A Null Ideal for Inaccessibles

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Abstract

In this paper we introduce a tree-like forcing notion extending some properties of the random forcing in the context of 2^{κ} , κ inaccessible, and study its associated ideal of null sets and notion of measurability. This issue was also addressed by Shelah ([11, Problem 0.5]) and concerns the definition of a forcing which is κ^{κ} -bounding, $< \kappa$ -closed and κ^+ -cc, for κ inaccessible. This also contributes to a line of research adressed in the survey paper [6].

1 Introduction

In [11, Problem 0.5(2)], Shelah addresses the following question:

Can one define a forcing which is κ^+ -cc, $<\kappa$ -closed and κ^{κ} -bounding, for κ inaccessible?

In [10], Shelah provides a positive answer when κ is weakly compact. More recently, Cohen and Shelah (see [2]) introduce a forcing notion satisfying the three properties mentioned for κ inaccessible, and not necessarily weakly compacy. A construction different from the one described by Cohen and Shelah [2] is presented here. For this purpose, we need a certain version of \Diamond , modelling our construction on work of Jensen [5] in the case $\kappa = \omega$. The main differences from [2] are the following: we use different versions of diamond; the use of diamond is different; and finally, our construction is somehow simpler, but also it is more restrictive, as our diamond hypothesis implies $2^{\kappa} = \kappa^+$.

Note that in the standard case, i.e., when $\kappa = \omega$, the random forcing fits with the three properties. So the first attempts to resolve such a problem might be to generalize it for κ inaccessible. Hence, since random forcing is usually defined by means of the Lebesgue measure in 2^{ω} , the natural way would be to define an appropriate measure in 2^{κ} as well. Nevertheless, there seem to be many obstacles in trying to do so. Our method for defining our forcing $\mathbb F$ will not use any notion of measure.

Note that even if we will not define a measure on 2^{κ} , we will define an ideal of \mathbb{F} -null sets and a notion of measurability associated to it, in a rather standard way. We will then investigate such a regularity property.

The paper is organized as follows. In section 3 we present the construction of our forcing \mathbb{F} . Section 4 is devoted to introducing the ideal of null sets and the notion of measurability, and to proving some (negative) results about Δ_1^1

 $^{^{1}\}mathrm{We}$ thank the referee to make us aware of [2], and to point these differences out.

sets and the club fiter. A final section is then dedicated to some concluding remarks and possible further developments.

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2 Preliminaries

In this preliminary section we simply introduce the basic notions and notation which is needed throughout the paper.

- A tree T is a subset of $2^{<\kappa}$, closed under initial segments. Stem(T) denotes the longest node of T compatible with all the other nodes of T; succ $(t,T):=\{\xi<\kappa:t^{\smallfrown}\xi\in T\}$; Split(T) is the set of splitting nodes of T (i.e., $t\in \text{Split}(T)$ if both $t^{\smallfrown}0$ and $t^{\smallfrown}1\in T$); we put $\text{ht}(T):=\sup\{\alpha:\exists t\in T(|t|=\alpha)\}$, while Term(T) denotes the terminal nodes of T (i.e., $t\in \text{Term}(T)$ if there is no $t'\supsetneq t$ such that $t'\in T$). For $\alpha<\kappa$, $T\!\!\upharpoonright\!\alpha:=\{t\in T:|t|<\alpha\}$. A branch through T of height κ is the limit of an increasing cofinal sequence $\{t_\xi:\xi<\kappa\}$ of nodes in T, and [T] will denote the set of all branches of T. For t in a tree T, T_t is the set of nodes in T compatible with t.
- Given $t \in \mathsf{Split}(T)$, we define the rank of t as the order type of $\{\alpha < |t| : t \upharpoonright \alpha \in \mathsf{Split}(T)\}$. Furthermore, we let $\mathsf{Split}_{\beta}(T)$ denote the set of splitting nodes in T of rank β . The forcing \mathbb{S}^Cub consists of trees T such that every node can be extended to a splitting node and for some club C in κ , the splitting nodes of T are exactly those with length in C. For T, T' in \mathbb{S}^Cub and $\gamma < \kappa$, we write $T \leq_{\gamma} T'$ iff T is a subtree of T' such that $\mathsf{Split}_{\gamma}(T) = \mathsf{Split}_{\gamma}(T'); \leq_0$ is simply denoted by \leq .

