On ultrafilter extensions of first-order models and ultrafilter interpretations

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Abstract

There exist two known types of ultrafilter extensions of first-order models, both in a certain sense canonical. One of them [16] comes from modal logic and universal algebra, and in fact goes back to [20]. Another one [27, 28] comes from model theory and algebra of ultrafilters, with ultrafilter extensions of semigroups [17] as its main precursor. By a classical fact of general topology, the space of ultrafilters over a discrete space is its largest compactification. The main result of [27, 28], which confirms a canonicity of this extension, generalizes this fact to discrete spaces endowed with an arbitrary first-order structure. An analogous result for the former type of ultrafilter extensions was obtained in [31]. Results of such kind are referred to as extension theorems.

After a brief introduction, we offer a uniform approach to both types of extensions based on the idea to extend the extension procedure itself. We propose a generalization of the standard concept of first-order interpretations in which functional and relational symbols are interpreted rather by ultrafilters over sets of functions and relations than by functions and relations themselves, and define ultrafilter models with an appropriate semantics for them. We provide two specific operations which turn ultrafilter models into ordinary models, establish necessary and sufficient conditions under which the latter are the two canonical ultrafilter extensions of some ordinary models, and obtain a topological characterization of ultrafilter models. We generalize a restricted version of the extension theorem to ultrafilter models. To formulate the full version, we propose a wider concept of ultrafilter models with their semantics based on limits of ultrafilters, and show that the former concept can be identified, in a certain way, with a particular case of the latter; moreover, the new concept absorbs the ordinary concept of models. We provide two more specific operations which turn ultrafilter models in the narrow sense into ones in the wide sense, and establish necessary and sufficient conditions under which ultrafilter models in the wide sense are the images of ones in the narrow sense under these operations, and also are two canonical ultrafilter extensions of some ordinary models. Finally, we establish three full versions of the extension theorem for ultrafilter models in the wide sense.

The results of the first three sections of this paper were partially announced in [25].

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1 Introduction

In this section, we recall main definitions and facts concerning ultrafilter extensions of arbitrary maps, relations, and first-order models. All results mentioned here are established in various previous papers, so we omit their proofs. The section provides also some (of necessity incomplete) historical information.

Given a set X, let $\boldsymbol{\beta} X$ be the set of ultrafilters over X. As usual, we let $X \subseteq \boldsymbol{\beta} X$ by identifying each $x \in X$ with the principal ultrafilter given by x. Fix a first-order language and consider an arbitrary model $\mathfrak{A}$ of the language:

$$\mathfrak{A} = (X, F, \ldots, R, \ldots)$$

with the universe X, operations $F, \ldots$, and relations $R, \ldots$.

Definition 1.1. An (*abstract*) *ultrafilter extension* of $\mathfrak{A}$ is any model $\mathfrak{A}'$ in the same language of form

$$\mathfrak{A}' = (\boldsymbol{\beta} X, F', \ldots, R', \ldots)$$

with the universe $\boldsymbol{\beta} X$ and operations $F', \ldots$ and relations $R', \ldots$ on $\boldsymbol{\beta} X$ that extend $F, \ldots$ and $R, \ldots$, respectively.

There are essentially *two* known ways to extend relations by ultrafilters, and *one* to extend maps. Particular instances of these extensions were discovered by various authors in different time and different areas, often without a knowledge of parallel studies in adjacent areas. It is convenient to describe these extensions in topological terms.

Recall that $\boldsymbol{\beta} X$ carries a natural topology generated by basic open sets

$$\widetilde{A} = \{\mathfrak{u} \in \boldsymbol{\beta} X : A \in \mathfrak{u}\}$$

for all $A \subseteq X$. Easily, the sets are also closed, so the space $\boldsymbol{\beta} X$ is zero-dimensional. Moreover, $\boldsymbol{\beta} X$ is compact, Hausdorff, extremally disconnected (the closure of any open set is open), and the largest compactification of the discrete space X. This means that X is dense in $\boldsymbol{\beta} X$ and every (trivially continuous) map h of X into any compact Hausdorff space Y uniquely extends to a continuous map $\widetilde{h}$ of $\boldsymbol{\beta} X$ into Y:

$$\begin{tikzcd} \boldsymbol{\beta} X \arrow[dr, dashed, "\widetilde{h}"] & \\ X \arrow[u] \arrow[r, "h"] & Y \end{tikzcd}$$

by letting for all $\mathfrak{u} \in \boldsymbol{\beta} X$,

$$\widetilde{h}(\mathfrak{u}) = y \text{ where } \{y\} = \bigcap_{A \in \mathfrak{u}} \operatorname{cl}_Y h``A.$$

(As usual, $\operatorname{cl}_S B$ is the closure of B in S, and $f``B$ is the image of B under f.) The largest compactification of Tychonoff spaces, usually referred to as the *Stone–Čech compactification*, was discovered independently by Čech [6] and M. Stone [34]; then Wallman [36] did the same for T_1 spaces (by using ultrafilters on lattices of closed sets); see [17, 8, 12] for more information.

The ultrafilter extension of a unary relation R on a set X is exactly the basic (cl)open set $\widetilde{R}$, and the ultrafilter extension of a unary map $F : X \to Y$, where Y is a compact Hausdorff space (for operations F on X we let $Y = \boldsymbol{\beta} X$ as $X \subseteq \boldsymbol{\beta} X$), is exactly its continuous extension $\widetilde{F}$. Thus in the unary case, the procedure gives classical objects known since 1930s. As for maps and relations of greater arities, several instances of their ultrafilter extensions were discovered only in 1960s.

Ultrafilter extensions of maps. Studying ultraproducts, Kochen [22] and Frayne, Morel, and Scott [13] considered a "multiplication" of ultrafilters, which actually is the ultrafilter extension of the n-ary operation of taking n-tuples. They shown that the successive iteration of ultrapowers by ultrafilters $\mathfrak{u}_1, \ldots, \mathfrak{u}_n$ is isomorphic to a single ultrapower by their "product". This has leaded to the general construction of iterated ultrapowers, invented by Gaifman and elaborated by Kunen, which has become common in model theory and set theory (see [7, 21]).

Ultrafilter extensions of semigroups appeared in 1960s as subspaces of function spaces. To the best of our knowledge, the first explicit construction of the semigroup that is the ultrafilter extension of a group is due to Ellis [11]; he also proved the existence of idempotents in compact Hausdorff semigroups with one-sided continuity [10]. In 1970s Galvin and Glazer applied these facts to give an easy proof of what now known as Hindman's Finite Sums Theorem; the key idea was to use ultrafilters that are idempotent w.r.t. the extended operation. Then the method was developed by Bergelson, Blass, van Douwen, Hindman, Protasov, Strauss, and many others, and provided numerous Ramsey-theoretic applications in number theory, algebra, topological dynamics, and ergodic theory. The book [17] is a comprehensive treatise of this area, with an historical information. This technique was recently applied for obtaining analogous results for certain non-associative algebras (see [26, 29]).

Ultrafilter extensions of arbitrary n-ary maps have been introduced independently in recent works by Goranko [16] and Saveliev [27, 28].

Definition 1.2. For a map $F : X_1 \times \ldots \times X_n \to Y$, the extended map $\widetilde{F} : \boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n \to \boldsymbol{\beta} Y$ is defined by letting

$$\widetilde{F}(\mathfrak{u}_1, \ldots, \mathfrak{u}_n) = \\ \big\{A \subseteq Y : \{x_1 \in X_1 : \ldots \{x_n \in X_n : F(x_1, \ldots, x_n) \in A\} \in \mathfrak{u}_n \ldots\} \in \mathfrak{u}_1\big\}.$$

One can simplify this cumbersome notation by introducing *ultrafilter quantifiers*. For every ultrafilter $\mathfrak{u}$ over a set X and formula $\varphi(x, \ldots)$ with parameters $x, \ldots$ valuated over X, let

$$(\forall^{\mathfrak{u}} x)\, \varphi(x, \ldots) \text{ mean } \{x : \varphi(x, \ldots)\} \in \mathfrak{u}.$$

In fact, such quantifiers are a special kind of second-order quantifiers: $(\forall^{\mathfrak{u}} x)$ is equivalent to $(\forall A \in \mathfrak{u})(\exists x \in A)$, and also (since $\mathfrak{u}$ is ultra) to $(\exists A \in \mathfrak{u})(\forall x \in A)$. Note also that ultrafilter quantifiers are self-dual, i.e. $\forall^{\mathfrak{u}}$ and $\exists^{\mathfrak{u}}$ coincide; they generally do not commute with each other, i.e. $(\forall^{\mathfrak{u}} x)(\forall^{\mathfrak{v}} y)$ and $(\forall^{\mathfrak{v}} y)(\forall^{\mathfrak{u}} x)$ are generally not equivalent; and if $\mathfrak{u}$ is the principal ultrafilter given by $a \in X$ then $(\forall^{\mathfrak{u}} x)\varphi(x, \ldots)$ is reduced to $\varphi(a, \ldots)$.

Now the definition above can be rewritten as follows:

$$\widetilde{F}(\mathfrak{u}_1, \ldots, \mathfrak{u}_n) = \big\{A \subseteq Y : (\forall^{\mathfrak{u}_1} x_1) \ldots (\forall^{\mathfrak{u}_n} x_n)\ F(x_1, \ldots, x_n) \in A\big\}.$$

The map $\widetilde{F}$ can be also described as the composition of the ultrafilter extension of taking n-tuples, which maps $\boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n$ into $\boldsymbol{\beta}(X_1 \times \ldots \times X_n)$, and the continuous extension of F considered as a unary map, which maps $\boldsymbol{\beta}(X_1 \times \ldots \times X_n)$ into $\boldsymbol{\beta} Y$.

Not many properties of original maps are preserved under their ultrafilter extensions. Specific identities preserved under ultrafilter extensions (e.g. associativity is so while commutativity and idempotency are not) are described in [28], Theorem 5.3.

Ultrafilter extensions of relations. One type of ultrafilter extensions of relations goes back to a seminal paper by Jónsson and Tarski [20] where they have been appeared implicitly, in terms of representations of Boolean algebras with operators. For binary relations, their representation theory was rediscovered in modal logic by Lemmon [23] who credited much of this work to Scott (see footnote 6 on p. 204); see also [24]. Goldblatt and Thomason [14] (where Section 2 was entirely due to Goldblatt) used this to characterize modal definability; the term "ultrafilter extension" has been coined probably in the subsequent work by van Benthem [2] (for modal definability see also [3, 35, 5]). Later Goldblatt [15] considered the extension of n-ary relations in the context of universal algebra and model theory.

The following definition is equivalent to one appeared in [16] (or [20, 15]):

Definition 1.3. For a relation $R \subseteq X_1 \times \ldots \times X_n$, the extended relation $R^* \subseteq \boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n$ is defined by letting

$$\begin{gathered} R^*(\mathfrak{u}_1, \ldots, \mathfrak{u}_n) \ \text{ iff } \\ (\forall A_1 \in \mathfrak{u}_1) \ldots (\forall A_n \in \mathfrak{u}_n)(\exists x_1 \in A_1) \ldots (\exists x_n \in A_n)\ R(x_1, \ldots, x_n). \end{gathered}$$

The first-order formulas corresponding to so-called canonical modal formulas (e.g. to all Sahlqvist formulas) are preserved under passing from $\mathfrak{A}$ to $\mathfrak{A}^*$, provided $\mathfrak{A}$ is a model of a relational language (see [2, 5]).

Another type of ultrafilter extensions of n-ary relations has been recently discovered in [27, 28]:

Definition 1.4. For a relation $R \subseteq X_1 \times \ldots \times X_n$, the extended relation $\widetilde{R} \subseteq \boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n$ is defined by letting

$$\begin{gathered} \widetilde{R}(\mathfrak{u}_1, \ldots, \mathfrak{u}_n) \ \text{ iff } \\ \big\{x_1 \in X_1 : \ldots \{x_n \in X_n : R(x_1, \ldots, x_n)\} \in \mathfrak{u}_n \ldots \big\} \in \mathfrak{u}_1. \end{gathered}$$

Rewriting this via ultrafilter quantifiers, we get an easier formulation:

$$\widetilde{R}(\mathfrak{u}_1, \ldots, \mathfrak{u}_n) \ \text{ iff } \ (\forall^{\mathfrak{u}_1} x_1) \ldots (\forall^{\mathfrak{u}_n} x_n)\ R(x_1, \ldots, x_n).$$

By decoding ultrafilter quantifiers, this also can be rewritten by

$$\begin{gathered} \widetilde{R}(\mathfrak{u}_1, \ldots, \mathfrak{u}_n) \ \text{ iff } \\ (\forall A_1 \in \mathfrak{u}_1)(\exists x_1 \in A_1) \ldots (\forall A_n \in \mathfrak{u}_n)(\exists x_n \in A_n)\ R(x_1, \ldots, x_n), \end{gathered}$$

whence it clearly follows that one of the two relations is included into another:

$$\widetilde{R} \subseteq R^*.$$

If R is a unary relation, both extensions, $\widetilde{R}$ and R^*, coincide with the basic open set given by R (and with $\mathrm{cl}_{\boldsymbol{\beta} X} R$, the closure of R in the space $\boldsymbol{\beta} X$). If a binary relation R is functional, then R^* (but not $\widetilde{R}$) coincides with the above-defined extension of R considered as a unary map; this does not work for relations of bigger arities. An easy instance of the $\widetilde{\ }$-extensions, where R are linear orders, was studied in [30].

A systematic comparative study of both extensions (for binary R) is undertaken in [31]. In particular, it is shown there that the *- and the $\widetilde{\ }$-extensions have a dual character w.r.t. relation-algebraic operations: the *-extension commutes with composition and inversion but not Boolean operations except for union, while the $\widetilde{\ }$-extension commutes with all Boolean operations but neither composition nor inversion. Also [31] provides topological characterizations of $\widetilde{R}$ and R^*

in terms of appropriate closure operations and in terms of Vietoris-type topologies (regarding R as multi-valued maps).

