A strong failure of $\aleph_0$-stability for atomic classes

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Abstract

We study classes of atomic models $\mathbf{At}_T$ of a countable, complete first-order theory T. We prove that if $\mathbf{At}_T$ is not pcl-small, i.e., there is an atomic model N that realizes uncountably many types over $\mathrm{pcl}(\bar{a})$ for some finite $\bar{a}$ from N, then there are $2^{\aleph_1}$ non-isomorphic atomic models of T, each of size $\aleph_1$.

1 Introduction

In a series of papers [2, 3, 4], Baldwin and the authors have begun to develop a model theory for complete sentences of $L_{\omega_1,\omega}$ that have fewer than $2^{\aleph_1}$ non-isomorphic models of size $\aleph_1$. By well known reductions, one can replace the reference to infinitary sentences by restricting to the class of *atomic* models of a countable, complete first-order theory.[1]

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[1]Specifically, for every complete sentence Φ of $L_{\omega_1,\omega}$, there is a complete first-order theory T in a countable vocabulary containing the vocabulary of Φ such that the models of Φ are precisely the reducts of the class of atomic models of T to the smaller vocabulary.

Fix, for the whole of this paper, a complete theory T in a countable language that has at least one atomic model[2] of size $\aleph_1$. By theorems of Vaught, these restrictions on T are well understood. Such a T has an atomic model if and only if every consistent formula can be extended to a complete formula. Furthermore, any two countable, atomic models of T are isomorphic, and a model is prime if and only if it is countable and atomic. Using a well-known union of chains argument, T has an atomic model of size $\aleph_1$ if and only if the countable atomic model is not minimal, i.e., it has a proper elementary substructure.

The analysis of $\mathbf{At}_T$, the class of atomic models of T, begins by restricting the notion of types to those that can be realized in an atomic model. Suppose M is atomic and $A \subseteq M$. We let $S_{at}(A)$ denote the set of complete types p over A for which Ab is an atomic set for some (equivalently, for every) realization b of p. It is easily checked that when A is countable, $S_{at}(A)$ is a G_δ subset of the Stone space $S(A)$, hence $S_{at}(A)$ is Polish with respect to the induced topology. We will repeatedly use the fact that any countable, atomic set A is contained in a countable, atomic model M. However, unlike the first-order case, some types in $S_{at}(A)$ need not extend to types in $S_{at}(M)$. Indeed, there are examples where the space $S_{at}(A)$ is uncountable (hence contains a perfect set) while $S_{at}(M)$ is countable. Thus, for analyzing types over countable, atomic sets $A \subseteq M$, we are led to consider

$$S^+_{at}(A, M) := \{p|A : p \in S_{at}(M)\}.$$

Equivalently, $S^+_{at}(A, M)$ is the set of $q \in S_{at}(A)$ that can be extended to a type $q^* \in S_{at}(M)$.

Next, we recall the notion of pseudo-algebraicity, which was introduced in [2], that is the correct analog of algebraicity in the context of atomic models. Suppose M is an atomic model, and $b, \bar{a}$ are from M. We say $b \in \mathrm{pcl}_M(\bar{a})$ if $b \in N$ for every elementary submodel $N \preceq M$ that contains $\bar{a}$. The seeming dependence on M is illusory – as is noted in [2], if $b', \bar{a}'$ are inside another atomic model M', and $\mathrm{tp}_{M'}(b'\bar{a}') = \mathrm{tp}_M(b\bar{a})$, then $b \in \mathrm{pcl}_M(\bar{a})$ if and only if $b' \in \mathrm{pcl}_{M'}(\bar{a}')$. It is easily seen that inside any atomic model M, $\mathrm{pcl}_M(\bar{a})$ is countable for any finite tuple $\bar{a}$. Moreover, if $f : M \to M'$ is an isomorphism of atomic models, then $f(\mathrm{pcl}_M(\bar{a})) = \mathrm{pcl}_{M'}(f(\bar{a}))$ setwise. As an important special case, if $\bar{a} \subseteq M' \preceq M$ and $f : M \to M'$ fixes $\bar{a}$ pointwise, then f

[2] A model M is *atomic* if, for every finite tuple $\bar{a}$ from M, $\mathrm{tp}_M(\bar{a})$ is *principal* i.e., is uniquely determined by a single formula $\varphi(\overline{x}) \in \mathrm{tp}_M(\bar{a})$.

induces an elementary permutation on $D = \mathrm{pcl}_M(\bar{a})$, which in turn induces a bijection between $S^+_{at}(D, M)$ and $S^+_{at}(D, M')$.

We now give the major new definition of this paper:

Definition 1.1 An atomic class $\mathbf{At}_T$ with an uncountable model is *pcl-small* if, for every atomic model N and for every finite $\bar{a}$ from N, N realizes only countably many complete types over $\mathrm{pcl}_N(\bar{a})$.

The name of this notion is by analogy with the first-order case – A complete, first-order theory T is small if and only if for every model N and every finite $\bar{a}$ from N, N realizes only countably many complete types over $\bar{a}$. The following proposition relates pcl-smallness with the spaces of types $S^+_{at}(D, M)$.

Proposition 1.2 *The atomic class* $\mathbf{At}_T$ *is* pcl-*small if and only if the space of types* $S^+_{at}(\mathrm{pcl}_M(\bar{a}), M)$ *is countable for every countable, atomic model* M *and every finite* $\bar{a}$ *from* M*.*

Proof. First, assume that some atomic model N and finite sequence $\bar{a}$ from N witness that $\mathbf{At}_T$ is not pcl-small. Choose $\{c_i : i \in \omega_1\} \subseteq N$ realizing distinct complete types over $D = \mathrm{pcl}_N(\bar{a})$. Also, choose a countable $M \preceq N$ that contains $\bar{a}$, and hence D. Then $\{\mathrm{tp}(c_i/D) : i \in \omega_1\}$ witness that $S^+_{at}(D, M)$ is uncountable.

For the converse, choose a countable, atomic model M and $\bar{a}$ from M such that $S^+_{at}(D, M)$ is uncountable, where $D = \mathrm{pcl}_M(\bar{a})$. We will inductively construct a continuous, increasing elementary chain $\langle M_\alpha : \alpha < \omega_1\rangle$ of countable, atomic models with $M = M_0$ and, for each ordinal α, there is an element $c_\alpha \in M_{\alpha+1}$ such that $\mathrm{tp}(c_\alpha/D)$ is not realized in M_α. Given such a sequence, it is evident that $N = \bigcup_{\alpha<\omega_1} M_\alpha$ and $\bar{a}$ witness that $\mathbf{At}_T$ is not pcl-small. To construct such a sequence, we have defined M_0 to be M and take unions at limit ordinals. For the successor step, assume M_α has been defined. As M and M_α are each countable atomic models that contain $\bar{a}$, choose an isomorphism $f : M \to M_\alpha$ fixing $\bar{a}$ pointwise. As noted above, f fixes D setwise. As M_α is countable, so is the set $\{\mathrm{tp}(c/D) : c \in M_\alpha\}$. As $S^+_{at}(D, M)$ is uncountable, choose an atomic type $p \in S_{at}(M)$, whose restriction to D is distinct from $\{f^{-1}(\mathrm{tp}(c/D)) : c \in M_\alpha\}$. Now choose c_α to realize $f(p)$. Then, as $M_\alpha c_\alpha$ is a countable atomic set, choose a countable elementary extension $M_{\alpha+1} \succeq M_\alpha$ containing c_α.

Recall that an atomic class $\mathbf{At}_T$ is $\aleph_0$-stable[3] if $S_{at}(M)$ is countable for all (equivalently, for some) countable atomic models M. As $S^+_{at}(A,M)$ is a set of projections of types in $S_{at}(M)$, it will be countable whenever $S_{at}(M)$ is. This observation makes the following corollary to Proposition 1.2 immediate:

Corollary 1.3 *If an atomic class* $\mathbf{At}_T$ *is* $\aleph_0$*-stable, then* $\mathbf{At}_T$ *is* pcl*-small.*

The converse to Corollary 1.3 fails. For example, the theory $T = REF(bin)$ of countably many, binary splitting equivalence relations is not $\aleph_0$-stable, yet $\mathrm{pcl}_M(\bar{a}) = \bar{a}$ for every model M and $\bar{a}$ from M. Thus, $S_{at}(\mathrm{pcl}_M(\bar{a}))$ and hence $S^+_{at}(\mathrm{pcl}(\bar{a}), M)$ is countable for every finite tuple $\bar{a}$ inside any atomic model M. The main theorem of this paper is:

Theorem 1.4 *Let T be a countable, complete theory T with an uncountable atomic model. If the atomic class* $\mathbf{At}_T$ *is not* pcl*-small, then there are* $2^{\aleph_1}$ *non-isomorphic models in* $\mathbf{At}_T$*, each of size* $\aleph_1$*.*

Section 2 sets the stage for the proof. It describes the spaces of types $S^+_{at}(A,M)$, states a transfer theorem for sentences of $L_{\omega_1,\omega}(Q)$, and details a non-structural configuration arising from non-pcl-smallness. In Section 3, the non-structural configuration is exploited to give a family of $2^{\aleph_0}$ non-isomorphic structures $(N, \bar{b}^*)$, where each of the reducts N is in $\mathbf{At}_T$ and has size $\aleph_1$. Theorem 1.4 is finally proved in Section 4. It is remarkable that whereas it is a ZFC theorem, the proof is non-uniform depending on the relative sizes of the cardinals $2^{\aleph_0}$ and $2^{\aleph_1}$.

2 Preliminaries

In this section, we develop some general tools that will be used in the proof of Theorem 1.4.

2.1 On $S^+_{at}(A,M)$

In this subsection we explore the space of types

$$S^+_{at}(A,M) = \{p|A : p \in S_{at}(M)\}$$

[3]Sadly, this usage of '$\aleph_0$-stability' is analogous, but distinct from, the familiar first-order notion.

where A is a subset of a countable, atomic model M.

Fix a countable, atomic model M and an arbitrary subset $A \subseteq M$. Let $\mathcal{P}$ denote the space of complete types in one free variable over finite subsets of M. As M is atomic, $\mathcal{P}$ can be identified with the set of complete formulas $\varphi(x, m)$ over M. Implication gives a natural partial order on $\mathcal{P}$, namely $p \leq q$ if and only if $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and $q \vdash p$. One should think of elements of $\mathcal{P}$ as 'finite approximations' of types in $S^+_{at}(A, M)$. We describe two conditions on $p \in \mathcal{P}$ that identify extreme behaviors in this regard.

Definition 2.1 We say a type $p^* \in S^+_{at}(A, M)$ *lies above* $p \in \mathcal{P}$ if there is some $\bar{p} \in S_{at}(M)$ extending $p \cup p^*$. As every $p \in \mathcal{P}$ extends to a type in $S_{at}(M)$, it follows that at least one $p^* \in S^+_{at}(A, M)$ lies above p.

- An element $p \in \mathcal{P}$ *determines a type in* $S^+_{at}(A, M)$ if exactly one $p^* \in S^+_{at}(A, M)$ lies above p.
- An element $p \in \mathcal{P}$ is *A-large* if $\{p^* \in S^+_{at}(A, M) : p^* \text{ lies above } p\}$ is uncountable.

To understand these extreme behaviors, we define a rank function $\mathrm{rk}_A : \mathcal{P} \to (\omega_1 + 1)$ as follows:

- $\mathrm{rk}_A(p) \geq 0$ for all $p \in \mathcal{P}$;
- For $\alpha \leq \omega_1$, $\mathrm{rk}_A(p) \geq \alpha$ if and only if for every $\beta < \alpha$ and for all finite F, $\mathrm{dom}(p) \subseteq F \subseteq M$, there is $q \in S_{at}(F)$ with $q \geq p$ that *β-A splits*, where:
  - A type $q \in S_{at}(F)$ *A-splits* if, for some $\varphi(x, \bar{a})$ with $\bar{a}$ from A, there are $q_1, q_2 \geq q$ with $q \cup \varphi(x, \bar{a}) \subseteq q_1$ and $q \cup \neg\varphi(x, \bar{a}) \subseteq q_2$; and $q \in S_{at}(F)$ *β-A splits* if, in addition, $\mathrm{rk}_A(q_1), \mathrm{rk}_A(q_2) \geq \beta$.
- For $\alpha < \omega_1$, we say $\mathrm{rk}_A(p) = \alpha$ if $\mathrm{rk}_A(p) \geq \alpha$, but $\mathrm{rk}_A(p) \not\geq \alpha + 1$.

