

$AD_{\mathbb{R}}$ implies that all sets of reals are Θ universally Baire

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The purpose of this note is to establish the following useful theorem.

Theorem 0.1 *Assume $AD_{\mathbb{R}}$. Then all set of reals are Θ universally Baire.*

We are interested in this theorem for two reasons. It is used in the author's construction of *generic generators* (see the forthcoming [11]). Secondly the authors of [6] introduced a derived model construction producing a model of determinacy in which all sets of reals are universally Baire. The above theorem shows how to build such models from determinacy rather than large cardinals¹ (see Corollary 5.4). Along the way of proving Theorem 0.1, we fill in some gaps in descriptive inner model theory. In particular, we outline the process of getting \mathcal{N}_x^* -like models used in [12] (see [12, Theorem 2.25]). We still leave one gap. The proof of Theorem 1.7 is not given as the full proof will go beyond the scope of this paper. However, we believe we have given enough outline so that the interested reader can prove it.

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1 Γ -Woodins

With the exception of Theorem 1.7, all of the material that we review in this section has been developed by the Cabal group, and among other places, can be found in [15] (especially [15, Chapter 3]), the Appendix of [12], [14] and [13].

Following [15, Chapter 3]), we say that Γ is a *good pointclass*² if

¹However, the large cardinal assumption used in [6] is probably optimal.

² \mathbb{R} is the Baire space.

1. Γ is closed under recursive substitution and number quantification,
2. Γ is ω -parametrized³,
3. Γ has the scale property,
4. Γ is closed under $\exists^{\mathbb{R}}$.

Σ_2^1 is the prototypical good pointclass. Every good pointclass has its boldface version $\tilde{\Gamma}$ which consists of those sets $A \subseteq \mathbb{R}^k$ such that for some $U \in \Gamma$ and $x \in \mathbb{R}$, $U \subseteq \mathbb{R}^{k+1}$ and $A = U_x = \{\vec{y} : (x, y) \in U\}$. One can also define the pointclass $\Gamma(x)$ for $x \in \mathbb{R}$ in a similar manner.

Each good pointclass has its associated C_Γ operator. Here for $x \in \mathbb{R}$,

$$C_\Gamma(x) = \{y \in \mathbb{R} : y \text{ is } \Gamma\text{-definable from } x \text{ and a countable ordinal}\}.$$

The C_Γ operator can be extended to sets in HC via the category quantifier. This is done, for instance, in [15, Chapter 3].

Below we explain how this is done. Given $a \in HC$ and $f : \omega \rightarrow a$ a surjection, we let $a_f = \{2^n 3^m : (n, m) \in \mathbb{N}^2 \wedge f(n) \in f(m)\}$. Given $b \subseteq a$, we let $b_f = \{n \in \mathbb{N} : f(n) \in b\}$. We can then easily decode b from (a_f, b_f) . Indeed, letting $c = \{(n, m) \in \mathbb{N}^2 : 2^n 3^m \in a_f\}$, (a, \in) is the transitive collapse of (ω, c) and (a, b, \in) is the transitive collapse of (ω, b_f, c) . We then set $C_\Gamma(a)$ to be the set of those $b \subseteq a$ such that for co-meager many $f : \omega \rightarrow a$ there is $b_f \in C_\Gamma(a_f)$. The Harrington-Kechris Theorem gives a nice description of the set $C_\Gamma(a)$.

Theorem 1.1 (Harrington-Kechris, [2]) *Assume AD. Suppose T is a tree of a Γ -scale on a universal Γ -set. Then for any transitive $a \in HC$,*

1. $C_\Gamma(a) = \{b \subseteq a : \text{for all } f : \omega \rightarrow a, b_f \in C_\Gamma(a_f)\}$, and
2. $C_\Gamma(a) = \wp(a) \cap L(T \cup \{a\}, a)$.

Next we introduce Γ -Woodins.

Notation 1.2 *Below we will use C_Γ^α for the α th iterate of C_Γ . Thus, $C_\Gamma^2(a) = C_\Gamma(C_\Gamma(a))$. We only need this notion for $\alpha \leq \omega$. We then set $C_\Gamma^\omega(a) = \bigcup_{n < \omega} C_\Gamma^n(a)$.*

Suppose T is the tree of a Γ -scale. For each $\alpha < \omega_1$, we let κ_α be the α th-infinite cardinal of $L[T, a]$. We can then simply set $C_\Gamma^\alpha = H_{\kappa_\alpha}^{L[T, a]}$. Then, using this definition, we have $C_\Gamma(a) = C_\Gamma^1(a)$.

³This means that there is a set $U^k \subseteq \omega \times \mathbb{R}^k$ such that $U^k \in \Gamma$ and for every set $A \in \Gamma$, if $A \subseteq \mathbb{R}^k$ then there is an integer n such that $x \in A \leftrightarrow (n, x) \in U^k$.

Definition 1.3 *Given a transitive P model of $ZFC - Replacement$, we say P is a Γ -Woodin if for some δ ,*

1. $P \models “\delta \text{ is the only Woodin cardinal}”$.
2. $P = C_\Gamma^\omega(P)$,
3. *for every P -inaccessible cardinal $\eta < \delta$, $C_\Gamma(V_\eta^P) \models “\eta \text{ is not a Woodin cardinal}”$.*

We let δ^P be the Woodin cardinal of P .

To make matters simple, we will work with prototypical good pointclass⁴. These are pointclass that correspond to Σ_1^2 of an initial segment of the universe. We say $(A_n : n < \omega) \subseteq \mathbb{R}^\omega$ is a self-justifying-system (sjs) if for each $n \in \omega$,

1. there is a sequence $(A_{m_k} : k \in \omega)$ that codes a scale on A_n ,
2. there is $m < \omega$ such that $A_n^c = A_m$.

Let T_0 be the theory

1. $ZF - Powerset$,
2. Θ exists⁵, and
3. $V = L_{\Theta^+}(C, \mathbb{R})$ for some $C \subseteq \mathbb{R}$.

Definition 1.4 *Suppose Γ is a good pointclass. We say Γ is a very good pointclass if there is a sjs $\vec{A} = (A_n : n \in \omega)$, $\gamma < \Theta^{L(\vec{A}, \mathbb{R})}$, a Σ_1 -formula ϕ and a real x such that $L_\gamma(\vec{A}, \mathbb{R})$ is the least initial segment of $L(\vec{A}, \mathbb{R})$ that satisfies $T_0 + \phi(x)$ and $\Gamma = (\Sigma_1^2(\vec{A}))^{L_\gamma(\vec{A}, \mathbb{R})}$.*

Recall that Martin showed (for instance, see [8] or [5]) that under $AD_{\mathbb{R}}$, every set of reals is Suslin. It follows that under $AD_{\mathbb{R}}$, for every set B there is a sjs \vec{A} such that $B = A_0$.

If γ is as in Definition 1.4 then $L_{\gamma+2}(\vec{A}, \mathbb{R})$ has a set of reals not in $L_{\gamma+1}(\vec{A}, \mathbb{R})$, namely the set of reals coding $L_\gamma(\vec{A}, \mathbb{R})$. Thus, γ ends a weak gap relative to \vec{A} . The reason for working with a sjs rather than any set A is that if \vec{A} is a sjs then $L(\vec{A}, \mathbb{R})$ has the same properties as $L(\mathbb{R})$. In particular, it admits a scales analysis (see [16]).

⁴The point of this move is to just avoid discussing $Env(\Gamma)$. Readers familiar with this notion do not have to make this move.

