for every function  $f: \kappa \to \kappa$ , there is an elementary embedding  $j: V \to M$  such that  $M^{j(f)(\kappa)} \subseteq M$ .

Note that an almost-high-jump cardinal is a uniform version of a Shelah for supercompactness cardinal — with an almost-high-jump cardinal, one embedding must be the witness for every f uniformly, whereas with a Shelah-for-supercompactness cardinal, each function f may have a separate witnessing embedding.

One might want to define an almost-Shelah-for-supercompactness cardinal by tweaking the above definition to require that the closure of the target model is only  $< j(f)(\kappa)$ . However, this definition is actually equivalent to a Shelah-for-supercompactness cardinal, because of the following argument. Let  $g:\kappa\to\kappa$  be given by  $g(\alpha)=f(\alpha)^+$ . If  $j:V\to M$  is an elementary embedding with critical point  $\kappa$  such that  $M^{< j(g)(\kappa)}\subseteq M$ , then  $M^{j(f)(\kappa)}\subseteq M$  as well.

In [13, p.201], Hamkins defines a high-jump function as follows. A **high-jump function** for a (partially) supercompact cardinal  $\kappa$  is a function  $f:\kappa \to \kappa$  such that  $j(f)(\kappa) > \lambda$  whenever j is a  $\lambda$ -supercompactness embedding on  $\kappa$ . Hamkins allows for partial functions, but any partial high-jump function can be extended to a total high-jump function, so I will assume without loss of generality that high-jump functions are total. Furthermore, I will extend the definition of a high-jump function to the vacuous case where  $\kappa$  has no supercompactness by saying that in this case, there exists a high-jump function for  $\kappa$ . The following proposition shows that the existence of a high-jump function for a cardinal  $\kappa$  is actually an anti-large-cardinal property.

**Proposition 2.7** Let  $\kappa$  be a cardinal. Then there exists a high-jump function for  $\kappa$  if and only if  $\kappa$  is not Shelah for supercompactness.

*Proof* The proof follows immediately from the definitions. The cardinal  $\kappa$  is Shelah for supercompactness if and only if

$$(\forall f: \kappa \to \kappa)(\exists j: V \to M \text{ with critical point } \kappa) \text{ such that } M^{j(f)(\kappa)} \subseteq M$$

The logical negation of this statement is

(\*) 
$$(\exists f : \kappa \to \kappa)(\forall j : V \to M \text{ with critical point } \kappa) M^{j(f)(\kappa)} \nsubseteq M$$

The formula (\*) asserts that 
$$f$$
 is a high-jump function for  $\kappa$ .<sup>3</sup>

<sup>&</sup>lt;sup>3</sup> If one requires that a  $\lambda$ -supercompactness embedding be generated by a normal fine measure on  $P_{\kappa}\lambda$  rather than simply defining such embeddings by the closure of the target model, then a factor embedding argument is required. See [18, proposition 59] for details.

The Shelah-for-supercompactness cardinals have an ultrafilter characterization similar to that for high-jump cardinals, given by the following corollary to lemma 2.4.

**Corollary 2.8** A cardinal  $\kappa$  is Shelah-for-supercompactness if and only if for every function  $f: \kappa \to \kappa$ , there is a cardinal  $\theta$  and a normal fine measure U on  $P_{\kappa}\theta$  such that the set  $\{A \in P_{\kappa}\theta \mid f(\operatorname{ot}(A \cap \kappa)) < \operatorname{ot}(A)\}$  is a member of U.

*Proof* The proof is very similar to that of lemma 2.4 and is given in [18, corollary 60].

# 3 The clearance, superstrongness embeddings, and related embeddings

In the large cardinal literature, a cardinal  $\kappa$  is **superstrong** if and only if there exists an elementary embedding  $j:V\to M$  such that  $V_{j(\kappa)}\subseteq M$ . A cardinal  $\kappa$  is **almost huge** if and only if there exists an elementary embedding  $j:V\to M$  such that  $M^{< j(\kappa)}\subseteq M$ . The chart in the introduction shows that an almost-huge cardinal is much stronger in consistency strength than a high-jump cardinal. Remarkably, the analogous situation does not hold in the case of strongness. In theorem 3.3, I will show that a superstrong cardinal is equivalent to a high-jump-for-strongness cardinal. Before proving this result, I will prove some facts about the clearance of an embedding and about almost-high-jump embeddings. I begin with the following lemma.

**Lemma 3.1** Let  $j: V \to M$  be an elementary embedding with critical point  $\kappa$  and clearance  $\theta$ . Then the following conclusions are true.

- There is no function  $f: \kappa \to \kappa$  such that  $j(f)(\kappa) = \theta$ .
- The ordinal  $\theta$  is a  $\beth$  fixed point in M, that is to say,  $\beth_{\theta}^{M} = \theta$ .
- The inequality  $\kappa^+ \leq \operatorname{cof}(\theta) \leq 2^{\kappa}$  holds in V.

*Proof* To prove the first conclusion, suppose to the contrary that f is a function such that  $j(f)(\kappa) = \theta$ . Let  $g : \kappa \to \kappa$  be defined by  $g(\alpha) = f(\alpha) + 1$ . Then  $j(g)(\kappa) = \theta + 1 > \theta$ , contradicting the definition of the clearance.

Next, I will show that  $\beth_{\beta}^{M} < \theta$  for all ordinals  $\beta < \theta$ , so that  $\beth_{\theta}^{M} = \theta$ . Let  $\beta < \theta$ . Then there exists a function  $f : \kappa \to \kappa$  such that  $j(f)(\kappa) \ge \beta$ . Let the function  $g : \kappa \to \kappa$  be given by  $g(\alpha) = \beth_{f(\alpha)}$ . Then  $\beth_{\beta}^{M} \le j(g)(\kappa) < \theta$ . It follows that  $\theta$  is a  $\beth$  fixed point in M.

The cofinality of the clearance  $\theta$  must be at most  $2^{\kappa}$ , because the clearance is defined as the supremum of a set indexed by functions from  $\kappa$  to  $\kappa$ , of which there are  $2^{\kappa}$  many.

Finally, I show that  $cof(\theta) \geq \kappa^+$  by a diagonalization argument. Suppose to the contrary that  $cof(\theta) \leq \kappa$ . Then there is a sequence  $\langle f_{\alpha} \rangle_{\alpha < \kappa}$  of functions on  $\kappa$  such that  $\theta = \sup\{j(f_{\alpha})(\kappa) \mid \alpha < \kappa\}$ . Define a function  $g: \kappa \to \kappa$  diagonalizing over these functions. That is to say, given  $\beta < \kappa$ , let  $g(\beta) = \sup\{f_{\alpha}(\beta) + 1 \mid \alpha \leq \beta\}$ . Then  $j(f_{\alpha})(\kappa) < j(g)(\kappa) < \theta$  for every  $\alpha < \kappa$ , contradicting the assumption that  $\theta = \sup\{j(f_{\alpha})(\kappa) \mid \alpha < \kappa\}$ .

The next lemma applies the result of lemma 3.1 in the case that j is an almost-high-jump embedding.

**Lemma 3.2** Suppose  $j:V\to M$  is an almost-high-jump embedding with critical point  $\kappa$  and clearance  $\theta$ . Then the following conclusions are true in both V and M.

- The cardinal  $\theta$  is a singular  $\beth$  fixed point.
- The inequality  $\kappa^+ \leq \operatorname{cof}(\theta) \leq 2^{\kappa}$  holds.
- The cardinal exponentiation identity  $\theta^{\kappa} = \theta$  holds.

*Proof* The proof follows from lemma 3.1, along with the fact that M is sufficiently closed so that it agrees with V on cofinalities less than  $\theta$  and on cardinal exponentiation below  $\theta$ .

To show that  $\theta^{\kappa} = \theta$  in both V and M, note that  $\theta$  is a strong limit in both V and M and  $\operatorname{cof}(\theta) > \kappa$  in both V and M. The fact that  $\theta^{\kappa} = \theta$  in both V and M then follows from a basic theorem of cardinal arithmetic (see [15, theorem 5.20]).

With these preliminaries out of the way, I now state the main theorem of this section.

**Theorem 3.3** A cardinal  $\kappa$  is high jump for strongness if and only if  $\kappa$  is superstrong. This fact follows from the following stronger but more technical result.

Let  $\kappa$  be a cardinal. Let  $j: V \to M$  be a high-jump-for-strongness embedding with critical point  $\kappa$  and clearance  $\theta$ . Then  $V_{\theta} \prec M_{j(\kappa)}$ , and j has a factor embedding  $h: V \to M'$  such that h is a superstrongness embedding with critical point  $\kappa$  and such tha  $h(\kappa) = \theta$ .

*Proof* Let  $j: V \to M$  be a high-jump-for-strongness embedding with critical point  $\kappa$  and clearance  $\theta$ . I define the seed hull of  $\theta$  in M, denoted by  $X_{\theta}$ , as follows.

$$X_{\theta} = \{ j(f)(\alpha) \mid \alpha < \theta \text{ and } f \in V \text{ is a function } \}.$$

The seed hull  $X_{\theta}$  is an elementary substructure of M, and setting M' equal to its Mostowski collapse yields the following commutative diagram of elementary embeddings of models of set theory, where k is the inverse of the collapse map.

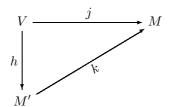


Figure 3.1: Factor embeddings of a high-jumpfor-strongness embedding

Next, I will show that the critical point of k is  $\theta$ , and  $k(\theta) = j(\kappa)$ . Since k is the inverse of the Mostowski collapse of  $X_{\theta}$ , it suffices to show that the supremum of the ordinals  $\beta$  of  $X_{\theta}$  below  $j(\kappa)$  is  $\theta$ . Every such ordinal  $\beta$  is of the form  $j(f)(\alpha)$  for some ordinal  $\alpha < \theta$  and some function  $f: \kappa \to \kappa$ . Fix such an ordinal  $\alpha$  and function f. Since  $\theta$  is the clearance of the embedding j, it follows that  $\alpha < j(g)(\kappa)$  for some other function  $g: \kappa \to \kappa$ . Define yet another function  $g': \kappa \to \kappa$  by  $g'(\beta) = \sup\{f(\gamma) \mid \gamma < g(\beta)\}$ . By the elementarity of j, and since  $\theta$  is the clearance of the embedding j, it follows that

$$j(g')(\kappa) = \sup\{j(f)(\gamma) \mid \gamma < j(g)(\kappa)\} < \theta \tag{3.1}$$

Considering the case  $\gamma = \alpha$  in equation 3.1 above, it follows that  $j(f)(\alpha) < \theta$ . It follows that  $\theta$  is the critical point of k and that  $k(\theta) = j(\kappa)$ , as claimed. Since the diagram above commutes, it further follows that  $h(\kappa) = \theta$ .

Next, I claim that  $V_{\theta} \subseteq M'$ . Towards the proof of this claim, first recall that since j is a high-jump-for-strongness embedding,  $V_{\theta} = M_{\theta}$ . Next, let  $f: \kappa \to V_{\kappa}$  be an enumeration of  $V_{\kappa}$  in V such that whenever  $\alpha < \kappa$ , it follows that  $V_{\alpha} \subseteq f$  "  $\beth_{\alpha}$ . By lemma 3.1, the ordinal  $\theta$  is a  $\beth$  fixed point in M, and so it follows from the definitions of f and of  $X_{\theta}$  that  $M_{\theta} \subseteq X_{\theta}$ . Furthermore  $M_{\theta} = V_{\theta}$ , so  $V_{\theta} \subseteq X_{\theta}$ . Since M' is the Mostowski collapse of  $X_{\theta}$  in M, it follows that  $V_{\theta} \subseteq M'$ , as claimed.