Ultrafilter extensions of models. Ultrafilter extensions of arbitrary first-order models were defined and studied for the first time independently in [16] and in [27] with two distinct versions of extended relations: Goranko considered models with the *-extensions of relations and Saveliev with their ${}^\sim$-extensions. Here we shall consider both types of extensions; for a given model $\mathfrak{A}$ denote them by $\mathfrak{A}^*$ and $\widetilde{\mathfrak{A}}$, respectively:[1]

Definition 1.5. For an arbitrary model $\mathfrak{A} = (X, F, \ldots, R, \ldots)$ we let

$$\mathfrak{A}^* = \big(\boldsymbol{\beta} X, \widetilde{F}, \ldots, R^*, \ldots\big) \quad \text{and} \quad \widetilde{\mathfrak{A}} = \big(\boldsymbol{\beta} X, \widetilde{F}, \ldots, \widetilde{R}, \ldots\big).$$

Since for any relation R we have $\widetilde{R} \subseteq R^*$, the following observation is obvious:

Theorem 1.6. *For any model $\mathfrak{A}$ with the universe X the identity map on $\boldsymbol{\beta} X$ is a homomorphism of $\widetilde{\mathfrak{A}}$ onto $\mathfrak{A}^*$:*

$$\begin{tikzcd}
\widetilde{\mathfrak{A}} \arrow[rr, "\mathrm{id}"] & & \mathfrak{A}^* \\
& \mathfrak{A} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Therefore, all positive formulas satisfied in $\widetilde{\mathfrak{A}}$ are also satisfied in $\mathfrak{A}^*$.

It follows from above mentioned facts that ultrafilter extensions are not elementary, except for certain degenerate cases. Even universal formulas are not preserved under these extensions, as seen from the example of a semigroup $\mathfrak{A}$ without idempotents: the semigroup $\widetilde{\mathfrak{A}}$ does have an idempotent by Ellis' theorem. On the other hand, idempotents in $\widetilde{\mathfrak{A}}$ is a key tool in obtaining various deep combinatorial results about the extended $\mathfrak{A}$, most of which have no known alternative (i.e. not using ultrafilter extensions) proofs (see [17]). More generally, some complex (typically, not first-order) assertions about the original model have counterparts about its ultrafilter extension which are easier to formulate and to prove; so, in a sense, the non-elementarity of ultrafilter extensions can be their advantage in studying the extended models.[2]

The following theorem has been appeared in [27] and called the First Extension Theorem in [28]:

Theorem 1.7. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two models of the same signature. If h is a homomorphism between $\mathfrak{A}$ and $\mathfrak{B}$, then the continuous extension $\widetilde{h}$ is a homomorphism between $\widetilde{\mathfrak{A}}$ and $\widetilde{\mathfrak{B}}$:*

$$\begin{tikzcd}
\widetilde{\mathfrak{A}} \arrow[r, dashed, "\widetilde{h}"] & \widetilde{\mathfrak{B}} \\
\mathfrak{A} \arrow[u] \arrow[r, "h"] & \mathfrak{B} \arrow[u]
\end{tikzcd}$$

Theorem 1.7 on the ${}^\sim$-extensions is a precise counterpart of Theorem 1.8 on the *-extensions, a principal result of [16]:

[1] Another notation was used in [16], where $\mathfrak{A}^*$ was denoted by $\mathbf{U}(\mathfrak{A})$, and in [27, 28, 32], where $\widetilde{\mathfrak{A}}$ was denoted by $\boldsymbol{\beta}\mathfrak{A}$.

[2] Compare this with non-standard extensions, also used to prove assertions about the extended model, which *are* elementary; it is unclear, however, whether this technique produces as many results with no known alternative proofs as the technique based on ultrafilter extensions does. Interestingly, a recent paper [9] combines both techniques to obtain results in number theory.

Theorem 1.8. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two models of the same signature. If h is a homomorphism between $\mathfrak{A}$ and $\mathfrak{B}$, then the continuous extension $\widetilde{h}$ is a homomorphism between $\mathfrak{A}^*$ and $\mathfrak{B}^*$:*

$$\begin{tikzcd} \mathfrak{A}^* \arrow[r, dashed, "\widetilde{h}"] & \mathfrak{B}^* \\ \mathfrak{A} \arrow[u] \arrow[r, "h"] & \mathfrak{B} \arrow[u] \end{tikzcd}$$

Both theorems remain true for isomorphic embeddings and some other model-theoretic interrelations (see [16, 27, 28]). On the other hand, it was shown in [32] that Theorem 1.7 does not hold for elementary embeddings, moreover, the ultrafilter extensions of a model and its elementary submodel do not need to be elementarily equivalent.

Theorem 1.7 is actually a particular case of a much stronger result of [27], called the Second Extension Theorem in [28]. To formulate this, we need the following concepts introduced in [27].

Definition 1.9. Let $X_1, \ldots, X_n, Y$ be topological spaces, and let $A_1 \subseteq X_1, \ldots, A_{n-1} \subseteq X_{n-1}$. An n-ary function $F : X_1 \times \ldots \times X_n \to Y$ is *right continuous w.r.t.* $A_1, \ldots, A_{n-1}$ iff for each i, $1 \leqslant i \leqslant n$, and every $a_1 \in A_1, \ldots, a_{i-1} \in A_{i-1}$ and $x_{i+1} \in X_{i+1}, \ldots, x_n \in X_n$, the unary map

$$x \mapsto F(a_1, \ldots, a_{i-1}, x, x_{i+1}, \ldots, x_n)$$

of X_i into Y is continuous. An n-ary relation $R \subseteq X_1 \times \ldots \times X_n$ is *right open* (*right closed*, *right clopen*, etc.) *w.r.t.* $A_1, \ldots, A_{n-1}$ iff for each i, $1 \leqslant i \leqslant n$, and every $a_1 \in A_1, \ldots, a_{i-1} \in A_{i-1}$ and $x_{i+1} \in X_{i+1}, \ldots, x_n \in X_n$, the set

$$\big\{x \in X_i : R(a_1, \ldots, a_{i-1}, x, x_{i+1}, \ldots, x_n)\big\}$$

is open (closed, clopen, etc.) in X_i.

Theorem 1.10 ([27, 28]) describes topological properties of the $\widetilde{\ }$-extensions and serves as a base of Theorem 1.11, the Second Extension Theorem of [28]. (A very particular instance of the latter theorem, in which the models under consideration are semigroups, has been appeared in [4], Theorem 4.5.3.)

Theorem 1.10. *Let $\mathfrak{A}$ be a model. In the extension $\widetilde{\mathfrak{A}}$, all operations are right continuous and all relations right clopen w.r.t. the universe of $\mathfrak{A}$.*

Theorem 1.11. *Let $\mathfrak{A}$ and $\mathfrak{C}$ be two models of the same signature, h a homomorphism of $\mathfrak{A}$ into $\mathfrak{C}$, and let $\mathfrak{C}$ be endowed with a compact Hausdorff topology in which all operations are right continuous, and all relations are right closed, w.r.t. the image of the universe of $\mathfrak{A}$ under h. Then $\widetilde{h}$ is a homomorphism of $\widetilde{\mathfrak{A}}$ into $\mathfrak{C}$:*

$$\begin{tikzcd} \widetilde{\mathfrak{A}} \arrow[dr, dashed, "\widetilde{h}"] & \\ \mathfrak{A} \arrow[u] \arrow[r, "h"] & \mathfrak{C} \end{tikzcd}$$

Theorem 1.7 (for homomorphisms) easily follows: take $\widetilde{\mathfrak{B}}$ as such a $\mathfrak{C}$. The main meaning of Theorem 1.11 is that it generalizes the mentioned classical Čech–Stone result to the case when the underlying discrete space X carries an arbitrary first-order structure.

A natural question is whether the *-extensions are also canonical in a similar sense. The answer is positive; two following theorems are counterparts of Theorems 1.10 and 1.11, respectively (essentially both have been proved in [31]). Recall that a set is *regular closed* iff it is the closure of an open set.

Theorem 1.12. *Let $\mathfrak{A}$ be a model. In the extension $\mathfrak{A}^*$, all relations are regular closed, namely, the closures of the relations in $\mathfrak{A}$ (while all operations are right continuous w.r.t. the universe of $\mathfrak{A}$ as before).*

Theorem 1.13. *Let $\mathfrak{A}$ and $\mathfrak{C}$ be two models of the same signature, h a homomorphism of $\mathfrak{A}$ into $\mathfrak{C}$, and let $\mathfrak{C}$ be endowed with a compact Hausdorff topology in which all operations are right continuous w.r.t. the image of the universe of $\mathfrak{A}$ under h, and all relations are closed. Then $\widetilde{h}$ is a homomorphism of $\mathfrak{A}^*$ into $\mathfrak{C}$.*

$$
\begin{tikzcd}
\mathfrak{A}^* \arrow[drr, dashed, "\widetilde{h}"] & & \\
\mathfrak{A} \arrow[u] \arrow[rr, "h"] & & \mathfrak{C}
\end{tikzcd}
$$

Similarly, Theorem 1.8 (for homomorphisms) follows from Theorem 1.13. The latter also generalizes the Stone–Čech result for discrete spaces to discrete models but with a narrow class of target models $\mathfrak{C}$: having relations rather closed than right closed in Theorem 1.11. In the sequel, we shall refer to Theorems 1.7 and 1.8 as the *First Extension Theorems*, and to stronger Theorems 1.11 and 1.13 as the *Second Extension Theorems*, for the *- and $^\sim$-types of ultrafilter extensions, respectively. Let us point out that in all these extension theorems the converse implication "if $\widetilde{h}$ is a homomorphism of an ultrafilter extension $\mathfrak{A}'$ then h is a homomorphism of $\mathfrak{A}$" is also true but trivial since $\mathfrak{A}$ is a submodel of $\mathfrak{A}'$. We note also that the Second Extension Theorems are based on an "abstract extension theorem" describing certain conditions on models, their submodels, homomorphisms, and topological properties, under which such a homomorphism lifts from such a submodel to the whole model. The theorem will be used in our paper, too; we shall formulate it later on (Theorem 4.21).

We end this introductory section with topological characterizations of both types of ultrafilter extensions of relations and ultrafilter extensions of maps into discrete spaces and into compact Hausdorff spaces.

Theorem 1.14. *Let $X_1, \ldots, X_n, Y$ be discrete spaces, Z a compact Hausdorff space, and let the sets $\boldsymbol{\beta} X_1, \ldots, \boldsymbol{\beta} X_n, Y$ be endowed with the standard topology on ultrafilters, and $\boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n$ with the usual product topology. Then*

(i) $Q \subseteq \boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n$ is $\widetilde{R}$ for some $R \subseteq X_1 \times \ldots \times X_n$ iff Q is right clopen w.r.t. $X_1, \ldots, X_{n-1}$;

(ii) $Q \subseteq \boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n$ is R^* for some $R \subseteq X_1 \times \ldots \times X_n$ iff Q is regular closed;

(iii) $G : \boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n \to \boldsymbol{\beta} Y$ is $\widetilde{F}$ for some $F : X_1 \times \ldots \times X_n \to Y$ iff G is right continuous and right open w.r.t. $X_1, \ldots, X_{n-1}$;

(iv) $G : \boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n \to Z$ is $\widetilde{H}$ for some $H : X_1 \times \ldots \times X_n \to Z$ iff G is right continuous w.r.t. $X_1, \ldots, X_{n-1}$.

Moreover, all the four extension operations: $R \mapsto \widetilde{R}$, $R \mapsto R^*$, $F \mapsto \widetilde{F}$, $H \mapsto \widetilde{H}$, are bijections.

This theorem shows that Theorems 1.10 and 1.12 in fact *characterize* the *- and $^\sim$-extensions via their topological properties (and the same will follow from Theorems 3.22 and 3.23 later).

The subsequent text is organized as follows.

In Section 2, we develop a topological technique that allows us to define an ultrafilter extension of the procedure of ultrafilter extension itself. This is a key tool for our article. Based on it, we provide a uniform approach to both types of ultrafilter extensions of relations (Theorem 2.12), and furthermore, in Section 3, we define an ultrafilter interpretation of first-order syntax, under which functional and relational symbols are interpreted rather by ultrafilters over sets of functions and relations than by their elements. We define ultrafilter models using ultrafilter evaluations of variables and ultrafilter interpretations and an appropriate semantics for them. We provide two specific operations, e and E, which turn ultrafilter models into ordinary ones, establish necessary and sufficient conditions under which the latter are two canonical ultrafilter extensions of some ordinary models (Theorem 3.22), and give a topological characterization of ultrafilter models (Theorem 3.23). Defining a natural concept of homomorphisms between ultrafilter models, we establish the First Extension Theorem for ultrafilter models (Theorem 3.27) and a stronger variant of it (Theorem 3.28).

In Section 4, we define an even wider concept of ultrafilter models together with their semantics based on limits of ultrafilters, and show that this new concept absorbs the ordinary concept of models with the usual semantics (Theorem 4.4) as well as our previous concept of ultrafilter models with their semantics (Theorem 4.5). We provide two more specific operations, i and I, which turn ultrafilter models in the narrow sense into ones in the wide sense, show how they relate to the operations e and E via their limits in appropriate topologies (Theorems 4.9 and 4.18), and establish necessary and sufficient conditions under which ultrafilter models in the wide sense are the images of ones in the narrow sense under i and I, and also are two canonical ultrafilter extensions of some ordinary models (Theorems 4.11 and 4.19). Finally, we define homomorphisms between ultrafilter models in the wide sense, and establish for them an "abstract extension theorem" (Theorem 4.23) and two Second Extension Theorems (Theorems 4.25 and 4.27). In Section 5, we conclude the article by posing some problems and tasks.