Note that $\mathbb{S}^{\mathsf{Cub}}$ is closed under \leq_{γ} -descending sequences of length less than κ for each $\gamma < \kappa$.

- If $\{\mathbb{F}_{\alpha} : \alpha < \kappa\}$ is a sequence of families of trees such that $\mathbb{F}_{\alpha} \subseteq \mathbb{F}_{\alpha+1}$, then for every tree $T \in \bigcup_{\alpha < \kappa} \mathbb{F}_{\alpha}$ define $\operatorname{Rank}(T)$ to be the least $\alpha < \kappa$ such that $T \in \mathbb{F}_{\alpha+1}$. $\mathbb{F}_{<\lambda}$ denotes the union of the \mathbb{F}_{α} for $\alpha < \lambda$.
- A forcing \mathbb{P} is called κ^{κ} -bounding iff for every $x \in \kappa^{\kappa} \cap V^{\mathbb{P}}$ there exists $z \in \kappa^{\kappa} \cap V$ such that $\Vdash \forall \alpha < \kappa(x(\alpha) < z(\alpha))$.
- In this paper \mathbb{C} refers to the κ -Cohen forcing, i.e., the poset consisting of $t \in 2^{<\kappa}$, ordered by extension. The elements of 2^{κ} and κ^{κ} are called κ -reals.

Under the assumption $2^{<\kappa} = \kappa$, \mathbb{C} is obviously κ^+ -cc, but also adds unbounded κ -reals, which means it is not κ^{κ} -bounding. For κ inaccessible $\mathbb{S}^{\mathsf{Cub}}$ is κ^{κ} -bounding, but one loses the κ^+ -cc. The next section is devoted to defining a refinement of $\mathbb{S}^{\mathsf{Cub}}$ in order to obtain the κ^+ -cc and maintain κ^{κ} -boundedness.

3 The main construction

Fix κ to be inaccessible. As we mentioned before, our first main goal is to define a tree-forcing $\mathbb F$ with the following three properties: κ^+ -cc, κ^{κ} -bounding and $< \kappa$ -closure. Assume $\lozenge_{\kappa^+}(S_{\kappa}^{\kappa^+})$, where $S_{\kappa}^{\kappa^+} := \{\lambda < \kappa^+ : \operatorname{cf}(\lambda) = \kappa\}$.

We construct an increasing sequence of tree forcings $\langle \mathbb{F}_{\lambda} : \lambda < \kappa^{+} \rangle$ by induction on $\lambda < \kappa^{+}$. We first remark that for all $\lambda < \kappa^{+}$ we will maintain the following:

- (P1) $\mathbb{F}_{\lambda} \subseteq \mathbb{S}^{\mathsf{Cub}} \text{ and } |\mathbb{F}_{\lambda}| \leq \kappa;$
- (P2) $\forall T \in \mathbb{F}_{<\lambda} \forall \gamma < \kappa \exists T' \leq_{\gamma} T \forall T'' \leq T' (T' \in \mathbb{F}_{\lambda} \wedge T'' \notin \mathbb{F}_{<\lambda});$
- (P3) $\forall T \in \mathbb{F}_{\lambda} \forall t \in T(T_t \in \mathbb{F}_{\lambda});$
- (P4) \mathbb{F}_{λ} is closed under descending $< \kappa$ -sequences;
- (P5) $\forall \alpha < \lambda \forall T \in \mathbb{F}_{\lambda} \setminus \mathbb{F}_{\alpha} \exists \bar{\gamma} < \kappa \forall \gamma \geq \bar{\gamma} \forall t \in \mathsf{Split}_{\gamma}(T) \exists S \in \mathbb{F}_{\alpha} \setminus \mathbb{F}_{\leq \alpha}(T_t \subseteq S).$

We remark that in P2 the property we are really interested in is P2bis: $\forall T \in \mathbb{F}_{<\lambda} \forall \gamma < \kappa \exists T' \leq_{\gamma} T(T' \in \mathbb{F}_{\lambda} \setminus \mathbb{F}_{<\lambda})$; the extra requirement on all $T'' \leq T'$ is only needed to make sure that such a property will be preserved in our recursive construction.

Furthermore, P5 will be used to help ensure the κ^+ -cc.