Proposition 2.2 *If $p \in \mathcal{P}$ and $\mathrm{rk}_A(p) = \alpha < \omega_1$, then some $r \geq p$ determines a type in $S^+_{at}(A, M)$.*

Proof. We prove this by induction on α. We begin with $\alpha = 0$. Suppose $\mathrm{rk}_A(p) = 0$. As $\mathrm{rk}_A(p) \not\geq 1$, there is a finite F, $\mathrm{dom}(p) \subseteq F \subseteq M$ for which there is no $q \in S_{at}(F)$ and $\varphi(x, \bar{a})$ with $\bar{a}$ from A for which $q \geq p$ and both

$q \cup \{\varphi(x,\bar{a})\}$ and $q \cup \{\neg\varphi(x,\bar{a})\}$ are consistent. So fix any $r \in S_{at}(F)$ with $r \geq p$. Any such r determines a type in $S^+_{at}(A, M)$.

Next, choose $0 < \alpha < \omega_1$ and assume the Proposition holds for all $\beta < \alpha$. Choose $p \in S_{at}(E)$ with $\mathrm{rk}_A(p) = \alpha$. As $\mathrm{rk}_A(p) \geq \alpha$, while $\mathrm{rk}_A(p) \not\geq \alpha + 1$, there is a finite F, $E \subseteq F \subseteq M$ for which there is no $q \in S_{at}(F)$ that both extends p and α-A splits. So choose any $q \in S_{at}(F)$ with $q \geq p$. If q determines a type in $S^+_{at}(A, M)$, then we finish, so assume otherwise. Thus, there is some $\varphi(x, \bar{a})$ with $\bar{a}$ from A such that both $q \cup \{\varphi(x, \bar{a})\}$ and $q \cup \{\neg\varphi(x, \bar{a})\}$ are consistent. Choose complete types $q_1, q_2 \in S_{at}(F\bar{a})$ extending these partial types. Clearly, both $q_1, q_2 \geq q$, but since q does not α-A split, at least one of them has $\mathrm{rk}_A(q_\ell) < \alpha$. But then by our inductive hypothesis, there is $r \geq q_\ell$ that determines a type in $S^+_{at}(A, M)$ and we finish.

Next, we turn our attention to A-large types and types of rank at least ω_1 and see that these coincide. We begin with two lemmas, the first involving types of rank at least ω_1 and the second involving A-large types.

Lemma 2.3 *Assume that $E \subseteq M$ is finite and $p \in S_{at}(E)$ has $\mathrm{rk}_A(p) \geq \omega_1$. Then:*

1. *For every finite F, $E \subseteq F \subseteq M$, there is $q \in S_{at}(F)$, $q \geq p$, with $\mathrm{rk}_A(q) \geq \omega_1$; and*
2. *There is some formula $\varphi(x, \bar{a})$ with $\bar{a}$ from A and $q_1, q_2 \in \mathcal{P}$ with $p \cup \{\varphi(x, \bar{a})\} \subseteq q_1$, $p \cup \{\neg\varphi(x, \bar{a})\} \subseteq q_2$, and both $\mathrm{rk}_A(q_1), \mathrm{rk}_A(q_2) \geq \omega_1$.*

Proof. (1) Fix a finite F satisfying $E \subseteq F \subseteq M$. As $\mathrm{rk}_A(p) \geq \omega_1$, for every $\beta < \omega_1$ there is some $q \geq p$ with $q \in S_{at}(F)$ for which certain extensions of q have rank at least β. It follows that $\mathrm{rk}_A(q) \geq \beta$ for any such witness. However, as $S_{at}(F)$ is countable, there is some $q \in S_{at}(F)$ which serves as a witness for uncountably many β. Thus, $\mathrm{rk}_A(q) \geq \omega_1$ for any such $q \geq p$.

(2) Assume that there were no such formula $\varphi(x, \bar{a})$. Then, for any formula $\varphi(x, \bar{a})$, since $\mathcal{P}$ is countable, there would be an ordinal $\beta^* < \omega_1$ such that **either** every $q \in \mathcal{P}$ extending $p \cup \{\varphi(x, \bar{a})\}$, $\mathrm{rk}_A(q) < \beta^*$ **or** every $q \in \mathcal{P}$ extending $p \cup \{\neg\varphi(x, \bar{a})\}$ has $\mathrm{rk}_A(q) < \beta^*$. Continuing, as there are only countably many formulas $\varphi(x, \bar{a})$, there would be an ordinal $\beta^{**} < \omega_1$ that works for all formulas $\varphi(x, \bar{a})$. Restating this, p does not β^{**}-A split, so no extension of p could β^{**}-A split either. This contradicts $\mathrm{rk}_A(p) \geq \beta^{**} + 1$.

Lemma 2.4 *Suppose $q \in S_{at}(F)$ is A-large. Then:*

1. *For every finite F', $F \subseteq F' \subseteq M$, there is some A-large $r \in S_{at}(F')$ with $r \geq q$; and*

2. *For some $\varphi(x, \bar{a})$, there are A-large extensions $r_1 \supseteq q \cup \{\varphi(x, \bar{a})\}$ and $r_2 \supseteq q \cup \{\neg\varphi(x, \bar{a})\}$.*

Proof. Fix such a q and let $\mathcal{S} = \{p^* \in S^+_{at}(A, M) : p^* \text{ lies above } q\}$.
(1) is immediate, since $\mathcal{S}$ is uncountable, while $S_{at}(F')$ is countable.
For (2), first note that if there is no such $\varphi(x, \bar{a})$, then there is at most one $p^* \in \mathcal{S}$ with the property that:

> For any formula $\varphi(x, \bar{a})$ with parameters from A, $\varphi(x, \bar{a}) \in p^*$ if and only if there is an A-large $r \in S_{at}(F\bar{a})$ extending $q \cup \{\varphi(x, \bar{a})\}$.

It follows that for any $q^* \in \mathcal{S} - \{p^*\}$, q^* lies over some $r \geq q$ that is not A-large. That is, using the fact that there are only countably many $r \geq q$, $\mathcal{S} - \{p^*\}$ is contained in the union of countably many countable sets. But this contradicts q being A-large.

Proposition 2.5 *For $p \in \mathcal{P}$, $\mathrm{rk}_A(p) \geq \omega_1$ if and only if p is A-large.*

Proof. First, assume that $\mathrm{rk}_A(p) \geq \omega_1$. Fix an enumeration $\{c_n : n \in \omega\}$ of M. Using Clauses (1) and (2) of Lemma 2.3, we inductively construct a tree $\{p_\nu : \nu \in 2^{<\omega}\}$ of elements of $\mathcal{P}$ satisfying:

1. $\mathrm{rk}_A(p_\nu) \geq \omega_1$ for all $\nu \in 2^{<\omega}$;

2. If $\lg(\nu) = n$, then $\{c_i : i < n\} \subseteq \mathrm{dom}(p_\nu)$;

3. $p_{\langle\rangle} = p$;

4. For $\nu \trianglelefteq \mu$, $p_\nu \leq p_\mu$;

5. For each ν there is a formula $\varphi(x, \bar{a})$ with $\bar{a}$ from A such that $\varphi(x, \bar{a}) \in p_{\nu 0}$ and $\neg\varphi(x, \bar{a}) \in p_{\nu 1}$.

Given such a tree, for each $\eta \in 2^\omega$, let $\bar{p}_\eta := \bigcup\{p_{\eta|n} : n \in \omega\}$ and let $p^*_\eta := \bar{p}_\eta | A$. By Clauses (2) and (4), each $\bar{p}_\eta \in S_{at}(M)$, so each $p^*_\eta \in S^+_{at}(A, M)$. By Clause (5), $p^*_\eta \neq p^*_{\eta'}$ for distinct $\eta, \eta' \in 2^\omega$. Finally, each of these types lies over p by Clause (3). Thus, p is A-large.

Conversely, we argue by induction on $\alpha < \omega_1$ that:

$(*)_\alpha$: If $p \in \mathcal{P}$ is A-large, then $\mathrm{rk}_A(p) \geq \alpha$.

Establishing $(*)_0$ is trivial, and for limit $\alpha < \omega_1$, it is easy to establish $(*)_\alpha$ given that $(*)_\beta$ holds for all $\beta < \alpha$. So assume $(*)_\alpha$ holds and we will establish $(*)_{\alpha+1}$. Choose any A-large $p \in \mathcal{P}$. Towards showing $\mathrm{rk}_A(p) \geq \alpha + 1$, choose any finite F, $\mathrm{dom}(p) \subseteq F \subseteq M$. As $S_{at}(F)$ is countable and uncountably many types in $S^+_{at}(A, M)$ lie above p, there is some A-large $q \in S_{at}(F)$ with $q \geq p$.

Next, by Lemma 2.4 choose a formula $\varphi(x, \bar{a})$ with $\bar{a}$ from A such that there are A-large extensions $r_1 \supseteq q \cup \{\varphi(x, \bar{a})\}$ and $r_2 \supseteq q \cup \{\neg\varphi(x, \bar{a})\}$. Applying $(*)_\alpha$ to both r_1, r_2 gives $\mathrm{rk}_A(r_1), \mathrm{rk}_A(r_2) \geq \alpha$. Thus, q α-A splits. Thus, by definition of the rank, $\mathrm{rk}_A(p) \geq \alpha + 1$.

We obtain the following Corollary, which is analogous to the statement 'If T is small, then the isolated types are dense' from the first-order context.

Corollary 2.6 *If $S^+_{at}(A, M)$ is countable, then every $p \in \mathcal{P}$ has an extension $q \geq p$ that determines a type in $S^+_{at}(A, M)$.*

Proof. If $S^+_{at}(A, M)$ is countable, then no $p \in \mathcal{P}$ is A-large. Thus, every $p \in \mathcal{P}$ has $\mathrm{rk}_A(p) < \omega_1$ by Proposition 2.5, so has an extension determining a type in $S^+_{at}(A, M)$ by Proposition 2.2.

We close with a complementary result about extensions of A-large types.

Definition 2.7 A type $r \in S_{at}(M)$ is *A-perfect* if $r\restriction_A$ is omitted in M and for every finite $\overline{m}$ from M, the restriction $r\restriction_{\overline{m}}$ is A-large.

The name *perfect* is chosen because, relative to the usual topology on $S_{at}(M)$, there are a perfect set of A-perfect types extending any A-large $p \in \mathcal{P}$. However, for what follows, all we need to establish is that there are uncountably many, which is notationally simpler to prove.

Proposition 2.8 *Suppose $p \in \mathcal{P}$ is A-large. Then there are uncountably many A-perfect $r \in S_{at}(M)$ extending p.*

Proof. Fix an A-large $p \in \mathcal{P}$. Choose a set $R \subseteq S_{at}(M)$ of representatives for $\{p^* \in S^+_{at}(A, M) : p^*$ lies above $p\}$, i.e., for every such p^*, there is exactly one $\bar{p} \in R$ whose restriction $\bar{p}\restriction_A = p^*$. As p is A-large, R is uncountable. Now, for each finite $\overline{m}$ from M, there are only countably many complete $q \in S_{at}(\overline{m})$, and if some $q \in S_{at}(\overline{m})$ is A-small, then only countably many $\bar{p} \in R$ extend q. As M is countable, there are only countably many $\overline{m}$, hence all but countably many $\bar{p} \in R$ satisfy $\bar{p}\restriction_{\overline{m}}$ A-large for every $\overline{m}$. Further, again since M is countable, at most countably many $\bar{p} \in R$ have restrictions to A that are realized in M. Thus, all but countably many $\bar{p} \in R$ are A-perfect.