Since  $V_{\theta} \subseteq M'$  and  $h(\kappa) = \theta$ , it follows that h is a superstrongness embedding. Furthermore, the embedding k witnesses that  $V_{\theta} \prec M_{j(\kappa)}$ .

A few easy corollaries to theorem 3.3 follow. I will label the first corollary as a lemma, because it is a key fact about almost-high-jump embeddings and will be used in many places in this paper.

**Lemma 3.4** Let  $j: V \to M$  be an almost-high-jump embedding for  $\kappa$  with clearance  $\theta$ . Then  $V_{\theta} \models \text{ZFC}$  and  $V_{\theta} \prec M_{j(\kappa)}$ .

*Proof* By lemma 3.2, the clearance  $\theta$  of j is a  $\square$  fixed point. Therefore, since j is a  $\theta$ -supercompactness embedding, it is also a  $\theta$ -strongness embedding, and thus a high-jump-for-strongness embedding. It follows immediately from theorem 3.3 that  $V_{\theta} \prec M_{j(\kappa)}$ . Moreover, since  $j(\kappa)$  is inaccessible in M, it follows that  $V_{\theta} \models \text{ZFC}$ .

**Corollary 3.5** Let  $j: V \to M$  be an elementary embedding with critical point  $\kappa$  and clearance  $\theta$ . Then  $M_{\theta} \models \text{ZFC}$  and  $M_{\theta} \prec M_{j(\kappa)}$ .

*Proof* The same line of reasoning as in the proof of theorem 3.3 shows that  $M_{\theta} \prec M_{j(\kappa)}$ , even without the assumption that the embedding j has additional strength.

**Corollary 3.6** Every almost-high-jump embedding has a superstrongness factor embedding.

*Proof* This follows immediately from theorem 3.3, since every almost-high-jump embedding is also a high-jump-for-strongness embedding, as was shown in the proof of lemma 3.4.

As a closing observation, note that analogues of many of the results in this section can be proven when V is replaced by a more general model, N.

### 4 Large cardinals strictly above a Vopěnka cardinal

In the next few sections, I define the remaining large cardinals mentioned in the chart from the introduction, and I prove results about their consistency and implicational strengths. The sections are organized in order of strength in the large cardinal hierarchy. In the present section, I consider cardinals stronger than a Vopěnka cardinal but no stronger than an almost-huge cardinal.

I begin by defining the large cardinal notions that I will be analyzing in this section, starting with the high-jump order and the super-high-jump cardinals. These definitions are somewhat analogous to the definitions of the many times huge and superhuge cardinals, which are defined in [8].

**Definition 4.1** Given an ordinal  $\eta$ , the cardinal  $\kappa$  has **high-jump order**  $\eta$  if and only if there exists a strictly increasing sequence  $\langle \theta_{\alpha} \mid \alpha < \eta \rangle$  of ordinals such that for each ordinal  $\alpha < \eta$ , there exists a high-jump embedding for  $\kappa$  with clearance  $\theta_{\alpha}$ . The cardinal  $\kappa$  is **super high jump** if and only if there exist high-jump embeddings for  $\kappa$  of arbitrarily high clearance. (In other words, a super-high-jump cardinal  $\kappa$  has high-jump order ORD.)

The almost-high-jump order and the super-almost-high-jump cardinals are defined similarly to the high-jump order and the super-high-jump cardinals, as follows.

**Definition 4.2** Given an ordinal  $\eta$ , the cardinal  $\kappa$  has almost-high-jump order  $\eta$  if and only if there exists a strictly increasing sequence  $\langle \theta_{\alpha} \mid \alpha < \eta \rangle$  of cardinals such that for each ordinal  $\alpha < \eta$ , there exists an almost-high-jump embedding for  $\kappa$  with clearance  $\theta_{\alpha}$ . The cardinal  $\kappa$  is super almost high jump if and only if there exist almost-high-jump embeddings of arbitrarily high clearance for  $\kappa$ .

It will also be interesting to consider high-jump embeddings with **excess closure**, that is, embeddings  $j:V\to M$  with clearance  $\theta$  such that the target model M is closed under sequences of length greater than  $\theta$ . For instance, high-jump embeddings with clearance  $\theta$  where the target model is closed under sequences of length  $2^{\theta}$  will be fruitful objects of study. An extreme example of excess closure is as follows.

**Definition 4.3** The cardinal  $\kappa$  is **high jump with unbounded excess closure** if and only if for some fixed clearance  $\theta$ , for all cardinals  $\lambda \geq \theta$ , there is a high-jump measure on  $P_{\kappa}\lambda$  generating an embedding with clearance  $\theta$ .

With all of the above definitions given, the time has come to prove many of the simpler consistency strength relations shown on the chart in the introduction, along with some additional related consistency strength relations that are not shown on the chart. I begin with the following proposition, which involves a high-jump embedding with a little bit of excess closure. This proposition is a simple example of the use of lemma 3.4, which will be used in more complicated arguments later.

**Proposition 4.4** Suppose that there exists a pair of cardinals  $(\kappa, \theta)$  such that there is a high-jump embedding  $j: V \to M$  with critical point  $\kappa$  and clearance  $\theta$  and such that  $M^{2^{\theta}} \subseteq M$ . Then the cardinal  $\kappa$  is super high jump in the model  $V_{\theta}$ , and the cardinal  $\kappa$  has high-jump order  $\theta$  in V. Furthermore, there are many super-high-jump cardinals in the models  $V_{\kappa}$ ,  $V_{\theta}$ , and  $M_{j(\kappa)}$ .

Proof By lemma 2.3, there is a factor embedding, h, of j such that h has clearance  $\theta$  and is generated by a high-jump measure U on  $P_{\kappa}\theta$ . By lemma 3.2, the cardinal exponentiation identity  $\theta^{\kappa} = \theta$  holds. It follows that the model M is sufficiently closed so that  $U \in M$ .

In the model  $M_{j(\kappa)}$ , consider the set of cardinals  $\lambda$  such that there is a high-jump measure generating an embedding with critical point  $\kappa$  and clearance  $\lambda$ . By lemma 3.4, the elementarity relation  $V_{\theta} \prec M_{j(\kappa)}$  holds. It follows that if this set of cardinals is bounded in the model  $M_{j(\kappa)}$ , then this bound is below  $\theta$ . But  $\theta$  is an element of this set, since  $U \in M$ . Therefore, the set is unbounded in both  $V_{\theta}$  and  $M_{j(\kappa)}$ , and in particular,  $\kappa$  is a super-high-jump cardinal in the model  $M_{j(\kappa)}$ . By reflection, there are many super-high-jump cardinals in the model  $V_{\kappa}$ . By the elementarity of j and since  $V_{\theta} \prec M_{j(\kappa)}$ , it follows that there are also many super-high-jump cardinals in  $M_{j(\kappa)}$  and in  $V_{\theta}$ . Finally, since  $V_{\theta} \models \text{ZFC}$  and since every high-jump measure of  $V_{\theta}$  is also a high-jump measure in V, it follows that the cardinal  $\kappa$  has high-jump order  $\theta$  in V.  $\square$ 

In later similar consistency proofs, I will finish the proof with a conclusion about one of  $M_{j(\kappa)}$ ,  $V_{\kappa}$ , or  $V_{\theta}$ , and leave it to the reader to work out the additional consequences in the other models. Note that the hypothesis of proposition 4.4 is equivalent to the hypothesis that there for some pair  $(\kappa, \theta)$ , such that there is a high-jump measure on  $P_{\kappa}2^{\theta}$ . This alternative hypothesis follows immediately from the hypothesis of proposition 4.4. For the converse, given a pair  $(\kappa, \theta)$  such that there is a high-jump measure on  $P_{\kappa}2^{\theta}$ , the clearance of the corresponding embedding must be at most  $\theta$ . If the clearance of this embedding is some  $\theta' < \theta$ , then take a  $2^{\theta'}$ -supercompactness factor embedding and apply lemma 2.3.

Next, I will consider elementary embeddings for which the closure of the target model is extremely large compared with the clearance of the embedding, beginning with the high-jump cardinals with unbounded excess closure.

**Proposition 4.5** Suppose the cardinal  $\kappa$  is almost huge. Then in the model  $V_{\kappa}$ , there are many cardinals  $\delta$  such that  $\delta$  is high jump with unbounded excess closure

*Proof* Suppose  $\kappa$  is almost huge, witnessed by an elementary embedding  $j:V\to M$  with clearance  $\theta$ . In particular, the embedding j is also a high-jump

embedding. Let  $\lambda$  be a cardinal such that  $\theta \leq \lambda < j(\kappa)$ . The cardinal  $j(\kappa)$  is a strong limit cardinal. Therefore, by lemma 2.3, the embedding j has a  $\lambda$ -supercompactness factor embedding with clearance  $\theta$  generated by a high-jump measure on  $P_{\kappa}\lambda$ . This high-jump measure is an element of  $M_{j(\kappa)}$ .

Consider a cardinal  $\kappa$  such that for all sufficiently large cardinals  $\lambda$ , there is a high-jump measure on  $P_{\kappa}\lambda$ . It may be possible that such a cardinal is not high jump with unbounded excess closure, because the high-jump measures may not all generate embeddings with the same closure. However, the following proposition shows that these two types of cardinals are equiconsistent.

**Proposition 4.6** The following two large cardinal axioms are equiconsistent over ZFC.

- 1. There exists a cardinal  $\kappa$  such that for all sufficiently large cardinals  $\lambda$ , there is a high-jump measure on  $P_{\kappa}\lambda$ .
- 2. There exists a cardinal that is high jump with unbounded excess closure

In particular if there are high-jump measures on  $P_{\kappa}\lambda$  for all sufficiently large cardinals  $\lambda$ , then either  $\kappa$  is high jump with unbounded excess closure or else there is a cardinal  $\theta$  such that  $\kappa$  is high jump with unbounded excess closure in the model  $V_{\theta}$ .

*Proof* It is immediate from the definitions that if  $\kappa$  is high jump with unbounded excess closure, then for all sufficiently large  $\lambda$ , there is a high-jump measure on  $P_{\kappa}\lambda$ .

For the converse, suppose that for all sufficiently large  $\lambda$ , there is a high-jump measure on  $P_{\kappa}\lambda$ , but the cardinal  $\kappa$  is not high jump with unbounded excess closure. Let  $\theta_0$  be the minimal cardinal such that for all cardinals  $\lambda \geq \theta_0$ , there is a high-jump measure on  $P_{\kappa}\lambda$ . Since the cardinal  $\kappa$  is not high jump with unbounded excess closure, these high-jump measures do not all generate embeddings with clearance  $\theta_0$ .

None of these measures generates a high-jump embedding with clearance less than  $\theta_0$ . If it did, then the minimality of  $\theta_0$  would be contradicted by taking factor embeddings and applying lemma 2.3.