⁵More precisely, “there is an ordinal which is not the surjective image of \mathbb{R} ”.

From now on we will write *vg* for very good. Suppose Γ is a *vg*-pointclass. We then let $M_\Gamma = L_\gamma(\vec{A}, \mathbb{R})$ be as in Definition 1.4 witnessing very goodness of Γ . We say that M_Γ is the *parent* of Γ . Clearly, M_Γ has many representations as $L_\gamma(\vec{A}, \mathbb{R})$. However, the resulting ambiguity is not problematic and we will not dwell on it.

Notice that we have that

Proposition 1.5 *if Γ is *vg*-pointclass and $M_\Gamma = L_\gamma(\vec{A}, \mathbb{R})$ is its parent then for any countable transitive a , $C_\Gamma(a) = OD^{M_\Gamma}(\vec{A}, a)$.*

The proof of the observation above is straightforward and uses the fact that

$$OD^{M_\Gamma}(\vec{A}, a) = \Sigma_1^2(\vec{A}, a).$$

Suppose now that Γ is a *vg*-pointclass. We then say that $\vec{B} \subseteq \mathbb{R}^\omega$ is a *weakly Γ -condensing sequence* if

1. B_0 codes a sjs such that $M_\Gamma = L_\gamma(B_0, \mathbb{R})$ and $\Gamma = (\Sigma_1^2(B_0))^{L_\gamma(B_0, \mathbb{R})}$,
2. $B_1 = \{(x, y) \in \mathbb{R}^2 : y \in C_\Gamma(x)\}$,
3. $B_2 = B_1^c$,
4. B_3 is any $OD^{M_\Gamma}(B_0)$ set⁶,
5. $(B_{2i+1} : i \in [2, \omega)) \subseteq \Gamma$ is a scale on B_1 ,
6. $(B_{2i} : i \in [2, \omega)) \subseteq M_\Gamma$ is a scale on B_2 ,
7. for every $i \in [2, \omega)$, $M_\Gamma \models "B_{2i} \text{ is ordinal definable from } B_0"$.

Suppose Γ is a *vg*-pointclass and $M_\Gamma = L_\gamma(\vec{A}, \mathbb{R})$ is its parent. Suppose $B \in M_\Gamma \cap \wp(\mathbb{R})$ is $OD^{M_\Gamma}(\vec{A})$ and suppose $a \in HC$ is a transitive set. Consider the term relation τ_B^a consisting of pairs (p, σ) such that

1. $p \in Coll(\omega, a)$,
2. $\sigma \in C_\Gamma(a)$ is a standard $Coll(\omega, a)$ -name for a real, and
3. for a co-meager many $g \subseteq Coll(\omega, a)$ (in the relevant topology) such that $p \in g$, $\sigma(g) \in B$ ⁷.

⁶We will need the freedom to include any $OD^{M_\Gamma}(B_0)$ set of reals into our condensing sequence.

⁷ $\sigma(g)$ is the realization of σ .

Then because τ_B^a is $OD^{M_\Gamma}(\vec{A}, a)$, $\tau_B^a \in C_\Gamma(C_\Gamma(a))$. Given $k \in \omega$, we let $\tau_{B,k}^a = \tau_{B,0}^{C_\Gamma^k}$. Thus, for every $k \in \omega$, $\tau_{B,k}^a \in C_\Gamma^{k+2}(a)$.

We say \vec{B} is a Γ -condensing sequence if it is a weakly condensing sequence with the additional property that for any transitive sets $a, b, M \in HC$ such that

1. $a \in M$ and
2. there is an embedding $\pi : M \rightarrow_{\Sigma_1} C_\Gamma^\omega(b)$ such that $\pi(a) = b$ and for every $i, k \in \omega$, $\tau_{B_i,k}^b \in \text{rng}(\pi)$,

$M = C^\omega(a)$ and for any $i, k \in \omega$, $\pi^{-1}(\tau_{B_i,k}^b) = \tau_{B_i,k}^a$.

If P is a Γ -Woodin and $B \in OD^{M_\Gamma}(\vec{A})$ then for $k \in \omega$ we let $\tau_{B,k}^P = \tau_{B,k}^{V_{\delta^P}^P}$.

Definition 1.6 Suppose Γ is a vg -pointclass and $M_\Gamma = L_\gamma(\vec{A}, \mathbb{R})$ is its parent. Suppose P is a Γ -Woodin and Σ is an ω_1 -iteration strategy for P . Suppose $B \in M_\Gamma \cap \wp(\mathbb{R})$ is $OD^{M_\Gamma}(\vec{A})$.

1. We say Σ is a Γ -fullness preserving strategy for P if whenever $i : P \rightarrow Q$ is an iteration of P via Σ , Q is a Γ -Woodin.
2. Given that Σ is Γ -fullness preserving, we say Σ respects B if whenever $i : P \rightarrow Q$ is an iteration of P via Σ , for every k , $i(\tau_{B,k}^P) = \tau_{B,k}^Q$.

The following theorem, which probably is originally due to Woodin, is unfortunately unpublished. The discussion that follows the theorem might be illuminating.

Theorem 1.7 Assume AD^+ and suppose Γ is a vg -pointclass. Let $M_\Gamma = L_\gamma(\vec{A}, \mathbb{R})$ ⁸ be its parent and $A \in OD(\vec{A})^{M_\Gamma}$. There is then a pair (P, Σ) and a Γ -condensing sequence \vec{B} such that

1. P is a Γ -Woodin,
2. Σ is a Γ -fullness preserving ω_1 -iteration strategy for P ,
3. for each i , Σ respects B_i ,
4. for every Σ -iterate Q of P , for every $i \in \omega$ and for every Q -generic $g \subseteq \text{Coll}(\omega, \delta^Q)$, $\tau_i^Q(g) = Q[g] \cap B_i$,
5. for any tree $\mathcal{T} \in \text{dom}(\Sigma)$, $\Sigma(\mathcal{T}) = b$ if and only if either

⁸Here \vec{A} is any sjs for which $M_\Gamma = L_\gamma(\vec{A}, \mathbb{R})$.

- (a) $C_\Gamma(\mathcal{M}(\mathcal{T})) \models \text{"}\delta(\mathcal{T}) \text{ is not a Woodin cardinal"}$ and b is the unique well-founded cofinal branch c of \mathcal{T} such that $C_\Gamma(\mathcal{M}(\mathcal{T})) \in \mathcal{M}_c^\mathcal{T}$, or
- (b) $C_\Gamma(\mathcal{M}(\mathcal{T})) \models \text{"}\delta(\mathcal{T}) \text{ is a Woodin cardinal"}$ and b is the unique well-founded cofinal branch c of \mathcal{T} such that letting $Q = C_\Gamma^\omega(\mathcal{M}(\mathcal{T}))$, $\mathcal{M}_c^\mathcal{T} = Q$ and for every $i \in \omega$, $\pi_c^\mathcal{T}(\tau_{B_i}^P) = \tau_{B_i}^Q$.

6. Σ respects A ,

Moreover, for any set $a \in HC$ there is (P, Σ) as above such that $a \in P$.

Definition 1.8 Suppose Γ is a vg-pointclass. We say that (P, Σ) is a Γ -excellent pair if for some Γ -condensing sequence \vec{B} , (P, Σ) has properties 1-5 described in Theorem 1.7 as witnessed by \vec{B} .

The author believes that there is enough facts that have appeared in the literature that a motivated reader can put together a proof of Theorem 1.7. We now give a general outline of how the proof of Theorem 1.7 might proceed.