Let $\{D_{\lambda} : \lambda < \kappa^{+}\}$ be a $\Diamond_{\kappa^{+}}(S_{\kappa^{+}}^{\kappa})$ -sequence. The recursive construction is developed as follows:

- 1. $\mathbb{F}_0 := \{(2^{<\kappa})_t : t \in 2^{<\kappa}\}.$
- 2. Case $\lambda + 1$:

For every $T \in \mathbb{F}_{\lambda} \setminus \mathbb{F}_{<\lambda}$ and $\gamma < \kappa$, pick $T' \in \mathbb{S}^{\mathsf{Cub}}$ such that $T' \leq_{\gamma} T$ and T' does not contain subtrees in \mathbb{F}_{λ} ; this is possible as \mathbb{F}_{λ} has cardinality κ . Then for all $t \in T'$ we add T'_t to $\mathbb{F}_{\lambda+1}$.

Then for every $T \in \mathbb{F}_{\lambda+1} \setminus \mathbb{F}_{\lambda}$, $S \in \mathbb{F}_{\lambda+1}$ and $\sigma : T \to 2^{<\kappa}$ the canonical isomorphism, put $\sigma^{-1}[S] \in \mathbb{F}_{\lambda+1}$. (This is indeed needed to have a suitable notion of \mathbb{F} -measurability as explained in Proposition 7.)

We finally close $\mathbb{F}_{\lambda+1}$ under descending $< \kappa$ -sequences, i.e., for every descending $\{T^i: i < \delta\}$ in $\mathbb{F}_{\lambda+1}$, with $\delta < \kappa$, we put $T^* := \bigcap_{i < \delta} T^i$ into $\mathbb{F}_{\lambda+1}$

- 3. Case cf(λ) < κ : Let \mathbb{F}_{λ} be the closure of $\mathbb{F}_{<\lambda}$ under descending < κ -sequences.
- 4. Case $cf(\lambda) = \kappa$, where $(\lambda_i : i < \kappa)$ is increasing and cofinal in λ :
 - (a) Suppose $D_{\lambda} \subseteq \lambda$ codes a maximal antichain A_{λ} in $\mathbb{F}_{<\lambda}$. For every $T \in \mathbb{F}_{<\lambda}$ and $\gamma < \kappa$, construct a " κ -fusion" sequence $\{T^i : i < \kappa\}$ of trees in $\mathbb{S}^{\mathsf{Cub}}$ such that
 - i. $T =: T^0 \ge_{\gamma} T^1 \ge_{\gamma+1} T^2 \ge_{\gamma+2} \dots \ge_{\gamma+i} T^{i+1} \ge_{\gamma+i+1} \dots$
 - ii. T^i_t belongs to $\mathbb{F}_{<\lambda}$ with $\mathsf{Rank}(T^i_t)$ at least λ_i for each t in $\mathsf{Split}_\gamma(T)$.

iii. $T^1 := \bigcup \{S_t : t \in \mathsf{Split}_{\gamma}(T)\}$, where each $S_t \leq T_t$ and S_t hits A_{λ} , i.e., there exists $S^* \in A_{\lambda}$ such that $S_t \leq S^*$.

Then add $T^* := \bigcap_{i < \kappa} T^i$ to \mathbb{F}_{λ} . Moreover, for every $t \in T^*$, add T^*_t to \mathbb{F}_{λ} too. Finally close \mathbb{F}_{λ} under descending $< \kappa$ -sequences.

- (b) Suppose that $D_{\lambda} \subseteq \lambda$ codes $\{A_{i,j} : i < \kappa, j < \kappa\}$, where for each $i < \kappa$, $\bigcup_{j < \kappa} A_{i,j}$ is a maximal antichain in $\mathbb{F}_{<\lambda}$ and $j_0 \neq j_1 \Rightarrow A_{i,j_0} \cap A_{i,j_1} = \emptyset$. For every $T \in \mathbb{F}_{<\lambda}$ and $\gamma < \kappa$, build a κ -fusion sequence $\{T^i : i < \kappa\}$ of trees in $\mathbb{S}^{\mathsf{Cub}}$ such that
 - i. $T=:T^0\geq_{\gamma}T^1\geq_{\gamma+1}T^2\geq_{\gamma+2}\cdots\geq_{\gamma+i}T^{i+1}\geq_{\gamma+i+1}\ldots$
 - ii. T_t^i belongs to $\mathbb{F}_{<\lambda}$ with $\mathsf{Rank}(T_t^i)$ at least λ_i for t in $\mathsf{Split}_{\gamma+i}(T^i)$.
 - iii. for every $i < \kappa$, $T^{i+1} := \bigcup \{S^{i+1}_t : t \in \mathsf{Split}_{\gamma+i}(T^i)\}$, where each $S^{i+1}_t \le T^i_t$ and S^{i+1}_t hits $\bigcup_{j < \kappa} A_{i,j}$.