Accordingly, let  $\theta_1$  be the least cardinal above  $\theta_0$  such that there is a high-jump embedding for  $\kappa$  with clearance  $\theta_1$ . Let  $j:V\to M$  be a high-jump embedding for  $\kappa$  with clearance  $\theta_1$ . Then the model  $V_{\theta_1}$  satisfies ZFC by lemma 3.4, and in this model, the cardinal  $\kappa$  is high jump with unbounded excess closure with respect to the clearance  $\theta_0$ .

Next, proposition 4.7 shows that the degrees of excess closure of high-jump embeddings form a hierarchy of consistency strength. In this hierarchy, there are many more cardinals above the ones described in proposition 4.7 and below the high-jump cardinals with unbounded excess closure. For further details, see [18, pp. 117-118].

**Proposition 4.7** Suppose that for some cardinals  $\kappa$  and  $\theta$  and for some ordinal  $\alpha < \theta$ , there exists a high-jump embedding  $j: V \to M$  with critical point  $\kappa$  and clearance  $\theta$  such that the model M is closed under sequences of length  $2^{\aleph_{\theta+\alpha}^{<\kappa}}$ . Then in the model  $M_{j(\kappa)}$ , there are unboundedly many cardinals  $\lambda$  such that there is a high-jump measure on  $P_{\kappa}(\aleph_{\lambda+\alpha})$  generating a high-jump embedding with critical point  $\kappa$  and clearance  $\lambda$ .

Proof In the model  $M_{j(\kappa)}$ , consider the set of cardinals  $\lambda$  such that there is a high-jump measure on  $P_{\kappa}(\aleph_{\lambda+\alpha})$  generating a high-jump embedding with critical point  $\kappa$  and clearance  $\lambda$ . The model  $M_{j(\kappa)}$  is sufficiently closed to see that  $\theta$  is an element of this set. By lemma 3.4, the elementarity relation  $V_{\theta} \prec M_{j(\kappa)}$  holds, so it follows that this set is unbounded in  $M_{j(\kappa)}$ .

Next, I move on to prove some results lower down in the hierarchy of highjump cardinals and related cardinals.

**Proposition 4.8** Let  $\eta$  and  $\eta'$  be ordinals such that  $\eta < \eta'$ . Suppose the cardinal  $\kappa$  has high-jump order  $\eta'$ . Then there is an elementary embedding  $j: V \to M$  with critical point  $\kappa$  such that the cardinal  $\kappa$  has high-jump order  $\eta$  in  $M_{j(\kappa)}$ .

Proof The cardinal  $\kappa$  has high-jump order  $\eta'$ , and this is witnessed by a sequence of clearances  $\langle \theta_{\alpha} \mid \alpha < \eta' \rangle$ . Let  $j: V \to M$  be a high-jump embedding for  $\kappa$  with clearance  $\theta$  for some  $\theta_{\alpha}$  such that  $\alpha \geq \eta$ . Then the model M is sufficiently closed so that in  $M_{j(\kappa)}$ , the cardinal  $\kappa$  has high-jump order  $\eta$ .  $\square$ 

I now move further down the large cardinal hierarchy, to the almost-high-jump cardinals. Recall from the introduction that almost-high-jump cardinals are characterized by combinatorially by coherent sequences of normal measures, which are described in detail in [18, lemma 57].

**Proposition 4.9** Suppose there is a high-jump embedding with critical point  $\kappa$  and clearance  $\theta$ . Then  $\kappa$  has almost-high-jump order  $\theta$ , and in the models  $V_{\theta}$ ,  $M_{j(\kappa)}$ , and  $V_{\kappa}$ , there are many super-almost-high-jump cardinals.

Proof Suppose  $j:V\to M$  is a high-jump embedding with critical point  $\kappa$  and clearance  $\theta$ . It follows immediately from definitions that the embedding j also witnesses that  $\kappa$  is almost high jump. By corollary 3.2, the cardinal  $\theta$  is a strong limit, and so it follows that the coherent sequence of measures witnessing that there is an almost-high-jump embedding for  $\kappa$  with clearance  $\theta$  is an element of  $H_{\theta^+}$ . This coherent sequence of measures is also an element of M, by the closure of M. Therefore, the cardinal  $\kappa$  is almost high jump in M with clearance  $\theta$ . Consider the set  $\{\delta \mid M_{j(\kappa)} \models \kappa \text{ almost high jump with clearance } \delta \}$ . By theorem 3.5, if this set has a bound in  $M_{j(\kappa)}$ , then the bound must be less than  $\theta$ . It follows that the set is unbounded in  $M_{j(\kappa)}$ , and so  $\kappa$  is super almost high jump in the model  $M_{j(\kappa)}$ . The other conclusions are immediate.  $\square$ 

**Proposition 4.10** Let  $\eta < \eta'$  be ordinals. Suppose the cardinal  $\kappa$  has almosthigh-jump order  $\eta'$ . Then there is an elementary embedding  $j: V \to M$  with critical point  $\kappa$  such that the cardinal  $\kappa$  is has almost-high-jump order  $\eta$  in  $M_{j(\kappa)}$ .

*Proof* The proof follows the same reasoning as the proof of proposition 4.8, replacing high-jump embeddings with almost-high-jump embeddings and high-jump measures with coherent sequences of measures.  $\Box$ 

Finally, I reach the Shelah-for-supercompactness cardinals.

**Proposition 4.11** Suppose the cardinal  $\kappa$  is almost-high-jump. Then there are many cardinals below  $\kappa$  that are Shelah for supercompactness.

Proof Suppose  $j:V\to M$  is an almost-high-jump embedding for  $\kappa$  with clearance  $\theta$ . I will show that  $\kappa$  is Shelah for supercompactness in M. Let  $f:\kappa\to\kappa$  be a function in M. Let  $j_0:V\to M_0$  be the  $\lambda$ -supercompactness factor embedding induced by j via the seed j "  $\lambda$ , where  $\lambda$  is the maximum of  $j(f)(\kappa)$  and  $\kappa$ . From corollary 3.2, it follows that  $2^{\lambda^{<\kappa}}<\theta$ . Therefore, the  $\lambda$ -supercompactness measure U that generates  $j_0$  is an element of M. By reasoning similar to the proof of lemma 2.3, it follows that  $j(f)(\kappa)=j_0(f)(\kappa)$ . Let  $j_0^M$  be the elementary embedding generated by U in M. The measure U is an element of  $V_\theta=M_\theta$ , which satisfies ZFC by theorem 3.5. It follows that  $j_0 \upharpoonright V_\theta=j_0^M \upharpoonright M_\theta$ . Therefore, in M, the elementary embedding  $j_0^M$  witnesses that  $\kappa$  is Shelah for supercompactness with respect to the function f. Since f was arbitrary, it follows that  $\kappa$  is Shelah for supercompactness in M.

I wind up the section with a few miscellaneous propositions. Proposition 4.12 shows that several direct implications are lacking from the large cardinal hierarchy.

**Proposition 4.12** The least high-jump cardinal is not  $\Sigma_2$ -reflecting. In particular, it is not supercompact and not even strong. The same is true for the least almost-huge cardinal, the least almost-high-jump cardinal, and the least Shelah-for-supercompactness cardinal.

*Proof* All of these cardinals can be characterized by  $\Sigma_2$  definitions — they are characterized by a measure or a set of measures with certain combinatorial properties, all of which can be seen from within a particular  $V_{\alpha}$ . Since supercompact and strong cardinals are  $\Sigma_2$ -reflecting, the theorem follows.

The proofs of propositions 4.13 and 4.14 are simple and are left as exercises for the reader. The complete proofs can be found in [18, propositions 81 and 83]

**Proposition 4.13** Suppose  $j: V \to M$  is an elementary embedding with clearance  $\theta$  witnessing that  $\kappa$  is almost-high-jump. Then in the model  $V_{\kappa}$  there are many supercompact cardinals.

The definition of super high jump makes it tempting to think that every cardinal that is both supercompact and high jump is super high jump. However, the following simple proposition shows that this is not the case.

**Proposition 4.14** If  $\kappa$  is the least cardinal that has high-jump order 2, then in  $V_{\kappa}$ , there are many cardinals that are both supercompact and high jump but not super high jump.

Proposition 4.14 shows that below a cardinal of high-jump order 2, there are many cardinals that are both high jump and tall. If a cardinal  $\kappa$  is both high jump and tall, then there are high-jump embeddings for  $\kappa$  such that  $j(\kappa)$  is arbitrarily large — this can be seen by taking a high-jump embedding for  $\kappa$  followed by a tallness embedding for  $j(\kappa)$ . It is easy to see that below a cardinal that is both high-jump and supercompact, there are many high-jump cardinals. But it is an open question whether the existence of a cardinal that is both high-jump and tall is equiconsistent with the existence of a high-jump cardinal.

Many more definitions could be made along the lines of the ones given in this section, and these definitions would lead to many more questions of consistency strength. In light of proposition 4.4, the following question comes to mind.

Question 4.15 What is the consistency strength of the existence of a high-jump embedding with critical point  $\kappa$  and clearance  $\theta$  generated by a high-jump measure on  $P_{\kappa}\theta^{+}$ ? Is the existence of such an embedding equiconsistent with the existence of a high-jump embedding with critical point  $\kappa$  and clearance  $\theta$  generated by a high-jump measure on  $P_{\kappa}2^{\theta}$ ?

Of course, under GCH, these two types of embeddings are equivalent.

# 5 The equivalence of Vopěnka cardinals and Woodin-for-supercompactness cardinals

In this section, I define a Woodin-for-supercompactness cardinal and show that it is strictly weaker than a Shelah-for-supercompactness cardinal. I then show that a cardinal is Woodin for supercompactness if and only if it is a Vopěnka cardinal.

The definition of a Woodin-for-supercompactness cardinal is as follows. This definition is taken from [4, 2]. These cardinals have also been studied by Foreman [9, p.31] and by Fuchs [10, p.1043] under the name of Woodinized supercompact cardinals. A Woodin-for-supercompactness cardinal is the analogue of a Woodin cardinal, with supercompactness in place of strongness.

**Definition 5.1** A cardinal  $\delta$  is **Woodin for supercompactness** if and only if for every function  $f:\delta\to\delta$ , there exists a cardinal  $\kappa<\delta$  such that  $\kappa$  is a closure point of f (i.e. f "  $\kappa\subseteq\kappa$ ), and there exists an elementary embedding  $j:V\to M$  with critical point  $\kappa$  such that  $M^{j(f)(\kappa)}\subseteq M$ .

Note in particular that every Woodin-for-supercompactness cardinal is also a Woodin cardinal.

Apter and Sargsyan require that f be defined so that  $f(\alpha)$  is always a cardinal and so that the elementary embedding j is generated by a supercompactness measure on  $P_{\kappa}\lambda$  for some  $\lambda < \delta$ , but this definition is equivalent to the definition given here. The first requirement does not make their definition any weaker, because any function f can be replaced with a function f' given by  $f'(\alpha) = |f(\alpha)|^+$ . Theorem 5.4 below shows that the second requirement does not make their definition any stronger.

The next theorem shows that a Shelah-for-supercompactness cardinal is strictly stronger than a Woodin-for-supercompactness cardinal. The proof is an improvement on lemma 1.1 of [4] and uses a similar line of reasoning.