2.2 A transfer result

In this brief subsection we state a transfer result that follows immediately by Keisler's completeness theorem for the logic $L_{\omega_1,\omega}(Q)$, given in [6]. Recall that $L_{\omega_1,\omega}(Q)$ is the logic obtained by taking the (usual) set of atomic L formulas and closing under boolean combinations, existential quantification, the 'Q-quantifier,' i.e., if $\theta(y, \overline{x})$ is a formula, then so is $Qy\theta(y, \overline{x})$; and countable conjunctions of formulas involving a finite set of free variables, i.e., if $\{\psi_i(\overline{x}) : i \in \omega\}$ is a set of formulas, then so is $\bigwedge_{i\in\omega} \psi_i(\overline{x})$. We are only interested in *standard interpretations* of these formulas, i.e., $M \models \bigwedge_{i\in\omega} \psi_i(\bar{a})$ if and only if $M \models \psi_i(\bar{a})$ for every $i \in \omega$; and $M \models Qy\theta(y, \bar{a})$ if and only if the solution set $\theta(M, \bar{a})$ is uncountable.

Throughout the discussion let ZFC^* denote a sufficiently large, finite subset of the ZFC axioms. In the notation of [8], Proposition 2.9 states that sentences of $L_{\omega_1,\omega}(Q)$ are *grounded.*

Proposition 2.9 *Suppose L is a countable language, and $\Phi \in L_{\omega_1,\omega}(Q)$ are given. There is a sufficiently large, finite subset ZFC^* of ZFC such that IF there is a countable, transitive model $(\mathcal{B}, \epsilon) \models ZFC^*$ with $L, \Phi \in \mathcal{B}$ and*

$$(\mathcal{B}, \epsilon) \models \text{ 'There is } M \models \Phi \text{ and } |M| = \aleph_1\text{'}$$

THEN (in V!) there is $N \models \Phi$ and $|N| = \aleph_1$.

Proof. This follows immediately from Keiser's completeness theorem for $L_{\omega_1,\omega}$, given that provability is absolute between transitive models of set theory. More modern, 'constructive' proofs can be found in [1] and [2]. These use the existence $\mathcal{B}$-normal ultrafilters. Given an arbitrary language $L^* \in \mathcal{B}$ and any countable L^*-structure $(\mathcal{B}, E, \ldots)$ where the reduct $(\mathcal{B}, E)$ is an ω-model of ZFC^*, for any $\mathcal{B}$-normal ultrafilter $\mathcal{U}$, the ultrapower $Ult(\mathcal{B},\mathcal{U})$ is a countable, ω-model that is an L^*-elementary extension of $(\mathcal{B}, E, \ldots)$. It has the additional property that for any L^*-definable subset D, $D^{Ult(\mathcal{B},\mathcal{U})}$ properly extends $D^{\mathcal{B}}$ if and only if $(\mathcal{B}, E, \ldots) \models$ 'D is uncountable'.

Using this, one constructs (in V!) a continuous, L^*-elementary ω_1-sequence $\langle \mathcal{B}_\alpha : \alpha < \omega_1 \rangle$ of ω-models, where each $\mathcal{B}_{\alpha+1} = Ult(\mathcal{B}_\alpha, \mathcal{U}_\alpha)$. Then the interpretation $M^{\mathcal{C}}$ where $\mathcal{C} = \bigcup_{\alpha \in \omega_1} \mathcal{B}_\alpha$ will be a suitable choice of N. More details of this construction are given in [1] or [2].

2.3 A configuration arising from non-pcl-smallness

The goal of this subsection is to prove the following Proposition, the data from which will be used throughout Section 3.

Proposition 2.10 *Assume T is a countable, complete theory for which $\mathbf{At}_T$ has an uncountable atomic model, but is not* pcl*-small. Then there are a countable, atomic $M^* \in \mathbf{At}_T$, finite sequences $\bar{a}^* \subseteq \bar{b}^* \subseteq M^*$, and complete 1-types $\{r_j(x, \bar{b}^*) : j \in \omega\}$ such that, letting $D^* = \mathrm{pcl}_{M^*}(\bar{a}^*)$, $A_n = \bigcup\{r_j(M^*, \bar{b}^*) : j < n\}$ and $A^* = \bigcup\{A_n : n \in \omega\}$ we have:*

1. *$A^* \subseteq D^*$;*
2. *$S^+_{at}(A_n, M^*)$ is countable for every $n \in \omega$; but*
3. *$S^+_{at}(A^*, M^*)$ is uncountable.*

Proof. Fix any countable, atomic $M^* \in \mathbf{At}_T$. Using Proposition 1.2 and the non-pcl-smallness of $\mathbf{At}_T$, choose a finite tuple $\bar{a}^* \subseteq M^*$ such that $S^+_{at}(D^*, M^*)$ is uncountable, where $D^* = \mathrm{pcl}_{M^*}(\bar{a}^*) \subseteq M^*$.

Fix any finite tuple $\bar{b} \supseteq \bar{a}^*$ from M^* and look at the complete 1-types $\mathcal{Q}_{\bar{b}} := \{r \in S_{at}(\bar{b})$ such that $r(M^*) \subseteq D^*\}$. These types visibly induce a partition D^*, and it is easily seen that if $\bar{b}' \supseteq \bar{b}$, the partition induced by $\bar{b}'$ refines the partition induced by $\bar{b}$. Let $\mathcal{Q} := \bigcup\{\mathcal{Q}_{\bar{b}} : \bar{a}^* \subseteq \bar{b} \subseteq M^*\}$.

Define a rank function $\mathrm{rk} : \mathcal{Q} \to ON \cup \{\infty\}$ as follows:

- $\mathrm{rk}(c/\bar{b}) \geq 0$ if and only if $\mathrm{tp}(c/\bar{b}) \in \mathcal{Q}$;
- $\mathrm{rk}(c/\bar{b}) \geq 1$ if and only if $\mathrm{tp}(c/\bar{b}) \in \mathcal{Q}$ and there are infinitely many $c' \in D^*$ realizing $\mathrm{tp}(c/D^*)$; and
- for an ordinal $\alpha \geq 2$, $\mathrm{rk}(c/\bar{b}) \geq \alpha$ if and only if for every $\beta < \alpha$ and every $\bar{b}'$ from M^*, there is $c' \in D^*$ realizing $\mathrm{tp}(c/\bar{b})$ such that $\mathrm{rk}(c'/\bar{b}\bar{b}') \geq \beta$.

- $\mathrm{rk}(c/\bar{b}) = \alpha$ if and only if $\mathrm{rk}(c/\bar{b}) \geq \alpha$ but $\mathrm{rk}(c/\bar{b}) \not\geq \alpha + 1$.

Claim 1. For every $r \in \mathcal{Q}$, $\mathrm{rk}(r)$ is a countable ordinal.

Proof. Assume by way of contradiction that $\mathrm{rk}(c/\bar{b}) \geq \omega_1$ for some type $c/\bar{b}$. Then, for any $\bar{b}'$ from M, as D^* is countable, there is an element $c' \in D^*$ such that $\mathrm{rk}(c'/\bar{b}\bar{b}') \geq \beta$ for uncountably many β's, hence $\mathrm{rk}(c'/\bar{b}\bar{b}') \geq \omega_1$ as well. Using this idea, if we let $\langle \bar{b}_n : n \in \omega \rangle$ be an increasing sequence of finite sequences from M^* whose union is all of M^*, then we can find a sequence $\langle c_n : n \in \omega \rangle$ of elements from D^* such that, for each n, $\mathrm{rk}(c_n/\bar{b}_n) \geq \omega_1$ and $\mathrm{tp}(c_n/\bar{b}_n) \subseteq \mathrm{tp}(c_{n+1}/\bar{b}_{n+1})$. The union of these 1-types yields a complete, atomic 1-type $q \in S_{at}(M^*)$ all of whose realizations are in $\mathrm{pcl}_{M^*}(\bar{a})$. However, since the type asserting that '$x = c$' has rank 0 for each $c \in D^*$, q is omitted in M^*. To obtain a contradiction, choose a realization e of q and, as M^*e is a countable, atomic set, construct a countable, elementary extension $M' \succeq M^*$ with $e \in M'$. But now, q implies that $e \in \mathrm{pcl}_{M'}(\bar{a})$, yet this is contradicted by the fact that M^* contains $\bar{a}$ but not e.

As notation, for a subset $\mathcal{S} \subseteq \mathcal{Q}_{\bar{b}}$, let $A_{\mathcal{S}} = \bigcup\{r(M^*) : r \in \mathcal{S}\}$, which is always a subset of D^*. Define the set of 'candidates' as

$$\mathcal{C} = \{(\mathcal{S}, \bar{b}) : \bar{b} \supseteq \bar{a}^*, \mathcal{S} \subseteq \mathcal{Q}_{\bar{b}}, \text{ and } S^+_{at}(A_{\mathcal{S}}, M^*) \text{ uncountable}\}$$

Note that $\mathcal{C}$ is non-empty as $(\mathcal{S}_0, \bar{a}^*) \in \mathcal{C}$, where $\mathcal{S}_0$ is an enumeration of all the complete, pseudo-algebraic types over $\bar{a}^*$. Among all candidates, choose $(\mathcal{S}^*, \bar{b}^*) \in \mathcal{C}$ such that

$$\alpha^* := \sup\{rk(r) + 1 : r \in \mathcal{S}^*\}$$

is as small as possible. Enumerate $\mathcal{S}^* = \{r_j : j \in \omega\}$ and put $A^* := A_{\mathcal{S}^*}$ and $A_n := \bigcup\{r_j(M^*, \bar{b}^*) : j < n\}$ for each $n \in \omega$. As Clauses (1) and (3) are immediate, it suffices to prove the following Claim:

Claim 2. For each $n \in \omega$, $S^+_{at}(A_n, M^*)$ is countable.

Proof. Fix any $n \in \omega$. First, note that if $\mathrm{rk}(r_j) = 0$ for every $j < n$, then A_n would be finite, which would imply $S_{at}(A_n)$ is countable. As $S_{at}(A_n)$ contains $S^+_{at}(A_n, M^*)$, the result follows.

Now assume $rk(r_j) > 0$ for at least one $j < n$. Let $\beta := \max\{rk(r_j) : j < n\}$ and let $F = \{j < n : rk(r_j) = \beta\}$. Clearly, $\beta < \alpha^*$. For each $j \in F$, as $\beta > 0$ but $rk(r_j) \not\geq \beta + 1$, there is a finite tuple $\bar{b}_j$ such that $\mathrm{rk}(c/\bar{b}^*\bar{b}_j) < \beta$ for all $c \in r_j(M^*)$.

Let $\bar{b}'$ be the concatenation of $\bar{b}^*$ with each $\bar{b}_j$ for $j \in F$ and let

$$\mathcal{S}' := \{r' \in \mathcal{Q}_{\bar{b}'} : r' \text{ extends some } r_j \text{ with } j < n\}$$

Subclaim. $\mathrm{rk}(r') < \beta$ for every $r' \in \mathcal{S}'$.

Proof. Fix $r' \in \mathcal{S}'$ and choose $c \in r'(M^*, \bar{b}')$. There are two cases. On one hand, if r' extends some r_j with $j \in F$, then $\mathrm{rk}(c/\bar{b}') \leq \mathrm{rk}(c/\bar{b}^*\bar{b}_j) < \beta$. On the other hand, if r' extends some r_j with $r_j \notin F$, then as $\mathrm{rk}(r_j) < \beta$, $\mathrm{rk}(c/\bar{b}') \leq \mathrm{rk}(c/\bar{b}^*) < \beta$.

Clearly $A_{\mathcal{S}'} = A_n$, so $S^+_{at}(A_n, M^*) = S^+_{at}(A_{\mathcal{S}'}, M^*)$. Thus, if $S^+_{at}(A_n, M^*)$ were uncountable, then $(\mathcal{S}', \bar{b}')$ would be a candidate, i.e., an element of $\mathcal{C}$. But, as $\beta < \alpha^*$, this is impossible by the Subclaim and the minimality of α^*.