A part of the results mentioned in Sections 1–3 was announced in [25]; here we provide complete proofs of all our results.[3]

2 Extending the ultrafilter extension procedure

A purpose of this section is to provide a uniform approach to both types of ultrafilter extensions: the smaller $\widetilde{\ }$-extensions and the larger *-extensions. For this, we shall develop some ideas and machinery which will lead us in the next section to certain structures, called there ultrafilter models, generalizing ultrafilter extensions of each of the two types.

We shall give an alternative description of the *-extension of relations in terms of the basic (cl)open sets and the continuous extension of maps. The crucial idea is to consider continuous extension of the procedure of ultrafilter extension itself, i.e. a *self-application* of the procedure. Let us clarify what is the idea precisely. For simplicity, consider firstly unary maps, for which the ultrafilter extensions are just the continuous extensions. To make the notation easier, let us denote the operation of continuous extension of maps by ext; i.e. $\mathrm{ext}(f)$ is another notation for $\widetilde{f}$:

$$\mathrm{ext}(f) = \widetilde{f}.$$

So if we consider (unary) maps of X into Y, then ext is a map of Y^X into $C(\boldsymbol{\beta}X, \boldsymbol{\beta}Y)$, the set of all continuous functions of $\boldsymbol{\beta}X$ into $\boldsymbol{\beta}Y$. If $C(\boldsymbol{\beta}X, \boldsymbol{\beta}Y)$ would be endowed with some compact Hausdorff topology, then we could extend the map ext to a (unique) continuous map $\widetilde{\mathrm{ext}}$ of $\boldsymbol{\beta}(Y^X)$

[3] In [25], it was erroneously stated that the set of right continuous maps forms a compact Hausdorff space w.r.t. the pointwise convergence topology; actually, the intended topology was a *restricted* pointwise convergence topology, as explained in details below.

into $C(\boldsymbol{\beta} X, \boldsymbol{\beta} Y)$:

$$\begin{tikzcd} \boldsymbol{\beta}(Y^X) \arrow[drr, dashed, "\widetilde{\mathrm{ext}}"] & & \\ Y^X \arrow[u] \arrow[rr, "\mathrm{ext}"] & & C(\boldsymbol{\beta} X, \boldsymbol{\beta} Y) \end{tikzcd}$$

We are going to show that such a topology on $C(\boldsymbol{\beta} X, \boldsymbol{\beta} Y)$ exists, and in fact, is a weaker version of the pointwise convergence topology (while the standard full version of the topology is not compact, as explained in Remark 2.7). Furthermore, as we shall see, the same approach will work in the case of n-ary maps (and relations, which can be reduced to maps).

Restricted pointwise convergence topology. Let X and Y be topological spaces and $A \subseteq X$. Define a topology on the set Y^X of all maps of X into Y by letting the family of sets $O_{a,B} = \{f \in Y^X : f(a) \in B\}$ for all $a \in A$ and all $B \subseteq Y$ which are open in Y, as an open subbase. We shall call it the A-*pointwise convergence* topology. Clearly, if $A = X$ then it is the usual pointwise convergence topology, which, as well-known (see e.g. [12]), coincides with the standard (Tychonoff) product topology.

Definition 2.1. Consider $Y^{X_1 \times \ldots \times X_n}$ as the set of n-ary maps, and choose subsets $A_1 \subseteq X_1, \ldots, A_n \subseteq X_n$. The topology with an open subbase consisting of sets

$$O_{a_1,\ldots,a_n,B} = \left\{ f \in Y^{X_1 \times \ldots \times X_n} : f(a_1, \ldots, a_n) \in B \right\}$$

for all $a_1 \in A_1, \ldots, a_n \in A_n$ and all $B \subseteq Y$ which are open in Y, will be called the $(A_1, \ldots, A_n)$-*pointwise convergence* topology.

Although it is the same that the set of unary maps of $X_1 \times \ldots \times X_n$ into Y endowed with the $A_1 \times \ldots \times A_n$-pointwise convergence topology, we shall use this terminology to emphasize when we shall say about n-ary maps.

Let $1 \leqslant i \leqslant j \leqslant n$. For any $f : X_1 \times \ldots \times X_n \to Y$, $\boldsymbol{a} \in X_1 \times \ldots \times X_{i-1}$, and $\boldsymbol{u} \in X_{j+1} \times \ldots \times X_n$, the map

$$f_{\boldsymbol{a}}^{\boldsymbol{u}} : X_i \times \ldots \times X_j \to Y$$

is defined by letting

$$f_{\boldsymbol{a}}^{\boldsymbol{u}}(x_i, \ldots, x_j) = f(\boldsymbol{a}, x_i, \ldots, x_j, \boldsymbol{u})$$

for all $x_i \in X_i, \ldots, x_j \in X_j$. We omit the sub- and superscripts whenever the sequences $\boldsymbol{a}$ and $\boldsymbol{u}$ respectively are empty.

Let cur be the *currying* (or *evaluation*) map taking any $f : X_1 \times \ldots \times X_n \to Y$ with $n \geqslant 2$ to the map $\mathrm{cur}(f) : X_n \to Y^{X_1 \times \ldots \times X_{n-1}}$ such that

$$\mathrm{cur}(f)(x) = f^x.$$

(A more precise term would be the *right currying* but we prefer the shorter one.) Clearly, the map cur is bijective.

Let for any positive $n < \omega$, topological spaces $X_1, \ldots, X_n, Y$, and sets $A_1 \subseteq X_1, \ldots, A_{n-1} \subseteq X_{n-1}$,

$$RC_{A_1,\ldots,A_{n-1}}(X_1, \ldots, X_n, Y)$$

denote the set of n-ary maps $f : X_1 \times \ldots \times X_n \to Y$ that are right continuous w.r.t. $A_1, \ldots, A_{n-1}$, which we consider with the $(A_1, \ldots, A_n)$-pointwise convergence topology.

Lemma 2.2. *If $f \in RC_{A_1,\dots,A_{n-1}}(X_1,\dots,X_n,Y)$ then*

$$\mathrm{cur}(f) \in C\big(X_n, RC_{A_1,\dots,A_{n-2}}(X_1,\dots,X_{n-1},Y)\big).$$

Proof. By definition of currying, $\mathrm{cur}(f)$ maps X_n into $RC_{A_1,\dots,A_{n-2}}(X_1,\dots,X_{n-1},Y)$. Let us verify that it is continuous. Pick any $\boldsymbol{a} \in X_1 \times \dots \times X_{n-1}$, open set B in Y, and consider the subbasic open set $O_{\boldsymbol{a},B} = \{h \in RC_{A_1,\dots,A_{n-2}}(X_1,\dots,X_{n-1},Y) : h(\boldsymbol{a}) \in B\}$ in the space $RC_{A_1,\dots,A_{n-1}}(X_1,\dots,X_n,Y)$. We have:

$$\mathrm{cur}(f)(u) \in O_{\boldsymbol{a},B} \ \text{ iff } \ \mathrm{cur}(f)(u)(\boldsymbol{a}) \in B \ \text{ iff } \ f_{\boldsymbol{a}}(u) \in B.$$

The set $\mathrm{cur}(f)^{-1}(O_{\boldsymbol{a},B}) = (f_{\boldsymbol{a}})^{-1}(B)$ is open since the map $f_{\boldsymbol{a}}$ is continuous. □

Lemma 2.3. *For any positive $n < \omega$, topological spaces $X_1,\dots,X_n$, their dense subsets $D_1 \subseteq X_1,\dots,D_n \subseteq X_n$, and Hausdorff space Y,*

(i) if maps $f,g \in RC_{D_1,\dots,D_{n-1}}(X_1,\dots,X_n,Y)$ coincide on $D_1 \times \dots \times D_n$, then they coincide everywhere,

(ii) the space $RC_{D_1,\dots,D_{n-1}}(X_1,\dots,X_n,Y)$ endowed with the $(D_1,\dots,D_{n-1})$-pointwise convergence topology, is Hausdorff.

Proof. For brevity, let us denote the space $RC_{D_1,\dots,D_{k-1}}(X_1\dots,X_k,Y)$ by RC_k. We argue by induction on n. For induction basis, see [12], Theorem 2.1.9. Assume we have already proven the claim for $n = k$. Let us prove this for $n = k+1$.

Let maps $f,g \in \mathrm{RC}_{k+1}$ coincide on $D_1 \times \dots \times D_{k+1}$. Then for each $a \in D_{k+1}$ the maps f^a and g^a coincide on $D_1 \times \dots \times D_k$ and are right continuous w.r.t. the D_i. By induction hypothesis, $f^a = g^a$. Hence, the maps $\mathrm{cur}(f), \mathrm{cur}(g) : X_{k+1} \to \mathrm{RC}_k$ coincide on D_{k+1}. By Lemma 2.2, the maps are continuous, while by induction hypothesis the space RC_k is Hausdorff. Hence, $\mathrm{cur}(f) = \mathrm{cur}(g)$ again by [12], Theorem 2.1.9. Therefore, $f = g$ since cur is bijective.

Furthermore, this shows that the space RC_{k+1} is Hausdorff. Indeed, let $f,g \in \mathrm{RC}_{k+1}$ and $f \neq g$. Then, by the just proven fact, $f(\boldsymbol{a}) \neq g(\boldsymbol{a})$ for some $\boldsymbol{a} \in D_1 \times \dots \times D_{k+1}$. Since Y is Hausdorff, pick any disjoint open neighborhoods $A,B \subseteq Y$ of $f(\boldsymbol{a})$ and $g(\boldsymbol{a})$. Then the sets $F = \{h \in \mathrm{RC}_{k+1} : h(\boldsymbol{a}) \in A\}$ and $G = \{h \in \mathrm{RC}_{k+1} : h(\boldsymbol{a}) \in B\}$ are disjoint open neighborhoods of $f \in F$ and $g \in G$. □

Lemma 2.4. *Let $X_1,\dots,X_n$ be discrete spaces and Y a compact Hausdorff space. The set*

$$RC_{X_1,\dots,X_{n-1}}(\boldsymbol{\beta}X_1,\dots,\boldsymbol{\beta}X_n,Y)$$

of n-ary maps of $\boldsymbol{\beta}X_1 \times \dots \times \boldsymbol{\beta}X_n$ into Y which are right continuous w.r.t. $X_1,\dots,X_{n-1}$, endowed with the $(X_1,\dots,X_n)$-pointwise convergence topology, is homeomorphic to the space $Y^{X_1\times\dots\times X_n}$ endowed with the usual pointwise convergence topology. Therefore, the space is compact Hausdorff; moreover, it is zero-dimensional iff so is Y.

Proof. Let us verify that the map ext, which takes each n-ary f in $Y^{X_1\times\dots\times X_n}$ to its extension $\mathrm{ext}(f) = \widetilde{f}$ in $RC_{X_1,\dots,X_{n-1}}(\boldsymbol{\beta}X_1,\dots,\boldsymbol{\beta}X_n,Y)$, is a homeomorphism. The fact that ext is injective is trivial, and that ext is surjective follows from Lemma 2.3 (since each X_i is dense in $\boldsymbol{\beta}X_i$ and Y is Hausdorff): whenever $g \in RC_{X_1,\dots,X_{n-1}}(\boldsymbol{\beta}X_1,\dots,\boldsymbol{\beta}X_n,Y)$ and $f = g \restriction (X_1 \times \dots \times X_n)$, then $\widetilde{f} = g$. Finally, the fact that it preserves in both directions open sets belonging to the subbases of the spaces, is immediate by the definition of the $(X_1,\dots,X_n)$-pointwise convergence topology. Therefore, the space is homeomorphic to the usual product space of Y, hence, by the Tychonoff theorem, is compact Hausdorff, and moreover, the zero-dimensionality iff so is Y (see e.g. [12]). $\square$

Lemma 2.5. *Let $X_1, \ldots, X_n$ be discrete spaces, Y a compact Hausdorff space, and $S \subseteq Y$ dense in Y. Then the set $\{\widetilde{f} : f \in S^{X_1 \times \ldots \times X_n}\}$ is dense in the space $RC_{X_1,\ldots,X_{n-1}}(\boldsymbol{\beta} X_1, \ldots, \boldsymbol{\beta} X_n, Y)$ endowed with the $(X_1, \ldots, X_n)$-pointwise convergence topology.*

Proof. Let $k < \omega$, pick for all $i < k$ arbitrary $\boldsymbol{a}_i \in X_1 \times \ldots \times X_n$ and $B_i \subseteq Y$ open in Y, and show that, whenever the basic open set $\bigcap_{i<k} O_{\boldsymbol{a}_i, B_i}$ is nonempty, then it contains a point $\widetilde{f}$ for some $f \in S^{X_1 \times \ldots \times X_n}$. Note that if some of the $\boldsymbol{a}_i$ coincide, say, $\boldsymbol{a}_i = \boldsymbol{a}_j$ for all $i, j \in A$ and some $A \subseteq k$, then $\bigcap_{i \in A} B_i$ is nonempty whenever so is $\bigcap_{i<k} O_{\boldsymbol{a}_i, B_i}$. So we can assume w.l.g. that all the $\boldsymbol{a}_i$ are distinct. Then any $f \in S^{X_1 \times \ldots \times X_n}$ satisfying $f(\boldsymbol{a}_i) \in B_i \cap S$ for all $i < k$, is as required. $\square$

Question 2.6. Does this remain true, moreover, for the full pointwise convergence topology? The answer is affirmative for unary maps, i.e. the set $\{\widetilde{f} : f \in S^X\}$ is dense in $C(\boldsymbol{\beta} X, Y)$. What happens for binary maps? (Problem 5.1.)

Remark 2.7. One may ask whether the usual (unrestricted) pointwise convergence topology on the set $RC_{X_1,\ldots,X_{n-1}}(\boldsymbol{\beta} X_1, \ldots, \boldsymbol{\beta} X_n, Y)$ is compact, or equivalently, whether the set forms a closed subspace of the compact Hausdorff space $Y^{\boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n}$ with the Tychonoff product topology. If this would be the case, we could use this more common topology for our purpose. Let us show that the answer is in the negative, even for unary maps.

