

A note on groups definable in the p -adic field

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Abstract

It is known [2] that a group G definable in the field \mathbb{Q}_p of p -adic numbers is definably locally isomorphic to the group $H(\mathbb{Q}_p)$ of p -adic points of a (connected) algebraic group H over \mathbb{Q}_p . We observe here that if H is commutative then G is commutative-by-finite. This shows in particular that any one-dimensional group G definable in \mathbb{Q}_p is commutative-by-finite. This extends to groups definable in p -adically closed fields. We situate the results in a *geometric structures* environment.

1 Introduction and preliminaries

We here consider analogous questions vis-a-vis the p -adic field (or more generally p -adically closed fields) to questions long studied for the real field (more generally real closed fields, \mathcal{o} -minimal structures). Namely the classification and description of definable groups. We emphasize *definable* rather than interpretable. These coincide in the real case because of elimination of imaginaries, but not in the p -adic case. Common features are that definable groups have naturally the structure of real and p -adic Lie groups [6], [7], as well as being locally isomorphic to real/ p -adic algebraic groups [2]. But from this point on, things diverge: the p -adic dimension on definable sets is less

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well-behaved, in particular definable groups are in general far from being definably connected, and arguments from the real case do not go through.

We prove here that when G is locally commutative, equivalently the (connected) algebraic group H over \mathbb{Q}_p such that G is definably locally isomorphic to $H(\mathbb{Q}_p)$, is commutative, then G is commutative-by-finite. When G is one-dimensional (in the sense of p -adic dimension) then H will be a connected algebraic group of algebraic-geometric dimension 1, so commutative. Thus we deduce from our results that one-dimensional groups definable in the p -adics are commutative-by-finite.

$Th(\mathbb{Q}_p)$ (i.e. the home sort) is dp -minimal (see [8]), hence any one-dimensional group G definable in \mathbb{Q}_p is also dp -minimal, so by a result of Simon [8], G is commutative-by-bounded exponent. So we could deduce commutative-by-finiteness of one-dimensional G definable in \mathbb{Q}_p if we knew that there are no bounded exponent groups *interpretable* in the p -adics other than finite groups. This must be true but we do not know a proof at the moment.

The proof of our main result, which is in effect passing from being locally commutative to being commutative-by-finite, is relatively soft, and we situate it in the context of geometric structures.

A commutative-by-finite group G is amenable (as a discrete group), hence definably amenable with respect to any ambient structure in which G happens to be definable. As also $Th(\mathbb{Q}_p)$ is *NIP* (see Section 4.2, [1]), this puts us in a position to be able to apply some nice results from [4] which refine the “algebraic group configuration theorem” of [2], and we (and others) intend to carry this out in a future work. Even though the current paper is short we thought it makes sense as a “stand-alone” paper as it is self-contained, the methods are elementary, and it may be useful for future work by the authors and others.

We use fairly basic model theory. We refer to the excellent survey [1] as well as [2] and [5] for the model theory of the p -adic field $(\mathbb{Q}_p, +, \times, 0, 1)$. In fact both [2] and [5] are also good references for the model theoretic background required for the current paper. A *geometric structure* (see Section 2 of [2]) is a one-sorted structure M such that in any model N of $Th(M)$, algebraic closure satisfies exchange (so gives a so-called pregeometry on N) and there is a finite bound on the sizes of finite sets in uniformly definable families. The structure $(\mathbb{Q}_p, +, \times, 0, 1)$ is an example of a geometric structure ([2], Proposition 2.11), as model-theoretic algebraic closure coincides with field-theoretic algebraic closure.

In a geometric structure M , if a is a finite tuple from M and B a subset of M then $\dim(a/B)$ denotes the size of a maximal algebraically independent over B subtuple of a . If M is saturated and X is a B -definable subset of M^n (where B is finite) then $\dim(X) = \max\{\dim(a/B) : a \in X\}$. It is important to know that when M is $(\mathbb{Q}_p, +, \times, 0, 1)$, and $X \subseteq M^n$ is definable, then its dimension in the above sense coincides with its “topological dimension”, namely the greatest $k \leq n$ such that the image of X under some projection from M^n to M^k contains an open set.