3 A family of $2^{\aleph_0}$ atomic models of size $\aleph_1$

Throughout the whole of this section, we assume that T is a complete theory in a countable language for which $\mathbf{At}_T$ has an uncountable atomic model, but is not pcl-small. Appealing to Proposition 2.10,

> **Fix, for the whole of this section, a countable atomic model $\mathbf{M}^*$, tuples $\bar{\mathbf{a}}^* \subseteq \bar{\mathbf{b}}^* \subseteq \mathbf{M}^*$ and sets A^* and A_n for each $n \in \omega$ as in Proposition 2.10.**

We work with this fixed configuration for the whole of this section and, in Subsection 3.3 eventually prove:

Proposition 3.1 *There is a family $\{(N_\eta, \bar{b}^*) : \eta \in 2^\omega\}$ of atomic models of T, each of size $\aleph_1$, that are pairwise non-isomorphic over $\bar{b}^*$.*

3.1 Colorings of models realizing many types over A^*

Definition 3.2 Call a structure $(N, \bar{b}^*)$ *rich* if $N \in \mathbf{At}_T$ has size $\aleph_1$, $M^* \preceq N$, and N realizes uncountably many 1-types over A^*.

Lemma 3.3 *For each $n \in \omega$, a rich $(N, \bar{b}^*)$ realizes only countably many distinct 1-types over A_n.*

Proof. Fix any $(N, \bar{b}^*)$ and $n < \omega$ as above. If $\{c_i : i \in \omega_1\}$ realize distinct types over A_n, then the types $\{\mathrm{tp}_N(c_i/M^*) : i \in \omega_1\}$ would be distinct, contradicting $S^+_{at}(A_n, M^*)$ countable.

How can we tell whether rich structures are non-isomorphic? We introduce the notion of $\mathcal{U}$-colorings and Corollary 3.6 gives a sufficient condition.

Definition 3.4 Fix a subset $\mathcal{U} \subseteq \omega$ and a rich $(N, \bar{b}^*)$.

- For elements $d, d' \in N$, define the *splitting number* $\mathrm{spl}(d, d') \in (\omega + 1)$ to be the least $k < \omega$ such that $\mathrm{tp}(d/A_k) \neq \mathrm{tp}(d'/A_k)$ if such exists; and $\mathrm{spl}(d, d') = \omega$ if $\mathrm{tp}(d/A^*) = \mathrm{tp}(d'/A^*)$.
- A *$\mathcal{U}$-coloring of a rich* $(N, \bar{b}^*)$ is a function

$$c : N \to \omega$$

  such that for all pairs $d, d' \in N$, at least one of the following hold:

  1. $\mathrm{tp}(d/A^*) = \mathrm{tp}(d'/A^*)$; or
  2. $c(d) \neq c(d')$; or
  3. $\mathrm{spl}(d, d') \in \mathcal{U}$.

- The *color filter* $\mathcal{F}(N, \bar{b}^*) := \{\mathcal{U} \subseteq \omega : \text{a } \mathcal{U}\text{-coloring of } (N, \bar{b}^*) \text{ exists}\}$.

Lemma 3.5 *Fix a rich $(N, \bar{b}^*)$. Then:*

1. *$\mathcal{F}(N, \bar{b}^*)$ is a filter;*
2. *$\mathcal{F}(N, \bar{b}^*)$ contains the cofinite subsets of ω; but*
3. *No finite $\mathcal{U} \subseteq \omega$ is in $\mathcal{F}(N, \bar{b}^*)$.*

Proof. (1) First, note that if $\mathcal{U} \subseteq \mathcal{U}' \subseteq \omega$, then every $\mathcal{U}$-coloring c is also a $\mathcal{U}'$-coloring. Thus, $\mathcal{F}(N, \bar{b}^*)$ is upward closed. Next, suppose $\mathcal{U}_1 \in \mathcal{F}(N, \bar{b}^*)$ via the coloring $c_1 : N \to \omega$ and $\mathcal{U}_2 \in \mathcal{F}(N, \bar{a}^*\bar{b}^*)$ via the coloring $c_2 : N \to \omega$. Fix any bijection $t : \omega \times \omega \to \omega$. It is easily checked that $c^* : N \to \omega$ defined by $c^*(d) = t(c_1(d), c_2(d))$ is a $\mathcal{U}_1 \cap \mathcal{U}_2$-coloring of $(N, \bar{b}^*)$. Thus, $\mathcal{U}_1 \cap \mathcal{U}_2 \in \mathcal{F}(N, \bar{b}^*)$. So $\mathcal{F}(N, \bar{b}^*)$ is a filter.

(2) As $\mathcal{F}(N, \bar{b}^*)$ is a filter, it suffices to show $(\omega - n) \in \mathcal{F}(N, \bar{b}^*)$ for each $n \in \omega$. So fix such an n. By Lemma 3.3, N realizes at most countably many

types over A_n. Thus, we can produce a map $c : N \to \omega$ such that $c(d) = c(d')$ if and only if $\mathrm{tp}(d/A_n) = \mathrm{tp}(d'/A_n)$. As any such c is an $(\omega - n)$-coloring, $(\omega - n) \in \mathcal{F}(N, \bar{b}^*)$.

(3) It suffices to show that no $n = \{0, \dots, n-1\}$ is in $\mathcal{F}(N, \bar{b}^*)$. To see this, let $c : N \to \omega$ be an arbitrary map. We will show that c is not an $\{0, \dots, n-1\}$-coloring. As N realizes $\aleph_1$ distinct types over A^*, there is some $m^* \in \omega$ and an uncountable subset $\{d_\alpha : \alpha < \omega_1\} \subseteq N$ that realize distinct types over A^*, yet $c(d_\alpha) = m^*$ for each α. However, as N realizes only countably many types over A_n, there are $\alpha \neq \beta$ such that $n \leq \mathrm{spl}(d_\alpha, d_\beta) < \omega$. Thus, c is not an $\{0, \dots, n-1\}$-coloring.

We close with a sufficient condition for non-isomorphism of rich models.

Corollary 3.6 *Suppose that for $\ell = 1, 2$, $(N_\ell, \bar{b}^*)$ is a $\mathcal{U}_\ell$-colored rich model, and $\mathcal{U}_1 \cap \mathcal{U}_2$ is finite. Then there is no isomorphism $f : N_1 \to N_2$ fixing $\bar{b}^*$ pointwise.*

Proof. If there were such an isomorphism, then $(N_2, \bar{b}^*)$ would be both $\mathcal{U}_1$-colored and $\mathcal{U}_2$-colored. Thus, both $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{F}(N_2, \bar{b}^*)$, which contradicts Lemma 3.5.

3.2 Constructing a colored rich model via forcing

Arguing as in the proof of Proposition 1.2, from the data of Lemma 2.10 we can construct a rich $(N, \bar{b}^*)$ as the union of a continuous, elementary chain $\langle M_\alpha : \alpha \in \omega_1 \rangle$ of countable, atomic models with $M_0 = M^*$ such that, for each $\alpha \in \omega_1$ there is a distinguished $b_\alpha \in M_{\alpha+1}$ such that $\mathrm{tp}(b_\alpha/A^*)$ is omitted in M_α.

Our goal is to construct a sufficiently generic rich $(N, \bar{b}^*)$, along with a coloring $c : N \to (\omega + 1)$ via forcing. Our forcing $(\mathbb{Q}, \leq_{\mathbb{Q}})$ encodes finite approximations of such an $(N, \bar{b}^*)$ and c. A fundamental building block is the notion of a *striated type* over a finite subset $\bar{a}$ satisfying $\bar{b}^* \subseteq \bar{a} \subseteq M^*$. As an atomic type over a finite subset is generated by a complete formula, we use the terms interchangeably.

Definition 3.7 Choose a finite tuple $\bar{a}$ with $\bar{b}^* \subseteq \bar{a} \subseteq M^*$. A *striated type over $\bar{a}$* is a complete formula $\theta(\overline{x}) \in S_{at}(\bar{a})$ whose variables are partitioned as $\overline{x} = \langle \overline{x}_j : j < \ell \rangle$ where, for each j, $\overline{x}_j = \langle x_{j,n} : n < n(j) \rangle$ is an $n(j)$-tuple of variable symbols that satisfy $\mathrm{tp}(x_{j,0}/\bar{a}\cup\{\overline{x}_i : i<j\})$ is A^*-large. The integer ℓ is the *length* of the striated type.

A *simple realization* of a striated type $\theta(\overline{x})$ of length ℓ is a sequence $\bar{b}=\langle \bar{b}_j : j<\ell\rangle$ of tuples from M^* such that $M^*\models\theta(\bar{b})$. A *perfect chain realization* of $\theta(\overline{x})$ is a pair $(\overline{M},\bar{b})$, consisting of a chain $M_0\preceq M_1\preceq M_{\ell-1}\preceq M^*$ of ℓ elementary submodels of M^* and a simple realization $\bar{b}=\langle\bar{b}_j : j<\ell\rangle$ from M^* that satisfy: For each $j<\ell$,

1. $\bar{a}\cup\{\bar{b}_i : i<j\}\subseteq M_j$; and
2. $\mathrm{tp}(b_{j,0}/M_j)$ is A^*-perfect (see Definition 2.7).

Lemma 3.8 *Every striated type $\theta(\overline{x})\in S_{at}(\bar{a})$ has a perfect chain realization.*

Proof. We argue by induction on ℓ, the length of the striation. For striations of length zero there is nothing to prove, so assume the Lemma holds for striated types of length ℓ and choose an $(\ell+1)$-striation $\theta(\overline{x})\in S_{at}(\bar{a})$. Let $\theta\restriction_\ell$ be the truncation of θ to the variables $\overline{x}\restriction_\ell=\langle\overline{x}_j : j<\ell\rangle$. As $\theta\restriction_\ell$ is clearly an ℓ-striation, it has a perfect chain realization, i.e., a chain $M_0\preceq M_1\preceq M_{\ell-1}\preceq M^*$ and a tuple $\bar{b}=\langle\bar{b}_j : j<\ell\rangle$ from M^* realizing $\theta\restriction_\ell$ such that $\bar{a}\cup\{\bar{b}_i : i<j\}\subseteq M_j$ and $\mathrm{tp}(b_{j,0}/M_j)$ is A^*-perfect for each $j<\ell$.

Now, since $\mathrm{tp}(x_{\ell,0}/\bar{a}\bar{b})$ is A^*-large, by applying Proposition 2.8 there is an A^*-perfect type $\bar{p}\in S_{at}(M^*)$ (in a single variable $x_{\ell,0}$) extending $\mathrm{tp}(x_{\ell,0}/\bar{a}\bar{b})$. Choose a countable, atomic $N\succeq M^*$ and $e\in N$ realizing $\bar{p}$. As N and M^* are both countable and atomic, choose an isomorphism $f:N\to M^*$ that fixes $\bar{a}\bar{b}$ pointwise. Then $f(M_0)\preceq f(M_1)\preceq\dots f(M_{\ell-1})\preceq f(M^*)\preceq M^*$ is a chain. Let $b_{\ell,0}:=f(e)$ and choose $\langle b_{\ell,1}\dots,b_{\ell,n(\ell)-1}\rangle$ arbitrarily from M^* so that, letting $\bar{b}_\ell=\langle\bar{b}_{\ell,n} : n<n(\ell)\rangle$, $\bar{b}\frown\bar{b}_\ell$ realizes $\theta(\overline{x})$. This chain and this sequence form a perfect chain realization of θ.

The following Lemma is immediate, and indicates the advantage of working with A^*-perfect types.

Lemma 3.9 *Let $(\overline{M},\bar{b})$ be any perfect chain realization of a striated type $\theta(\overline{x})\in S_{at}(\bar{a})$. Then for every $\bar{c}\subseteq M_0$, $\mathrm{tp}(\bar{b}/\bar{a}\bar{c})\in S_{at}(\bar{a}\bar{c})$ is a striated type extending $\theta(\overline{x})$, and $(\overline{M},\bar{b})$ is a perfect chain realization of it.*

The Lemma below, whose proof simply amounts to unpacking definitions, demonstrate that striated types are rather malleable.