**Theorem 5.2** Suppose the cardinal  $\kappa$  is Shelah for supercompactness. Then  $\kappa$  is Woodin for supercompactness, and there are many cardinals below  $\kappa$  that are Woodin for supercompactness in both  $V_{\kappa}$  and V.

Proof Let  $\kappa$  be Shelah for supercompactness. The main difficulty is to show that  $\kappa$  is Woodin for supercompactness. Once this has been shown, it is immediate that there are many cardinals below  $\kappa$  that are Woodin for supercompactness in the model  $V_{\kappa}$ , because "The cardinal  $\kappa$  is Woodin for supercompactness" is  $\Pi_1^1$ -definable over  $V_{\kappa}$ , and Shelah-for-supercompactness cardinals are  $\Pi_1^1$ -indescribable, since they are weakly compact. Furthermore, it is easily seen that any cardinal that is Woodin for supercompactness in the model  $V_{\kappa}$  must also be Woodin for supercompactness in V.

Towards showing that  $\kappa$  is Woodin for supercompactness, let  $f: \kappa \to \kappa$  be an arbitrary function. I will show that  $\kappa$  satisfies the definition of Woodin for supercompactness with respect to f. Without loss of generality, I assume that f is nowhere regressive, that is to say, for all  $\alpha < \kappa$  the inequality  $\alpha \leq f(\alpha)$  holds. It follows that  $\kappa \leq j(f)(\kappa)$ . Let  $g: \kappa \to \kappa$  be given by  $g(\alpha) = 2^{f(\alpha)^{<\alpha}}$ . Let  $j: V \to M$  witness that  $\kappa$  is Shelah for supercompactness with respect to the function g, that is, the embedding j has critical point  $\kappa$  and  $M^{j(g)(\kappa)} \subseteq M$ . Note that  $j(g)(\kappa) = 2^{(j(f)(\kappa))^{<\kappa}}$ , as calculated in both M and V.

Let U be the normal fine measure on  $P_{\kappa}(j(f)(\kappa))$  induced via j by the seed  $j(f)(\kappa)$ . Let  $j_U:V\to N$  be the  $j(f)(\kappa)$ -supercompactness factor embedding induced by U. Let  $k:N\to M$  be the elementary embedding such that  $k\circ h=j$ . The model M is sufficiently closed so that  $U\in M$ , and the closure of M further guarantees that every function from  $P_{\kappa}(j(f)(\kappa))$  to M is an element of M. It follows that the elementary embedding induced by U in M, as calculated in M, is equal to  $j_U\upharpoonright M$ , as calculated in V.

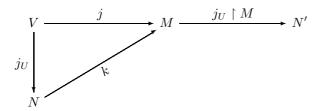


Figure 5.1: Factor embeddings of a Shelah-for-supercompactness embedding

Next, I will show that

$$j(f)(\kappa) = j_U(j(f))(\kappa) \tag{5.1}$$

First of all,  $f = j(f) \upharpoonright \kappa$ . Applying  $j_U$  to both sides, it follows that  $j_U(f) = j_U(j(f)) \upharpoonright j_U(\kappa)$ , and in particular,  $j_U(f)(\kappa) = j_U(j(f))(\kappa)$ . Furthermore, by reasoning similar to the proof of lemma 2.3, it follows that  $j(f)(\kappa) = j_U(f)(\kappa)$ , and so statement 5.1 is proven.

The embedding  $j_U \upharpoonright M$  is generated in M by the measure  $U \in M$ . Therefore, in M, the cardinal  $j(\kappa)$  satisfies the definition of Woodin for supercompactness with respect to the function j(f) — this fact is witnessed by the embedding  $j_U \upharpoonright M$  along with statement 5.1, since  $\kappa$  is a closure point of j(f). It follows from the elementarity of j that in V, the cardinal  $\kappa$  satisfies the definition of Woodin for supercompactness with respect to the function f. But f was chosen arbitrarily, so  $\kappa$  is Woodin for supercompactness in V.  $\square$ 

Like Woodin cardinals, Woodin-for-supercompactness cardinals have several alternative characterizations, as described below in theorem 5.4. The proof that these characterizations hold is essentially the same as for the case of Woodin cardinals, (see [16, Theorem 26.14]). In order to state them, I need the following definition of  $(\gamma, A)$ -supercompactness.

**Definition 5.3** Given a set A and a cardinal  $\gamma$ , the cardinal  $\kappa$  is  $(\gamma, \mathbf{A})$ supercompact if and only if there is an elementary embedding  $j: V \to M$ with critical point  $\kappa$  such that

- 1.  $\gamma < j(\kappa)$ ,
- 2.  $M^{\gamma} \subseteq M$ , and
- 3.  $A \cap V_{\gamma} = j(A) \cap V_{\gamma}$ .

Given cardinals  $\kappa$  and  $\delta$  such that  $\kappa < \delta$ , the notation  $\kappa$  is  $(< \delta, A)$ -supercompact denotes that  $\kappa$  is  $(\gamma, A)$ -supercompact for all cardinals  $\gamma < \delta$ . The alternative characterizations of Woodin-for-supercompactness cardinals are as follows.

**Theorem 5.4** Given a cardinal  $\delta$ , the following are equivalent.

- 1. For every function  $f: \delta \to \delta$ , there exists a cardinal  $\kappa < \delta$  such that  $\kappa$  is a closure point of f (i.e.  $f "\kappa \subseteq \kappa$ ), and there exists an elementary embedding  $h: V \to M$  with critical point  $\kappa$  such that  $M^{h(f)(\kappa)} \subseteq M$ . Furthermore, the embedding h is generated by a normal fine measure on  $P_{\kappa}\lambda$  for some cardinal  $\lambda < \delta$ .
- The cardinal δ is Woodin for supercompactness. That is to say, for every function f: δ → δ, there exists a cardinal κ < δ such that κ is a closure point of f (i.e. f " κ ⊆ κ), and there exists an elementary embedding j: V → M with critical point κ such that M<sup>j(f)(κ)</sup> ⊆ M.
- 3. For every set  $A \subseteq V_{\delta}$ , the set

$$\{ \kappa < \delta \mid \kappa \text{ is } (<\delta, A)\text{-supercompact} \}$$

is stationary in  $\delta$ .

4. For every set of ordinals  $A \subseteq \delta$ , the set

$$\{ \kappa < \delta \mid \kappa \text{ is } (<\delta, A)\text{-supercompact} \}$$

is nonempty.

*Proof* This is the analogue of a standard theorem for Woodin cardinals, and the proof is essentially the same as for that theorem. See [18, theorem 88] for details.  $\Box$ 

I now shift my attention to the Vopěnka cardinals, which I will eventually show are equivalent to the Woodin-for-supercompactness cardinals.

**Definition 5.5** The cardinal  $\delta$  is **Vopěnka** if and only if for every  $\delta$ -sequence of model-theoretic structures  $\langle M_{\alpha} \mid \alpha < \delta \rangle$  over the same language, with each structure  $M_{\alpha}$  an element of  $V_{\delta}$ , there exists an elementary embedding  $j: M_{\alpha} \to M_{\beta}$  for some ordinals  $\alpha < \beta < \delta$ .

The following result is well-established in the large cardinal literature.

**Proposition 5.6** The least cardinal that is Vopěnka is not weakly compact.

*Proof* "The cardinal  $\kappa$  is Vopěnka" is definable by a  $\Pi_1^1$  formula over  $V_{\kappa}$ , but weakly compact cardinals are  $\Pi_1^1$ -indescribable.

It is convenient to have a characterization of Vopěnka cardinals in terms of a more limited class of model-theoretic structures. Towards this end, following [16], I make the following definition.

**Definition 5.7** Let  $\delta$  be a cardinal. A sequence of model-theoretic structures,  $\langle M_{\alpha} \mid \alpha < \delta \rangle$  is a **natural**  $\delta$ -sequence if and only if the following properties are satisfied. There is a function  $f: \delta \to \delta$  such that the domain of  $M_{\alpha}$  is  $V_{f(\alpha)}$  and such that whenever  $\alpha < \beta < \delta$ , are ordinals, it follows that  $\alpha < f(\alpha) \le f(\beta) < \delta$ . Furthermore, each structure  $M_{\alpha}$  is of the form  $(V_{f(\alpha)}, \in, \{\alpha\}, R_{\alpha})$ , for some unary relation  $R_{\alpha} \subseteq V_{f(\alpha)}$ .

Without loss of generality,  $R_{\alpha}$  may be taken to encode finitely many constants, functions, and relations of any finite arity on  $V_{f(\alpha)}$ . I will show in proposition 5.9 below that it suffices to consider only natural sequences when determining whether the cardinal  $\kappa$  is Vopěnka.

It is immediately clear that any sequence of the type specified in definition 5.5 can be encoded as a natural sequence if the language involved is countable. For larger languages, one might be concerned that the critical point of the embedding would be small enough to mess up the encoding.

To clear up this concern, I will discuss the critical point of an elementary embedding between two elements of a natural sequence. This analysis will also play an important role in the proof of the equivalence between Vopěnka cardinals and Woodin-for-supercompactness cardinals. The inclusion of the constant  $\{\alpha\}$  in the definition of a natural sequence ensures that any elementary embedding between members of a natural sequence has a critical point.

I begin the analysis of these critical points with the definition of the Vopěnka filter, given below in definition 5.8. The Vopěnka filter is actually a filter on  $\delta$  if and only if  $\delta$  is a Vopěnka cardinal. In this case, the Vopěnka filter is a normal filter and contains every club.<sup>4</sup> These facts are proven in [16, pp.336-337] and [19, proposition 6.3].

**Definition 5.8** ([16, p.336]) Let  $\delta$  be an inaccessible cardinal. Then the set  $X \subseteq \delta$  is a member of the **Vopěnka filter** on  $\delta$  if and only if there exists a natural  $\delta$ -sequence  $\langle M_{\alpha} \mid \alpha < \delta \rangle$  such that whenever  $j : M_{\alpha} \to M_{\beta}$  is an elementary embedding, then the critical point of j is an element of X.

Given an elementary embedding  $j: V_{\alpha} \to V_{\beta}$ , the critical point of j must be inaccessible, and so if  $\delta$  is a Vopěnka cardinal, then the set of inaccessible cardinals below  $\delta$  is a member of the Vopěnka filter on  $\delta$ . In particular, this implies that  $\delta$  is Mahlo, since the Vopěnka filter contains every club.

The next proposition states that it suffices to consider only natural sequences in defining Vopěnka cardinals.

**Proposition 5.9** The cardinal  $\delta$  is Vopěnka if and only if for every normal  $\delta$ -sequence  $\langle M_{\alpha} \mid \alpha < \delta \rangle$ , there is an elementary embedding  $j: M_{\alpha} \to M_{\beta}$  for some ordinals  $\alpha, \beta < \delta$ .

*Proof* The forwards direction is immediate. For the converse, the essential observation is that the Vopěnka filter on  $\delta$  contains every tail, so that after encoding a sequence of model-theoretic structures as a natural sequence, it is possible to choose an embedding with critical point much larger than the size of the language of the original structures, so that the embedding does not interfere with the encoding.

Kanamori suggests the equivalence of a Woodin-for-supercompactness cardinal to a Vopěnka cardinal in [16, p.364]. However, he does not formally define

<sup>&</sup>lt;sup>4</sup> Every normal ultrafilter on a cardinal  $\delta$  contains every club. But a normal filter on a cardinal  $\delta$  contains every club if and only if it contains every tail. The Vopěnka filter on a Vopěnka cardinal is not in general an ultrafilter, by proposition 5.6.

a Woodin-for-supercompactness cardinal, nor does he work out the details of the equivalence.