First and foremost, at the very heart of Theorem 1.7 is Woodin's theorem that for a set of ordinals T and for T -cone of d , $\omega_2^{L[T, d]}$ is a Woodin cardinal in $\text{HOD}^{L[T, d]}(T)$ (see [4, Theorem 5.4]). The use of this theorem is as follows. Let Γ be a vg-pointclass. Find a bigger vg-pointclass Γ^* such that $\Gamma \subseteq \Delta_{\Gamma^*}$. Let T be the tree of a Γ^* -scale on a Γ^* -universal set. Theorem 5.4 of [4] shows that for any $x \in \mathbb{R}$, we can find transitive Q such that $x \in Q$ and letting $\nu = \text{Ord} \cap Q$, $L[T, Q] \models \text{"}\nu \text{ is a Woodin cardinal"}$. ν in question is $\omega_2^{L[T, z]}$ for some z and Q is $V_\nu^{\text{HOD}^{L[T, z]}(T)}$. If we choose z big enough then $C_\Gamma(Q) \in L[T, Q]$.

The Γ -condensing sequence \vec{B} can be produced via analyzing scales in M_Γ . It is a theorem of Martin that any set in M_Γ has a semi-scale consisting of sets in Γ (see [3, Chapter 3]). We can then let \vec{B} be any sequence such that for each $i \in \omega$, some subsequence $(B_{n_j} : j \in \omega)$ is a semi-scale on B_i . The proof of [15, Lemma 3.7] shows that \vec{B} is Γ -condensing.

At this point, we can aim at obtaining a strategy for a Γ -Woodin. Let δ be least such that $C_\Gamma(V_\delta^Q) \models \text{"}\delta \text{ is a Woodin cardinal"}$. Our choice of z and Γ^* guarantee that $\delta < \eta$. To see this, notice that $\{\tau_{B_i, k}^Q : i, k \in \omega\} \in L[T, Q]$ (consequence of choosing Γ^* bigger than Γ). We now take a transitive below η Skolem hull of $L[T, Q]$ such that the set $\{\tau_{B_i, k}^Q : i, k \in \omega\}$ is in it. Considering the collapse of the hull, we have $\pi : N \rightarrow L_\xi[T, Q]$. We now need to use the condensation properties of $\{\tau_{B_i, k}^Q : i, k \in \omega\}$. These properties imply that if $\delta = \text{crit}(\pi)$ then $C_\Gamma(V_\delta^N) \models \text{"}\delta \text{ is a Woodin cardinal"}$ as $C_\Gamma(V_\delta^N) \in N$. To gain more intuition on how exactly

such condensation properties might work consider [15, Lemma 3.7]. There are such condensation lemmas at various parts of [13] and [12]. This finishes our general outline of Theorem 1.7.

Theorem 1.7 has an interesting consequence.

Theorem 1.9 (Steel-Woodin) *Assume AD^+ . Then all uncountable regular cardinals $< \Theta$ are measurable.*

Theorem 1.9 can be proved via exactly the same method that is used to prove [17, Theorem 8.27]. However, both theorems need a comparison theory which does not exist for Γ -Woodins (this is because Γ -Woodins are coarse models of a fragment of set theory not fine structural models). To actually implement the above mentioned proofs one needs to use fine structural models. Theorem 3.1 demonstrates how to obtain such models.

2 Θ -extensions of iteration strategies

The following is the main theorem of this section. Assume $AD_{\mathbb{R}}$ and suppose Σ is an ω_1 -iteration strategy for some P . Given a set A such that

1. $A \in L[A]$,
2. $L[A] \models ZFC$,
3. $A \in L_{\beta}[A]$ for some $\beta < \Theta$,

we let β_A be the least $L[A]$ -cardinal γ such that $A \in L_{\gamma}[A]$ and μ_A be the ω_1 -supercompactness measure on $\wp_{\omega_1}(L_{\beta_A}[A])$ ⁹.

We say Σ^+ is a Θ -extension of Σ if for any \mathcal{T} that is according to Σ^+ and is of length $< \Theta$, for a $\mu_{\mathcal{T}}$ measure one set of countable $X \prec L_{\beta}[\mathcal{T}]$, letting \mathcal{T}_X be the transitive collapse of X , \mathcal{T}_X is according to Σ .

Recall the definition of hull condensation from [12, Definition 1.30]. Essentially hull condensation for a strategy Σ is the statement that if \mathcal{T} is according to Σ and \mathcal{U} is a Skolem hull of \mathcal{T} then \mathcal{U} is according to Σ .

Theorem 2.1 *Assume $AD_{\mathbb{R}}$ and suppose Σ is an ω_1 -iteration strategy for some P with hull condensation. Then Σ can be uniquely extended to a Θ -strategy. Moreover, if Σ is an (ω_1, ω_1) -iteration strategy with hull condensation then Σ can be uniquely extended to a (Θ, Θ) -strategy.*

⁹Recall that $AD_{\mathbb{R}}$ implies that ω_1 is supercompact.

Proof. We only prove the first part of the claim. Given \mathcal{T} on \mathcal{M} such that $lh(\mathcal{T}) < \Theta$, set

Correct(\mathcal{T}): for a $\mu_{\mathcal{T}}$ measure one set of $X \prec L_{\beta}[\mathcal{T}]$, letting \mathcal{T}_X be the transitive collapse of \mathcal{T} , \mathcal{T}_X is according to Σ

Suppose *Correct*(\mathcal{T}) holds and \mathcal{T} is of limit length.

Lemma 2.2 *There is a unique well-founded branch b of \mathcal{T} such that *Correct*($\mathcal{T} \cap \{b\}$) holds.*

Proof. Suppose $cf(\mathcal{T}) > \omega$. It follows from Theorem 1.9 that $cf(\mathcal{T})$ is a measurable cardinal, and hence \mathcal{T} has a unique branch b . We claim that *Correct*($\mathcal{T} \cap \{b\}$) holds. Let $\beta = \beta_{\mathcal{T} \cap \{b\}}$ and fix a countable $X \prec L_{\beta}[\mathcal{T} \cap \{b\}]$. Let $\gamma = \sup(X \cap lh(\mathcal{T}))$ and let $Y \prec L_{\beta}[\mathcal{T} \cap \{b\}]$ be countable and such that $X \cup \{\gamma\} \subseteq Y$. Let $c_X : X \rightarrow N_X$ and $c_Y : Y \rightarrow N_Y$ be the transitive collapses of X and Y . Let $c_{X,Y} : N_X \rightarrow N_Y$ be given by $c_{X,Y}(c_X(u)) = c_Y(u)$ for $u \in X$. Because *Correct*(\mathcal{T}) holds, we can find Y as above such that $c_Y(\mathcal{T})$ is according to Σ . Notice now that $c_{X,Y} \upharpoonright (c_X(\mathcal{T} \cap \{b\}))$ witnesses that $c_X(\mathcal{T} \cap \{b\})$ is a hull of $c_Y(\mathcal{T}) \upharpoonright c_Y(\gamma + 1)$. Therefore, $\Sigma(c_X(\mathcal{T})) = c_X(b)$.

Assume now that $cf(\mathcal{T}) = \omega$. Set $\beta = \beta_{\mathcal{T}}$ and $\mu = \mu_{\mathcal{T}}$. Given a countable $X \prec L_{\beta}[\mathcal{T}]$, we let $c_X : X \rightarrow N_X$ be the transitive collapse of X and $b_X = \Lambda(c_X(\mathcal{T}))$. Define $b \subseteq lh(\mathcal{T})$ by setting: $\alpha \in b$ if and only if for a μ -measure one many X ,

1. $X \prec L_{\beta}[\mathcal{T}]$,
2. $\alpha \in X$, and
3. $c_X(\alpha) \in b_X$.