Then add $T^* := \bigcap_{i < \kappa} T^i$ to \mathbb{F}_{λ} . Moreover, for every $t \in T^*$, add T^*_t to \mathbb{F}^*_{λ} too. Finally close \mathbb{F}_{λ} under descending $< \kappa$ -sequences.

(c) If D_{λ} neither codes a maximal antichain (case (a)) nor an instance of κ^{κ} -bounding (case (b)), then proceed as in case (a) without its item iii

Finally let $\mathbb{F} := \bigcup_{\lambda < \kappa^+} \mathbb{F}_{\lambda}$.

Proposition 1. The construction of \mathbb{F} satisfies the five properties P1-P5.

Proof. P1 is clear, since at any stage we only add κ many new trees which are in $\mathbb{S}^{\mathsf{Cub}}$. Also P3 and P4 follow immediately from the construction.

For P2, note that the successor case $\lambda+1$ follows easily from the construction; for $\operatorname{cf}(\lambda)<\kappa$, start with $T\in\mathbb{F}_{<\lambda}$ and use induction to build a descending \leq_{γ} -descending sequence $\{T^i:i<\operatorname{cf}(\lambda)\}$ such that $\{\operatorname{Rank}(T^i):i<\operatorname{cf}(\lambda)\}$ is cofinal in λ and put $T^*:=\bigcap_{i<\operatorname{cf}(\lambda)}T^i$. We may additionally require that T^i contains no subtree in $\mathbb{F}_{<\operatorname{Rank}(T^i)}$ and so T^* contains no subtree in $\mathbb{F}_{<\lambda}$ (in particular, T^* does not belong to $\mathbb{F}_{<\lambda}$); finally for the case $\operatorname{cf}(\lambda)=\kappa$ we argue similarly, using the fact that we take fusion sequences with tree-ranks cofinal in λ .

For P5, we distinguish again the three different situations. In the successor case $\lambda+1$, we can have: case 1) $\alpha<\lambda$, so simply pick some $T_0\supseteq T$ in \mathbb{F}_{λ} and use the inductive hypothesis; case 2) $\alpha=\lambda$, so pick T_0 as in case 1) and use it as the S needed to satisfy P5. In case $\mathrm{cf}(\lambda)<\kappa$, as above start with $T\in\mathbb{F}_{<\lambda}$ and use induction to build a descending γ -sequence of length $\mathrm{cf}(\lambda)$ such that $\{\mathrm{Rank}(T^i):i<\mathrm{cf}(\lambda)\}$ is cofinal in λ and put $T^*:=\bigcap_{i<\mathrm{cf}(\lambda)}T^i$. Let $\alpha<\lambda$ and for $i<\mathrm{cf}(\lambda)$ such that $\alpha<\mathrm{Rank}(T^i)$ choose γ_i sufficiently large so that for all $t\in\mathrm{Split}_{\geq\gamma_i}(T^i)$ one has T^i_t is contained in a tree of $\mathbb{F}_{\alpha}\setminus\mathbb{F}_{<\alpha}$; then, if $\gamma^*:=\sup\{\gamma_i:i<\mathrm{cf}(\lambda)\}$, for every $t\in\mathrm{Split}_{\geq\gamma^*}(T)$ one has T_t is contained in a tree in $\mathbb{F}_{\alpha}\setminus\mathbb{F}_{<\alpha}$. The case $\mathrm{cf}(\lambda)=\kappa$ is treated similarly, by using a sequence with tree-ranks cofinal in λ .

Proposition 2. \mathbb{F} is $< \kappa$ -closed, κ^+ -cc and κ^{κ} -bounding.

Proof. The $< \kappa$ -closure follows from point 3 of the construction.