Lemma 3.10 *1. If* $\mathrm{tp}(\bar{c}/\bar{a})$ *is a striated type of length* k *and* $\mathrm{tp}(\bar{d}/\bar{a}\bar{c})$ *is a striated type of length* ℓ*, then* $\mathrm{tp}(\bar{c}\bar{d}/\bar{a})$ *is a striated type of length* $k+\ell$*.*

2. *Suppose* $\mathrm{tp}(\bar{b}/\bar{a})$ *is a striated type of length* ℓ *and* $k < \ell$*. Let* $\bar{b}_{<k}$ *and* $\bar{b}_{\geq k}$ *be the induced partition of* $\bar{b}$*. Then* $\mathrm{tp}(\bar{b}_{<k}/\bar{a})$ *is a striated type of length* ℓ *and* $\mathrm{tp}(\bar{b}_{\geq k}/\bar{a}\bar{b}_{<k})$ *is a striated type of length* $(\ell-k)$*. Moreover, if* $(\overline{M}, \bar{b})$ *is a perfect chain realization of* $\mathrm{tp}(\bar{b}/\bar{a})$*, then* $(\overline{M}_{<k}, \bar{b}_{<k})$ *is a perfect chain realization of* $\mathrm{tp}(\bar{b}_{<k}/\bar{a})$ *and* $(\overline{M}_{\geq k}, \bar{b}_{\geq k})$ *is a perfect chain realization of* $\mathrm{tp}(\bar{b}_{\geq k}/\bar{a}\bar{b}_{<k})$*.*

We begin by defining a partial order $(\mathbb{Q}_0, \leq_{\mathbb{Q}_0})$ of 'preconditions'. Then our forcing $(\mathbb{Q}, \leq_{\mathbb{Q}})$ will be a dense suborder of these preconditions.

Definition 3.11 $\mathbb{Q}_0$ is the set of all $\mathbf{p} = (\bar{\mathbf{a}}_{\mathbf{p}}, u_{\mathbf{p}}, \overline{n}_{\mathbf{p}}, \theta_{\mathbf{p}}(\overline{x}_{\mathbf{p}}), k_{\mathbf{p}}, \mathcal{U}_{\mathbf{p}}, c_{\mathbf{p}})$, where

1. $\bar{\mathbf{a}}_{\mathbf{p}}$ is a finite subset of M^* containing $\bar{b}^*$;
2. $u_{\mathbf{p}}$ is a finite subset of ω_1;
3. $\overline{n}_{\mathbf{p}} = \langle n_t : t \in u_{\mathbf{p}} \rangle$ is a sequence of positive integers;
4. $\overline{x}_{\mathbf{p}} = \langle \overline{x}_{t,\mathbf{p}} : t \in u_{\mathbf{p}} \rangle$, where each $\overline{x}_{t,\mathbf{p}} = \langle x_{t,n} : n < \overline{n}_t \rangle$ is a finite sequence from the set $X = \{x_{t,n} : t \in \omega_1, n \in \omega\}$ of variable symbols;
5. $\theta_{\mathbf{p}}(\overline{x}_{\mathbf{p}}) \in S_{at}(\bar{\mathbf{a}}_{\mathbf{p}})$ is a striated type of length $|u_{\mathbf{p}}|$ (see Definition 3.7);
6. $k_{\mathbf{p}} \in \omega$;
7. $\mathcal{U}_{\mathbf{p}} \subseteq k_{\mathbf{p}} = \{0, \ldots, k_{\mathbf{p}} - 1\}$;
8. $c_{\mathbf{p}} : \overline{x}_{\mathbf{p}} \to \omega$ is a function such that for all pairs $x_{t,n}, x_{s,m}$ from $\overline{x}_{\mathbf{p}}$ with $c_{\mathbf{p}}(x_{t,n}) = c_{\mathbf{p}}(x_{s,m})$
   (a) **either** $\mathrm{spl}(b_{t,n}, b_{s,m}) \geq k_{\mathbf{p}}$ for all perfect chain realizations $(\overline{M}, \bar{b})$ of $\theta_{\mathbf{p}}(\overline{x}_{\mathbf{p}})$;
   (b) **or** there is some $k \in \mathcal{U}_{\mathbf{p}}$ such that $\mathrm{spl}(b_{t,n}, b_{s,m}) = k$ for all perfect chain realizations $(\overline{M}, \bar{b})$ of $\theta_{\mathbf{p}}(\overline{x}_{\mathbf{p}})$.

We order elements of $\mathbb{Q}_0$ by: $\mathbf{p} \leq_{\mathbb{Q}_0} \mathbf{q}$ if and only if

- $\bar{\mathbf{a}}_{\mathbf{p}} \subseteq \bar{\mathbf{a}}_{\mathbf{q}}$;

- $u_{\mathbf{p}} \subseteq u_{\mathbf{q}}$ and $n_{t,\mathbf{p}} \leq n_{t,\mathbf{q}}$ for all $t \in u_{\mathbf{p}}$, hence $\overline{x}_{\mathbf{p}}$ is a subsequence of $\overline{x}_{\mathbf{q}}$;
- $\theta_{\mathbf{q}}(\overline{x}_{\mathbf{q}}) \vdash \theta_{\mathbf{p}}(\overline{x}_{\mathbf{p}})$;
- $k_{\mathbf{p}} \leq k_{\mathbf{q}}$;
- $\mathcal{U}_{\mathbf{p}} = \mathcal{U}_{\mathbf{q}} \cap k_{\mathbf{p}}$ (hence, for $j < k_{\mathbf{p}}$, $j \in \mathcal{U}_{\mathbf{p}}$ if and only if $j \in \mathcal{U}_{\mathbf{q}}$);
- $c_{\mathbf{p}} = c_{\mathbf{q}} \restriction_{\overline{x}_{\mathbf{p}}}$.

Visibly, $(\mathbb{Q}_0, \leq_{\mathbb{Q}_0})$ is a partial order. Call a precondition $\mathbf{p} \in \mathbb{Q}_0$ *unarily decided* if, for every $x_{t,n} \in \overline{x}_{\mathbf{p}}$, $p(\overline{x}_{\mathbf{p}})$ determines a type in $S^+_{at}(A_{k_{\mathbf{p}}}, M^*)$ (see Definition 2.1). That the unarily decided preconditions are dense follows easily from the fact that $S^+_{at}(A_{k_{\mathbf{p}}}, M^*)$ is countable.

Lemma 3.12 *The set $\{\mathbf{p} \in \mathbb{Q}_0 : \mathbf{p}$ is unarily decided$\}$ is dense in $(\mathbb{Q}_0, \leq_{\mathbb{Q}_0})$. Moreover, given any $\mathbf{p} \in \mathbb{Q}_0$, there is a unarily decided $\mathbf{q} \geq_{\mathbb{Q}_0} \mathbf{p}$ with $\overline{x}_{\mathbf{q}} = \overline{x}_{\mathbf{p}}$ and $k_{\mathbf{q}} = k_{\mathbf{p}}$ (hence $\mathcal{U}_{\mathbf{q}} = \mathcal{U}_{\mathbf{p}}$).*

Proof. Fix $\mathbf{p} \in \mathbb{Q}_0$ and let $k := k_{\mathbf{p}}$. Arguing by induction on the size of the finite set $\overline{x}_{\mathbf{p}}$, it is enough to strengthen $p(x_{t,n})$ individually for each $x_{t,n} \in \overline{x}_{\mathbf{p}}$. So fix $x_{t,n} \in \overline{x}_{\mathbf{p}}$. By Corollary 2.6 there is an $\bar{a}' \supseteq \bar{a}_{\mathbf{p}}$ and a 1-type $q_1(x_{t,n}) \in S_{at}(\bar{a}')$ extending $\mathrm{tp}(x_{t,n}/\bar{a}_{\mathbf{p}})$ that determines a type in $S^+_{at}(A_{k_{\mathbf{p}}}, M^*)$. Then, using Lemma 3.10(1) we can choose a striated type $p'(\overline{x}_{\mathbf{p}}) \in S_{at}(\bar{a}')$ extending $p(\overline{x}_{\mathbf{p}}) \cup q_1$.

We iterate the above procedure for each of the (finitely many) elements of $\overline{x}_{\mathbf{p}}$. We then get a unarily decided precondition $\mathbf{p}' \geq_{\mathbb{Q}_0} \mathbf{p}$ whose type $p'(\overline{x}_{\mathbf{p}})$ still has the same free variables, and each of $k_{\mathbf{p}}$, $\mathcal{U}_{\mathbf{p}}$, $c_{\mathbf{p}}$ are unchanged.

Next, call a precondition $\mathbf{p} \in \mathbb{Q}_0$ *fully decided* if, it is unarily decided and, for each pair $x_{t,n}, x_{s,m}$ from $\overline{x}_{\mathbf{p}}$ with $c_{\mathbf{p}}(x_{t,n}) = c_{\mathbf{p}}(x_{s,m})$, if $\mathrm{spl}(b_{t,n}, b_{s,m}) \geq k_{\mathbf{p}}$ for some perfect chain realization $(\overline{M}, \bar{b})$, then $\mathrm{tp}(b_{t,n}/A^*) = \mathrm{tp}(b_{s,m}/A^*)$ for all perfect chain realizations $(\overline{M}, \bar{b})$ of $\theta_{\mathbf{p}}(\overline{x}_{\mathbf{p}})$.

Lemma 3.13 *The set $\{\mathbf{p} \in \mathbb{Q}_0 : \mathbf{p}$ is fully decided$\}$ is dense in $(\mathbb{Q}_0, \leq_{\mathbb{Q}_0})$. Moreover, given any $\mathbf{p} \in \mathbb{Q}_0$, there is a fully decided $\mathbf{q} \geq_{\mathbb{Q}_0} \mathbf{p}$ with $\overline{x}_{\mathbf{q}} = \overline{x}_{\mathbf{p}}$.*

Proof. It suffices to handle each pair $x_{t,n}, x_{s,m}$ from $\overline{x}_{\mathbf{p}}$ with $c(x_{t,n}) = c(x_{s,m})$ separately. Given such a pair, suppose there is some perfect chain realization $(\overline{M}, \bar{b})$ of $\theta(\overline{x}_{\mathbf{p}}) \in S_{at}(\bar{\mathbf{a}}_{\mathbf{p}})$ with $k_{\mathbf{p}} \leq \mathrm{spl}(b_{t,n}, b_{s,m}) < \omega$. Among all such perfect chain realizations, choose one that minimizes $k^* = \mathrm{spl}(b_{t,n}, b_{s,m})$.

Choose a formula $\varphi(x, \bar{c})$ with $\bar{c}$ from A_{k^*+1} witnessing that $\mathrm{tp}(b_{t,n}/A_{k^*+1}) \neq \mathrm{tp}(b_{s,m}/A_{k^*+1})$. As $A_{k^*+1} \subseteq M_0$, by applying Lemma 3.9, let $\theta^*(\overline{x}_{\mathbf{p}})$ be a complete formula over $\overline{\mathbf{a}}_{\mathbf{p}}\bar{c}$ isolating $\mathrm{tp}(\bar{b}/\overline{\mathbf{a}}_{\mathbf{p}}\bar{c})$. Form the precondition $\mathbf{p'} \in \mathbb{Q}_0$ by putting $\overline{\mathbf{a}}_{\mathbf{p'}} = \overline{\mathbf{a}}_{\mathbf{p}}\bar{c}$; $\theta_{\mathbf{p'}} = \theta^*$; $k_{\mathbf{p'}} = k^* + 1$; and $\mathcal{U}_{\mathbf{p'}} = \mathcal{U}_{\mathbf{p}} \cup \{k^*\}$; while leaving $\overline{x}_{\mathbf{p}}$ and $c_{\mathbf{p}}$ unchanged. It is evident that $\mathrm{spl}(b'_{t,n}, b'_{s,m}) = k^* \in \mathcal{U}_{\mathbf{p'}}$ for all perfect chain realizations $(\overline{M}, \bar{b}')$ of $\theta_{\mathbf{p'}}$. Continuing this process for each of the (finitely many) relevant pairs gives us a fully decided extension of $\mathbf{p}$.