Sublemma 2.3 *b is a cofinal well-founded branch of \mathcal{T} .*

Proof. Assume b is not cofinal. Let $\gamma < lh(\mathcal{T})$ be such that $b \subseteq \gamma$. For each $X \prec L_{\beta}[\mathcal{T}]$ such that $\gamma \in X$, let $\gamma_X = \min(b_X - c_X(\gamma))$. Let $\alpha_X = c_X^{-1}(\gamma_X)$. Then $\alpha_X \in X$. Because μ is normal, we have that for a some $\alpha < lh(\mathcal{T})$, for μ -measure one set of X , $\alpha_X = \alpha$. But then $\alpha \in b$.

Towards showing that b is a well-founded branch, notice that we have that for a μ -measure one many X , $c_X[b \cap X]$ is cofinal in b_X . Indeed, let $(\alpha_i : i < \omega)$ be an increasing cofinal sequence in $b \cap lh(\mathcal{T})$. For $i < \omega$, let $B_i \in \mu$ be such that for $X \in B_i$,

1. $X \prec L_\beta[\mathcal{T}]$,
2. $(\alpha_i : i < \omega) \subseteq X$ and
3. $c_X(\alpha_i) \in b_X$.

Let $B = \cap B_i$. Fix now $X \in B$. Then $c_X(\alpha_i) \in b_X$ for every $i < \omega$. Moreover, $(c_X(\alpha_i) : i < \omega)$ is cofinal in $c_X(lh(\mathcal{T}))$.

Fix now $B \in \mu$ such that for every $X \in B$, $c_X[b \cap X]$ is cofinal in b_X . Let $X \prec L_\beta[\mathcal{T} \cap \{b\}]$ be countable and such that $X \cap L_\beta[\mathcal{T}] \in B$. Set $Z = X \cap L_\beta[\mathcal{T}]$. Let $c : X \rightarrow N$ be the transitive collapse of X . It is enough to see that $c(b)$ is a well-founded branch of $c(\mathcal{T})$. Notice that $c(\mathcal{T}) = c_Z(\mathcal{T})$ and $c(b) = c_Z[b \cap Z]$. It then follows that $c(b) = b_Z$. Since $b_Z = \Lambda(c_Z(\mathcal{T}))$, we are done. \square

The proof of Sublemma 2.3 also verifies that $Correct(\mathcal{T} \cap \{b\})$ holds. Indeed, the set B witnesses it as shown by the last computation, namely that $c(b) = b_Z$. \square

We now define Σ^+ by setting

1. $dom(\Sigma^+) = \{\mathcal{T} : \mathcal{T} \text{ has a limit length } < \Theta \text{ and } Correct(\mathcal{T}) \text{ holds}\}$ and
2. $\Sigma^+(\mathcal{T}) = b$ if and only if b is the unique branch of \mathcal{T} such that $Correct(\mathcal{T} \cap \{b\})$ holds.

\square

3 $\mathcal{M}_n^{\#, \Sigma}$

Suppose Γ is a vg-pointclass and (P, Σ) is a Γ -excellent pair. For $n \in \omega$ and $x \in HC$ such that $L[x] \models "x \text{ is well-ordered}"$ (such x are called swo¹⁰), we say that $\mathcal{M}_n^{\#, \Sigma}(x)$ exists if there is a sound active Σ -premouse \mathcal{M} that projects to ω , has n Woodin cardinals and is ω_1 -iterable (as a Σ -premouse¹¹).

Theorem 3.1 *Assume AD^+ . Suppose Γ is a vg-pointclass and (P, Σ) is a Γ -excellent pair. Then for every $n \in \omega$, $\mathcal{M}_n^{\#, \Sigma}$ exists.*

Proof. The proof is via induction on n . We first prove the inductive step.

¹⁰self-well-ordered

¹¹i.e., the iterates of \mathcal{M} are also Σ -premise

Sublemma 3.2 *Assume for every $x \in HC$, $\mathcal{M}_n^{\#, \Sigma}(x)$ exists. Then for every swo x , $\mathcal{M}_{n+1}^{\#, \Sigma}(x)$ exists.*

Proof. Towards a contradiction assume that $\mathcal{M}_{n+1}^{\#, \Sigma}(x)$ doesn't exist for some swo $x \in HC$. Notice that the set $A_n = \{(z, y) \in \mathbb{R}^2 : y \text{ codes } \mathcal{M}_n^{\#, \Sigma}(z)\}$ is projective in Σ . Let Π be a vg-pointclass such that $\Gamma \subseteq \Delta_\Pi$. The key fact that allows us to prove our claim is that all sets of reals that are projective in $Code(\Sigma)$ are in Δ_Π . This immediately follows from the definition of vg-pointclass. Let $M_\Pi = L_\nu(D, \mathbb{R})$ be the parent of Π . We then have that $Code(\Sigma) \in (\Delta_1^2(D))^{M_\Pi}$. Because $(\Delta_1^2(D))^{M_\Pi}$ is closed under the real quantification, we have that all sets projective in $Code(\Sigma)$ are in $(\Delta_1^2(D))^{M_\Pi}$ and moreover, there is a set $U \in (\Delta_1^2(D))^{M_\Pi}$ that codes a sequence $(U_n : n < \omega)$ such that for each $n \in \omega$, U_n is a universal $\Sigma_n^1(Code(\Sigma))$ set of reals.

Let (Q, Λ) be a Π -excellent pair such that $x \in Q$ and if \vec{B} is Π -qsjs then B_0 codes the pair $(Code(\Sigma), A_n)$. Let $\delta = \delta^Q$. Notice that we have that $P \in HC^Q$. Let \mathcal{M} be the fully backgrounded $L[\vec{E}][x]$ -constructions relative to Σ done inside V_{δ^Q} . \mathcal{M} inherits an ω_1 -strategy Ψ from Λ (see [9, Chapter 12]).

Because B_0 codes A_n , we have that $\mathcal{M}_n^{\#, \Sigma}(V_{\delta^Q}) \in Q$. Therefore, $\mathcal{M}_n^{\#, \Sigma}(V_{\delta^Q}) \models \text{"}\delta \text{ is a Woodin cardinal"}$. It follows that $\mathcal{M}_n^{\#, \Sigma}(\mathcal{M}) \models \text{"}\delta \text{ is a Woodin cardinal"}$.¹²

However, because \mathcal{M} is ω_1 -iterable as witnessed by Ψ , there is no active initial segment of \mathcal{M} that reaches $n + 1$ Woodin cardinals. Because $\mathcal{M}_n^{\#, \Sigma}(\mathcal{M})$ has $n + 1$ many Woodins and is active, we have that $\rho(\mathcal{M}_n^{\#, \Sigma}(\mathcal{M})) = \omega$. However, as we argued in the footnote below, $\rho(\mathcal{M}_n^{\#, \Sigma}(\mathcal{M})) = \delta$. \square

To finish the proof of the lemma, we need to show that $\mathcal{M}_0^{\#, \Sigma}(x)$ exists for every x . The proof of the sublemma above shows how to do this. All we need to do is get any (Q, Λ) as in the proof of sublemma and consider \mathcal{M} defined the same way as in the proof of the sublemma. Because there are measurable cardinals in Q , we must have that \mathcal{M} has measurable cardinals. It follows that $\mathcal{M}_0^{\#, \Sigma}(x)$ exists. \square