**Theorem 5.10** The cardinal  $\delta$  is Woodin for supercompactness if and only if  $\delta$  is a Vopěnka cardinal. Furthermore, if  $\delta$  is a Vopěnka cardinal, then for every set  $A \subseteq V_{\delta}$ , the set  $\{ \kappa < \delta \mid \kappa \text{ is } (<\delta, A)\text{-supercompact} \}$  is a member of the Vopěnka filter on  $\delta$ .

*Proof* For the forward direction, let  $\delta$  be Woodin for supercompactness, and let  $A = \langle A_{\alpha} \mid \alpha < \delta \rangle$  be a natural  $\delta$ -sequence. I will show that for some ordinals  $\alpha < \beta < \delta$ , there exists an elementary embedding  $j : A_{\alpha} \to A_{\beta}$ .

Since  $\delta$  is Woodin for supercompactness, there is a cardinal  $\kappa < \delta$  such that  $\kappa$  is  $(<\delta,A)$ -supercompact. Therefore, by choosing a large enough degree of A-supercompactness, there is an elementary embedding  $j:V\to N$  such that  $j(A)_{\kappa}=A_{\kappa}$  and  $j\upharpoonright A_{\kappa}\in N$ . In N, the map  $j\upharpoonright A_{\kappa}$  is an elementary embedding from  $j(A)_{\kappa}$  to  $j(A)_{j(\kappa)}$ . So in N, there exists an elementary embedding between two elements of j(A). By the elementarity of j, there exists an elementary embedding between two elements of A in A. It follows that A is Vopěnka.

The proof of the converse direction uses some of the same ideas as the proof of proposition 24.14 of [16], which shows that  $V_{\delta}$  contains many extendible cardinals if  $\delta$  is Vopěnka. Suppose that the cardinal  $\delta$  is Vopěnka, and let  $A \subseteq V_{\delta}$ . I will show that there exists a cardinal  $\kappa < \delta$  such that  $\kappa$  is  $(<\delta,A)$ -supercompact, thereby showing that the cardinal  $\delta$  is Woodin for supercompactness. Indeed, I will show that the set of such  $\kappa$  is an element of the Vopěnka filter on  $\delta$ .

Let  $g: \delta \to \delta$  be the variation of the failure-of-A-supercompactness function described as follows. Given  $\xi < \delta$ , let  $g(\xi)$  be the least cardinal  $\eta > \xi$  such that  $\xi$  is not  $(\eta, A)$ -supercompact. In case no such  $\eta$  exists, then set  $g(\xi) = \xi$ .

Let  $C \subseteq \delta$  be the club of closure points of g, i.e.  $C = \{ \rho < \delta \mid g " \rho \subseteq \rho \}$ . Since the Vopěnka filter on  $\delta$  contains every club, it follows that the club C is a member of this filer. Therefore, there exists a natural  $\delta$ -sequence  $\langle M_{\alpha} \mid \alpha < \delta \rangle$  such that whenever  $j: M_{\alpha} \to M_{\beta}$  is an elementary embedding, the critical point of j is an element of C.

For each ordinal  $\alpha < \delta$ , let  $\gamma_{\alpha}$  be the least inaccessible element of C above all the ordinals of  $M_{\alpha}$ , and for each ordinal  $\alpha < \delta$ , let

$$N_{\alpha} = (V_{\gamma_{\alpha}}, \in, \{\alpha\}, M_{\alpha}, C \cap \gamma_{\alpha}, A \cap V_{\gamma_{\alpha}}).$$

Let  $j: N_{\alpha} \to N_{\beta}$  be an elementary embedding. It suffices to show that the critical point,  $\kappa$ , of j is  $(<\delta, A)$ -supercompact.

Assume to the contrary that  $\kappa$  is not  $(<\delta, A)$ -supercompact. Then  $\kappa < g(\kappa)$ . Furthermore,  $g(\kappa) < \gamma_{\alpha}$ , because  $\gamma_{\alpha} \in C$ . Since  $M_{\alpha}$  is encoded in  $N_{\alpha}$ , it follows from the definition of the sequence  $\langle M_{\alpha} \rangle$  that  $\kappa \in C$ . By the elementarity of j, it follows that  $j(\kappa) \in C$  as well.

Let U be the normal fine measure on  $P_{\kappa}(g(\kappa))$  induced via j by the seed j "  $g(\kappa)$ , and let  $j_U: V \to N$  be the ultrapower generated by U. Using the

fact that  $\gamma_{\alpha}$  is inaccessible, the theory of factor embeddings shows that there exists an elementary embedding k so that the following diagram commutes.

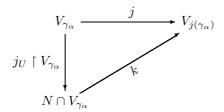


Figure 5.2: Factor embeddings of a Vopěnka embedding

I claim that the map  $j_U: V \to N$  witnesses that  $\kappa$  is  $(g(\kappa), A)$ -supercompact. This will contradict the definition of g, thereby completing the proof. Clearly, this map is a  $g(\kappa)$ -supercompactness embedding with critical point  $\kappa$ , and so it suffices to show that  $j_U(A) \cap V_{q(\kappa)} = A \cap V_{q(\kappa)}$ .

it suffices to show that  $j_U(A) \cap V_{g(\kappa)} = A \cap V_{g(\kappa)}$ . First of all, since  $A \cap V_{\gamma_\alpha}$  is encoded in  $N_\alpha$ , it follows that  $j(A) \cap V_{g(\kappa)} = A \cap V_{g(\kappa)}$ . Since the critical point of k is an inaccessible cardinal above  $g(\kappa)$ , it follows that  $j(A) \cap V_{g(\kappa)} = j_U(A) \cap V_{g(\kappa)}$ .

#### 6 There are no excessively hypercompact cardinals.

In definition 1.2 of [2], Apter defined an excessively hypercompact cardinal as follows.  $^5$ 

**Definition 6.1 (Apter, [2])** A cardinal  $\kappa$  is excessively 0-hypercompact iff  $\kappa$  is supercompact. For  $\alpha > 0$ , a cardinal  $\kappa$  is excessively  $\alpha$ -hypercompact iff for any cardinal  $\delta \geq \kappa$ , there is an elementary embedding  $j: V \to M$  witnessing the  $\delta$ -supercompactness of  $\kappa$  (i.e.  $\operatorname{cp}(j) = \kappa, j(\kappa) > \delta$ , and  $M^{\delta} \subseteq M$ ) generated by a supercompact ultrafilter over  $P_{\kappa}(\delta)$  such that  $M \models \text{``}\kappa$  is excessively  $\beta$ -hypercompact for every  $\beta < \alpha$ ''. A cardinal  $\kappa$  is excessively hypercompact iff  $\kappa$  is excessively  $\alpha$ -hypercompact for every ordinal  $\alpha$ .

Postulating the existence of an excessively hypercompact cardinal leads to a contradiction.

**Theorem 6.2** There are no excessively hypercompact cardinals. In particular, there is no cardinal  $\kappa$  such that  $\kappa$  is excessively  $(2^{\kappa})^+$ -hypercompact.

Proof Suppose towards a contradiction that  $\kappa$  is least such that  $\kappa$  is excessively  $(2^{\kappa})^+$ -hypercompact. Apply the definition of excessive hypercompactness in the case  $\delta = \kappa$  to obtain an elementary embedding  $j: V \to M$  which is witnessed by a normal fine measure on  $P_{\kappa}\kappa$  (which is isomorphic to a normal measure on  $\kappa$ ) such that  $\kappa$  is excessively  $\alpha$ -hypercompact in M for all  $\alpha < ((2^{\kappa})^+)^V$ . This includes all  $\alpha < j(\kappa)$ , since  $j(\kappa)$  has cardinality  $2^{\kappa}$  in V.

<sup>&</sup>lt;sup>5</sup> At that time, Apter called these cardinals hypercompact rather than excessively hypercompact. But in light of theorem 6.2, we now call them excessively hypercompact.

In particular, it includes the case of  $\alpha = \left((2^{\kappa})^{+}\right)^{M}$ , since this is less than  $j(\kappa)$ . By reflection, there are many  $\gamma < \kappa$  such that  $\gamma$  is excessively  $(2^{\gamma})^{+}$ -hypercompact, and this contradicts the minimality of  $\kappa$ .

In definition 6.3, I describe a hypercompact cardinal. Apter had erroneously believed that this definition was equivalent to the definition of an excessively hypercompact cardinal.<sup>6</sup> However, the existence of a hypercompact cardinal is strictly weaker in consistency strength than the existence of a Woodin-for-supercompactness cardinal; I prove this fact in theorem 6.4. The proofs in [2] all work using hypercompact cardinals in place of excessively hypercompact cardinals, so the error in the definition given in that paper did not have severe consequences.<sup>7</sup>

**Definition 6.3** The hypercompact cardinals are defined recursively as follows. Given any ordinal  $\alpha$ , the cardinal  $\kappa$  is  $\alpha$ -hypercompact if and only if for every ordinal  $\beta < \alpha$  and for every cardinal  $\lambda \geq \kappa$ , there exists a cardinal  $\lambda' \geq \lambda$  and there exists an elementary embedding  $j: V \to M$  generated by a normal fine measure on  $P_{\kappa}\lambda'$  such that the cardinal  $\kappa$  is  $\beta$ -hypercompact in M. (In particular, every cardinal is 0-hypercompact, and 1-hypercompact is equivalent to supercompact.) The cardinal  $\kappa$  is hypercompact if and only if it is  $\beta$ -hypercompact for every ordinal  $\beta$ .

The key difference between the definitions of hypercompact and excessively hypercompact is that in the definition of hypercompact, the embedding j need not be witnessed by a normal fine measure on  $P_{\kappa}\lambda$ , but can be witnessed instead by a larger supercompactness measure.<sup>8</sup>

Note that both the hypercompact cardinals and the excessively hypercompact cardinals are first-order definable in ZFC. Formally, the definition of a hypercompact cardinal is by recursion on  $\kappa$  as follows. Assuming recursively that the set

$$HC_{<\kappa} := \{ (\alpha, \eta) \mid \eta \text{ is } \alpha\text{-hypercompact and } \eta < \kappa \}$$

is already defined, define that  $\kappa$  is  $\alpha$ -hypercompact if and only if for every  $\beta < \alpha$  and for every  $\lambda \geq \kappa$  there exists  $\lambda' \geq \lambda$  and there exists an elementary embedding  $j: V \to M$  generated by a normal fine measure on  $P_{\kappa}\lambda'$  such that  $(\beta, \kappa) \in j(H_{<\kappa})$ . This in turn can be stated formally as a first-order proposition using the Loś theorem, without referring explicitly to the embedding j.

I now establish the consistency of a hypercompact cardinal relative to a Woodin-for-supercompactness cardinal. The bold part of the proof emphasizes why the proof would not work to establish the consistency of an excessively hypercompact cardinal.

<sup>&</sup>lt;sup>6</sup> personal communication with Apter, 2012

<sup>&</sup>lt;sup>7</sup> personal communication with Apter, 2012.