To prove κ^+ -cc we argue as follows. Let $A \subseteq \mathbb{F}$ be a maximal antichain and pick λ such that $\mathrm{cf}(\lambda) = \kappa$ and $A \cap \mathbb{F}_{<\lambda}$ is a maximal antichain of $\mathbb{F}_{<\lambda}$ and it is coded by D_{λ} , using $\Diamond_{\kappa^+}(S_{\kappa^+}^{\kappa})$. By 4.(a) of the construction, for every

 $T \in \mathbb{F}_{\lambda} \setminus \mathbb{F}_{<\lambda}$, there is γ' such that for every $\gamma \geq \gamma'$ for every $t \in \mathsf{Split}_{\gamma}(T)$, T_t is a subtree of some element of $A \cap \mathbb{F}_{<\lambda}$. By P5, if $T \in \mathbb{F} \setminus \mathbb{F}_{\lambda}$, there is $\gamma'' \geq \gamma'$ such that for every $\gamma \geq \gamma''$ for every $t \in \mathsf{Split}_{\gamma}(T)$, T_t is a subtree of some element of $\mathbb{F}_{\lambda} \setminus \mathbb{F}_{<\lambda}$. It follows that for any $T \in \mathbb{F}_{\lambda} \setminus \mathbb{F}_{<\lambda}$ there is $t \in T$ such that T_t is a subtree of some element of $A \cap \mathbb{F}_{<\lambda}$, and therefore $A \cap \mathbb{F}_{<\lambda}$ is a maximal antichain in \mathbb{F} . So $A \cap \mathbb{F}_{<\lambda} = A$, which finishes the proof as $|\mathbb{F}_{<\lambda}| = \kappa$.

For κ^{κ} -bounding we argue as follows. Let \dot{x} be an \mathbb{F} -name for an element of κ^{κ} and $T \in \mathbb{F}$. Choose $\{A_{i,j} : i < \kappa, j < \kappa\}$ so that for each $i < \kappa$, $\bigcup_{j < \kappa} A_{i,j}$ is a maximal antichain and elements of $A_{i,j}$ force $\dot{x}(i) = j$. Pick $\lambda < \kappa$ such that T belongs to $\mathbb{F}_{<\lambda}$, $\mathrm{cf}(\lambda) = \kappa$ and D_{λ} codes such $\langle A_{i,j} \cap \mathbb{F}_{<\lambda} : i, j \in \kappa \rangle$. and they are maximal antichains as in 4(b). Hence, we can build a κ -fusion sequence in order to get $T' \leq T$ such that for each $i < \kappa$, T' forces the generic to hit $\bigcup_{j \in J_i} A_{ij}$, where each $J_i \subseteq \kappa$ has size $\leq 2^i$. Define $z \in \kappa^{\kappa} \cap V$ by $z(i) = \sup J_i$; then $T' \Vdash \forall i < \kappa, \dot{x}(i) \leq z(i)$.

4 Ideal and measurability

Once we have a tree forcing notion we can introduce a related ideal of *small* sets.

Definition 3. A set $X \subseteq 2^{\kappa}$ is said to be \mathbb{F} -null iff for all $T \in \mathbb{F}$ there exists $T' \in \mathbb{F}$, $T' \leq T$ such that $[T'] \cap X = \emptyset$. Further let $\mathcal{I}_{\mathbb{F}}$ be the ideal consisting of all \mathbb{F} -null sets. A set is \mathbb{F} -conull if its complement is in $\mathcal{I}_{\mathbb{F}}$.

Remark 4. $\mathcal{I}_{\mathbb{F}}$ is a κ^+ -ideal; let $\{X_{\alpha}: \alpha < \kappa\}$ be a sequence of \mathbb{F} -null sets, and fix $T \in \mathbb{F}$. Using 4(b) of the construction of \mathbb{F} , build a κ -fusion sequence $\{T_{\alpha}: \alpha < \kappa\}$ such that for all $\alpha < \kappa$, for all $\beta \leq \alpha$, $[X_{\beta}] \cap [T_{\alpha}] = \emptyset$. Then $T' := \bigcap_{\alpha < \kappa} T_{\alpha}$ has the desired property.

One of the main properties of the null ideal in the standard framework is that of being *orthogonal* to the meager ideal, i.e., the space can be partitioned into a meager piece and a null piece. We now prove that the same holds for $\mathcal{I}_{\mathbb{F}}$.

Proposition 5. There is $X \subseteq 2^{\kappa}$ such that $X \in \mathcal{M}$ and $2^{\kappa} \setminus X \in \mathcal{I}_{\mathbb{F}}$.