1. The set $C(\boldsymbol{\beta} X, \boldsymbol{\beta} Y)$ endowed with the pointwise convergence topology is not compact.

It suffices to verify that for an arbitrary map $h : \boldsymbol{\beta} X \to \boldsymbol{\beta} Y$ there exists an ultrafilter $\mathfrak{f}$ over $C(\boldsymbol{\beta} X, \boldsymbol{\beta} Y)$ converging to h (to recall related facts the reader can look at the beginning of Section 4). Since h can be discontinuous, this will show that $C(\boldsymbol{\beta} X, \boldsymbol{\beta} Y)$ is not closed in $\boldsymbol{\beta} Y^{\boldsymbol{\beta} X}$. Consider the family

$$\begin{aligned} \mathcal{F} &= \{O_{\mathfrak{u}, \widetilde{S}} : \mathfrak{u} \in \boldsymbol{\beta} X \text{ and } h(\mathfrak{u}) \in \widetilde{S}\} \\ &= \{\{f \in C(\boldsymbol{\beta} X, \boldsymbol{\beta} Y) : S \in f(\mathfrak{u})\} : \mathfrak{u} \in \boldsymbol{\beta} X \text{ and } S \in h(\mathfrak{u})\}. \end{aligned}$$

Let us check that $\mathcal{F}$ is centered. It suffices to show that for any positive $n \in \omega$, ultrafilters $\mathfrak{u}_0, \ldots, \mathfrak{u}_{n-1}$ over X, and non-empty subsets $S_0, \ldots, S_{n-1}$ of X, there exists a map $f \in C(\boldsymbol{\beta} X, \boldsymbol{\beta} Y)$ satisfying

$$S_0 \in f(\mathfrak{u}_0), \ldots, S_{n-1} \in f(\mathfrak{u}_{n-1}).$$

To see, pick arbitrary pairwise disjoint sets $A_0, \ldots, A_{n-1}$ such that $A_0 \in \mathfrak{u}_0, \ldots, A_{n-1} \in \mathfrak{u}_{n-1}$, elements $s_0 \in S_0, \ldots, s_{n-1} \in S_{n-1}$, and consider a map $g : X \to Y$ such that $g(x) = s_i$ whenever $x \in A_i$, $i < n$, and $g(x) = y$, where y is a fixed element of Y, otherwise. (Actually, g on the set $X \setminus (A_0 \cup \ldots \cup A_{n-1})$ could be defined arbitrarily.) Let $f = \widetilde{g}$, so $f \in C(\boldsymbol{\beta} X, \boldsymbol{\beta} Y)$. For each $i < n$ we have $A_i \subseteq g^{-1}(S_i)$ and $A_i \in \mathfrak{u}_i$, therefore, $g^{-1}(S_i) \in \mathfrak{u}_i$, and so, $S_i \in \widetilde{g}(\mathfrak{u}_i) = f(\mathfrak{u}_i)$. Thus the map f witnesses that the family $\mathcal{F}$ is centered.

Now extend $\mathcal{F}$ to an ultrafilter $\mathfrak{f} \in \boldsymbol{\beta} C(\boldsymbol{\beta} X, \boldsymbol{\beta} Y)$. It is clear that $\mathfrak{f}$ converges to the map h, as required.

2. Since we know that $C(\boldsymbol{\beta} X, \boldsymbol{\beta} Y)$ with the X-pointwise convergence topology is compact while with the (full) pointwise convergence topology is not, we may ask what is the map $f \in C(\boldsymbol{\beta} X, \boldsymbol{\beta} Y)$ such that the ultrafilter $\mathfrak{f}$ defined above converges to f in the weaker (restricted) topology. It is not difficult to show that $f = \widetilde{h \restriction X}$. Note that $(\boldsymbol{\beta} Y)^{\boldsymbol{\beta} X}$ with the X-pointwise convergence

topology is compact (since it is compact even with the stronger pointwise convergence topology). The map r of this compact space onto its compact subspace $C(\boldsymbol{\beta}X, \boldsymbol{\beta}Y)$, defined by letting for all $h \in (\boldsymbol{\beta}Y)^{\boldsymbol{\beta}X}$

$$r(h) = \widetilde{h \restriction X},$$

is a natural retraction. However, $(\boldsymbol{\beta}Y)^{\boldsymbol{\beta}X}$ with the X-pointwise convergence topology is not Hausdorff nor even a T_0-space since, whenever $h \in (\boldsymbol{\beta}Y)^{\boldsymbol{\beta}X}$ is discontinuous, then the points h and $r(h)$ are distinct but have the same neighborhoods (it suffices to consider subbasic neighborhoods, and for any $a \in X$ and open $B \subseteq \boldsymbol{\beta}Y$ we have $h \in O_{a,B}$ iff $r(h) \in O_{a,B}$).

3. These observations hold in a general setting, for n-ary maps into any compact Hausdorff space Y: the full pointwise convergence topology on $RC_{X_1,\ldots,X_{n-1}}(\boldsymbol{\beta}X_1, \ldots, \boldsymbol{\beta}X_n, Y)$ is not compact, while the $(X_1, \ldots, X_n)$-pointwise convergence topology on $Y^{\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n}$ is compact but not T_0, and the map r defined by letting for all $h \in Y^{\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n}$

$$r(h) = \mathrm{ext}(h \restriction (X_1 \times \ldots \times X_n))$$

is a natural retraction of $Y^{\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n}$ onto $RC_{X_1,\ldots,X_{n-1}}(\boldsymbol{\beta}X_1, \ldots, \boldsymbol{\beta}X_n, Y)$.

Self-application of the extension operation. Now we are ready to define the continuous extension $\widetilde{\mathrm{ext}}$ of the map ext in a general form. Let $X_1, \ldots, X_n$ be discrete spaces and Y a compact Hausdorff space. Recall that for any n-ary map f of $X_1 \times \ldots \times X_n$ into Y, $\mathrm{ext}(f)$ is $\widetilde{f}$, the extension of f to ultrafilters which is right continuous w.r.t. principal ultrafilters:

$$\mathrm{ext} : Y^{X_1 \times \ldots \times X_n} \to RC_{X_1,\ldots,X_{n-1}}(\boldsymbol{\beta}X_1, \ldots, \boldsymbol{\beta}X_n, Y).$$

By Lemma 2.4, the set $RC_{X_1,\ldots,X_{n-1}}(\boldsymbol{\beta}X_1, \ldots, \boldsymbol{\beta}X_n, Y)$ endowed with the $(X_1, \ldots, X_n)$-pointwise convergence topology is a compact Hausdorff space. Therefore, ext extends to a unique continuous map $\widetilde{\mathrm{ext}}$ on ultrafilters over the set $Y^{X_1 \times \ldots \times X_n}$:

$$\begin{tikzcd}
\boldsymbol{\beta}\big(Y^{X_1 \times \ldots \times X_n}\big) \arrow[drr, dashed, "\widetilde{\mathrm{ext}}"] & & \\
Y^{X_1 \times \ldots \times X_n} \arrow[u] \arrow[rr, "\mathrm{ext}"] & & RC_{X_1,\ldots,X_{n-1}}(\boldsymbol{\beta}X_1, \ldots, \boldsymbol{\beta}X_n, Y)
\end{tikzcd}$$

Remark 2.8. Alternatively, we can first define $\widetilde{\mathrm{ext}}$ on ultrafilters over the set of unary maps and then extend it to $\widetilde{\mathrm{ext}}$ on ultrafilters over the set of n-ary maps by induction on n by using currying.

For this, we first note that the one-to-one correspondence between the sets $Y^{X_1 \times \ldots \times X_n \times X_{n+1}}$ and $(Y^{X_1 \times \ldots \times X_n})^{X_{n+1}}$ given by cur induces the one-to-one correspondence between the sets of ultrafilters over them, which takes each ultrafilter $\mathfrak{f} \in \boldsymbol{\beta}(Y^{X_1 \times \ldots \times X_n \times X_{n+1}})$ to the ultrafilter $\mathfrak{f}' = \{\mathrm{cur}``A : A \in \mathfrak{f}\} \in \boldsymbol{\beta}((Y^{X_1 \times \ldots \times X_n})^{X_{n+1}})$. Or else, $\mathfrak{f}'$ can be defined via the continuous extension of currying:

$$\begin{tikzcd}
\boldsymbol{\beta}\big(Y^{X_1 \times \ldots \times X_{n+1}}\big) \arrow[r, dashed, "\widetilde{\mathrm{cur}}"] & \boldsymbol{\beta}\Big(\big(Y^{X_1 \times \ldots \times X_n}\big)^{X_{n+1}}\Big) \\
Y^{X_1 \times \ldots \times X_{n+1}} \arrow[u] \arrow[r, "\mathrm{cur}"] & \big(Y^{X_1 \times \ldots \times X_n}\big)^{X_{n+1}} \arrow[u]
\end{tikzcd}$$

Since cur is a bijection, it is easily follows that so is $\widetilde{\mathrm{cur}}$, and for all $\mathfrak{f} \in \boldsymbol{\beta}(Y^{X_1 \times \ldots \times X_n \times X_{n+1}})$ we have $\widetilde{\mathrm{cur}}(\mathfrak{f}) = \mathfrak{f}'$.

Now, for $n = 1$, we extend $\mathrm{ext} : Y^X \to C(\boldsymbol{\beta}X, Y)$ to $\widetilde{\mathrm{ext}} : \boldsymbol{\beta}(Y^X) \to C(\boldsymbol{\beta}X, Y)$. And assuming that $\widetilde{\mathrm{ext}}$ has been already defined for n, we can define $\widetilde{\mathrm{ext}}(\mathfrak{f})$ by letting

$$\widetilde{\mathrm{ext}}(\mathfrak{f})(\mathfrak{u}_1, \ldots, \mathfrak{u}_n, \mathfrak{u}_{n+1}) = \widetilde{\mathrm{ext}}\big(\widetilde{\mathrm{ext}}(\mathfrak{f}')(\mathfrak{u}_{n+1})\big)(\mathfrak{u}_1, \ldots, \mathfrak{u}_n)$$

since $\widetilde{\mathrm{ext}}$ has been already defined on $\mathfrak{f}'$ and $\widetilde{\mathrm{ext}}(\mathfrak{f}')(\mathfrak{u}_{n+1})$ by induction hypothesis.

Question 2.9. One can offer another, alternative way to extend the ultrafilter extension procedure by considering it as the map not into the space of right continuous maps but into set of all maps with the usual product topology. Thus for any discrete $X_1, \ldots, X_n$ and compact Hausdorff Y, let ext be a map of the discrete space $Y^{X_1 \times \ldots \times X_n}$ into $Y^{\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n}$ endowed with the usual product topology (or equivalently, the usual, unrestricted pointwise convergence topology). As the range is a compact Hausdorff space, the map ext continuously extends to $\widetilde{\mathrm{ext}}$ (in the new sense):

$$\begin{tikzcd}
\boldsymbol{\beta}\big(Y^{X_1 \times \ldots \times X_n}\big) \arrow[drr, dashed, "\widetilde{\mathrm{ext}}"] & & \\
Y^{X_1 \times \ldots \times X_n} \arrow[u] \arrow[rr, "\mathrm{ext}"] & & Y^{\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n}
\end{tikzcd}$$

Unlike the previous construction, now some ultrafilters $\mathfrak{f} \in \boldsymbol{\beta}(Y^{X_1 \times \ldots \times X_n})$ are mapped into maps $\widetilde{\mathrm{ext}}(\mathfrak{f}) \in Y^{\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n}$ that no longer are right continuous w.r.t. principal ultrafilters (as explained in the remarks above). However, these maps $\widetilde{\mathrm{ext}}(\mathfrak{f})$ are still close to those: any neighborhood of $\widetilde{\mathrm{ext}}(\mathfrak{f})$ contains some right continuous map; this is because

$$\widetilde{\mathrm{ext}}\text{“}\ \boldsymbol{\beta}\big(Y^{X_1 \times \ldots \times X_n}\big) = \mathrm{cl}_{Y^{\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n}}\big(\mathrm{ext}\text{“}\ Y^{X_1 \times \ldots \times X_n}\big).$$

Is this version of $\widetilde{\mathrm{ext}}$ surjective? This is the case iff the image of ext is dense in the space; see Question 2.6.

Can this version of $\widetilde{\mathrm{ext}}$ lead to some interesting possibilities? (Problem 5.2.)

Lemma 2.10. *For any positive $n < \omega$, discrete spaces $X_1, \ldots, X_n$, and compact Hausdorff space Y, the continuous map*

$$\widetilde{\mathrm{ext}} : \boldsymbol{\beta}(Y^{X_1 \times \ldots \times X_n}) \to RC_{X_1, \ldots, X_{n-1}}(\boldsymbol{\beta}X_1, \ldots, \boldsymbol{\beta}X_n, Y)$$

is surjective and, whenever at least one of the X_i is infinite, non-injective.

Proof. To simplify the notation, let RC denote the space $RC_{X_1, \ldots, X_{n-1}}(\boldsymbol{\beta}X_1, \ldots, \boldsymbol{\beta}X_n, Y)$ endowed with the $(X_1, \ldots, X_n)$-pointwise convergence topology. Pick any $f \in \mathrm{RC}$, let

$$Z = \big\{(\boldsymbol{a}, B) : \boldsymbol{a} \in X_1 \times \ldots \times X_n\,,\ B \text{ is open in } Y, \text{ and } f(\boldsymbol{a}) \in B\big\},$$

and consider the following family $\mathcal{F}$ of subsets of $Y^{X_1 \times \ldots \times X_n}$:

$$\mathcal{F} = \big\{\big\{g \in Y^{X_1 \times \ldots \times X_n} : g(\boldsymbol{a}) \in B\big\} : (\boldsymbol{a}, B) \in Z\big\}.$$

The family $\mathcal{F}$ is centered; this can be stated by arguments similar to those in the first remark after Lemma 2.5. We are going to prove the following key property of the family $\mathcal{F}$:

$$\widetilde{\mathrm{ext}}(\mathfrak{f}) = f \ \text{ for all } \mathfrak{f} \in \boldsymbol{\beta}\big(Y^{X_1 \times \ldots \times X_n}\big) \text{ such that } \mathcal{F} \subseteq \mathfrak{f}.$$

The lemma will be deduced from this property: since the argument works for all $f \in \mathrm{RC}$, the property shows that $\widetilde{\mathrm{ext}}$ is surjective; and that $\widetilde{\mathrm{ext}}$ is non-injective will be shown once two distinct such ultrafilters $\mathfrak{f}', \mathfrak{f}''$ will be constructed.