<sup>&</sup>lt;sup>8</sup> An additional minor difference is that the definition of hypercompact handles limit stages differently from the definition of excessively hypercompact. I made this change in order to unify the definition for the successor and limit stages, and also to define hypercompact cardinals analogously to the Mitchell order.

**Theorem 6.4** If the cardinal  $\delta$  is Woodin for supercompactness, then in the model  $V_{\delta}$ , there is a proper class of hypercompact cardinals.

Proof Suppose  $\delta$  is Woodin for supercompactness. Suppose towards a contradiction that the hypercompact cardinals of  $V_{\delta}$  are bounded above by some cardinal  $\eta$ . Let the function  $f:\delta\to\delta$  be the failure-of-hypercompactness function as defined in the model  $V_{\delta}$ . That is to say, for an ordinal  $\xi<\delta$ , let  $f(\xi)$  be the least ordinal  $\beta$  such that  $\xi$  is not  $\beta$ -hypercompact in  $V_{\delta}$  if such a  $\beta$  exists, and let  $f(\xi)=0$  otherwise.

By theorem 5.4, there is a  $(<\delta,f)$ -supercompact cardinal  $\kappa$  above  $\eta$ , and this fact is witnessed by a collection of elementary embeddings  $j_{\gamma}:V\to M_{\gamma}$  for  $\gamma<\delta$ . (The subscripted  $\gamma$  serves to index the target model, not to refer to a rank-initial cut thereof. ) If  $\gamma$  is taken to be sufficiently large, then  $(\kappa, f(\kappa)) \in j_{\gamma}(f)$ , and so  $j_{\gamma}(f)(\kappa) = f(\kappa)$ . That is to say, in  $M_{\gamma}$ , the cardinal  $\kappa$  is  $\beta$ -hypercompact for every  $\beta < f(\kappa)$ . By taking a factor embedding if necessary, assume that  $j_{\gamma}$  is generated by a normal fine measure U on  $P_{\kappa}\gamma$  such that  $U \in V_{\delta}$ . Thus in  $V_{\delta}$ , the collection of embeddings  $\langle j_{\gamma} \rangle$  witness that  $\kappa$  is  $f(\kappa)$ -hypercompact, contradicting the definition of f. This contradiction completes the proof.

Finally, I consider the extent to which the hierarchy of  $\beta$ -hypercompactness and the hierarchy of excessive  $\beta$ -hypercompactness coincide for particular small values of  $\beta$ .

**Theorem 6.5** Let  $\kappa$  be a cardinal, and let  $\beta \leq \kappa^+$  be an ordinal. If  $\kappa$  is  $\beta$ -hypercompact, then for every ordinal  $\alpha < \beta$  and for every cardinal  $\lambda \geq \kappa$ , there is an elementary embedding  $j: V \to M$  generated by a normal fine measure on  $P_{\kappa}\lambda$  such that  $\kappa$  is  $\alpha$ -hypercompact in M. Thus, the  $\beta$ -hypercompactness and excessive  $\beta$ -hypercompactness hierarchies align below  $\kappa^+$ .

*Proof* The proof is by induction on ordinals  $\beta$ . Suppose that the cardinal  $\kappa$  is  $\beta$ -hypercompact and that the theorem is true for all  $\beta' < \beta$ . Let  $\lambda \geq \kappa$  be a cardinal, and let  $\alpha < \beta$ . It suffices to show that there is an elementary embedding  $j: V \to M$  generated by a normal fine measure in V on  $P_{\kappa}\lambda$  such that in M, the cardinal  $\kappa$  is  $\alpha$ -hypercompact.

By hypothesis, the cardinal  $\kappa$  is  $\beta$ -hypercompact. So for some cardinal  $\theta \geq \lambda$ , there exists an elementary embedding  $j: V \to M$  such that in M, the cardinal  $\kappa$  is  $\alpha$ -hypercompact.

Let  $j_{\lambda}: V \to M_{\lambda}$  be the  $\lambda$ -supercompactness factor embedding induced via j by the seed j "  $\lambda$ , and let  $k: M_{\lambda} \to M$  be the elementary embedding such that  $k \circ j_{\lambda} = j$ , as in the following commutative diagram. To be precise, the embedding  $j_{\lambda}$  is the ultrapower generated by  $U_{\lambda}$ , where  $U_{\lambda}$  is the normal fine measure on  $P_{\kappa}\lambda$  given by  $A \in U \iff j$  "  $\lambda \in J(A)$ . (The subscript  $\lambda$  in  $M_{\lambda}$  serves to index the model  $M_{\lambda}$ , not to denote a level of its cumulative

<sup>&</sup>lt;sup>9</sup> Actually, this alignment is off by one, because the definitions of these hierarchies handle limit stages differently. But this fact is a technical detail not germane to the main idea.

hierarchy.)

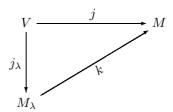


Figure 6.1: Factor embeddings of a hypercompactness embedding

If  $M_{\lambda} = M$ , then the existence of the embedding  $j_{\lambda}$  suffices to complete the proof. If  $M_{\lambda} \neq M$ , then the elementary embedding k must be nontrivial, and its critical point must be an greater than  $\kappa$  and inaccessible in  $M_{\lambda}$ . The model  $M_{\lambda}$  agrees with V on  $\kappa^+$ , so this critical point must be greater than  $\kappa^+$ . Therefore, it follows from the elementarity of k that  $\kappa$  is  $\alpha$ -hypercompact in  $M_{\lambda}$ .

# 7 Enhanced supercompact cardinals

In this brief section, I analyze the consistency strength of an enhanced supercompact cardinal. The definition of an enhanced supercompact cardinal comes from Apter's paper, [1].

**Definition 7.1** A cardinal  $\kappa$  is **enhanced supercompact** if and only if there exists a strong cardinal  $\theta > \kappa$  such that for every cardinal  $\lambda > \theta$ , there exists a  $\lambda$ -supercompactness embedding  $j: V \to M$  such that  $\theta$  is strong in M.

Apter required that the embedding j be generated by a normal fine measure on  $P_{\kappa}\lambda$ ,. This requirement provides a first-order characterization, but it adds no strength, because one can take a factor embedding.

This next theorem shows that a Woodin-for-supercompactness cardinal is strictly stronger in consistency than an enhanced supercompact cardinal.

**Theorem 7.2** Suppose the cardinal  $\delta$  is Woodin for supercompactness. Then there are unboundedly many cardinals  $\kappa < \delta$  such that  $\kappa$  is a limit of cardinals  $\eta$  such that there exists an inaccessible cardinal  $\beta$  such that  $\eta < \beta < \kappa$ , and

$$V_{\beta} \models \eta$$
 is enhanced supercompact.

Proof The proof follows the same general line of reasoning as theorem 5 of [1]. Suppose  $\delta$  is Woodin for supercompactness. Let  $f: \delta \to \delta$  be given by taking  $f(\alpha)$  to be the second strong cardinal of  $V_{\delta}$  greater than  $\alpha$ . This function is well-defined, since the strong cardinals of  $V_{\delta}$  are unbounded, since  $\delta$  is Woodin.

Let  $\kappa$  be a closure point of f, and let  $j: V \to M$  be an elementary embedding such that  $M^{j(f)(\kappa)} \subseteq M$  and  $j(f)(\kappa) < \delta$ , i.e. the embedding j witnesses that  $\delta$  is Woodin for supercompactness with respect to the function f. By theorem 5.4, assume without loss of generality that the embedding j

is generated by a normal fine measure on  $P_{\kappa}\lambda$  for some cardinal  $\lambda < \delta$ . It follows that  $j(\delta) = \delta$ . By the definition of f and the elementarity of j, there is a cardinal  $\kappa_0$  such that  $\kappa < \kappa_0 < j(f)(\kappa)$ , and the cardinal  $\kappa_0$  is strong in the model  $M_{j(\delta)} = M_{\delta}$ , and furthermore, the cardinal  $j(f)(\kappa)$  is strong in the model  $M_{\delta}$ .<sup>10</sup>

For each cardinal  $\lambda$  such that  $\kappa_0 < \lambda < j(f)(\kappa)$ , let  $U_{\lambda}$  be the normal fine measure on  $P_{\kappa}\lambda$  given by  $A \in U \iff j \text{ "} \lambda \in j(A)$ . Let  $j_{\lambda}: V \to M_{\lambda}$  be the  $\lambda$ -supercompactness embedding generated by  $U_{\lambda}$ , and let  $i: M_{\lambda} \to M$  be the elementary embedding such that  $i \circ j_{\lambda} = j$ . (The subscripted  $\lambda$  serves to index the model  $M_{\lambda}$ , not to denote a level of its cumulative hierarchy.)

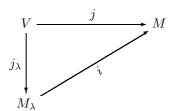


Figure 7.1: Factor embeddings of a Woodin-for-supercompactness embedding Suppose towards a contradiction that for some cardinal  $\gamma$  with  $\kappa_0 < \gamma < j(f)(\kappa)$ , and for some cardinal  $\lambda$  such that  $\kappa_0 < \lambda < j(f)(\kappa)$ ,

$$M_{\lambda} \models \kappa_0$$
 is not  $\gamma$ -strong.

Then by elementarity,

$$M \models i(\kappa_0)$$
 is not  $i(\gamma)$ -strong.

But i fixes  $\kappa_0$ , and so this contradicts the fact that the cardinal  $\kappa_0$  is strong in  $M_{\delta}$ , since  $i(\gamma) < i(j(f)(\kappa)) \le j(j(f)(\kappa)) < \delta$ . From this contradiction, I conclude that for all cardinals  $\gamma$  and  $\lambda$ , if  $\kappa_0 < \gamma < j(f)(\kappa)$  and  $\kappa_0 < \lambda < j(f)(\kappa)$  then

$$M_{\lambda} \models \kappa_0$$
 is  $\gamma$ -strong.

Finally, from the closure of M, it follows that  $U_{\lambda} \in M_{j(f)(\kappa)}$  for each cardinal  $\lambda$  such that  $\kappa_0 < \lambda < j(f)(\kappa)$ . Furthermore, for each such cardinal  $\lambda$ , the elementary embedding generated by  $U_{\lambda}$  in the model M is equal to  $j_{\lambda} \upharpoonright M$ . Since  $\lambda$  was taken to be an arbitrary cardinal between  $\kappa_0$  and  $j(f)(\kappa)$ , it follows that in the model  $M_{j(f)(\kappa)}$ , the cardinal  $\kappa$  is enhanced supercompact.

By reflection, in  $V_{\kappa}$ , there are unboundedly many cardinals  $\eta$  such that for some inaccessible cardinal  $\beta$  with  $\eta < \beta < \kappa$ ,

$$V_{\beta} \models \eta$$
 is enhanced supercompact.

By a simple modification to the function f, the cardinal  $\kappa$  can be made arbitrarily large below  $\delta$ . The conclusion of the theorem follows.

<sup>&</sup>lt;sup>10</sup> Actually, it suffices for the proof that  $j(f)(\kappa)$  is inaccessible.

# 8 High-jump cardinals and forcing

In this section, I prove some results about the preservation and destruction of high-jump cardinals by forcing.

Suppose  $j:V\to M$  is a high-jump embedding, and V[G] is a forcing extension of V. Under what conditions does j lift to a high-jump embedding  $j^*:V[G]\to M[H]$ ? The conditions under which a supercompactness embedding lifts to a supercompactness embedding have been well-studied in the literature. The following lemma extends these conditions to provide conditions for which a high-jump embedding lifts to a high-jump embedding.