Definition 3.14 *The forcing $(\mathbb{Q}, \leq_{\mathbb{Q}})$ is the set of fully decided $\mathbf{p} \in \mathbb{Q}_0$ with the inherited order.*

Lemma 3.15 *The forcing $(\mathbb{Q}, \leq_{\mathbb{Q}})$ has the countable chain condition (c.c.c.).*

Proof. Suppose $\{\mathbf{p}_i : i \in \omega_1\}$ is an uncountable subset of $\mathbb{Q}$. In light of Lemma 3.13, it suffices to find $i \neq j$ for which there is some precondition $\mathbf{q} \in \mathbb{Q}_0$ satisfying $\mathbf{p}_i \leq_{\mathbb{Q}_0} \mathbf{q}$ and $\mathbf{p}_j \leq_{\mathbb{Q}_0} \mathbf{q}$. First, by the Δ-system lemma applied to the finite sets $\{u_{\mathbf{p}_i}\}$, we may assume that $|u_{\mathbf{p}_i}|$ is constant and there is some fixed u^* that is an initial segment of each $u_{\mathbf{p}_i}$ and, moreover, whenever $i < j$, every element of $(u_{\mathbf{p}_i} \setminus u^*)$ is less than every element of $(u_{\mathbf{p}_j} \setminus u^*)$. By further trimming, but preserving uncountability, we may assume that the integer $k_{\mathbf{p}}$, the subset $\mathcal{U}_{\mathbf{p}} \subseteq k_{\mathbf{p}}$, and the parameter $\overline{\mathbf{a}}_{\mathbf{p}}$ remain constant. As notation, for $i < j$, let $f : u_{\mathbf{p}_i} \to u_{\mathbf{p}_j}$ be the unique order-preserving bijection. We may additionally assume that $n_{\mathbf{p}_i}(t) = n_{\mathbf{p}_j}(f(t))$, hence f has a natural extension (also called f): $\overline{x}_{\mathbf{p}_i} \to \overline{x}_{\mathbf{p}_j}$ given by $f(x_{t,n}) = x_{f(t),n}$. With this identification, we may assume $\theta_{\mathbf{p}_i}(\overline{x}_{\mathbf{p}_i}) = \theta_{\mathbf{p}_j}(f(\overline{x}_{\mathbf{p}_i}))$. As well, we may also assume $\mathrm{tp}(x_{t,n}/A_{k_{\mathbf{p}}}) = \mathrm{tp}(x_{f(t),n}/A_{k_{\mathbf{p}}})$ for every $x_{t,n} \in \overline{x}_{\mathbf{p}_i}$. As well, the colorings match up as well, i.e., $c(x_{t,n}) = x_{f(t),n}$.

Now fix $i < j$. Define $\mathbf{q}$ by $k_{\mathbf{q}} := k_{\mathbf{p}}$; $\mathcal{U}_{\mathbf{q}} := \mathcal{U}_{\mathbf{p}}$; and $\overline{\mathbf{a}}_{\mathbf{q}} := \overline{\mathbf{a}}_{\mathbf{p}}$ (the common values). Let $u_{\mathbf{q}} := u_{\mathbf{p}_i} \cup u_{\mathbf{p}_j}$, and, for $t \in u_{\mathbf{p}_i}$, $n_{t,\mathbf{q}} = n_{t,\mathbf{p}_i}$ while $n_{t,\mathbf{q}} = n_{t,\mathbf{p}_j}$ for $t \in u_{\mathbf{p}_j}$. To produce the striated type $\theta_{\mathbf{q}} \in S_{at}(\overline{\mathbf{a}}_{\mathbf{q}})$, first choose a perfect chain realization $(\overline{M}, \bar{b})$ of $\theta_{\mathbf{p}_i}(\overline{x}_{\mathbf{p}_i})$. Say $|u_{\mathbf{p}_i}| = \ell = |u_{\mathbf{p}_j}|$, while $|u^*| = k < \ell$. By Lemma 3.10(2), $\mathrm{tp}(\bar{b}_{<k}/\overline{\mathbf{a}}_{\mathbf{p}})$ is a striated type of length k and $(\overline{M}_{\geq k}, \bar{b}_{\geq k})$ is a perfect chain realization of the striated type $\mathrm{tp}(\bar{b}_{\geq k}/\overline{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k})$ of length $(\ell - k)$. Choose $\bar{d}$ from M_k such that $\mathrm{tp}(\bar{d}/\overline{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k}) = \mathrm{tp}(\bar{b}_{\geq k}/\overline{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k})$. Then by Lemma 3.9 (with M_k playing the role of M_0 there), $(\overline{M}_{\geq k}, \bar{b}_{\geq k})$ is a perfect chain realization of the striated type $\mathrm{tp}(\bar{b}_{\geq k}/\overline{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k}\bar{d})$. So, by Lemma 3.10(1), $\mathrm{tp}(\bar{d}\bar{b}_{\geq k}/\overline{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k})$ is a striated type of length $2(\ell - k)$. Thus, a second application of Lemma 3.10(1) implies that $\mathrm{tp}(\bar{b}_{<k}\bar{d}\bar{b}_{\geq k}/\overline{\mathbf{a}}_{\mathbf{p}})$ is a striated

type of length $2\ell - k$. Let $\theta_{\mathbf{q}}$ be a complete formula over $\overline{\mathbf{a}}_{\mathbf{p}}$ generating this type.

In order to show that $\mathbf{q}$ is a precondition (i.e., an element of $\mathbb{Q}_0$) only Clause (8) requires an argument. Fix any $x_{t,n}, x_{s,m}$ in $\overline{x}_{\mathbf{q}}$ with $c_{\mathbf{q}}(x_{t,n}) = c_{\mathbf{q}}(x_{s,m})$. As both $\mathbf{p}_i, \mathbf{p}_j \in \mathbb{Q}_0$, the verification is immediate if $\{t, s\}$ is a subset of either $u_{\mathbf{p}_i}$ or $u_{\mathbf{p}_j}$, so assume otherwise. By symmetry, assume $t \in u_{\mathbf{p}_i} - u^*$ and $s \in u_{\mathbf{p}_j} - u^*$. The point is that by our trimming, $x_{f(t),n} \in \overline{x}_{\mathbf{p}_j}$, $c_{\mathbf{p}_j}(x_{f(t),n}) = c_{\mathbf{p}_i}(x_{t,n})$, and $\mathrm{tp}(x_{t,n}/A_{k_{\mathbf{p}}}) = \mathrm{tp}(x_{f(t),n}/A_{k_{\mathbf{p}}})$. There are now two cases: First, if $\mathrm{tp}(x_{f(t),n}/A^*) = \mathrm{tp}(x_{s,m}/A^*)$, then it follows that $\mathrm{tp}(x_{t,n}/A_{k_{\mathbf{p}}}) = \mathrm{tp}(x_{s,m}/A_{k_{\mathbf{p}}})$, hence $\mathrm{spl}(e_{t,n}, e_{s,m}) \geq k_{\mathbf{p}}$ for any perfect chain realization $(\overline{N}, \bar{e})$ of $\theta_{\mathbf{q}}$. On the other hand, if $\theta_{\mathbf{p}_j}$ 'says' $\mathrm{spl}(x_{f(t),n}, x_{s,m}) = k \in \mathcal{U}_{\mathbf{p}}$, then $\theta_{\mathbf{q}}$ 'says' $\mathrm{spl}(x_{t,n}, x_{s,m}) = k \in \mathcal{U}_{\mathbf{q}}$ as well. Thus, $\mathbf{q} \in \mathbb{Q}_0$, which suffices by Lemma 3.13.

Lemma 3.16 *Each of the following sets are dense and open in* $(\mathbb{Q}, \leq_{\mathbb{Q}})$.

1. *For every* $t \in \omega_1$, $D_t = \{\mathbf{p} \in \mathbb{Q} : t \in u_{\mathbf{p}}\}$;
2. *For every* $(t, n) \in \omega_1 \times \omega$, $D_{t,n} = \{\mathbf{p} \in \mathbb{Q} : x_{t,n} \in \overline{x}_{\mathbf{p}}\}$; *and*
3. **Henkin witnesses:** *For all* $t \in \omega_1$, *all* $\langle x_{s_i,n_i} : i < m\rangle$ *with each* $s_i \leq t$ *and all* $\varphi(y, v_i : i < m)$, $\{\mathbf{p} \in \mathbb{Q} :$ **either** $\theta_{\mathbf{p}}(\overline{x}_{\mathbf{p}}) \vdash \forall y \neg\varphi(y, x_{s_i,n_i} : i < m)$ **or** *for some* n^*, $\theta_{\mathbf{p}}(\overline{x}_{\mathbf{p}}) \vdash \varphi(x_{t,n^*}, x_{s_i,n_i} : i < m)\}$.
4. *For all* $e \in M^*$, $D_e = \{\mathbf{p} \in \mathbb{Q} : e \in \overline{\mathbf{a}}_{\mathbf{p}}$ *and* $\theta(\overline{x}_{\mathbf{p}}) \vdash x_{0,n} = e$ *for some* $n \in \omega\}$.

Proof. That each of these sets is open is immediate. As for density, in all four clauses we will show that given some $\mathbf{p} \in \mathbb{Q}$, we will find an extension $\mathbf{q} \geq_{\mathbb{Q}} \mathbf{p}$ with $\overline{x}_{\mathbf{q}}$ a one-point extension of $\overline{x}_{\mathbf{p}}$. In all cases, we will put $k_{\mathbf{q}} := k_{\mathbf{p}}$, $\mathcal{U}_{\mathbf{q}} = \mathcal{U}_{\mathbf{p}}$ and since $\overline{x}_{\mathbf{p}}$ is finite, we can choose the color $c_{\mathbf{q}}$ of the 'new element' to be distinct from the other colors. Because of that, Clause (8) for $\mathbf{q}$ follows immediately from the fact $\mathbf{p} \in \mathbb{Q}$. Thus, for all four clauses, all of the work is in finding a striated type $\theta_{\mathbf{q}}$ extending $\theta_{\mathbf{p}}$.

(1) Fix $t \in \omega_1$ and choose an arbitrary $\mathbf{p} \in \mathbb{Q}$. If $t \in u_{\mathbf{p}}$ then there is nothing to prove, so assume otherwise. Let $\ell = |u_{\mathbf{p}}|$ and let $k = |\{s \in u_{\mathbf{p}} : s < t\}|$. Assume that $k < \ell$, as the case of $k = \ell$ is similar, but easier. Choose a perfect chain realization $(\overline{M}, \bar{b})$ of $\theta_{\mathbf{p}}(\overline{x}_{\mathbf{p}})$. By Lemma 3.10(2), $\mathrm{tp}(\bar{b}_{<k}/\overline{\mathbf{a}}_{\mathbf{p}})$ is a striated type of length k. By Lemma 2.4(1), choose an A^*-large type $r \in$

$S_{at}(\overline{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k})$ and choose a realization e of r in M_k. One checks immediately that $\operatorname{tp}(\bar{b}_{<k}e/\overline{\mathbf{a}}_{\mathbf{p}})$ is a striated type of length $(k+1)$. Now, also by Lemma 3.10(2), $(\overline{M}_{\geq k}, \bar{b}_{\geq k})$ is a perfect chain realization of $\operatorname{tp}(\bar{b}_{\geq k}/\overline{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k})$. So, by Lemma 3.9, $(\overline{M}_{\geq k}, \bar{b}_{\geq k})$ is also a perfect chain realization of $\operatorname{tp}(\bar{b}_{\geq k}/\overline{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k}e)$. In particular, $\operatorname{tp}(\bar{b}_{\geq k}/\overline{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k}e)$ is a striated type of length $(\ell - k)$. Thus, by Lemma 3.10(1), $\operatorname{tp}(\bar{b}_{<k}e\bar{b}_{\geq k}/\overline{\mathbf{a}}_{\mathbf{p}})$ is a striated type of length $(\ell+1)$. Take $\overline{\mathbf{a}}_{\mathbf{q}} := \overline{\mathbf{a}}_{\mathbf{p}}$, $\overline{x}_{\mathbf{q}} := \overline{x}_{\mathbf{p}} \cup \{x_{t,0}\}$, and take $\theta_{\mathbf{q}}(\overline{x}_{\mathbf{q}})$ to be a complete formula in $\operatorname{tp}(\bar{b}_{<k}e\bar{b}_{\geq k}/\overline{\mathbf{a}}_{\mathbf{q}})$.