Let us verify the following equality:

$$\bigcap_{A \in \mathcal{F}} \mathrm{cl}_{\mathrm{RC}}\big\{\widetilde{g} : g \in A\big\} = \{f\}.$$

First note that by Lemma 2.5, for every $A = \{g \in Y^{X_1 \times \ldots \times X_n} : g(\boldsymbol{a}) \in B\}$ in $\mathcal{F}$ we have

$$\mathrm{cl}_{\mathrm{RC}}\{\widetilde{g} : g \in A\} = \mathrm{cl}_{\mathrm{RC}}\{h \in \mathrm{RC} : h(\boldsymbol{a}) \in B\}.$$

Therefore,

$$f \in \bigcap_{A \in \mathcal{F}} \mathrm{cl}_{\mathrm{RC}}\{\widetilde{g} : g \in A\}.$$

Next, toward a contradiction, assume that there exists $f' \in \mathrm{RC}$ such that $f' \neq f$ and $f' \in \bigcap_{A \in \mathcal{F}} \mathrm{cl}_{\mathrm{RC}}\{\widetilde{g} : g \in A\}$. By Lemma 2.3, there exists $\boldsymbol{b} \in X_1 \times \ldots \times X_n$ such that $f(\boldsymbol{b}) \neq f'(\boldsymbol{b})$. As Y is Hausdorff, pick disjoint open neighborhoods U and U' of the points $f(\boldsymbol{b})$ and $f'(\boldsymbol{b})$, respectively. We have:

$$\begin{aligned}
\bigcap_{A \in \mathcal{F}} \mathrm{cl}_{\mathrm{RC}}\{\widetilde{g} : g \in A\} &= \bigcap_{(\boldsymbol{a},B) \in Z} \mathrm{cl}_{\mathrm{RC}}\{h \in \mathrm{RC} : h(\boldsymbol{a}) \in B\} \\
&\subseteq \mathrm{cl}_{\mathrm{RC}}\big\{h \in \mathrm{RC} : h(\boldsymbol{b}) \in U\big\} \\
&\subseteq \mathrm{cl}_{\mathrm{RC}}\big\{h \in \mathrm{RC} : h(\boldsymbol{b}) \in Y \backslash U'\big\} = \big\{h \in \mathrm{RC} : h(\boldsymbol{b}) \in Y \backslash U'\big\},
\end{aligned}$$

(where the last equality holds since the set $\{h \in \mathrm{RC} : h(\boldsymbol{b}) \in Y \backslash U'\}$ is the complement in RC of the subbasic open set $\{h \in \mathrm{RC} : h(\boldsymbol{b}) \in U'\} = O_{\boldsymbol{b},U'}$). Therefore,

$$f' \notin \bigcap_{A \in \mathcal{F}} \mathrm{cl}_{\mathrm{RC}}\{\widetilde{g} : g \in A\},$$

a contradiction. Thus we have verified that the equality is true.

Now the required key property of the family $\mathcal{F}$, i.e. that we have $\widetilde{\mathrm{ext}}(\mathfrak{f}) = f$ whenever $\mathfrak{f} \supseteq \mathcal{F}$, is clearly follows from this equality. As observed above, this property immediately implies that $\widetilde{\mathrm{ext}}$ is surjective; and to show that $\widetilde{\mathrm{ext}}$ is also non-injective, it remains to construct two distinct ultrafilters $\mathfrak{f}', \mathfrak{f}'' \supseteq \mathcal{F}$.

Pick a family $\{B_{\boldsymbol{a}} : \boldsymbol{a} \in X_1 \times \ldots \times X_n\}$ of subsets of Y such that $B_{\boldsymbol{a}} \neq Y$ and $f(\boldsymbol{a}) \in B_{\boldsymbol{a}}$ for all $\boldsymbol{a} \in X_1 \times \ldots \times X_n$. The families

$$\begin{aligned}
\mathcal{F}' &= \mathcal{F} \cup \big\{\big\{g \in Y^{X_1 \times \ldots \times X_n} : (\forall \boldsymbol{a} \in X_1 \times \ldots \times X_n)\ g(\boldsymbol{a}) \in B_{\boldsymbol{a}}\big\}\big\}, \\
\mathcal{F}'' &= \mathcal{F} \cup \big\{\big\{g \in Y^{X_1 \times \ldots \times X_n} : (\exists \boldsymbol{a} \in X_1 \times \ldots \times X_n)\ g(\boldsymbol{a}) \notin B_{\boldsymbol{a}}\big\}\big\}
\end{aligned}$$

are both centered (the fact that $\mathcal{F}''$ is centered uses that one of the X_i is infinite). We extend them to two (automatically distinct) ultrafilters $\mathfrak{f}'$ and $\mathfrak{f}''$, respectively. By the key property of $\mathcal{F}$, we obtain

$$\widetilde{\mathrm{ext}}(\mathfrak{f}') = \widetilde{\mathrm{ext}}(\mathfrak{f}'') = f,$$

thus showing that $\widetilde{\mathrm{ext}}$ is not injective. Note also that, since $f \in \mathrm{RC}$ was choosen arbitrary, we have established a bit more: the preimage of *each* point in RC under the map $\widetilde{\mathrm{ext}}$ consists of more than one point.

The lemma is proved. □

Lemma 2.11. *Let $X_1, \ldots, X_n$ be discrete spaces, Y a compact Hausdorff space, $S \subseteq Y$, and $R \subseteq S^{X_1 \times \ldots \times X_n}$. Then $\widetilde{\mathrm{ext}}$ maps the closure of R in the space $\boldsymbol{\beta}(S^{X_1 \times \ldots \times X_n})$ onto the closure of R in the space $RC_{X_1,\ldots,X_{n-1}}(\boldsymbol{\beta}X_1, \ldots, \boldsymbol{\beta}X_n, Y)$ endowed with the $(X_1, \ldots, X_n)$-pointwise convergence topology:*

$$\widetilde{\mathrm{ext}}``\,\mathrm{cl}_{\boldsymbol{\beta}(S^{X_1 \times \ldots \times X_n})} R = \mathrm{cl}_{RC_{X_1,\ldots,X_{n-1}}(\boldsymbol{\beta}X_1,\ldots,\boldsymbol{\beta}X_n,Y)}\, \mathrm{ext}``\,R.$$

Proof. Again, to simplify notation, we temporarily let:

$$Z = \boldsymbol{\beta}\big(S^{X_1 \times \ldots \times X_n}\big),$$
$$\mathrm{RC} = RC_{X_1,\ldots,X_{n-1}}(\boldsymbol{\beta}X_1, \ldots, \boldsymbol{\beta}X_n, Y).$$

We consider Z with the standard topology on the space of ultrafilters, so it actually does not depend on the topology on S as a subspace of Y. The fact that Y is compact Hausdorff is used only to get the same properties of the topology on RC, which are essential to extend ext to $\widetilde{\mathrm{ext}}$.

To prove the inclusion

$$\widetilde{\mathrm{ext}}``\,\mathrm{cl}_Z\, R \subseteq \mathrm{cl}_{\mathrm{RC}}\, \mathrm{ext}``\,R,$$

recall that by the general definition of continuous extensions of unary maps, for any $\mathfrak{f} \in Z$ we have $\{\widetilde{\mathrm{ext}}(\mathfrak{f})\} = \bigcap_{A \in \mathfrak{f}} \mathrm{cl}_{\mathrm{RC}}\, \mathrm{ext}``\,A$. Therefore,

$$\widetilde{\mathrm{ext}}``\,\mathrm{cl}_Z R = \big\{\widetilde{\mathrm{ext}}(\mathfrak{f}) : \mathfrak{f} \in \mathrm{cl}_Z R\big\} = \bigcup_{\mathfrak{f} \in \mathrm{cl}_Z R}\ \bigcap_{A \in \mathfrak{f}} \mathrm{cl}_{\mathrm{RC}}\, \mathrm{ext}``\,A.$$

But for any $\mathfrak{f} \in \mathrm{cl}_Z R$ we have $R \in \mathfrak{f}$ and hence $\bigcap_{A \in \mathfrak{f}} \mathrm{cl}_{\mathrm{RC}}\, \mathrm{ext}``\,A \subseteq \mathrm{cl}_{\mathrm{RC}}\, \mathrm{ext}``\,R$, whence it follows

$$\bigcup_{\mathfrak{f} \in \mathrm{cl}_Z R}\ \bigcap_{A \in \mathfrak{f}} \mathrm{cl}_{\mathrm{RC}}\, \mathrm{ext}``\,A \subseteq \mathrm{cl}_{\mathrm{RC}}\, \mathrm{ext}``\,R,$$

which gives the required inclusion.

To prove the converse inclusion

$$\mathrm{cl}_{\mathrm{RC}}\, \mathrm{ext}``\,R \subseteq \widetilde{\mathrm{ext}}``\,\mathrm{cl}_Z R,$$

note that $\mathrm{cl}_{\mathrm{RC}}\, \widetilde{\mathrm{ext}}``\,R \subseteq \widetilde{\mathrm{ext}}``\,\mathrm{cl}_Z R$ since the map $\widetilde{\mathrm{ext}} : Z \to RC$ is closed as a continuous map of a compact space into a Hausdorff space (see e.g. [12], Corollary 3.1.11), and that $\widetilde{\mathrm{ext}}``\,R = \mathrm{ext}``\,R$ since R consists of principal ultrafilters over the set $S^{X_1 \times \ldots \times X_n}$ (under our customary identification of elements with principal ultrafilters given by them).

The lemma is proved. ☐

Now we are ready to give the promised alternative description of the *-extension of relations. For simplicity, we formulate it only for the case when $X_1 = \ldots = X_n = X$; nevertheless, this formulation does not lose generality since for given X_i we can take their union as such an X.

Theorem 2.12. *Let $R \subseteq X \times \ldots \times X$ be any n-ary relation on a set X. Then its extension $R^* \subseteq \boldsymbol{\beta}X \times \ldots \times \boldsymbol{\beta}X$ is (identified with) the image under $\widetilde{\mathrm{ext}}$ of the basic set $\widetilde{R}$ in the space $\boldsymbol{\beta}(X^n)$ where R is considered as a unary relation on X^n:*

$$R^* = \big\{\widetilde{\mathrm{ext}}(\mathfrak{r}) : R \in \mathfrak{r}\big\} = \widetilde{\mathrm{ext}}``\,\widetilde{R}.$$

Proof. By Theorem 1.12, $R^* = \mathrm{cl}_{\boldsymbol{\beta}X\times\ldots\times\boldsymbol{\beta}X} R$. As usual, the product space $\boldsymbol{\beta}X \times \ldots \times \boldsymbol{\beta}X$ (n times) is identified with $(\boldsymbol{\beta}X)^n$, so up to this identification we can let

$$R^* = \mathrm{cl}_{(\boldsymbol{\beta}X)^n} R.$$

We are going to use Lemma 2.11 by choosing appropriate discrete $X_1, \ldots, X_m$, a compact Hausdorff Y, and $S \subseteq Y$. Let $m = 1$, let the space X_1 be n with the discrete topology, so $\boldsymbol{\beta}X_1 = X_1 = n$, let the space Y be $\boldsymbol{\beta}X$ with the standard topology on the space of ultrafilters, and let S be X, so we have:

$$\boldsymbol{\beta}\big(S^{X_1}\big) = \boldsymbol{\beta}(X^n) \ \text{ and } \ C(X_1, Y) = (\boldsymbol{\beta}X)^n$$

(clearly, the X_1-pointwise convergence topology on the latter set is the same that the full pointwise convergence topology). Now Lemma 2.11 gives us

$$\widetilde{\mathrm{ext}}\,``\,\mathrm{cl}_{\boldsymbol{\beta}(X^n)} R = \mathrm{cl}_{(\boldsymbol{\beta}X)^n}\, \mathrm{ext}\,``\,R.$$

But $\mathrm{cl}_{\boldsymbol{\beta}(X^n)} R = \widetilde{R}$ where R is considered as a unary relation on X^n (recall that if Z is discrete and $A \subseteq Z$, then the basic open set $\widetilde{A}$ equals the closure $\mathrm{cl}_Z A$), and furthermore, $\mathrm{ext}\,``\,R = R$ (since $\widetilde{f} = f$ for all $f \in Y^n$ as $\boldsymbol{\beta}\,n = n$). Putting all this together, we obtain:

$$R^* = \mathrm{cl}_{(\boldsymbol{\beta}X)^n} R = \mathrm{cl}_{(\boldsymbol{\beta}X)^n}\, \mathrm{ext}\,``\,R = \widetilde{\mathrm{ext}}\,``\,\mathrm{cl}_{\boldsymbol{\beta}(X^n)} R = \widetilde{\mathrm{ext}}\,``\,\widetilde{R},$$

as required. □

Although this description of the *-extension of relations is not simpler than one given by Theorem 1.12, it provides some connection of this larger extension with the smaller $\widetilde{\ }$-extension of relations (by using also continuous extensions of maps). Other interrelations between the $\widetilde{\ }$- and *-extensions of relations are established via Vietoris-type topologies in [31].