**Lemma 8.1** Suppose  $j: V \to M$  is a high-jump embedding for  $\kappa$  with clearance  $\theta$ . Let V[G] be a forcing extension of V, and suppose that j lifts to a  $\theta$ -supercompactness embedding  $j^*: V[G] \to M[H]$ . Let U be the normal measure on  $\kappa$  given by  $A \in U \iff \kappa \in j(A)$ . If the family of functions  $(\kappa^{\kappa})^V$  is  $\leq_U$ -unbounded in  $(\kappa^{\kappa})^{V[G]}$ , then the lifted embedding  $j^*$  is a high-jump embedding. Furthermore, if  $M[H]^{\theta^+} \nsubseteq M[H]$  in V[G], then the conclusion can be strengthened to a biconditional: the lifted embedding  $j^*$  is a high-jump embedding if and only if the family of functions  $(\kappa^{\kappa})^V$  is  $\leq_U$ -unbounded in  $(\kappa^{\kappa})^{V[G]}$ .

Proof Note that since U is an ultrafilter, the family of functions  $(\kappa^{\kappa})^{V}$  is  $\leq_{U}$ -unbounded in  $(\kappa^{\kappa})^{V[G]}$  if and only this family is a dominating family, which is true if and only if the forcing does not add a U-dominating function.

To prove the first part of the theorem, assume that the family of functions  $(\kappa^{\kappa})^V$  is  $\leq_U$ -unbounded in  $(\kappa^{\kappa})^{V[G]}$ . In V[G], let  $f:\kappa\to\kappa$ . It suffices to show that  $j^*(f)(\kappa)<\theta$ . Since  $(\kappa^{\kappa})^V$  is a dominating family, there is a function  $g\in(\kappa^{\kappa})^V$  such that  $f\leq_U g$ . It follows that

$$j(f)(\kappa) \le j(g)(\kappa) < \theta$$
,

and so the lifted embedding is a high-jump embedding.

To prove the second part of the theorem, suppose that  $M[H]^{\theta+} \nsubseteq M[H]$  in V[G], and that  $(\kappa^{\kappa})^V$  is  $\leq_U$ -bounded by some function  $g \in (\kappa^{\kappa})^{V[G]}$ . Then  $j^*(g)(\kappa) \geq j^*(f)(\kappa)$  for every  $f \in (\kappa^{\kappa})^V$ , and so in particular  $j^*(g)(\kappa) \geq \theta$ , and so the function g witnesses that  $j^*$  is not a high-jump embedding.  $\square$ 

One particular important instance where the class of functions  $(\kappa^{\kappa})^{V}$  is unbounded in  $(\kappa^{\kappa})^{V[G]}$  is if the forcing satisfies the  $\kappa$ -chain condition.

The biconditional version of the lemma actually holds even holds in many cases where M[H] is closed under sequences of length greater than  $\theta$  — given a g such that  $j^*(g)(\kappa) \geq \theta$ , one can easily modify the function g to produce another function h such that  $j^*(h)(\kappa)$  is much larger than  $\theta$ . For instance, let  $h(\alpha)$  be the least measurable cardinal above  $g(\alpha)$ , so that  $j^*(h)(\kappa)$  is the least measurable cardinal of M[H] above  $\theta$ .

<sup>&</sup>lt;sup>11</sup> By a  $\theta$ -supercompactness embedding, I simply mean that M[H] is sufficiently closed, not that the embedding is generated by a normal fine measure.

The next theorem addresses the preservation of high-jump cardinals in the downwards direction.

**Theorem 8.2** Suppose  $V \subseteq \overline{V}$  satisfies the  $\delta$  approximation and cover properties, and for some cardinals  $\kappa, \theta > \delta$  there is a high-jump measure U on  $P_{\kappa}\theta$  in  $\overline{V}$ . Then there is a high-jump measure on  $P_{\kappa}\theta$  in V as well.

Proof Let  $j: \overline{V} \to \overline{N}$  be the elementary embedding generated by U in  $\overline{V}$ . By the proof of corollary 26 of [11], the restricted embedding  $j \upharpoonright V : V \to N$  is amenable with V, and  $N^{\theta} \subseteq N$  in V. In particular,  $j \upharpoonright V$  is a high-jump embedding. Let  $j_0: V \to M$  be the  $\theta$ -supercompactness factor embedding induced via  $j \upharpoonright V$  by the seed j "  $\theta$ . Let  $f: \kappa \to \kappa$  be a function. It follows from lemma 2.3 applied in V to the embedding construction ensures that  $j_0$  is generated by a measure that is an element of V, so the proof is complete.  $\square$ 

Next, I show that the previous two results together prove the analogue of the Levy-Solovay theorem for high-jump cardinals.

**Theorem 8.3** Let  $\mathbb{P}$  be a forcing notion such that  $|\mathbb{P}| < \kappa$ . Let  $G \subseteq \mathbb{P}$  be V-generic. Then in V[G], the cardinal  $\kappa$  is high jump if and only if  $\kappa$  is high jump in V.

Proof Since the forcing is small, in particular it satisfies the  $\kappa$ -chain condition, and so every function  $f: \kappa \to \kappa$  in V[G] is bounded by such a function in V. Thus, the upwards direction of the proof follows from lemma 8.1. By lemma 13 of [11], the forcing  $\mathbb P$  satisfies the  $\delta$  approximation and cover properties for some cardinal  $\delta < \kappa$ . Thus, the downwards direction of the proof follows immediately from theorem 8.2.

Next, I apply lemma 8.1 to show that the canonical forcing of the GCH preserves high-jump cardinals.

**Theorem 8.4** Every high-jump cardinal is preserved by the canonical forcing  $\mathbb{P}$  of the GCH. To be precise, the forcing  $\mathbb{P}$  is defined as the Easton support product over all infinite cardinals  $\delta$  of  $Add(\delta^+, 1)$ .

Proof Let  $G \subseteq \mathbb{P}$  be V-generic. In V, let U be a high-jump measure on  $P_{\kappa}\theta$  for some cardinals  $\kappa$  and  $\theta$ . Let  $j_U$  be the high-jump embedding generated by U. It follows from a standard argument that the embedding  $j_U$  lifts to a  $\theta$ -supercompactness embedding  $j_U^*:V[G]\to M[H].^{12}$  To complete the proof that high-jump cardinals are preserved by  $\mathbb{P}$ , it suffices to show that every function on  $f:\kappa\to\kappa$  in V[G] is dominated by such a function in V and then apply lemma 8.1. Towards this end, note that the forcing  $\mathbb{P}$  factors as  $\mathbb{P}_{\leq\kappa}*\mathbb{P}_{\geq\kappa}$ . The first factor satisfies the  $\kappa$ -chain condition, and so every function  $f:\kappa\to\kappa$  added by it is dominated by a ground model function. The second factor is  $\leq\kappa$ -closed, and so it adds no new function  $f:\kappa\to\kappa$ .

 $<sup>^{12}</sup>$  For details, see theorem 105 of Hamkins's unpublished book, Forcing and Large Cardinals.

High-jump cardinals are in general much more fragile than supercompact cardinals, as is shown by the following theorem.

**Theorem 8.5** Let  $\kappa$  be a high-jump cardinal. After forcing with  $Add(\kappa, 1)$  or with  $Add(\kappa^+, 1)$ , the cardinal  $\kappa$  is no longer a high-jump cardinal.

*Proof* A recent theorem of Bagaria, Hamkins, and Tsaprounis shows that superstrong cardinals are destroyed by these forcings, among others [7]. By corollary 3.6, every high-jump cardinal is also superstrong, so it follows that these forcings also destroy high-jump cardinals.

Finally, I show that if the cardinal  $\kappa$  is high jump, then there is a forcing extension where  $\kappa$  is still high jump but is not supercompact.

**Theorem 8.6** Suppose there exists a high-jump measure on  $P_{\kappa}\theta$ , and furthermore, the cardinal  $\kappa$  is supercompact. Let  $\mathbb{P}$  be any forcing smaller than  $\kappa$ . Let  $g \subseteq \mathbb{P}$  be V-generic. Let  $\mathbb{Q}$  be any nontrivial forcing that is  $\leq 2^{\theta^{<\kappa}}$ -closed in V[g], and let  $G \subseteq \mathbb{Q}$  be V[g]-generic. Then in V[g][G], there is still a high-jump measure on  $P_{\kappa}\theta$ , but the cardinal  $\kappa$  is not supercompact.

Proof Since the forcing  $\mathbb{P}$  is small relative to  $\kappa$ , the cardinal  $\kappa$  is still both supercompact and high jump in V[g]. Because of the closure condition on the forcing  $\mathbb{Q}$ , this forcing does not add any subsets or elements to  $P_{\kappa}\theta$ , nor does it add any new functions  $f: \kappa \to \kappa$ . Therefore, in V[g][G], the cardinal  $\kappa$  is still high jump. However, since the forcing  $\mathbb{Q}$  is nontrivial, there is a cardinal  $\lambda > 2^{\theta^{<\kappa}}$  such that  $\mathbb{Q}$  adds a subset to  $\lambda$ . By a theorem of Hamkins and Shelah ([14, p.551]), the cardinal  $\kappa$  is no longer  $\lambda$ -supercompact in V[g][G].

Some open questions on the topics of this section are as follows.

Question 8.7 Suppose  $j: V \to M$  is a high-jump embedding for  $\kappa$  with clearance  $\theta$ . What types of forcing, if any, preserve the  $\theta$ -supercompactness of  $\kappa$  while destroying the high-jump cardinal property of  $\kappa$ ?

Question 8.7 can be further refined to a question about individual embeddings rather than about cardinals.

Question 8.8 Let  $j:V\to M$  be a high-jump embedding for  $\kappa$  generated by a high-jump measure. Let  $\mathbb P$  be a forcing notion, and suppose that the embedding j lifts over  $\mathbb P$  such that the lift is a supercompactness embedding. Under what conditions does the lift fail to be a high-jump embedding?

### 9 Laver functions for high-jump cardinals

In this section, I define Laver functions for super-high-jump cardinals and establish their existence under suitably strong hypotheses. Laver functions were originally defined for supercompact cardinals in [17].

Given a supercompact cardinal  $\kappa$ , a supercompactness Laver function for  $\kappa$  is a partial function  $\ell:\kappa\to V_\kappa$  such that for every cardinal  $\lambda$  and for every set  $x\in H_{\lambda^+}$  there is a  $\lambda$ -supercompactness embedding generated by a normal fine measure on  $P_\kappa\lambda$  such that  $j(\ell)(\kappa)=x$ . One can also put additional requirements on the domain of a supercompactness Laver function. For instance, one can require that each  $\gamma\in\mathrm{dom}(\ell)$  is an inaccessible cardinal such that the closure property  $\ell$  "  $\gamma\subseteq V_\gamma$  is satisfied. A super-high-jump Laver function is defined similarly to a supercompactness Laver function, as follows.

**Definition 9.1** Given a super-high-jump cardinal  $\kappa$ , a **super-high-jump** Laver function for  $\kappa$  is a partial function  $\ell : \kappa \to V_{\kappa}$  satisfying the following properties. For every set x, for unboundedly many cardinals  $\delta$ , there is high-jump embedding with critical point  $\kappa$  and clearance  $\delta$ , generated by a high-jump measure, such that  $j(\ell)(\kappa) = x$ . Furthermore, for every ordinal  $\gamma \in \text{dom}(\ell)$ , the closure property  $\ell$  "  $\gamma \subseteq V_{\gamma}$  holds.