The proofs of (2) and (3) are extremely similar. We prove (2) and indicate the adjustment necessary for (3). Fix $(t,n) \in \omega_1 \times \omega$. By (1) and an inductive argument, we may assume we are given $\mathbf{p} \in \mathbb{Q}$ with $t \in u_{\mathbf{p}}$ and $x_{t,n-1} \in \overline{x}_{\mathbf{p}}$. Say $|u_{\mathbf{p}}| = \ell$ and assyne t is the $(k-1)$st element of u_p in ascending order. Choose a perfect chain realization $(\overline{M}, \bar{b})$ of $\theta_{\mathbf{p}}(\overline{x}_{\mathbf{p}})$. By Lemma 3.10(2), $\operatorname{tp}(\bar{b}_{<k}/\overline{\mathbf{a}}_{\mathbf{p}})$ is striated of length k. Choose an arbitrary $e \in M_k$[4] and adjoin it to $\bar{b}_{k-1}$. More formally, let $\bar{b}^*_{<k} := \langle \bar{b}^*_j : j < k\rangle$, where $\bar{b}^*_j = \bar{b}_j$ for $j < k-2$, while $\bar{b}^*_{k-1} := \bar{b}_{k-1}e$. Note that $\operatorname{tp}(\bar{b}^*_{<k}/\overline{\mathbf{a}}_{\mathbf{p}})$ remains a striated type of length k. By Lemma 3.10(2), $(\overline{M}_{\geq k}, \bar{b}_{\geq k})$ is a perfect chain realization of $\operatorname{tp}(\bar{b}_{\geq k}/\overline{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k})$. So, by Lemma 3.9 it is also a perfect chain realization of $\operatorname{tp}(\bar{b}_{\geq k}/\overline{\mathbf{a}}_{\mathbf{p}}\bar{b}^*_{<k})$. In particular, $\operatorname{tp}(\bar{b}_{\geq k}/\overline{\mathbf{a}}_{\mathbf{p}}\bar{b}^*_{<k})$ is a striated type of length $(\ell - k)$, so $\operatorname{tp}(\bar{b}^*_{<k}\bar{b}_{\geq k}/\overline{\mathbf{a}}_{\mathbf{p}})$ is a striated type of length ℓ extending $\theta_{\mathbf{p}}(\overline{x}_{\mathbf{p}})$. Put $\overline{x}_{\mathbf{q}} := \overline{x}_{\mathbf{p}} \cup \{x_{t,n}\}$ and let $\theta_{\mathbf{q}}(\overline{x}_{\mathbf{q}})$ be a complete formula isolating this type.

(4) is also similar and is left to the reader.

The following Proposition follows immediately from the density conditions described above.

Proposition 3.17 *Let G be a $\mathbb{Q}$-generic filter. Then, in $V[G]$, a rich, $\mathcal{U}_G$-colored atomic model of T exists, where $\mathcal{U}_G = \{k \in \omega : k \in \mathcal{U}_{\mathbf{p}}$ for some $\mathbf{p} \in G\}$.*

Proof. There is a congruence $\sim_G$ defined on $X = \{x_{t,n} : t \in \omega_1, n \in \omega\}$ defined by $x_{t,n} \sim_G x_{s,m}$ if and only if $\theta_{\mathbf{p}} \vdash x_{t,n} = x_{s,m}$ for some $\mathbf{p} \in G$. Let M_G be the model of T with universe $X/\sim_G$ and relations $M_G \models \varphi(a_1, \ldots, a_k)$ if and only if there are $(x_{t_1,n_1}, \ldots, x_{t_k,n_k}) \in X^k$ such that $[x_{t_i,n_i}] = a_i$ for each i and $\theta_{\mathbf{p}} \vdash \varphi(x_{t_1,n_1}, \ldots, x_{t_k,n_k})$ for some $\mathbf{p} \in G$. Since $(\mathbb{Q}, \leq_{\mathbb{Q}})$ has c.c.c., M_G has size $\aleph_1$. As notation, for each $t \in \omega_1$, let $M_{\leq t}$ be the substructure of M_G with universe $\{[x_{s,m}] : s \leq t, m \in \omega\}$. Then $M^* \preceq M_0$ and $M_{\leq s} \preceq M_{\leq t} \preceq M_G$ whenever $s \leq t < \omega_1$. The definition of a striated type implies that

[4] In the proof of (3), e would be a realization of $\varphi(y, b_{s_i,n_i} : i < m)$ in M_k, if one existed.

$\mathrm{tp}([x_{t,0}]/A^*)$ is omitted in $M_{<t}$, hence the set $\{[x_{t,0}] : t \in \omega_1\}$ witnesses that $(M_G, \bar{b}^*)$ is rich. Also, define $c_G := \bigcup\{c_\mathbf{p} : \mathbf{p} \in G\}$. Using the fact that each $\mathbf{p} \in \mathbb{Q}$ is fully decided, check that c_G is a $\mathcal{U}_G$-coloring of $(M_G, \bar{b}^*)$.

Note that in the Conclusion below, such a $G \in V$ always exists, since $\mathcal{B}$ is countable.

Conclusion 3.18 *Suppose $\mathcal{B}$ is a countable, transitive model of ZFC^*, with $\{M^*, T, L\} \subseteq \mathcal{B}$, and let $G \in V$, $G \subseteq \mathbb{Q}$ be any filter meeting every dense $D \subseteq \mathbb{Q}$ with $D \in \mathcal{B}$. Then: Let $\mathcal{U}_G = \{k \in \omega : k \in \mathcal{U}_\mathbf{p}$ for some $\mathbf{p} \in G\}$. Then:*

1. *$\mathcal{U}_G \in V$; and*
2. *In V, there is a $\mathcal{U}_G$-colored, rich atomic model $(N, \bar{b}^*)$ of T.*

Proof. That $\mathcal{U}_G \in V$ is immediate, since both $\mathcal{B}$ and G are. As for (2), as G meets every dense set in $\mathcal{B}$, $\mathcal{B}[G]$ is a countable, transitive model of ZFC^*, and by applying Proposition 3.17,

$$\mathcal{B}[G] \models \text{`There is a rich, } \mathcal{U}_G\text{-colored } (M_G, \bar{b}^*) \text{ of size } \aleph_1\text{'}$$

Let $L' = L \cup \{c, R\} \cup \{c_m : m \in M^*\}$ Working in $\mathcal{B}[G]$, expand M_G to an L'-structure M', interpreting each c_m by m, interpreting the unary function $c^{M'}$ as $c_G = \bigcup\{c_\mathbf{p} : \mathbf{p} \in G\}$, and the unary predicate $R^{M'} = \{[x_{t,0}] : t \in \omega_1\}$.

Now, for each $d, d' \in M'$ and $k \in \omega$, the relation $\mathrm{tp}_{M'}(d/A_k) = \mathrm{tp}_{M'}(d'/A_k)$ is definable by an $L'_{\omega_1,\omega}$-formula. Thus, the binary function $\mathrm{spl} : (M')^2 \to (\omega + 1)$ is also $L'_{\omega_1,\omega}$-definable, hence, using the coloring c, there is an $L'_{\omega_1,\omega}$-sentence Ψ stating that 'c induces a $\mathcal{U}_G$-coloring.' Finally, using the Q-quantifier to state that R is uncountable, there is an $L'_{\omega_1,\omega}$-sentence $\Phi \in \mathcal{B}[G]$ stating that the $L(\bar{b}^*)$-reduct of a given L'-structure is a rich, atomic model of T, that is $\mathcal{U}_G$-colored via c. We finish by applying Proposition 2.9 to M' and Φ.

3.3 Mass production

In this subsection we define a forcing $(\mathbb{P}, \leq_\mathbb{P})$ such that a $\mathbb{P}$-generic filter G produces a perfect set $\{G_\eta : \eta \in 2^\omega\}$ of $\mathbb{Q}$-generic filters such that the associated subsets $\{\mathcal{U}_{G_\eta} : \eta \in 2^\omega\}$ of ω are almost disjoint. Although the application there is very different, the argument in this subsection is similar to one appearing in [7].

We begin with one easy density argument concerning the partial $(\mathbb{Q}, \leq_{\mathbb{Q}})$. Fundamentally, it allows us to 'stall' the construction for any fixed, finite length of time.

Lemma 3.19 *For every $\mathbf{p} \in \mathbb{Q}$ and every $k^* > k_{\mathbf{p}}$, there is $\mathbf{q} \geq_{\mathbb{Q}} \mathbf{p}$ such that $\overline{x}_{\mathbf{q}} = \overline{x}_{\mathbf{p}}$, (hence $c_{\mathbf{q}} = c_{\mathbf{p}}$); but $k_{\mathbf{q}} = k^*$ and $\mathcal{U}_{\mathbf{q}} = \mathcal{U}_{\mathbf{p}}$, i.e., $\mathcal{U}_{\mathbf{q}} \cap [k_{\mathbf{p}}, k^*) = \emptyset$.*

Proof. Simply define $\mathbf{q}$ as above and then verify that $\mathbf{q} \in \mathbb{Q}$.

Definition 3.20 For $n \in \omega$, let

$$\mathbb{P}_n = \{(k, \bar{p}) : k \in \omega, \bar{p} = \langle p_\eta : \eta \in 2^n \rangle, \text{ where each } p_\nu \in \mathbb{Q} \text{ and every } k_{p_\nu} = k\}$$

As notation, for $\mathbf{p} \in \mathbb{P}_n$, we let $k(\mathbf{p})$ denote the (integer) first coordinate of $\mathbf{p}$. For each $\ell < k(\mathbf{p})$, define the *trace of ℓ*, $\mathrm{tr}_\ell(\mathbf{p}) = \{\nu \in 2^n : \ell \in \mathcal{U}_{p_\nu}\}$.

Let $\mathbb{P} = \bigcup_{n \in \omega} \mathbb{P}_n$. As notation, for $\mathbf{p} \in \mathbb{P}$, $n(\mathbf{p})$ is the unique n for which $\mathbf{p} \in \mathbb{P}_n$.

Definition 3.21 Define an order $\leq_{\mathbb{P}}$ on $\mathbb{P}$ by $\mathbf{p} \leq_{\mathbb{P}} \mathbf{q}$ if and only if

1. $n(\mathbf{p}) \leq n(\mathbf{q})$, $k(\mathbf{p}) \leq k(\mathbf{q})$;
2. $p_\nu \leq_{\mathbb{Q}} q_\mu$ for all pairs $\nu \in 2^{n(\mathbf{p})}, \mu \in 2^{n(\mathbf{q})}$ satisfying $\nu \trianglelefteq \mu$; and
3. For all $\ell \in [k(\mathbf{p}), k(\mathbf{q}))$, the set $\{\mu\restriction_{n(\mathbf{p})} : \mu \in \mathrm{tr}_\ell(\mathbf{q})\}$ is either empty or is a singleton.

It is easily checked that $(\mathbb{P}, \leq_{\mathbb{P}})$ is a partial order, hence a notion of forcing. The following Lemma describes the dense subsets of $\mathbb{P}$.

Lemma 3.22 *1. For each n and k, $\{\mathbf{p} \in \mathbb{P} : n(\mathbf{p}) \geq n\}$ and $\{\mathbf{p} \in \mathbb{P} : k(\mathbf{p}) \geq k\}$ are dense;*

2. Suppose D is a dense, open subset of $\mathbb{Q}$. Then for every n and every $\mathbf{p} \in \mathbb{P}_n$, there is $\mathbf{q} \in \mathbb{P}_n$ such that $\mathbf{q} \geq_{\mathbb{P}} \mathbf{p}$ and, for every $\nu \in 2^n$, $\mathbf{q}_\nu \in D$.