4 The internal theory of $\mathcal{M}_1^{\#, \Sigma}$

Fix a vg-pointclass Γ and a Γ -excellent pair (P, Σ) . Fix \vec{B} witnessing the Γ -excellence of (P, Σ) and let u be a real coding the sequence $(\tau_{B_i}^P : i < \omega)$. Let $\mathcal{M} = \mathcal{M}_1^{\#, \Sigma}(u)$

¹²To see this, first because δ is inaccessible, condensation implies that $\rho(\mathcal{M}_n^{\#, \Sigma}(\mathcal{M})) = \delta$ as otherwise we could take a transitive below δ hull \mathcal{N} of $\mathcal{M}_n^{\#, \Sigma}(\mathcal{M})$, and by condensation $\mathcal{N} \trianglelefteq \mathcal{M}$. Next, fix $f : \delta \rightarrow \delta$ such that $f \in \mathcal{M}_n^{\#, \Sigma}(\mathcal{M})$. Then following the proof of [9, Theorem on page 115] find an extender E such that $\pi_E^Q(f)(\text{crit}(E)) < \text{strength}(E)$ and $E \cap \mathcal{M} \in \mathcal{M}$. This E witnesses Woodinness of δ with respect to f .

and let Λ be the ω_1 -iteration strategy of \mathcal{M} . The following is the main theorem of this section.

Theorem 4.1 *Let δ be the Woodin cardinal of \mathcal{M} . There are trees $(T, S) \in \mathcal{M}$ on $\omega \times (\delta^+)^{\mathcal{M}}$ such that $\mathcal{M} \models “(T, S) \text{ are } \delta\text{-complementing}”$ and whenever $i : \mathcal{M} \rightarrow \mathcal{N}$ is an iteration according to Λ and $g \subseteq \text{Coll}(\omega, i(\delta))$ is \mathcal{N} -generic, $\mathcal{N}[g] \cap p[i(T)] = \text{Code}(\Sigma) \cap \mathcal{N}[g]$.*

Proof. The proof uses the idea of *generic genericity* iterations. Fixing a cardinal η of \mathcal{M} , there is a tree $\mathcal{T}_\eta \in \mathcal{M}$ on P that is according to Σ with last model P_η such that for any real x that is generic over \mathcal{M} for a poset in $\mathcal{M}|\eta$, x is generic over $\mathbb{W}_{\delta^{P_\eta}}^{P_\eta}$ ¹³. To obtain \mathcal{T}_η first iterate the least measure of P , $\eta + 1$ times and then follow the usual construction of genericity iterations with a slight modification; namely that at a successor step of the genericity iteration we pick the least extender E such that there is a condition $p \in \text{Coll}(\omega, \eta)$ that forces that some real x violates some axiom generated by E .

Let $\mathcal{T} = \mathcal{T}_{\delta^+}$ and Q be the last model of \mathcal{T} . Working in \mathcal{M} , we want to find a formula ϕ , a formula ψ and a parameter $r \in \mathcal{M}|(\delta^{+2})^{\mathcal{M}}$ such that

$\dagger(\phi, \psi, r) :$ for a club of countable $X \prec \mathcal{M}|(\delta^{+2})^{\mathcal{M}}$ letting $\pi_X : \mathcal{N}_X \rightarrow \mathcal{M}|(\delta^+)^{\mathcal{M}}$ be the transitive collapse of X and setting $\delta_X = \pi_X^{-1}(\delta)$ and $r_X = \pi_X^{-1}(r)$, for any $\mathbb{P} \in \mathcal{N}_X|(\delta_X^+)^{\mathcal{N}_X}$ whenever $g \subseteq \mathbb{P}$ is \mathcal{N}_X -generic the following conditions hold:

1. For every $y \in \mathbb{R} \cap \mathcal{N}_X[g]$, $\mathcal{N}_X[g] \models \phi[y, r_X]$ if and only if y codes a tree $\mathcal{U} \in \text{dom}(\Sigma)$.
2. For every $(y, z) \in \mathbb{R}^2 \cap \mathcal{N}_X[g]$, $\mathcal{N}_X[g] \models \psi[y, z, r_X]$ if and only if y codes a tree $\mathcal{U} \in \text{dom}(\Sigma)$ and z codes $\Sigma(\mathcal{U})$.
3. $\mathcal{N}_X[g] \models \forall y \in \mathbb{R}(\phi(y, r_X) \rightarrow \exists z \in \mathbb{R}\psi(y, z, r_X))$.

We will take $r = \mathcal{T}$. We describe ϕ and ψ by first describing another formula σ . Notice that the language of \mathcal{M} already has names for $\vec{E}^{\mathcal{M}}$, P , u and Σ . Below we describe σ , ϕ and ψ .

The description of σ : We let $\sigma(y, r)$ be the formula expressing the following statement: $y \in \mathbb{R}$ codes a tree \mathcal{U} on P such that for every limit $\alpha < lh(\mathcal{U})$ letting $b = [0, \alpha)_{\mathcal{U}}$, b is the unique branch c of $\mathcal{U} \restriction \alpha$ such that either

¹³ \mathbb{W}_η^S is the extender algebra of S associated to η , a Woodin cardinal of S .

1. $C_\Gamma(\mathcal{M}(\mathcal{U} \restriction \alpha)) \models \text{"}\delta(\mathcal{U} \restriction \alpha) \text{ is not a Woodin cardinal"}$, $\mathcal{M}_c^{\mathcal{U} \restriction \alpha}$ is well-founded and $C_\Gamma(\mathcal{M}(\mathcal{U} \restriction \alpha)) \in \mathcal{M}_c^{\mathcal{U} \restriction \alpha}$ or
2. $C_\Gamma(\mathcal{M}(\mathcal{U} \restriction \alpha)) \models \text{"}\delta(\mathcal{U} \restriction \alpha) \text{ is a Woodin cardinal"}$, $\mathcal{M}_c^{\mathcal{U} \restriction \alpha}$ is well-founded, and letting $R = C_\Gamma(C_\Gamma(\mathcal{M}(\mathcal{U} \restriction \alpha)))$, $R = \mathcal{M}_c^{\mathcal{U} \restriction \alpha}$ and $\pi_c^{\mathcal{U} \restriction \alpha}(\tau_{B_i}^P) = \tau_{B_i}^R$ for every i .

The description of ϕ : We let $\phi(y, r)$ be the formula expressing the following statement: $\sigma(y, r)$ and if \mathcal{U} is the tree coded by y then \mathcal{U} has a limit length.

The description of ψ : We let $\psi(y, z, r)$ be the formula expressing the following statement: $\phi(y, r)$ and if \mathcal{U} is the tree coded by y then $z \in \mathbb{R}$ codes a cofinal well-founded branch b of \mathcal{U} such that letting w be any real coding $\mathcal{U} \cap \{b\}$, $\sigma(w, r)$ holds.

Clearly the definition of σ is vague as it refers to C_Γ operator and the sequence of terms $(\tau_{B_i}^R : i \in \omega)$. We now describe formulas σ_0 and σ_1 that define the functions $x \rightarrow C_\Gamma(x)$ and $R \rightarrow (\tau_{B_i}^R : i \in \omega)$. The appropriate structures for evaluating σ_0 and σ_1 are again small generic extensions of Skolem hulls of $(\mathcal{M} | (\delta^{+2})^\mathcal{M}, \mathcal{T}, \in)$.