I will prove that every supercompact cardinal has a supercompactness Laver function anticipating every set, and whenever the cardinal  $\kappa$  is  $2^{\theta^{<\kappa}}$ -supercompact, there is a supercompactness Laver function for  $\kappa$  anticipating every set in  $H_{\theta}^+$ .

The analysis for high-jump cardinals is more complicated than in the case of supercompact cardinals, because a supercompactness factor embedding of a high-jump embedding is not in general a high-jump embedding. For this reason, the high-jump cardinals with excess closure are a useful tool. As a warm-up exercise before reading the proof of the existence of super-high-jump Laver functions, the reader may wish to review proposition 4.4, which uses a related technique.

**Theorem 9.2** Let  $\kappa$  be a cardinal. Then there exists a partial function  $\ell : \kappa \to V_{\kappa}$  such that for all cardinals  $\theta$ , if there is a high-jump measure on  $P_{\kappa}2^{\theta}$  generating an ultrapower embedding with clearance  $\theta$ , then in the model  $V_{\theta}$ , the function  $\ell$  is a super-high-jump Laver function for  $\kappa$ .

*Proof* Define the function  $\ell$  recursively as follows. Suppose that  $\ell \upharpoonright \gamma$  has been defined. Define  $\ell(\gamma)$  as described in the next paragraph if the relevant hypotheses hold. Otherwise, leave  $\gamma$  out of the domain of  $\ell$ .

Suppose that  $\ell$  "  $\gamma \subseteq V_{\gamma}$  and that furthermore, in the model  $V_{\kappa}$ , some set x witnesses that the function  $\ell \upharpoonright \gamma$  is not a super-high-jump Laver function for  $\gamma$ . That is to say, in the model  $V_{\kappa}$ , there is a cardinal  $\delta_0$  such that for all cardinals  $\delta > \delta_0$ , there is no elementary embedding  $j: V \to M$  with critical point  $\gamma$  and clearance  $\delta$ , generated by a high-jump measure, such that  $j(\ell \upharpoonright \gamma)(\gamma) = x$ . Then pick a set  $x \in V_{\kappa}$  of minimal  $\in$ -rank among all sets with this property, and let  $\ell(\gamma) = x$ .

I now verify that the function  $\ell$  has the desired feature. Suppose not. Then there is some cardinal  $\theta$  such that there is a high-jump measure  $\mu$  on  $P_{\kappa}2^{\theta}$  generating an ultrapower embedding with clearance  $\theta$ , but in the model  $V_{\theta}$ , some set x witnesses that the function  $\ell$  fails to be a super-high-jump Laver

function for  $\kappa$ . Let  $j:V\to M$  be the ultrapower generated by  $\mu$ . By lemma 3.4, the elementarity relation  $V_{\theta}\prec M_{j(\kappa)}$  holds. Therefore, in the model  $M_{j(\kappa)}$ , the set x witnesses that the function  $\ell$  is not a super-high-jump Laver function for  $\kappa$ . That is to say, in the model  $M_{j(\kappa)}$ , there is some cardinal  $\delta_0$  such that for all cardinals  $\delta > \delta_0$ , there does not exist a high-jump embedding h, generated by a high-jump measure, with critical point  $\kappa$  and clearance  $\delta$ , such that  $h(\ell)(\kappa) = x$ .

Accordingly, since  $j(\ell) \upharpoonright \kappa = \ell$ , it follows from the definition of the function  $\ell$  and from the elementarity of the embedding j that  $j(\ell)(\kappa)$  is defined and equal to some set  $y \in M_{j(\kappa)}$  such that in the model  $M_{j(\kappa)}$ , the set y witnesses that the function  $\ell$  is not a super-high-jump Laver function for  $\kappa$ . Furthermore, this set y is of minimal  $\in$ -rank, and so  $y \in V_{\theta}$ , since  $V_{\theta} \prec M_{j(\kappa)}$ .

Let U be the  $\theta$ -supercompactness measure on  $P_{\kappa}\theta$  induced by j via the seed j "  $\theta$ . Let  $j_U:V\to N$  be the supercompactness embedding generated by U, and let the elementary embedding k be such that the diagram below commutes.

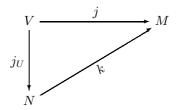


Figure 9.1: Factor embeddings of a high-jump embedding with excess closure

By reasoning similar to the proof of lemma 2.3, the measure U is a high-jump measure, the clearance of  $j_U$  is  $\theta$ , and  $j_U(\ell)(\kappa) = y$ . Furthermore, since the model  $M_{j(\kappa)}$  is closed in V under sequences of length  $2^{\theta}$ , and since  $\theta^{\kappa} = \theta$  by lemma 3.2, it follows that  $U \in M_{j(\kappa)}$ , and the model  $M_{j(\kappa)}$  correctly computes that  $j_U(\ell)(\kappa) = y$ . (In the previous sentence,  $j_U$  denotes the embedding generated by the measure U in the model  $M_{j(\kappa)}$ .) However,  $\theta > \delta_0$ , so this computation contradicts the fact that y is not anticipated in  $M_{j(\kappa)}$  by  $\ell$  with respect to any high-jump embedding with clearance greater than  $\delta_0$  that is generated by a high-jump measure.

Corollary 9.3 Suppose that for some cardinal  $\kappa$ , there is an unbounded set of cardinals  $\theta$  such that there is a high-jump measure on  $P_{\kappa}2^{\theta}$  generating an ultrapower embedding with clearance  $\theta$ . Then there exists a super-high-jump Laver function for  $\kappa$  in V.

Proof Define the function  $\ell$  as in theorem 9.2. It follows from theorem 9.2 that for arbitrarily large cardinals  $\theta$ , the function  $\ell$  is a super-high-jump Laver function for  $V_{\theta}$ . Furthermore, for such a cardinal  $\theta$ , if  $U \in V_{\theta}$  is a high-jump measure, generating an embedding  $j_U : V \to M$ , then the embedding generated by the measure U in the model  $V_{\theta}$  is  $j_U \upharpoonright V_{\theta}$ . It follows that  $\ell$  is a super-high-jump Laver function for  $\kappa$ .

It is also possible to define Laver functions for other large cardinal notions related to high-jump cardinals, for instance for particular types of high-jump cardinals with excess closure. For an example, see [18, theorem 116].

I close the section with a question.

Question 9.4 Is it possible to prove the existence or the consistency of a superhigh-jump Laver function from a hypothesis substantially weaker than that of theorem 9.2?

For instance, it may be possible to force the existence of a super-high-jump Laver function for  $\kappa$ , beginning in a model where  $\kappa$  is only super-high-jump.

#### 10 Ideas for further research

In this section, I review some of the areas for further research discussed in previous sections, and I also suggest a few additional areas for further research.

One relationship between cardinals in the chart is unresolved. I do not know the relationship between enhanced supercompact cardinals and hypercompact cardinals. One established large cardinal is conspicuously missing from my analysis. An extendible cardinal is known to be intermediate in consistency strength between a supercompact cardinal and a Vopěnka cardinal. But I don't know the relationship between an extendible cardinal and a hypercompact cardinal or an enhanced supercompact cardinal. Furthermore, the  $C^{(n)}$ -extendible cardinals, introduced by Bagaria in [6], fall into the large cardinal hierarchy between an extendible cardinal and a Vopěnka cardinal.

Another possible direction for further research would be to define more large cardinal notions by modifying the definitions that I have already given. One possibility would be to modify the definition of a high-jump cardinal so that M is closed under  $j(f)(\kappa)$ -sequences for all  $f:\kappa\to ORD$  rather than just for  $f:\kappa\to\kappa$ . Such a cardinal would be huge, to say the least. This hugeness is witnessed by the case where f is the function with constant value  $\kappa$ .

As discussed in the conclusion of section 8, more work can also be done on the relationships between high-jump cardinals and forcing. More generally, there is more work to be done proving forcing results for all of the cardinals discussed in this paper, and in particular, many of the results from section 8 could be extended to apply to other large cardinals in this paper.

Keeping in mind that many of the large cardinals in this paper were first applied towards universal indestructibility results, it is an interesting goal to use the new large cardinals to weaken the hypotheses for universal indestructibility results, with the goal of eventually proving an equiconsistency between a universal indestructibility result and a large cardinal notion. One could also prove new universal indestructibility results. More generally, the large cardinals that I have studied here could be used to weaken the hypotheses or find equiconsistencies for other set-theoretic results as well.

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## References

- Arthur W. Apter. Reducing the consistency strength of an indestructibility theorem. Math. Log. Q., 54:288–293, 2008.
- Arthur W. Apter. Some applications of Sargsyan's equiconsistency method. Fund. Math., 216:207–222, 2012.
- 3. Arthur W. Apter and Joel David Hamkins. Universal indestructibility. *Kobe J. Math.*, 16(2):119–130, 1999.
- 4. Arthur W. Apter and Grigor Sargsyan. A reduction in consistency strength for universal indestructibility. Bull. Pol. Acad. Sci. Math., 55:1–6, 2007.
- Arthur W. Apter and Grigor Sargsyan. An equiconsistency for universal indestructibility. J. Symbolic Logic, 75:314–322, 2010.
- 6. Joan Bagaria.  $C^{(n)}$ -cardinals. Arch. Math. Logic.
- 7. Joan Bagaria, Joel David Hamkins, and Konstantinos Tsaprounis. Superstrong cardinals are never Laver indestructible. In preparation.
- 8. Julius B. Barbanel, Carlos A. Diprisco, and It Beng Tan. Many-times huge and superhuge cardinals. *J. Symbolic Logic*.
- Matthew Foreman. Generic elementary embeddings, lecture 3. Appalachian Set Theory, June 2007. http://www.math.cmu.edu/~eschimme/Appalachian/Foreman.html (Accessed Dec. 19, 2012).
- 10. Gunter Fuchs. Combined maximality principles up to large cardinals. J. Symbolic Logic.
- 11. Joel David Hamkins. Extensions with the approximation and cover properties have no new large cardinals. Fund. Math.
- 12. Joel David Hamkins. Forcing and large cardinals. book manuscript in preparation.
- Joel David Hamkins. Destruction or preservation as you like it. Ann. Pure Appl. Logic, 91(2-3):191–229, 1998.
- Joel David Hamkins and Saharon Shelah. Superdestructibility: a dual to Laver's indestructibility. J. Symbolic Logic, 63(2):549–554, 1998. [HmSh:618].
- 15. Thomas Jech. Set Theory. Springer Monographs in Mathematics, 3rd edition, 2003.
- Akihiro Kanamori. The Higher Infinite, Corrected Second Edition. Springer-Verlag, 2004
- 17. Richard Laver. Making the supercompactness of  $\kappa$  indestructible under  $\kappa$ -directed closed forcing. Israel J. Math.
- Norman Lewis Perlmutter. Inverse limits and high-jump cardinals. PhD dissertation, CUNY Graduate Center, Department of Mathematics, May 2013.
- Robert M. Solovay, William N. Reinhardt, and Akihiro Kanamori. Strong axioms of infinity and elementary embeddings. Ann. Math. Logic.