Proof. Let $A := \{A_i : i < \kappa\}$ be a maximal antichain in \mathbb{F} . Clearly, $X := \bigcup_{i < \kappa} [A_i]$ is \mathbb{F} -conull, since for every $T \in \mathbb{F}$, there is $i < \kappa$ such that A_i and T are compatible, and so there is $T' \leq A_i$ such that $T' \leq T$. It is then sufficient to show that we can find such an antichain A with the further property that any $[A_i]$ is nowhere dense. But note that by property P2, any $T \in \mathbb{F}$ can be extended to contain no subtree of the form $(2^{<\kappa})_s$ for $s \in 2^{<\kappa}$ and [T] is nowhere dense for such a tree T. Now let $\mathbb{F}^* \subseteq \mathbb{F}$ be the dense set of such trees, and pick A a maximal antichain in \mathbb{F}^* . Then A remains a maximal antichain in \mathbb{F} as well, and it is then enough for our purpose.

Measurability. There are essentially two possible notions of regularity related to \mathbb{F} .

Definition 6. A set $X \subseteq 2^{\kappa}$ is said to be:

- 1. \mathbb{F} -measurable iff for every $T \in \mathbb{F}$ there exists $T' \in \mathbb{F}$, $T' \leq T$ such that $[T'] \setminus X \in \mathcal{I}_{\mathbb{F}}$ or $X \cap [T'] \in \mathcal{I}_{\mathbb{F}}$.
- 2. \mathbb{F} -regular iff there exists a Borel set B such that $X \triangle B \in \mathcal{I}_{\mathbb{F}}$.

Concerning definition 6.1, we could equivalently require " $[T'] \subseteq X$ or $[T'] \cap X = \emptyset$ ", as $\mathcal{I}_{\mathbb{F}}$ is a κ^+ -ideal.

Proposition 7. Let $X \subseteq 2^{\kappa}$. X is \mathbb{F} -measurable iff X is \mathbb{F} -regular.

Proof. The proof is just as the general case of \mathbb{P} -measurability in the standard case. We give it here for completeness.

 \Rightarrow : by assumption, the set $E:=\{T\in\mathbb{F}:[T]\cap X\in\mathcal{I}_{\mathbb{F}}\vee[T]\cap X^c\in\mathcal{I}_{\mathbb{F}}\}$ is dense in \mathbb{F} . Then pick a maximal antichain A in E and put

$$B := \bigcup \{ [T] : T \in A \land [T] \cap X^c \in \mathcal{I}_{\mathbb{F}} \}.$$

Note that B is Borel (Σ_2^0) , since $|A| \leq \kappa$. We claim that $X \triangle B \in \mathcal{I}_{\mathbb{F}}$. Indeed, for every $T \in \mathbb{F}$ we have two cases: there is $S \leq T$ such that $[S] \cap X \in \mathcal{I}_{\mathbb{F}}$; if we pick the unique $S' \in A$ such that $S \leq S'$ we get $[S] \cap (X \triangle B) \in \mathcal{I}_{\mathbb{F}}$, as $S' \in A$ implies $[S] \cap X^c \in \mathcal{I}_{\mathbb{F}}$; otherwise, the specular situation occurs, i.e., there is $S \leq T$ such that $[S] \cap X^c \in \mathcal{I}_{\mathbb{F}}$; so if we pick the unique $S' \in A$ such that $S \leq S'$ we get $[S] \cap (X \triangle B) \in \mathcal{I}_{\mathbb{F}}$, as $S' \in A$ implies $[S] \subseteq B$.

 \Leftarrow : by assumption, it is enough to show that any Borel set is \mathbb{F} -measurable. We do that by induction on the Borel hierarchy. By point 2 in the main construction of \mathbb{F} , we can generalize a result of Brendle and Löwe (see [1]), proving that all Borel sets are \mathbb{F} -measurable is equivalent to proving that for every Borel set B there is $T \in \mathbb{F}$ such that $[T] \subseteq B$ or $[T] \cap B = \emptyset$. For $s \in 2^{<\kappa}$, [s] is trivially \mathbb{F} -measurable, and if B is \mathbb{F} -measurable, then by symmetry B^c is \mathbb{F} -measurable too. Finally, if B is the union of $\leq \kappa$ many Borel sets $\{C_\alpha : \alpha < \delta\}$, with $\delta \leq \kappa$, then we have two cases: there is $\alpha < \delta$ and $T \in \mathbb{F}$ such that $[T] \subseteq C_\alpha \subseteq B$; or for all $\alpha < \delta$, $C_\alpha \in \mathcal{I}_{\mathbb{F}}$, and so $B \in \mathcal{I}_{\mathbb{F}}$ as well.