3 Ultrafilter interpretations

In this section, we define our main concepts: ultrafilter interpretations (of functional and relational symbols) and ultrafilter models (involving ultrafilter evaluations and ultrafilter interpretations) together with their semantics. Then we provide two specific operations turning ultrafilter models into ordinary ones, establish necessary and sufficient conditions under which the latter are two canonical ultrafilter extensions of some ordinary models, and give a topological characterization of ultrafilter models. Finally, we define homomorphisms of ultrafilter models and prove for them a version of the First Extension Theorem and an its refinement.

Ultrafilter models. Using ultrafilters over maps in our previous considerations leads us to the following concept.

Definition 3.1. Given a signature τ, we define an *ultrafilter interpretation* as a map $\imath$ that takes each n-ary functional symbol F in τ to an ultrafilter over the set of n-ary operations on X, and each n-ary predicate symbol R in τ to an ultrafilter over the set of n-ary relations on X; let also v be an *ultrafilter valuation* of variables, i.e. a valuation which takes each variable x to an ultrafilter over a given set X:

$$v(x) \in \boldsymbol{\beta}X, \quad \imath(F) \in \boldsymbol{\beta}\big(X^{X\times\ldots\times X}\big), \quad \imath(R) \in \boldsymbol{\beta}\,\mathcal{P}(X \times \ldots \times X).$$

We refer to the structure

$$(\boldsymbol{\beta} X, \imath(F), \ldots, \imath(R), \ldots)$$

as an *ultrafilter model* of τ.[4]

Now we are going to define an appropriate satisfiability relation between ultrafilter models and first-order formulas, which we shall denote by the symbol $\Vdash$.

First, given an interpretation $\imath$ of non-logical symbols, we expand any valuation v of variables to the map $v_\imath$ defined on all terms as follows. Let $\mathrm{app} : X_1 \times \ldots \times X_n \times Y^{X_1 \times \ldots \times X_n} \to Y$ be the *application* operation:

$$\mathrm{app}(a_1, \ldots, a_n, f) = f(a_1, \ldots, a_n).$$

Extend it to the map $\widetilde{\mathrm{app}} : \boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n \times \boldsymbol{\beta}(Y^{X_1 \times \ldots \times X_n}) \to \boldsymbol{\beta} Y$ right continuous w.r.t. the principal ultrafilters, in the usual way:

$$\begin{tikzcd}
\boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n \times \boldsymbol{\beta}(Y^{X_1 \times \ldots \times X_n}) \arrow[r, dashed, "\widetilde{\mathrm{app}}"] & \boldsymbol{\beta} Y \\
X_1 \times \ldots \times X_n \times Y^{X_1 \times \ldots \times X_n} \arrow[u] \arrow[r, "\mathrm{app}"] & Y \arrow[u]
\end{tikzcd}$$

Let $v_\imath$ coincide with v on variables, and if $v_\imath$ has been already defined on terms $t_1, \ldots, t_n$, we let

$$v_\imath(F(t_1, \ldots, t_n)) = \widetilde{\mathrm{app}}(v_\imath(t_1), \ldots, v_\imath(t_n), \imath(F)).$$

Remark 3.2. We can consider, more generally, for any compact Hausdorff space Y the extension $\widetilde{\mathrm{app}} : \boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n \times \boldsymbol{\beta}(Y^{X_1 \times \ldots \times X_n}) \to Y$ right continuous w.r.t. the principal ultrafilters:

$$\begin{tikzcd}
\boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n \times \boldsymbol{\beta}(Y^{X_1 \times \ldots \times X_n}) \arrow[dr, dashed, "\widetilde{\mathrm{app}}"] & \\
X_1 \times \ldots \times X_n \times Y^{X_1 \times \ldots \times X_n} \arrow[u] \arrow[r, "\mathrm{app}"] & Y
\end{tikzcd}$$

though this is redundant for our immediate purposes.

Further, given an ultrafilter model $\mathfrak{A} = (\boldsymbol{\beta} X, \imath(F), \ldots, \imath(R), \ldots)$, define the satisfiability in $\mathfrak{A}$ as follows. Let $\mathrm{in} \subseteq X_1 \times \ldots \times X_n \times \mathcal{P}(X_1 \times \ldots \times X_n)$ be the *membership* predicate:

$$\mathrm{in}\,(a_1, \ldots, a_n, R) \ \text{ iff } \ (a_1, \ldots, a_n) \in R.$$

Extend it to the relation $\widetilde{\mathrm{in}} \subseteq \boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n \times \boldsymbol{\beta}\mathcal{P}(X_1 \times \ldots \times X_n)$ right clopen w.r.t. principal ultrafilters.

Definition 3.3. The *satisfiability* of a formula φ in $\mathfrak{A}$ is defined by induction on the construction of φ: If $t_1 = t_2$ is an identity, we let

$$\mathfrak{A} \Vdash t_1 = t_2 \ [v] \ \text{ iff } \ v_\imath(t_1) = v_\imath(t_2).$$

If $R(t_1, \ldots, t_n)$ is an atomic formula in which R is not the equality predicate, we let

$$\mathfrak{A} \Vdash R(t_1, \ldots, t_n) \ [v] \ \text{ iff } \ \widetilde{\mathrm{in}}\,(v_\imath(t_1), \ldots, v_\imath(t_n), \imath(P)).$$

[4]Ultrafilter models were introduced in [25] under the name of generalized models. In Section 4, we shall introduce a wider concept of ultrafilter models (Definition 4.1); the ultrafilter models defined here will be referred to as those "in the narrow sense".

(Equivalently, we could define the satisfiability of atomic formulas by identifying predicates with their characteristic functions and using the satisfiability of equalities of the resulting terms.) Finally, if $\varphi(t_1, \ldots, t_n)$ is obtained by negation, conjunction, or quantification from formulas for which $\Vdash$ has been already defined, we define $\mathfrak{A} \Vdash \varphi\,[v]$ in the standard way.

When needed, we shall use variants of notation commonly used for ordinary models and satisfiability, for our generalized variants. E.g. for an ultrafilter model $\mathfrak{A}$ with the universe $\boldsymbol{\beta} X$, a formula $\varphi(x_1, \ldots, x_n)$, and elements $\mathfrak{u}_1, \ldots, \mathfrak{u}_n$ of $\boldsymbol{\beta} X$, the notation $\mathfrak{A} \Vdash \varphi\,[\mathfrak{u}_1, \ldots, \mathfrak{u}_n]$ means that φ is satisfied in $\mathfrak{A}$ under a valuation taking the variables $x_1, \ldots, x_n$ to the ultrafilters $\mathfrak{u}_1, \ldots, \mathfrak{u}_n$, respectively.

Ultrafilter models actually generalize not all ordinary models but those that are ultrafilter extensions of some models. It is worth also pointing out that whenever an ultrafilter interpretation is *principal*, i.e. all non-logical symbols are interpreted by principal ultrafilters, we naturally identify it with the obvious ordinary interpretation with the same universe $\boldsymbol{\beta} X$; however, not every ordinary interpretation with the universe $\boldsymbol{\beta} X$ is of this form. Precise relationships between ultrafilter models, ordinary models, and ultrafilter extensions will be described in Theorems 3.22 and 3.23. Let us also note in advance that ultrafilter models in the wide sense, which we shall define in Section 4, will cover (up to some natural identification) not ultrafilter extensions only but all ordinary models.

An ultrafilter valuation v is *principal* iff it takes any variable to a principal ultrafilter.

Lemma 3.4. *Let $\mathfrak{A} = (\boldsymbol{\beta} X, \imath(F), \ldots, \imath(R), \ldots)$ and $\mathfrak{B} = (\boldsymbol{\beta} X, \jmath(F), \ldots, \jmath(R), \ldots)$ be two ultrafilter models of the same signature and having the same universe $\boldsymbol{\beta} X$. If for all functional symbols F, predicate symbols R, variables $x_1, \ldots, x_n$, and principal valuations v,*

$$\begin{gathered}\widetilde{\mathrm{app}}(v(x_1), \ldots, v(x_n), \imath(F)) = \widetilde{\mathrm{app}}(v(x_1), \ldots, v(x_n), \jmath(F)),\\ \widetilde{\mathrm{in}}\,(v(x_1), \ldots, v(x_n), \imath(R)) \;\text{ iff }\; \widetilde{\mathrm{in}}\,(v(x_1), \ldots, v(x_n), \jmath(R)),\end{gathered}$$

then for all formulas φ, terms $t_1, \ldots, t_n$, and valuations v,

$$\mathfrak{A} \Vdash \varphi(t_1, \ldots, t_n)\,[v] \;\text{ iff }\; \mathfrak{B} \Vdash \varphi(t_1, \ldots, t_n)\,[v].$$

Proof. By induction on construction of formulas using the right continuity of $\widetilde{\mathrm{app}}$ and the right clopenness of $\widetilde{\mathrm{in}}$ w.r.t. X. ☐

Corollary 3.5. *Given an ultrafilter model $\mathfrak{A} = (\boldsymbol{\beta} X, \imath(F), \ldots, \imath(R), \ldots)$, define an ultrafilter model $\mathfrak{B} = (\boldsymbol{\beta} X, \jmath(F), \ldots, \jmath(R), \ldots)$ of the same signature as follows: let $\mathfrak{B}$ have the same universe $\boldsymbol{\beta} X$, let $\jmath$ coincide with $\imath$ on functional symbols, and for each predicate symbol R let $\jmath(R)$ be the principal ultrafilter given by the relation*

$$\big\{(a_1, \ldots, a_n) \in X^n : \widetilde{\mathrm{in}}\,(a_1, \ldots, a_n, \imath(R))\big\}.$$

Then for all valuations v, formulas φ, and terms $t_1, \ldots, t_n$,

$$\mathfrak{A} \Vdash \varphi(t_1, \ldots, t_n)\,[v] \;\text{ iff }\; \mathfrak{B} \Vdash \varphi(t_1, \ldots, t_n)\,[v].$$

Proof. Lemma 3.4. ☐

Definition 3.6. If $X_1, \ldots, X_n, Y$ are discrete spaces, let us say that an ultrafilter $\mathfrak{f}$ over the set $Y^{X_1 \times \ldots \times X_n}$ of n-ary maps is *pseudo-principal* iff $\widetilde{\mathrm{app}}$ takes any n-tuple consisting of principal ultrafilters together with $\mathfrak{f}$ to a principal ultrafilter:

$$a_1 \in X_1, \ldots, a_n \in X_n \;\text{ implies }\; \widetilde{\mathrm{app}}(a_1, \ldots, a_n, \mathfrak{f}) \in Y.$$

Clearly, if the space Y is finite, then all ultrafilters in $\boldsymbol{\beta}(Y^{X_1\times\ldots\times X_n})$ are pseudo-principal. (More generally, if we would defined $\widetilde{\mathrm{app}}$ with the range in any compact Hausdorff Y, as proposed in the remark above, then all ultrafilters in $\boldsymbol{\beta}(Y^{X_1\times\ldots\times X_n})$ were pseudo-principal.)

Lemma 3.7. *Let $X_1,\ldots,X_n,Y$ be discrete spaces. In $\boldsymbol{\beta}(Y^{X_1\times\ldots\times X_n})$, every principal ultrafilter is pseudo-principal, and if Y and at least one of the X_i are infinite, then there exist pseudo-principal ultrafilters that are not principal as well as ultrafilters that are not pseudo-principal.*

Proof. Pick any $f\in Y^{X_1\times\ldots\times X_n}$. Let $\mathcal{F}$ be the following family of subsets of the space $Y^{X_1\times\ldots\times X_n}$:

$$\mathcal{F}=\big\{\{g\in Y^{X_1\times\ldots\times X_n}:g(\boldsymbol{a})=f(\boldsymbol{a})\}:\boldsymbol{a}\in X_1\times\ldots\times X_n\big\}\cup\big\{\{g\in Y^{X_1\times\ldots\times X_n}:g\neq f\}\big\}.$$

The family $\mathcal{F}$ is centered (as at least one of the X_i is infinite), so pick any ultrafilter $\mathfrak{f}$ over the set $Y^{X_1\times\ldots\times X_n}$ such that $\mathcal{F}\subseteq\mathfrak{f}$. Since $\bigcap\mathcal{F}$ is empty, the ultrafilter $\mathfrak{f}$ is non-principal. On the other hand, for every $\boldsymbol{a}=(a_1,\ldots,a_n)\in X_1\times\ldots\times X_n$ we have:

$$\begin{aligned}S\in\widetilde{\mathrm{app}}(a_1,\ldots,a_n,\mathfrak{f})\ &\text{ iff }\ (\forall^{a_1}x_1)\ldots(\forall^{a_n}x_n)(\forall^{\mathfrak{f}}g)\ \mathrm{app}(x_1,\ldots,x_n,g)\in S\\ &\text{ iff }\ (\forall^{a_1}x_1)\ldots(\forall^{a_n}x_n)(\forall^{\mathfrak{f}}g)\ g(x_1,\ldots,x_n)\in S\\ &\text{ iff }\ (\forall^{\mathfrak{f}}g)\ g(a_1,\ldots,a_n)\in S\\ &\text{ iff }\ (\exists F\in\mathfrak{f})(\forall g\in F)\ g(a_1,\ldots,a_n)\in S.\end{aligned}$$

(The first equivalence follows from the definition of extensions of maps via ultrafilter quantifiers, the second holds by the definition of app, the third since $a_1,\ldots,a_n$ are principal, and the fourth decodes the definition of the $\forall^{\mathfrak{f}}$ quantifier.) Letting $S=\{f(a_1,\ldots,a_n)\}$, we have $\widetilde{\mathrm{app}}(a_1,\ldots,a_n,\mathfrak{f})=f(a_1,\ldots,a_n)\in Y$, thus witnessing that $\mathfrak{f}$ is pseudo-principal.