The description of σ_0 : We let $\sigma_0(y, z, \mathcal{T})$ be the formula expressing the following statement: $(y, z) \in \mathbb{R}^2$ and z codes the set of all reals w such that

$$Q[y, w] \models_{\text{Coll}(\omega, \delta^Q)} (\check{y}, \check{w}) \in \pi^\mathcal{T}(\tau_{B_0}^P).$$

The description of σ_1 : We let $\sigma_1(y, i, z, w, \mathcal{T})$ be the formula expressing the following statement: $i \in \omega$, $(y, z, w) \in \mathbb{R}^3$, $\sigma_0(y, z, \mathcal{T})$ holds, and letting a be the set of reals coded by z , w codes the set of those reals $t \in a$ such that

$$Q[t] \models_{\text{Coll}(\omega, \delta^Q)} t \in \pi^\mathcal{T}(\tau_{B_i}^P).$$

We then have the following two lemmas.

Lemma 4.2 *Suppose $X \prec (\mathcal{M} | (\delta^{+2})^\mathcal{M}, \mathcal{T}, \in)$ is countable. Let $\pi_X : \mathcal{N}_X \rightarrow \mathcal{M} | (\delta^+)^\mathcal{M}$ be the transitive collapse of X and set $\delta_X = \pi_X^{-1}(\delta)$, $\mathcal{T}_X = \pi_X^{-1}(\mathcal{T})$ and $Q_X = \pi_X^{-1}(Q)$. Let $\mathbb{P} \in \mathcal{N}_X | (\delta_X^+)^\mathcal{N}_X$ be a poset and $g \subseteq \mathbb{P}$ be \mathcal{N}_X -generic. Suppose further that $a \in HC^{Q_X[g]}$ is an swo.*

1. *Assume $a \subseteq \omega$. Working in $\mathcal{N}_X[g]$, let b the set of of all those reals that are coded by any z such that $\mathcal{N}_X[g] \models \sigma_0[a, z, \mathcal{T}_X]$. Then $b = C_\Gamma(a)$.*

2. Assume a is a transitive model of some fragment of ZFC. Working in $\mathcal{N}_X[g]$, let b be the set of all those sets c^* that are coded by some $c \subseteq a$ with the property that for a comeager many $h \subseteq \text{Coll}(\omega, a)$, for any z such that $\mathcal{N}_X[g] \models \sigma_0[a_h, z, \mathcal{T}_X]$, c_h is one of the reals coded by z . Then $b = C_\Gamma(a)$.

Proof. Clause 1 is easy and follows from the fact that \vec{B} is a Γ -condensing sequence. Decoding σ_0 shows that what we have is that b is the set of all those sets c^* that are coded by some $c \subseteq a$ with the property that $\mathcal{N}_X[g] \models$ “for a comeager many $h \subseteq \text{Coll}(\omega, a)$, $c_h \in C_\Gamma(a_h)$ ”. This latter fact is absolute between $\mathcal{N}_x[g]$ and V . Hence, $b = C_\Gamma(a_h)$. \square

The proof of the next lemma is similar and we leave it to the reader.

Lemma 4.3 *Suppose $X \prec (\mathcal{M} | (\delta^{+2})^\mathcal{M}, \mathcal{T}, \in)$ is countable. Let $\pi_X : \mathcal{N}_X \rightarrow \mathcal{M} | (\delta^+)^\mathcal{M}$ be the transitive collapse of X and set $\delta_X = \pi_X^{-1}(\delta)$, $\mathcal{T}_X = \pi_X^{-1}(\mathcal{T})$ and $Q_X = \pi_X^{-1}(Q)$. Let $\mathbb{P} \in \mathcal{N}_X | (\delta_X^+)^\mathcal{N}_X$ be a poset and $g \subseteq \mathbb{P}$ be \mathcal{N}_X -generic. Suppose further that $a \in HC^{Q_X}[g]$ is an swo and $b = C_\Gamma(a)$. Fix $i \in \omega$ and let $\tau \in b^{\text{Coll}(\omega, a)}$ be a name consisting of pairs (θ, p) such that in $\mathcal{N}_X[g]$,*

1. θ is a standard name for a real and
2. for comeager many $h \subseteq \text{Coll}(\omega, a)$, for any z coding b , $\mathcal{N}_X[g] \models \sigma_1(i, a_h, z, \theta(h))$ holds.

Then $\tau = \tau_{B_i}^b$ and $\tau \in C_\Gamma(b)$.

We now resume the proof of Theorem 4.1. First we verify that $\dagger(\phi, \psi, \mathcal{T})$ holds. To see this, fix $X \prec (\mathcal{M} | (\delta^{+2})^\mathcal{M}, \mathcal{T}, \in)$ and let $\pi_X : \mathcal{N}_X \rightarrow (\mathcal{M} | (\delta^{+2})^\mathcal{M}, \mathcal{T}, \in)$ be the transitive collapse of X . We will use subscript X to denote π_X -preimages of objects in X . Let $\mathbb{P} \in \mathcal{N}_X | (\delta_X^+)^\mathcal{N}_X$ be a poset and $g \subseteq \mathbb{P}$ be \mathcal{N}_X -generic in \mathcal{M} . The following is the main claim towards the proof of Lemma 4.1.

Claim 1. Suppose $\mathcal{U} \in HC^{\mathcal{N}_X[g]} \cap \text{dom}(\Sigma)$. Then $\Sigma(\mathcal{U}) \in \mathcal{N}_X[g]$.

Proof. Let $\Sigma(\mathcal{U}) = b$. We then know that b is the unique well-founded branch c of \mathcal{U} such that for every $i < \omega$, $\pi_c^\mathcal{U}(\tau_{B_i}^P) = \tau_{B_i}^Q$ where $Q = C_\Gamma(C_\Gamma(\mathcal{M}(\mathcal{U}))$. Lemma 4.2 and Lemma 4.3 imply that $Q, b \in \mathcal{N}_X[g]$. \square

Fix now $y \in \mathbb{R}^{\mathcal{N}_X[g]}$.

Claim 2. Suppose y codes a tree $\mathcal{U} \in \text{dom}(\Sigma)$ and $\mathcal{N}_X[g] \models \phi[y, \mathcal{T}]$. Then letting $b = \Sigma(\mathcal{U})$, $b \in \mathcal{N}_X[g]$ is the unique well-founded branch c of \mathcal{U} such that whenever $z \in \mathcal{N}_X[g] \cap \mathbb{R}$ codes $\mathcal{U} \smallfrown \{c\}$, $\mathcal{N}_X \models \phi[z, \mathcal{T}]$.

Proof. We only need to verify that b satisfies clauses 1 and 2 of $\sigma[z, \mathcal{T}]$. But this immediately follows from the fact that $\Sigma(\mathcal{U}) = b$, Lemma 4.2 and Lemma 4.3. \square

Claim 3. $\mathcal{N}_X[g] \models \phi[y, \mathcal{T}]$ if and only if y codes a tree $\mathcal{U} \in \text{dom}(\Sigma)$.

Proof. That y codes a tree on P is absolute between $\mathcal{N}_X[g]$ and \mathcal{M} . We then assume that y indeed codes a tree \mathcal{U} of limit length.

Assume first that $\mathcal{N}_X[g] \models \phi[y, \mathcal{T}]$. Fix a limit $\alpha < lh(\mathcal{U})$ and assume that $\mathcal{U} \restriction \alpha \in \text{dom}(\Sigma)$. Let $b = [0, \alpha)_{\mathcal{U}}$. It follows from Lemma 4.2 and Lemma 4.3 that $\Sigma(\mathcal{U} \restriction \alpha) = b$.