In [7] and [9] it was shown that for some tree forcing notions one can force all projective sets to be measurable. Nevertheless this is not the case for \mathbb{F} . Indeed, next result shows that there is no hope for $\Sigma_1^1(\mathbb{F})$ to be consistent.

Proposition 8. The club filter Cub is not \mathbb{F} -measurable.

Proof. Standard. Given an arbitrary $T \in \mathbb{F}$ it suffices to show that [T] contains branches both in Cub and in NS. We argue as follows:

- let $t_0 = \mathsf{Stem}(T)$
- for $\alpha < \kappa$ successor, pick $t_{\alpha} \supset t_{\alpha-1} \hat{\ } 1$ such that $t_{\alpha} \in \mathsf{Split}(T)$ and $|t_{\alpha}|$ is a limit ordinal.
- for $\alpha < \kappa$ limit, pick $t_{\alpha} \supset (\bigcup_{\xi < \alpha} t_{\xi})^{\hat{}} 1$ such that $t_{\alpha} \in \mathsf{Split}(T)$ and $|t_{\alpha}|$ is a limit ordinal.

Finally put $x := \bigcup_{\alpha < \lambda} t_{\alpha}$. Clearly, $x \in [T] \cap \mathsf{Cub}$.

Analogously, if in the previous choices of the t_{α} 's we replace 1 by 0, we get $x \in [T] \cap NS$.

We conclude by remarking that a standard construction shows that $\Delta_1^1(\mathbb{F})$ is consistently false (e.g., if V = L).

5 Concluding remarks and open questions

It would be interesting to prove that $\Delta_1^1(\mathbb{F})$ is consistently true. The usual way for forcing such a statement for a tree-forcing \mathbb{P} is to take a κ^+ -iteration of \mathbb{P} with κ -support. The main point is to make sure that κ^+ is preserved. For our forcing \mathbb{F} it is not clear whether it is the case. So we leave the following as an open question.

Question. Does a κ^+ -iteration of \mathbb{F} with κ -support preserve κ^+ ? Or, can we modify \mathbb{F} is order to let the latter work?

What remains also open is the last part of [6, Question 3.1].

Question. Can one define a tree forcing that is $< \kappa$ -closed, κ^{κ} -bounding and κ^+ -cc, for κ successor?

We remark that our construction can in fact be applied to successor κ , yielding $< \kappa$ -closure and the κ^+ -cc, but κ^{κ} -bounding will fail.

References

- [1] Jörg Brendle, Benedikt Löwe, Solovay-Type characterizations for Forcing-Algebra, Journal of Simbolic Logic, Vol. 64 (1999), pp. 1307-1323.
- [2] Shani Cohen, Saharon Shelah, On a parallel of random real forcing for in-accessible cardinals, preprint, arxiv:math.LO/1603.08362.
- [3] Sy D. Friedman, Lyubomir Zdomskyy, Measurable cardinals and the cofinality of the symmetric group, Fundamenta Matematicae, Vol. 207 (2010), pp 101-122.
- [4] Sy D. Friedman, Yurii Khomskii, Vadim Kulikov, Regularity properties on the generalized reals, Annals of Pure and Applied Logic 167 (4), pp. 408-430 (2016).
- [5] Ronald B. Jensen, *Definable sets of minimal degree*, Mathematical Logic and Foundation of Set Theory, North Holland (1968).
- [6] Yurii Khomskii, Giorgio Laguzzi, Benedikt Löwe and Ilya Sharankou, Questions on generalized Baire spaces, Mathematical Logic Quarterly, Volume 62, Issue 4-5, (2016).
- [7] Giorgio Laguzzi, Generalized Silver and Miller measurability, Mathematical Logic Quarterly, Volume 61, Issue 1-2, pp 91-102 (2015).
- [8] Andrzej Roslanowski, Saharon Shelah, *Sheva-Sheva-Sheva: Large Creatures*, Israel Journal of Mathematics, Vol. 159 (2007), pp 109-174.
- [9] Philipp Schlicht, Perfect subsets of generalized Baire spaces and long games, accepted for publication, Journal of Symbolic Logic.

- [10] Saharon Shelah, A parallel to the null ideal for inaccessible λ , (preprint, 2014) available online at: http://shelah.logic.at/files/1004.pdf.
- [11] Saharon Shelah, $On\ Con(cov(\mathcal{M}) < \mathfrak{d}),$ Transaction of the AMS, (2014).