To construct a non-pseudo-principal ultrafilter, pick any $\boldsymbol{a}=(a_1,\ldots,a_n)\in X_1\times\ldots\times X_n$ and $\mathfrak{u}\in\boldsymbol{\beta}Y\setminus Y$ (as Y is infinite), and expand the centered family

$$\begin{aligned}\mathcal{G}&=\big\{\{f\in Y^{X_1\times\ldots\times X_n}:f(\boldsymbol{a})\in S\}:S\in\mathfrak{u}\big\}\\ &=\big\{O_{\boldsymbol{a},\widetilde{S}}:S\in\mathfrak{u}\big\}\end{aligned}$$

to an ultrafilter $\mathfrak{g}\supseteq\mathcal{G}$ over $Y^{X_1\times\ldots\times X_n}$. Calculations similar to those in the above give us

$$\widetilde{\mathrm{app}}(a_1,\ldots,a_n,\mathfrak{g})=\mathfrak{u},$$

thus witnessing that $\mathfrak{g}$ is not pseudo-principal. □

Definition 3.8. An ultrafilter interpretation $\imath$ is *pseudo-principal on functional symbols* iff $\imath(F)$ is a pseudo-principal ultrafilter for each functional symbol F (and then, for each term t).

Corollary 3.9. *Given an ultrafilter model $\mathfrak{A}=(\boldsymbol{\beta}X,\imath(F),\ldots,\imath(R),\ldots)$ with $\imath$ pseudo-principal on functional symbols, define an ultrafilter model $\mathfrak{B}=(\boldsymbol{\beta}X,\jmath(F),\ldots,\jmath(R),\ldots)$ of the same signature as follows: let $\mathfrak{B}$ have the same universe $\boldsymbol{\beta}X$, let $\jmath$ coincide with $\imath$ on predicate symbols, and for each functional symbol F let $\jmath(F)$ be the principal ultrafilter given by the operation $f:X^n\to X$ defined by letting*

$$f(a_1,\ldots,a_n)=\widetilde{\mathrm{app}}(a_1,\ldots,a_n,\imath(F)).$$

Then for all valuations v, formulas φ, and terms $t_1,\ldots,t_n$,

$$\mathfrak{A}\Vdash\varphi(t_1,\ldots,t_n)\,[v]\ \ \textit{iff}\ \ \mathfrak{B}\Vdash\varphi(t_1,\ldots,t_n)\,[v].$$

Proof. Lemma 3.4. □

It follows that for every ultrafilter model $\mathfrak{A}$ whose interpretation is pseudo-principal on functional symbols, by replacing its relations as in Corollary 3.5 and its operations as in Corollary 3.9, one obtains an ordinary model $\mathfrak{B}$ with the same universe such that for all formulas φ and elements $\mathfrak{u}_1, \ldots, \mathfrak{u}_n$ of the universe, $\mathfrak{A} \Vdash \varphi\,[\mathfrak{u}_1, \ldots, \mathfrak{u}_n]$ iff $\mathfrak{B} \vDash \varphi\,[\mathfrak{u}_1, \ldots, \mathfrak{u}_n]$.

We do not formulate this fact as a separate theorem since we shall be able to establish stronger facts soon. In Theorem 3.16, we shall establish that for any ultrafilter model $\mathfrak{A}$, not necessarily with a pseudo-principal interpretation, one can construct a certain ordinary model $e(\mathfrak{A})$ satisfying the same formulas; and then, in Theorem 3.22, that whenever $\mathfrak{A}$ has a pseudo-principal interpretation, $e(\mathfrak{A})$ is nothing but the $\widetilde{\ }$-extension of some model. In fact, in the latter case, $e(\mathfrak{A})$ coincides with $\mathfrak{B}$ from the previous paragraph.

***Operations** e **and** E.* Let us now define two operations, e and E, which turn ultrafilter models into certain ordinary models that (as we shall see soon) generalize the *- and $\widetilde{\ }$-extensions. Both operations take ultrafilters over n-ary maps to n-ary maps over ultrafilters, and ultrafilters over n-ary relations to n-ary relations over ultrafilters. Both operations are surjective and non-injective (Lemma 3.21).

The map e on ultrafilters over maps will be the map $\widetilde{\mathrm{ext}}$ defined and discussed in Section 2. Now we extend $\widetilde{\mathrm{ext}}$ to ultrafilters over relations by identifying n-ary relations with their n-ary characteristic functions into the discrete space $2 = \{0, 1\}$:

$$\begin{tikzcd}
\boldsymbol{\beta}\mathcal{P}(X_1 \times \ldots \times X_n) \arrow[dr, dashed, "\widetilde{\mathrm{ext}}"] & \\
\mathcal{P}(X_1 \times \ldots \times X_n) \arrow[u] \arrow[r, "\mathrm{ext}"] & \{Q \subseteq \boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n : Q \text{ is right clopen w.r.t. } X_1, \ldots, X_{n-1}\}
\end{tikzcd}$$

(Recall that by Theorem 1.14(i), a subset of $\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n$ is right clopen w.r.t. $X_1, \ldots, X_{n-1}$ iff it is of form $\widetilde{R}$ for some n-ary subset R of $X_1 \times \ldots \times X_n$.) Let the map e on ultrafilters over relations also coincide with the map $\widetilde{\mathrm{ext}}$ on them. So in result we have:

$$e``\boldsymbol{\beta}\big(Y^{X_1 \times \ldots \times X_n}\big) \subseteq \boldsymbol{\beta}Y^{\,\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n},$$
$$e``\boldsymbol{\beta}\,\mathcal{P}(X_1 \times \ldots \times X_n) \subseteq \mathcal{P}(\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n).$$

We observe that e and $\widetilde{\mathrm{app}}$ (or $\widetilde{\mathrm{in}}$) are expressed via each other:

Lemma 3.10. *Let $X_1, \ldots, X_n, Y$ be discrete spaces. For all $\mathfrak{f} \in \boldsymbol{\beta}(Y^{X_1 \times \ldots \times X_n})$, $\mathfrak{r} \in \boldsymbol{\beta}\,\mathcal{P}(X_1 \times \ldots \times X_n)$, and $\mathfrak{u}_1 \in \boldsymbol{\beta}X_1, \ldots, \mathfrak{u}_n \in \boldsymbol{\beta}X_n$,*

$$e(\mathfrak{f})(\mathfrak{u}_1, \ldots, \mathfrak{u}_n) = \widetilde{\mathrm{app}}(\mathfrak{u}_1, \ldots, \mathfrak{u}_n, \mathfrak{f}),$$
$$e(\mathfrak{r})(\mathfrak{u}_1, \ldots, \mathfrak{u}_n) \ \textit{ iff } \ \widetilde{\mathrm{in}}\,(\mathfrak{u}_1, \ldots, \mathfrak{u}_n, \mathfrak{r}).$$

In other words,

$$e(\mathfrak{f}) = \big\{(\mathfrak{u}_1, \ldots, \mathfrak{u}_n, \mathfrak{v}) \in \boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n \times \boldsymbol{\beta}Y : \widetilde{\mathrm{app}}(\mathfrak{u}_1, \ldots, \mathfrak{u}_n, \mathfrak{f}) = \mathfrak{v}\big\},$$
$$e(\mathfrak{r}) = \big\{(\mathfrak{u}_1, \ldots, \mathfrak{u}_n) \in \boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n : \widetilde{\mathrm{in}}\,(\mathfrak{u}_1, \ldots, \mathfrak{u}_n, \mathfrak{r})\big\}.$$

Proof. To simplify the notation, let RC be the space $RC_{X_1,\ldots,X_{n-1}}(\boldsymbol{\beta}X_1, \ldots, \boldsymbol{\beta}X_n, \boldsymbol{\beta}Y)$ of n-ary maps on $\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n$ into $\boldsymbol{\beta}Y$ that are right continuous w.r.t. $X_1, \ldots, X_{n-1}$, endowed with the

$(X_1, \ldots, X_n)$-pointwise convergence topology. By Lemma 2.4, RC is compact Hausdorff. Recall that for any $\mathfrak{f} \in \boldsymbol{\beta}(Y^{X_1 \times \ldots \times X_n})$ we have

$$e(\mathfrak{f}) = \widetilde{\mathrm{ext}}(\mathfrak{f}) = g \in \mathrm{RC} \;\text{ such that }\; \{g\} = \bigcap_{A \in \mathfrak{f}} \mathrm{cl}_{\mathrm{RC}} \{\widetilde{f} : f \in A\},$$

and that $\widetilde{\mathrm{app}}^{\mathfrak{f}}$ is the n-ary map on $\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n$ into $\boldsymbol{\beta}Y$ defining by letting $\widetilde{\mathrm{app}}^{\mathfrak{f}}(\mathfrak{u}_1, \ldots, \mathfrak{u}_n) = \widetilde{\mathrm{app}}(\mathfrak{u}_1, \ldots, \mathfrak{u}_n, \mathfrak{f})$ for all $\mathfrak{u}_1 \in \boldsymbol{\beta}X_1, \ldots, \mathfrak{u}_n \in \boldsymbol{\beta}X_n$.

Note that both maps $\widetilde{\mathrm{app}}^{\mathfrak{f}}$ and $e(\mathfrak{f})$ are in RC (the first follows from the fact that $\widetilde{\mathrm{app}}$ is right continuous w.r.t. $X_1, \ldots, X_{n-1}$, the second holds since $\widetilde{\mathrm{ext}}$ is a map into RC). Therefore, by Lemma 2.3, in order to show that they coincide, it suffices to verify that they coincide on principal ultrafilters.

For this, pick any $a_1 \in X_1, \ldots, a_n \in X_n$ and $S \subseteq Y$. We have:

$$\begin{aligned} S \in \widetilde{\mathrm{app}}^{\mathfrak{f}}(a_1, \ldots, a_n) \;&\text{ iff }\; (\forall^{a_1} x_1) \ldots (\forall^{a_n} x_n)(\forall^{\mathfrak{f}} f)\, \mathrm{app}(x_1, \ldots, x_n, f) \in S \\ &\text{ iff }\; (\forall^{\mathfrak{f}} f)\, \mathrm{app}(a_1, \ldots, a_n, f) \in S \\ &\text{ iff }\; (\forall^{\mathfrak{f}} f)\; f(a_1, \ldots, a_n) \in S \\ &\text{ iff }\; \left\{ f \in Y^{X_1 \times \ldots \times X_n} : f(a_1, \ldots, a_n) \in S \right\} \in \mathfrak{f}. \end{aligned}$$

(The first equivalence follows from the definition of extensions of maps via ultrafilter quantifiers, the second holds since $a_1, \ldots, a_n$ are principal, the third by the definition of app, and the fourth by the definition of the $\forall^{\mathfrak{f}}$ quantifier.) Therefore,

$$e(\mathfrak{f}) \in \mathrm{cl}_{\mathrm{RC}} \left\{ \widetilde{f} \in \mathrm{RC} : f \in Y^{X_1 \times \ldots \times X_n} \text{ and } f(a_1, \ldots, a_n) \in S \right\}.$$

As stated in Lemma 2.4, the space RC is zero-dimensional; in particular, the open set $O_{a_1, \ldots, a_n, \widetilde{S}} = \left\{ g \in \mathrm{RC} : g(a_1, \ldots, a_n) \in \widetilde{S} \right\}$ is closed (since its complement $\mathrm{RC} \setminus O_{a_1, \ldots, a_n, \widetilde{S}} = O_{a_1, \ldots, a_n, \boldsymbol{\beta}Y \setminus \widetilde{S}}$ is open too). It follows that

$$\mathrm{cl}_{\mathrm{RC}} \left\{ \widetilde{f} \in \mathrm{RC} : f \in Y^{X_1 \times \ldots \times X_n} \text{ and } f(a_1, \ldots, a_n) \in S \right\} \subseteq O_{a_1, \ldots, a_n, \widetilde{S}}\,.$$

Therefore, we obtain

$$e(\mathfrak{f}) \in O_{a_1, \ldots, a_n, \widetilde{S}}\,,$$

or, in other words, $e(\mathfrak{f})(a_1, \ldots, a_n) \in \widetilde{S}$. The latter is clearly equivalent to $S \in e(\mathfrak{f})(a_1, \ldots, a_n)$. Thus we get the inclusion $\widetilde{\mathrm{app}}(a_1, \ldots, a_n, \mathfrak{f}) \subseteq e(\mathfrak{f})(a_1, \ldots, a_n)$. But since both $\widetilde{\mathrm{app}}(a_1, \ldots, a_n, \mathfrak{f})$ and $e(\mathfrak{f})(a_1, \ldots, a_n)$ are *ultra*filters, the inclusion actually gives the equality $\widetilde{\mathrm{app}}(a_1, \ldots, a_n, \mathfrak{f}) = e(\mathfrak{f})(a_1, \ldots, a_n)$.

This proves the lemma for ultrafilters over sets of maps. The remaining claim about ultrafilters over sets of relations follows by replacing the relations with their characteristic functions. □

Question 3.11. For which compact Hausdorff space Y, instead of $\boldsymbol{\beta}Y$ with a discrete Y, does Lemma 3.10 remain true (providing that $\widetilde{\mathrm{app}}$ is defined as in Remark 3.2)? Does this hold at least for all zero-dimensional, or all extremally disconnected compact Hausdorff Y? (Problem 5.3.)