Suppose next that $\mathcal{U} \in \text{dom}(\Sigma)$. Let $\alpha < lh(\mathcal{U})$ be a limit ordinal and let $b = [0, \alpha)_{\mathcal{U}}$. Because $b = \Sigma(\mathcal{U} \restriction \alpha)$, we have that the two clauses of formula σ are satisfied in \mathcal{M} . Lemma 4.2 and Lemma 4.3 imply that the two clauses of σ are also satisfied in $\mathcal{N}_X[g]$. \square

The next claim is an easy corollary of Claim 1-3.

Claim 4. Suppose $\mathcal{N}_X[g] \models \phi[y, \mathcal{T}]$. Then there is a real $z \in \mathcal{N}_X[g]$ such that $\mathcal{N}_X \models \psi[y, z, \mathcal{T}]$.

We now have that Claim 1-4 imply that $\dagger(\phi, \psi, \mathcal{T})$ holds. It follows that we can find $(T, S) \in \mathcal{M}^{(\delta^{+2})^M}$ such that

1. $\mathcal{M} \models \text{"}(T, S) \text{ are } (\delta^+)^\mathcal{M}\text{-complementing"}\text{"}$, and
2. for every $< (\delta^+)^\mathcal{M}$ -generic g , in $\mathcal{M}[g]$, $(y, z) \in p[T]$ if and only if there is a countable $X \prec (\mathcal{M}[g], \mathcal{T}, \in)$ such that $(y, z) \in X$ and letting

$$\pi_X : \mathcal{N}_X \rightarrow (\mathcal{M}[g], \mathcal{T}, \in)$$

be the transitive collapse of X , $\mathcal{N}_X \models \psi[y, z, \pi_X^{-1}(\mathcal{T})]$.

Repeating the proofs of Claim 1-4 and using elementarily we see that whenever $i : \mathcal{M} \rightarrow \mathcal{N}$ is an iteration via Λ ,

(*) for every $< i((\delta^+)^\mathcal{M})$ -generic g , $p[i(T)] \cap \mathcal{N}[g] = \text{Code}(\Sigma) \cap \mathcal{N}[g]$.

Clearly (*) finishes the proof of Lemma 4.1. □

5 All sets are Θ -uB

In this section, we prove Theorem 0.1. Assume $AD_{\mathbb{R}}$. We will describe two approaches and complete only one leaving the other to the reader. Both approaches are the same in the sense that they use the same ideas. However, approach 2 uses the theorems and lemmas that we have already proved. Nevertheless, Approach 1 is easier and doesn't use the material on the existence of iterable models with Woodin cardinals.

Approach 1

Let μ be the supercompactness measure on $\wp_{\omega_1}(\mathbb{R})$. Fix a set of reals A and find a vg-pointclass Γ_0 such that it has a parent $M_{\Gamma_0} = L_{\gamma}(\vec{A}, \mathbb{R})$ with $A = A_0$. Let Γ be a vg-pointclass such that $\Gamma_0 \subseteq \Delta_{\Gamma}$ and Γ has a parent of the form $L_{\xi}(\vec{C}, \mathbb{R})$ such that C_0 codes the sequence \vec{B} . Let \vec{B} be a Γ -condensing sequence. Thus A is coded into \vec{B} .

Let T be a tree of a Γ -scale on a Γ -universal set. We say $Q \in HC$ is Γ -full if letting $o(Q) = Ord \cap Q$, $Q = V_{o(Q)}^{L[T, Q]}$.

Suppose now that Q is Γ -full and is a model of ZF . We then have that

- (1) for every set $x \in Q$, $C_{\Gamma}^{\omega}(x) \in Q$,
- (2) for every set $x \in Q$, $\{\tau_{B_i, k}^x : i, k \in \omega\} \in Q$.

Give a transitive set $a \in HC$, let $\xi_a < \omega_1$ be least such that $L_{\xi_a}(T, a) \models ZF$ and set $Q_a = L_{\xi_a}(T, a)$. We have that Q_a is Γ -full (see Theorem 1.1). Using μ , we can extend the function $a \rightarrow Q_a$ to all transitive sets in $L_{\Theta}(\wp(\mathbb{R}))$ by repeating the proof of Theorem 2.1. By repeating the proof of Theorem 4.1, we can show that

Lemma 5.1 *for every transitive $a \in HC$, there are trees $(T_a, S_a) \in Q_a$ such that letting $\nu_a = \Theta^{Q_a}$ and $\lambda_a = (\nu_a^+)^{Q_a}$,*

- 1. $Q_a \models "(T_a, S_a) \text{ are } \nu_a\text{-complementing}"$,
- 2. *for any $\mathbb{P} \in Q_a$ such that $Q_a \models "\mathbb{P} \text{ is a surjective image of } \mathbb{R}^{Q_a}"$ and for any Q_a -generic $g \subseteq \mathbb{P}$,*

$$Q_a[g] \cap p[T_a] = C_0 \cap Q_a[g]$$

Let now $Q^+ : L_\Theta(\wp(\mathbb{R})) \rightarrow L_\Theta(\wp(\mathbb{R}))$ be the extension of Q to all transitive sets in $L_\Theta(\wp(\mathbb{R}))$. The method extending functions defined on HC to $L_\Theta(\wp(\mathbb{R}))$ given in the proof of Lemma 2.1 guarantees that

(3) for a a transitive set in $L_\Theta(\wp(\mathbb{R}))$, letting μ^* be the ω_1 -supercompactness measure on $\wp_{\omega_1}(Q_a^+)$ obtained as a projection of μ , for μ^* -measure one many $X \prec (Q_a^+, \in)$ letting $\tau_X : a \rightarrow N_X$ be the transitive collapse of X , $N_X = Q_{\tau_X(a)}$.

It follows from Lemma 5.1 that for every transitive $a \in L_\Theta(\wp(\mathbb{R}))$, there are trees $(T_a^+, S_a^+) \in Q_a^+$ such that

(4) letting μ^* be the ω_1 -supercompactness measure on $\wp_{\omega_1}(Q_a^+)$ obtained as a projection of μ , for μ^* -measure one many $X \prec (Q_a^+, \in)$ letting $\tau_X : a \rightarrow N_X$ be the transitive collapse of X , $N_X = Q_{\tau_X(a)}$ and $\tau_X(T_a^+, S_a^+) = (T_a, S_a)$.

Combining (4) with Lemma 5.1, we get that

(5) for any transitive $a \in L_\Theta(\mathbb{R})$ model of ZF that contains \mathbb{R} , for any $\mathbb{P} \in Q_a^+$ such that $Q_a^+ \models \text{“}\mathbb{P} \text{ is a surjective image of } \mathbb{R}\text{”}$ and for any Q_a^+ -generic $g \subseteq \text{Coll}(\omega, \mathbb{R})$, $(p[T])^{Q_a^+[g]} = (p[S]^{Q_a^+[g]})^c$ and $p[T] = C_0$.

Clearly (5) implies that C_0 is universally Baire. Hence, A is also universally Baire.

Approach 2

As we said above, approach 2 is very similar. Let μ be the supercompactness measure on $\wp_{\omega_1}(\mathbb{R})$. Fix a set of reals A and find a vg-pointclass Γ such that it has a parent $M_\Gamma = L_\gamma(\vec{A}, \mathbb{R})$ with $A = A_0$. Let \vec{B} be a Γ -condensing sequence such that B_0 codes \vec{A} and let (P, Σ) be a Γ -excellent pair as witnessed by \vec{B} .