In one of the general results below we make an assumption on the existence of G^0 . So we explain what this means, although it is discussed in the first section of [5]. Work in a saturated structure \bar{M} and suppose G is definable in \bar{M} . Let A be a small (of cardinality strictly less than the degree of saturation of \bar{M}) subset of \bar{M} such that G is defined over A . Then by G_A^0 we mean the intersection of all A -definable subgroups of G of finite index. We say that G^0 exists if G_A^0 does not depend on A , namely cannot get smaller by increasing A . Note that the existence of G^0 is equivalent to the nonexistence of an infinite uniformly definable family of subgroups of G of some fixed finite index, which amounts to the *DCC* on intersections of uniformly definable subgroups of (a given) finite index. As mentioned in [5], the existence of G^0 follows from the ambient theory having *NIP*.

Finally let us state clearly the local isomorphism results alluded to earlier.

Fact 1.1. ([7]) *Let G be a group definable in the field \mathbb{Q}_p . Then G has definably the structure of a p -adic Lie group. Moreover if G has dimension k as a definable group then it has dimension k as a p -adic Lie group.*

Fact 1.2. ([2]) *Suppose G is a group definable in the field \mathbb{Q}_p . Consider G with its topology given by Fact 1.1. Then there is a connected algebraic group H over \mathbb{Q}_p , where the algebraic-geometric dimension of H equals the dimension of G , and a definable homeomorphism f between an open neighbourhood U of the identity in G and an open neighbourhood V of the identity of (the p -adic Lie group) $H(\mathbb{Q}_p)$ such that $f(ab) = f(a)f(b)$ whenever $a, b \in U$ and $ab \in U$.*

Remark 1.3. (i) *In Fact 1.2 we can actually choose f to be a definable isomorphism (and homeomorphism) between definable open subgroups of G and $H(\mathbb{Q}_p)$, because any (definable) p -adic Lie group has a (definable) compact open subgroup, and a compact p -adic Lie group is profinite.*

(ii) *In fact both Fact 1.1 and Fact 1.2 hold (with appropriate definitions of*

definable Lie group over a p -adically closed field K) for groups G definable in an elementary extension K of \mathbb{Q}_p . See Step 1 of the proof of Theorem 2.1 in [3] in the real closed field situation which works word for word in the p -adically closed field case.

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2 Results

We start with an easy lemma about geometric structures.

Lemma 2.1. *Suppose M is a geometric structure, $X \subseteq M^n$ a definable set of dimension k say, and f is a definable function from X to M^m . Suppose that $f^{-1}(b)$ has dimension k for all $b \in \text{Im}(f)$ then $\text{Im}(f)$ is finite.*

Proof. We may assume M to be saturated and work over the parameters over which X and f are defined. Suppose for a contradiction $\text{Im}(f)$ to be infinite. Then we can find $b \in \text{Im}(f)$ such that $\dim(b) \geq 1$. As $\dim(f^{-1}(b)) = k$ we can find $a \in f^{-1}(b)$ such that $\dim(a/b) = k$. Hence by subadditivity, $\dim(a, b) > k$. As $b \in \text{dcl}(a)$ it follows that $\dim(a) > k$, contradicting that $a \in X$ and $\dim(X) = k$. \square

Remark 2.2. *The conclusion of the lemma is weaker than stating that there is no definable equivalence relation on a definable k -dimensional set with infinitely many classes of dimension k . The latter is false in \mathbb{Q}_p*

Proposition 2.3. *Let M be a geometric structure, and let $G \subseteq M^n$ be a group definable in M with $\dim(G) = k$. Assume that (working in a saturated model) G^0 exists. Suppose that G contains a definable subset X of dimension k such that $ab = ba$ for all $a, b \in X$. Then G is commutative-by-finite, namely G has a (definable) subgroup H of finite index such that H is commutative.*

Proof. For $a \in X$, let f_a be the (definable) function from G to G defined by $f_a(g) = gag^{-1}a^{-1}$.