Proof. Arguing by induction, it suffices to prove that for any given $\mathbf{p} \in \mathbb{P}$, there is $\mathbf{q} \geq_{\mathbb{P}} \mathbf{p}$ with $n(\mathbf{q}) = n(\mathbf{p})+1$ and an $\mathbf{r} \geq_{\mathbb{P}} \mathbf{p}$ with $k(\mathbf{r}) > k(\mathbf{p})$. Fix $\mathbf{p} \in \mathbb{P}$. Say $\mathbf{p} \in \mathbb{P}_n$ and $\mathbf{p} = (k, \bar{p})$. To construct $\mathbf{q}$, for each $\nu \in 2^n$, define $q_{\nu 0} = q_{\nu 1} = p_\nu$. Let $\bar{q} := \langle q_\mu : \mu \in 2^{n+1} \rangle$ and $\mathbf{q} = (k, \bar{q})$. Then $\mathbf{q} \in \mathbb{P}_{n+1}$ and $\mathbf{q} \geq_{\mathbb{P}} \mathbf{p}$ (note that Clause (3) in the definition of $\leq_{\mathbb{P}}$ is vacuously satisfied since $k(\mathbf{p}) = k(\mathbf{q})$).

To construct $\mathbf{r}$, simply apply Lemma 3.19 to each p_ν to produce an extension $r_\nu \geq_{\mathbb{Q}} p_\nu$ with $k_{r_\nu} = k+1$, but $\mathcal{U}_{r_\nu} = \mathcal{U}_{p_\nu}$. Then let $\bar{r} := \langle r_\nu : \nu \in 2^n \rangle$ and $\mathbf{r} = (k+1, \bar{r})$. Then $\mathbf{r} \geq_{\mathbb{P}} \mathbf{p}$ as required.

(2) Fix such a D and n. As we are working exclusively in $\mathbb{P}_n$ and because 2^n is a fixed finite set, it suffices to prove that for any chosen $\nu \in 2^n$,

For every $\mathbf{p} \in \mathbb{P}_n$ there is $\mathbf{q} \in \mathbb{P}_n$ with $\mathbf{q} \geq_{\mathbb{P}} \mathbf{p}$ and $q_\nu \in D$.

To verify this, fix $\nu \in 2^n$ and $\mathbf{p} \in \mathbb{P}_n$. Concentrating on p_ν, as D is dense, choose $q_\nu \in D \cap \mathbb{Q}$ with $q_\nu \geq_{\mathbb{Q}} p_\nu$. Let $k^* := k_{q_\nu}$. Next, for each $\delta \in 2^n$ with $\delta \neq \nu$, apply Lemma 3.19 to p_δ, obtaining some $q_\delta \in \mathbb{Q}$ satisfying $q_\delta \geq_{\mathbb{Q}} p_\delta$, $k_{q_\delta} = k^*$, but $\mathcal{U}_{q_\delta} = \mathcal{U}_{p_\delta}$. Now, collect all of this data into a condition $\mathbf{q} \in \mathbb{P}_n$ defined by $k(\mathbf{q}) = k^*$ and $\bar{q} = \langle q_\gamma : \gamma \in 2^n \rangle$, where each q_γ is as above. To see that $\mathbf{q} \geq_{\mathbb{P}} \mathbf{p}$, Clause (3) is verified by noting that for every $\ell \in [k(\mathbf{p}), k^*)$, $\mathrm{tr}_\ell(\mathbf{q})$ is either empty, or equals $\{\nu\}$, depending on whether or not $\ell \in \mathcal{U}_{q_\nu}$.

Notation 3.23 Suppose $\mathcal{B} \models ZFC^*$ and let $G^* \subseteq \mathbb{P}$, $G^* \in V$ be a filter meeting every dense subset $D^* \subseteq \mathbb{P}$ with $D^* \in \mathcal{B}$. For each n and $\nu \in 2^n$, let

$$G_\nu := \{\mathbf{p} \in \mathbb{Q} : \text{ for some } \mathbf{p}^* = (k, \bar{p}) \in G^*,\ \mathbf{p} = \mathbf{p}^*_\nu\}$$

Then, for each $\eta \in 2^\omega$, let

$$G_\eta := \bigcup \{G_{\eta|n} : n \in \omega\} \ \text{ and } \ \mathcal{U}_\eta := \{\ell \in \omega : \ell \in \mathcal{U}_{\mathbf{q}} \text{ for some } \mathbf{q} \in G_\eta\}$$

Proposition 3.24 *In the notation of 3.23:*

1. *For every $\eta \in 2^\omega$, $G_\eta \subseteq \mathbb{Q}$ is a filter meeting every dense $D \subseteq \mathbb{Q}$ with $D \in \mathcal{B}$;*
2. *The sets $\{\mathcal{U}_\eta : \eta \in 2^\omega\}$ are an almost disjoint family of infinite subsets of ω.*

Proof. (1) follows immediately from Lemma 3.22(2).

(2) Choose distinct $\eta, \eta' \in 2^\omega$. Choose n_0 such that $\eta|n \neq \eta'|n$ whenever $n \geq n_0$. By Lemma 3.22(1), choose $\mathbf{p}^* \in G^*$ with $n(\mathbf{p}^*) \geq n_0$. We show that $\mathcal{U}_\eta \cap \mathcal{U}_{\eta'}$ is finite by establishing that if $\ell \in \mathcal{U}_\eta \cap \mathcal{U}_{\eta'}$, then $\ell \leq k(\mathbf{p}^*)$.

To establish this, choose $\ell \in \mathcal{U}_\eta \cap \mathcal{U}_{\eta'}$. By unpacking the definitions, choose $\mathbf{q}^*, \mathbf{r}^* \in G^*$ such that, letting $\mu := \eta|n(\mathbf{q}^*)$ and $\mu' := \eta'|n(\mathbf{r}^*)$, we have $\ell \in \mathcal{U}_{\mathbf{q}^*_\mu} \cap \mathcal{U}_{\mathbf{r}^*_{\mu'}}$. As G^* is a filter, choose $\mathbf{s}^* \in G^*$ with $\mathbf{s}^* \geq_{\mathbb{P}} \mathbf{p}^*, \mathbf{q}^*, \mathbf{r}^*$. As notation, let $\delta := \eta|n(\mathbf{s}^*)$ and $\delta' := \eta'|n(\mathbf{s}^*)$.

Claim: $\ell \in \mathcal{U}_{\mathbf{s}^*_\delta} \cap \mathcal{U}_{\mathbf{s}^*_{\delta'}}$.

Proof. As $\ell \in \mathcal{U}_{\mathbf{q}^*_\mu}$, $\ell < k(\mathbf{q}^*)$. From $\mathbf{q}^* \leq_{\mathbb{P}} \mathbf{s}^*$ we conclude $k(\mathbf{q}^*) \leq k(\mathbf{s}^*)$, so $\ell < k(\mathbf{s}^*)$ as well. From $\mathbf{q}^* \leq_{\mathbb{P}} \mathbf{s}^*$ and $\mu \trianglelefteq \delta$ we obtain $\mathbf{q}^*_\mu \leq_{\mathbb{Q}} \mathbf{s}^*_\delta$. But then, as $\ell \in \mathcal{U}_{\mathbf{q}^*_\mu}$, it follows that $\ell \in \mathcal{U}_{\mathbf{s}^*_\delta}$. That $\ell \in \mathcal{U}_{\mathbf{s}^*_{\delta'}}$ is analogous, using $\mathbf{r}^*$ in place of $\mathbf{q}^*$.

Finally, assume by way of contradiction that $\ell \geq k(\mathbf{p}^*)$. The Claim implies that $\{\delta, \delta'\} \subseteq \mathrm{tr}_\ell(\mathbf{s}^*)$. As $\ell \in [k(\mathbf{p}^*), k(\mathbf{s}^*))$, Clause (3) of $\mathbf{p}^* \leq_{\mathbb{P}} \mathbf{s}^*$ implies that $\delta|n(\mathbf{p}^*) = \delta'|n(\mathbf{p}^*)$. But, as $\eta|n(\mathbf{p}^*) = \delta|n(\mathbf{p}^*)$ and $\eta'|n(\mathbf{p}^*) = \delta'|n(\mathbf{p}^*)$, we contradict our choice of $\mathbf{p}^*$.

We close this section with the proof of Proposition 3.1, which we restate for convenience.

Conclusion 3.25 *There is a family $\{(N_\eta, \bar{b}^*) : \eta \in 2^\omega\}$ of $2^{\aleph_0}$ rich, atomic models of T, each of size $\aleph_1$, that are pairwise non-isomorphic over $\bar{b}^*$.*

Proof. Choose any countable, transitive model $\mathcal{B}$ of ZFC^* and choose any $G^* \in V$, $G^* \subseteq \mathbb{P}$, G^* meets every dense subset $D^* \in \mathcal{B}$ (as $\mathcal{B}$ is countable, such a G^* exists). For each $\eta \in 2^\omega$, choose G_η and $\mathcal{U}_\eta$ as in Proposition 3.24, and apply Conclusion 3.18 to get a rich $\mathcal{U}_\eta$-colored $(N_\eta, \bar{b}^*)$ in V. That this family is pairwise non-isomorphic over $\bar{b}^*$ follows immediately from Corollary 3.6, since the sets $\{\mathcal{U}_\eta : \eta \in 2^\omega\}$ are almost disjoint.

4 The proof of Theorem 1.4

Assume that the class $\mathbf{At}_T$ is not pcl-small, as witnessed by an (uncountable) model N^* containing a finite tuple $\bar{a}^*$. Fix a countable, elementary substructure $M^* \preceq N^*$ that contains $\bar{a}^*$. To aid notation, let $D^* := \mathrm{pcl}_{N^*}(\bar{a}^*)$. We now split into cases, depending on the relationship between the cardinals $2^{\aleph_0}$ and $2^{\aleph_1}$.

Case 1. $2^{\aleph_0} < 2^{\aleph_1}$.

In this case, expand the language of T to $L(D^*)$, adding a new constant symbol for each $d \in D^*$. Then, the natural expansion $N^*_{D^*}$ N^* to an $L(D^*)$-structure is a model of the infinitary $L(D^*)$-sentence Φ that entails $Th(N^*_{D^*})$ and ensures that every finite tuple is L-atomic with respect to T. As $N^*_{D^*}$ is a model of Φ that realizes uncountably many types over the empty set (after fixing D^*!), it follows from [5], Theorem 45 of Keisler that there are $2^{\aleph_1}$ pairwise non-$L(D^*)$-isomorphic models Φ, each of size $\aleph_1$. As $2^{\aleph_0} < 2^{\aleph_1}$, it follows that there is a subfamily of $2^{\aleph_1}$ pairwise non-L-isomorphic reducts to the original language L. As each of these models are L-atomic, we conclude that $\mathbf{At}_T$ has $2^{\aleph_1}$ non-isomorphic models of size $\aleph_1$.

Case 2. $2^{\aleph_0} = 2^{\aleph_1}$.

Choose $\bar{b}^*$ from M^* as in Proposition 2.10 and apply Conclusion 3.25 to get a set $\mathcal{F}^* = \{(N_\eta, \bar{b}^*) : \eta \in 2^\omega\}$ of atomic models, each of size $\aleph_1$, that are pairwise non-isomorphic over $\bar{b}^*$. Let $\mathcal{F} = \{N_\eta : \eta \in 2^\omega\}$ be the set of reducts of elements from $\mathcal{F}^*$. By our cardinal hypothesis, $\mathcal{F}$ has size $2^{\aleph_1}$. The relation of L-isomorphism is an equivalence relation on $\mathcal{F}$, and each L-isomorphism equivalence class has size at most $\aleph_1$ (since $\aleph_1^{<\omega} = \aleph_1$). As $\aleph_1 < 2^{\aleph_1}$ we conclude that $\mathcal{F}$ has a subset of size $2^{\aleph_1}$ of pairwise non-isomorphic atomic models of T, each of size $\aleph_1$.

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