Given a countable $\sigma \subseteq \mathbb{R}$ and $C \subseteq \mathbb{R}$, let $\mathcal{N}(C, \sigma) = \mathcal{M}_1^{\#, \Sigma}((C \cap \sigma, \sigma)^\#)$. Recall that under $AD_\mathbb{R}$ every set of reals has a sharp¹⁴.

Lemma 5.2 *Let $F : \wp(\mathbb{R}) \times \wp_{\omega_1}(\mathbb{R}) \rightarrow HC$ be given by $F(C, \sigma) = \mathcal{N}(C, \sigma)$. Then there is a total $F^+ : \wp(\mathbb{R}) \rightarrow L_\Theta(\wp(\mathbb{R}))$ such that for every $C \in \text{dom}(F^+)$, for a*

¹⁴ $C^\#$ can be thought as the minimal active mouse over C, \mathbb{R} that is ω_1 -iterable

club of $X \prec (F^+(C), C, \in)$, letting $\pi_X : (F^+(C), C) \rightarrow (N_C^+, N_C)$ be the transitive collapse of X , $N_C^+ = \mathcal{N}(C, \pi_X[\mathbb{R}])$.

Proof. Given $C \subseteq \mathbb{R}$, let μ_C be the supercompactness measure on $\wp_{\omega_1}(C^\#)$. Set $F^+(C) = (\prod \mathcal{N}(C, C \cap \pi_X[\mathbb{R}])) / \mu_C$ where the product ranges over countable $X \prec C^\#$ and $\pi_X : C^\# \rightarrow N_X$ is the transitive collapse of X .

We want to show that F^+ has the desired properties. Fix $C \subseteq \mathbb{R}$. Suppose now that $Y \prec (F^+(C), C, \in)$ is countable. Let (M_Y^+, M_Y, \in) be the transitive collapse of Y . We can then fix $(f_i : i < \omega)$ such that for every i , $f_i : \wp_{\omega_1}(C^\#) \rightarrow HC$ is such that $\{[f_i]_\mu : i < \omega\} = Y$. There is then a μ_C -measure one set S_0 of $X \in \wp_{\omega_1}(C^\#)$ such that for each i , $f_i(X) \in \mathcal{N}(C, \pi_X[\mathbb{R}])$ where $\pi_X : X \rightarrow N_X$ is the transitive collapse of X . It follows that for any such X , there is an embedding $\sigma_X : N_Y^+ \rightarrow \mathcal{N}(C, \pi_X[\mathbb{R}])$ given by $\sigma_X(\pi_Y(a)) = f_i(X)$ where $a = [f_i]_{\mu_C}$. By shrinking S_0 further to some $S \in \mu_C$ we can make sure that σ_X is Σ_1 -elementary for every $X \in S$. It then follows that $M_Y^+ = \mathcal{N}(C, \pi_Y[\mathbb{R}])$ \square

For each $C \subseteq \mathbb{R}$, let $(T_C, S_C) \in F^+(C)$ be the trees described in Lemma 4.1. For each C and $X \prec C^\#$ we let $f(X) = (T, S)$ where $(T, S) \in \mathcal{N}(C, \pi_X[\mathbb{R}])$ are the trees described in the proof of Theorem 4.1 and $\pi_X : X \rightarrow N_X$ is the transitive collapse of X . Let then $(T_C, S_C) = [f]_{\mu_C}$. It follows from the definition of $F^+(C)$ that

(1) For every poset $\mathbb{P} \in F^+(C)$ such that $F^+(C) \models \text{“}\mathbb{P} \text{ is a surjective image of } \mathbb{R}\text{”}$, whenever $g \subseteq \mathbb{R}$ is $F^+(C)$ -generic, $F^+(C)[g] \models \text{“}p[T_C] = (p[S_C])^c\text{”}$.

Notice that if $C \subseteq \mathbb{R}$ is such that $C_0 \in C^\#$ ¹⁵ then for a μ_C -measure one many $X \prec C^\#$, $\text{Code}(\Sigma) \in X$ and letting $\pi_X : X \rightarrow N_X$ be the transitive collapse of X ,

$$(p[T])^{\mathcal{N}(C, \pi_X[\mathbb{R}])} = \text{Code}(\Sigma) \cap \mathcal{N}(C, \pi_X[\mathbb{R}]).$$

It follows that

(2) for every $C \subseteq \mathbb{R}$, $p[T_C] = \text{Code}(\Sigma)$.

Thus,

(3) for every C , (T_C, S_C) is a universally Baire representation of $\text{Code}(\Sigma)$ for posets $\mathbb{P} \in C^\#$ with the property that $C^\# \models \text{“}\mathbb{P} \text{ is a surjective image of } \mathbb{R}\text{”}$.

¹⁵This could happen if for instance, C is Wadge reducible to C .

As for every poset \mathbb{P} that is a surjective image of a reals there is in some $C^\#$ such that $\mathbb{P} \in C^\#$ and $C^\# \models$ “ \mathbb{P} is a surjective image of \mathbb{R} ”, it follows from (3) that

(4) $Code(\Sigma)$ is Θ -uB.

To conclude that A is also Θ -uB we use the fact that Σ -respects A . More precisely, the set U consisting of $(x, y) \in \mathbb{R}^2$ such that

1. x codes a tree \mathcal{T} on P with last model Q ,
2. y is generic over the extender algebra of Q at $\pi^{\mathcal{T}}(\delta)$ and
3. $Q[y] \models \Vdash_{Coll(\omega, i(\delta^P))} \check{y} \in \pi^{\mathcal{T}}(\tau_{A,0}^P)$,

is $\Pi_1^1(u)$ for some real u . It is the easy to extract a Θ -uB representation for A from one for $Code(\Sigma)$ and U . This finishes the proof of Theorem 0.1.

We finish this section with an application. The authors of [6] produced models of determinacy that satisfy “All sets of reals are universally Baire”. They worked under large cardinal assumptions and the proof used the proof of the derived model theorem. Here we use determinacy. First we record the following easy corollary to our construction of the universally Baire representation of iteration strategies.

Corollary 5.3 *Assume $AD_{\mathbb{R}}$ and suppose Γ is a vg -pointclass. Let (P, Σ) be a Γ -excellent pair. Then $Code(\Sigma)$ is Θ -uB as witnessed by a pair of trees (T, S) that is ordinal definable from $Code(\Sigma)$ and a real.*

Here is our second corollary.

Corollary 5.4 *Assume $AD_{\mathbb{R}}$. Let $(\theta_\alpha : \alpha \leq \Omega)$ be the Solovay sequence, and fixing $\alpha < \Omega$, set $\Gamma = \{A : w(A) < \alpha\}$. Then for all $\beta \in [\alpha + 1, \Omega)$ such that $HOD \models$ “ θ_β is regular”, $V_{\theta_\beta}^{HOD(\Gamma)} \models ZF + AD^+ +$ “All sets are universally Baire”.*

Proof. It is known that if $HOD \models$ “ θ_β is regular” then $HOD(\Gamma_\beta) \models$ “ θ_β is regular” where $\Gamma_\beta = \{A \subseteq \mathbb{R} : w(A) < \theta_\beta\}$ (see for instance [1, Theorem 2.3]). It follows that for each β as in the statement of the corollary, $V_{\theta_\beta}^{HOD(\Gamma)} \models ZF + AD^+$. That $V_{\theta_\beta}^{HOD(\Gamma)} \models$ “All sets are universally Baire” follows from Corollary 5.3 and the short argument given after bullet point (4) in the proof of Theorem 0.1. \square

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