Claim I. Fix $a \in X$. For any $g_1, g_2 \in G$, $f_a(g_1) = f_a(g_2)$ iff $g_1C_G(a) = g_2C_G(a)$.

Proof of Claim I.

This is obvious but we do it anyway. $g_1ag_1^{-1}a^{-1} = g_2ag_2^{-1}a^{-1}$ iff $g_2^{-1}g_1ag_1^{-1}g_2 = a$, namely $g_1g_2^{-1} \in C_G(a)$.

Claim II. For $a \in X$, $Im(f_a)$ is finite.

Proof of Claim II.

Note that $C_G(a)$, the centralizer of a in G contains X (by assumption) so has dimension k . Hence for any $g \in G$, $dim(gC_G(a)) = k$. By the right implies left implication in Claim I, f_a is constant on $gC_G(a)$ for all $g \in G$. So we conclude by Lemma 2.1.

Claim III. For any $a \in X$, $C_G(a)$ has finite index in G .

Proof of Claim III.

By Claim I, $Im(f)$ is in bijection with $G/C_G(a)$, so by Claim II, $C_G(a)$ has finite index in G .

We may assume that we have been working in a saturated model M . So by compactness there is a finite bound on the index of $C_G(a)$ in G for $a \in X$. Our assumption that G^0 exists (as discussed in the previous section) implies that $C_G(X) = \bigcap_{a \in X} C_G(a)$ is a finite subintersection, so is a definable subgroup H say of G , of finite index.

Now for any $a \in H$, $C_G(a)$ contains X , so has dimension k . So repeating Claims I, II, and III, for a in H rather than X we conclude that $C_G(a)$ has finite index in G for all $a \in H$, and so again that $C_G(H)$ has finite index in G . So $H \cap C_G(H)$ has finite index in G and is commutative. \square

Remark 2.4. *One really needs only a finite-valued subadditive dimension on types of real tuples, for Lemma 2.1 and Proposition 2.3. And for Proposition 2.3, one only needs a definable set $X \subseteq G$ of dimension k such that for all $a \in X$, $C_G(a)$ has dimension k .*

In any case, from Proposition 2.3 we conclude:

Theorem 2.5. *(i) Let G be a group definable in \mathbb{Q}_p . Let H be a connected algebraic group over \mathbb{Q}_p as in Fact 1.1. Suppose that H is commutative, then G is commutative-by-finite.*

(ii) Let G be a group of dimension 1 definable in \mathbb{Q}_p . Then G is commutative-by-finite.

Proof. (i) Let U, V be definable open neighbourhoods of the identity of $G, H(\mathbb{Q}_p)$ respectively and $f : U \rightarrow V$ as given by Fact 1.2. By choosing a smaller definable open neighbourhood U_1 of the identity contained in G such that $ab \in U$ for $a, b \in U_1$, we see from the assumptions that $ab = ba$ for $a, b \in U_1$. Now the p -adic topological dimension of U_1 coincides with that of G , but both coincide with the dimension with respect to \mathbb{Q}_p as a geometric structure. So, bearing in mind that G^0 exists (working in a saturated model), as remarked in the introduction. We can apply Proposition 2.3 to conclude that G is commutative-by-profinite.

(ii) If G has dimension 1 then by Fact 1.2 the connected algebraic group H has dimension 1 as an algebraic group, so is commutative. So part (i) implies. \square

Remark 2.6. *By Remark 1.3 (ii), Theorem 2.4 goes through for groups definable in arbitrary p -adically closed fields (i.e. models of $\text{Th}(\mathbb{Q}_p)$).*

Question 2.7. (i) *Do we need the assumption that G^0 exists in Proposition 2.3? We presume yes, namely there is a counterexample without it.*

(ii) *Let F be a geometric field in the sense of Definition 2.9 of [2]. Let G be a group definable in F and let H be a connected algebraic group over F given by Proposition 3.1' of [2]. Suppose H is commutative. Can one find a definable subset X of G of dimension equal to $\dim(G)$ such that for all $a \in X$, $C_G(a)$ has dimension equal to $\dim(G)$?*

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