Corollary 3.12. *Let $X_1, \ldots, X_n, Y$ be discrete spaces. The set of pseudo-principal ultrafilters is the preimage of the set $\{\widetilde{f} : f \in Y^{X_1 \times \ldots \times X_n}\}$ under the map e:*

$$\left\{ \mathfrak{f} \in \boldsymbol{\beta}\big(Y^{X_1 \times \ldots \times X_n}\big) : \mathfrak{f} \textit{ is pseudo-principal} \right\} = e^{-1} \left\{ \widetilde{f} : f \in Y^{X_1 \times \ldots \times X_n} \right\}.$$

Recalling that $e = \widetilde{\mathrm{ext}}$, that on the set $Y^{X_1\times\ldots\times X_n}$ (identified with principal ultrafilters) $\widetilde{\mathrm{ext}}$ is just ext, and that $e``Y^{X_1\times\ldots\times X_n} = \{\widetilde{f} : f \in Y^{X_1\times\ldots\times X_n}\}$, we can rewrite the set of pseudo-principal ultrafilters also by

$$e^{-1}\, e``\, Y^{X_1\times\ldots\times X_n} = \widetilde{\mathrm{ext}}^{\,-1}\, \mathrm{ext}``\, Y^{X_1\times\ldots\times X_n}.$$

Proof. Show first that if $\mathfrak{f}$ is pseudo-principal, then $e(\mathfrak{f}) = \widetilde{f}$ for some $f \in Y^{X_1\times\ldots\times X_n}$. By the definition of e $(= \widetilde{\mathrm{ext}})$, always $e(\mathfrak{f})$ is a map belonging to the set $RC_{X_1,\ldots,X_{n-1}}(\boldsymbol{\beta}X_1,\ldots,\boldsymbol{\beta}X_n,Y)$. Since by Lemma 3.10 we have $e(\mathfrak{f})(\mathfrak{u}_1,\ldots,\mathfrak{u}_n) = \widetilde{\mathrm{app}}(\mathfrak{u}_1,\ldots,\mathfrak{u}_n,\mathfrak{f})$, we see that the map e takes principal ultrafilters to principal ultrafilters whenever $\mathfrak{f}$ is pseudo-principal. But then it follows from Lemma 2.3 that $e(\mathfrak{f})$ coincides with $\widetilde{f}$ if the map f is the restriction of $e(\mathfrak{f})$ to principal ultrafilters:

$$e(\mathfrak{f}) = \widetilde{f} \ \text{ for } \ f = e(\mathfrak{f}) \restriction (X_1 \times \ldots \times X_n).$$

It remains to show the converse implication, i.e. that for every $\widetilde{f}$ there exists a pseudo-principal ultrafilter $\mathfrak{f}$ with $e(\mathfrak{f}) = \widetilde{f}$. For this, it clearly suffices to let $\mathfrak{f}$ equal to the principal ultrafilter given by f. □

Question 3.13. What are topological properties of the set of pseudo-principal ultrafilters in the space $\boldsymbol{\beta}(Y^{X_1\times\ldots\times X_n})$? topological properties of its preimage under e, the set $\{\widetilde{f} : f \in Y^{X_1\times\ldots\times X_n}\}$, in the space $RC_{X_1,\ldots,X_{n-1}}(\boldsymbol{\beta}X_1,\ldots,\boldsymbol{\beta}X_n,\boldsymbol{\beta}Y)$ with the $(X_1,\ldots,X_n)$-pointwise convergence topology (besides the fact that it is dense there, as stated in Lemma 2.5), or with the (usual) pointwise convergence topology? in the space $(\boldsymbol{\beta}Y)^{\boldsymbol{\beta}X_1\times\ldots\times\boldsymbol{\beta}X_n}$ with the pointwise convergence topology?

Let us point out that objects naturally defined in terms of ultrafilter extensions often have rather hardly definable topological properties, cf. [18, 19]. (Problem 5.4.)

Corollary 3.14. *For all ultrafilter models $\mathfrak{A} = (\boldsymbol{\beta}X, \imath(F),\ldots,\imath(R),\ldots)$ and valuations v,*

$$v_\imath(F(t_1,\ldots,t_n)) = e(\imath(F))(v_\imath(t_1),\ldots,v_\imath(t_n)),$$
$$\mathfrak{A} \Vdash R(t_1,\ldots,t_n)\ [v] \ \text{ iff } \ e(\imath(R))(v_\imath(t_1),\ldots,v_\imath(t_n)).$$

Proof. Lemma 3.10 with $X_1 = \ldots = X_n = Y = X$. □

Definition 3.15. For an ultrafilter model $\mathfrak{B} = (\boldsymbol{\beta}X, \mathfrak{f},\ldots,\mathfrak{r},\ldots)$, let

$$e(\mathfrak{B}) = (\boldsymbol{\beta}X, e(\mathfrak{f}),\ldots,e(\mathfrak{r}),\ldots).$$

Note that $e(\mathfrak{B})$ is an ordinary model.

The following theorem is the first of the three main results of this section, it states that in point of view of the satisfaction of formulas, any ultrafilter model $\mathfrak{A}$ is not distinguished from the ordinary model $e(\mathfrak{A})$.

Theorem 3.16. *If $\mathfrak{A}$ is an ultrafilter model, then for all formulas φ and elements $\mathfrak{u}_1,\ldots,\mathfrak{u}_n$ of the universe of $\mathfrak{A}$,*

$$\mathfrak{A} \Vdash \varphi\,[\mathfrak{u}_1,\ldots,\mathfrak{u}_n] \ \text{ iff } \ e(\mathfrak{A}) \vDash \varphi\,[\mathfrak{u}_1,\ldots,\mathfrak{u}_n].$$

Proof. Induction on φ starting from Corollary 3.14. □

Now we define the map E, which has the same domain that the map e does and also satisfying

$$E``\boldsymbol{\beta}\big(Y^{X_1\times\ldots\times X_n}\big) \subseteq \boldsymbol{\beta}Y^{\,\boldsymbol{\beta}X_1\times\ldots\times\boldsymbol{\beta}X_n},$$
$$E``\boldsymbol{\beta}\,\mathcal{P}(X_1\times\ldots\times X_n) \subseteq \mathcal{P}(\boldsymbol{\beta}X_1\times\ldots\times\boldsymbol{\beta}X_n),$$

as follows: E and e coincide on $\boldsymbol{\beta}(Y^{X_1\times\ldots\times X_n})$, and whenever $\mathfrak{r}\in\boldsymbol{\beta}\mathcal{P}(X_1\times\ldots\times X_n)$ then we define

$$E(\mathfrak{r}) = \widetilde{\mathrm{ext}}\text{“}\,\widetilde{\mathrm{ext}}(\mathfrak{r}) = \big\{\widetilde{\mathrm{ext}}(\mathfrak{q}) : \mathfrak{q}\in\widetilde{\mathrm{ext}}(\mathfrak{r})\big\}$$

where $\mathfrak{r}$ is considered as an ultrafilter over unary relations on $X_1\times\ldots\times X_n$ while $\mathfrak{q}$ is considered as an ultrafilter over unary maps on n (and $\widetilde{\mathrm{ext}}$ has the corresponding meaning). Let us now explain the construction in details.

First, we consider $\mathcal{P}(X_1\times\ldots\times X_n)$ as the set of *unary* relations on $X_1\times\ldots\times X_n$. Then the map ext takes any subset R of $X_1\times\ldots\times X_n$ to the clopen subset $\widetilde{R}$ of $\boldsymbol{\beta}(X_1\times\ldots\times X_n)$. Therefore, the extended map $\widetilde{\mathrm{ext}}$ takes any ultrafilter $\mathfrak{r}$ over $\mathcal{P}(X_1\times\ldots\times X_n)$ to some clopen subset $Q=\widetilde{\mathrm{ext}}(\mathfrak{r})$ of $\boldsymbol{\beta}(X_1\times\ldots\times X_n)$:

$$\begin{tikzcd}
\boldsymbol{\beta}\mathcal{P}(X_1\times\ldots\times X_n) \arrow[dr, dashed, "\widetilde{\mathrm{ext}}"] & \\
\mathcal{P}(X_1\times\ldots\times X_n) \arrow[u] \arrow[r, "\mathrm{ext}"] & \{Q\subseteq\boldsymbol{\beta}(X_1\times\ldots\times X_n) : Q \text{ is clopen}\}
\end{tikzcd}$$

Next, we identify the product $X_1\times\ldots\times X_n$ with the set of unary maps f from the set n into $\bigcup_i X_i$ satisfying $f(i)\in X_{i+1}$ (for all $i<n$). Then the map ext takes any such f to the unary continuous map $\widetilde{f}$ from n into $\bigcup_i\boldsymbol{\beta}X_i$ satisfying $f(i)\in\boldsymbol{\beta}X_{i+1}$, and we identify the set of such maps $\widetilde{f}$ backwards with the product $\boldsymbol{\beta}X_1\times\ldots\times\boldsymbol{\beta}X_n$. Therefore, the extended map $\widetilde{\mathrm{ext}}$ takes any ultrafilter $\mathfrak{q}$ over $X_1\times\ldots\times X_n$ to some n-tuple $(\mathfrak{u}_1,\ldots,\mathfrak{u}_n)=\widetilde{\mathrm{ext}}(\mathfrak{q})$ in $\boldsymbol{\beta}X_1\times\ldots\times\boldsymbol{\beta}X_n$:

$$\begin{tikzcd}
\boldsymbol{\beta}(X_1\times\ldots\times X_n) \arrow[dr, dashed, "\widetilde{\mathrm{ext}}"] & \\
X_1\times\ldots\times X_n \arrow[u] \arrow[r, "\mathrm{ext}"] & \boldsymbol{\beta}X_1\times\ldots\times\boldsymbol{\beta}X_n
\end{tikzcd}$$

(An analogous construction was previously used in Theorem 2.12.) In result, the set $Q=\widetilde{\mathrm{ext}}(\mathfrak{r})\subseteq\boldsymbol{\beta}(X_1\times\ldots\times X_n)$ is mapped onto the set $\widetilde{\mathrm{ext}}\text{“}Q=E(\mathfrak{r})\subseteq\boldsymbol{\beta}X_1\times\ldots\times\boldsymbol{\beta}X_n$. Since Q is clopen and the map $\widetilde{\mathrm{ext}}$ is closed (as a continuous map between compact Hausdorff spaces), the resulting $E(\mathfrak{r})$ is a closed subset of the space $\boldsymbol{\beta}X_1\times\ldots\times\boldsymbol{\beta}X_n$.

Lemma 3.17. *Let* $\mathfrak{r}\in\boldsymbol{\beta}\mathcal{P}(X_1\times\ldots\times X_n)$. *Then*

$$e(\mathfrak{r})=\widetilde{R}\ \ \textit{and}\ \ E(\mathfrak{r})=R^*$$

for $R=e(\mathfrak{r})\cap(X_1\times\ldots\times X_n)=E(\mathfrak{r})\cap(X_1\times\ldots\times X_n)=\bigcap_{S\in\mathfrak{r}}\bigcup S$. *Consequently,*

$$e(\mathfrak{r})\subseteq E(\mathfrak{r}).$$

We can write up this R more explicitly:

$$R=\big\{(a_1,\ldots,a_n)\in X_1\times\ldots\times X_n : (\forall S\in\mathfrak{r})\,(\exists Q\in S)\;Q(a_1,\ldots,a_n)\big\}.$$

Proof. For $e(\mathfrak{r})=\widetilde{R}$, apply Lemma 3.10. For $E(\mathfrak{r})=R^*$, note that $\mathfrak{q}\in\widetilde{\mathrm{ext}}(\mathfrak{r})$ iff $\bigcap_{S\in\mathfrak{r}}\bigcup S\in\mathfrak{q}$. $\square$

Definition 3.18. For an ultrafilter model $\mathfrak{B}=(\boldsymbol{\beta}X,\mathfrak{f},\ldots,\mathfrak{r},\ldots)$, let

$$E(\mathfrak{B})=(\boldsymbol{\beta}X,E(\mathfrak{f}),\ldots,E(\mathfrak{r}),\ldots).$$

Then $E(\mathfrak{B})$, like $e(\mathfrak{B})$, is an ordinary model.

The following easy observation is similar to Theorem 1.6, and moreover, it turns out to be that theorem whenever the interpretation of $\mathfrak{B}$ is pseudo-principal on functional symbols, as we shall see after Theorem 3.22.

Theorem 3.19. *For any ultrafilter model $\mathfrak{B}$ the identity map on its universe is a homomorphism of $e(\mathfrak{B})$ onto $E(\mathfrak{B})$:*

$$\begin{tikzcd} & \mathfrak{B} \arrow[dl, "e"'] \arrow[dr, "E"] & \\ e(\mathfrak{B}) \arrow[rr, "\mathrm{id}"] & & E(\mathfrak{B}) \end{tikzcd}$$

Therefore, all positive formulas satisfied in $e(\mathfrak{B})$ are also satisfied in $E(\mathfrak{B})$.

Proof. Immediate from Lemma 3.17 since $\widetilde{R} \subseteq R^*$ for all relations R. □

Now we are going to establish two remaining main results of this section, Theorems 3.22 and 3.23. The first of them characterizes ultrafilter models such that their e- and E-images are ultraextensions of ordinary models, while the second one characterizes ordinary models that are the e- and E-images of ultrafilter models. Before this we prove two more auxiliary lemmas, which actually follow from the previously stated facts.

Lemma 3.20. *Let $\mathfrak{A}$ be an ultrafilter model with a pseudo-principal interpretation of functional symbols, and $\mathfrak{B}$ the ultrafilter model with a principal interpretation of functional symbols constructed from $\mathfrak{A}$ as in Corollary 3.9. Then $e(\mathfrak{A}) = e(\mathfrak{B})$.*

Proof. Let $\imath$ and $\jmath$ be the interpretations in $\mathfrak{A}$ and $\mathfrak{B}$, respectively. If F is a functional symbol, then the operations $e(\imath(F))$ and $e(\jmath(F))$ are right continuous w.r.t. principal ultrafilters. Therefore, by Lemma 2.3, in order to show that they coincide, it suffices to verify that they coincide on principal ultrafilters.

If the symbol F is n-ary, let $\boldsymbol{a}$ be any n-tuple of principal ultrafilters. We have:

$$\widetilde{\mathrm{app}}(\boldsymbol{a}, \imath(F)) = \jmath(F)(\boldsymbol{a}) = \mathrm{app}(\boldsymbol{a}, \jmath(F)) = \widetilde{\mathrm{app}}(\boldsymbol{a}, \jmath(F))$$

(the first equality holds by the definition of $\jmath$ from Corollary 3.9, the second as $\jmath(F)$ is principal, and the third as $\widetilde{\mathrm{app}}$ extends app). By Lemma 3.10,

$$e(\imath(F))(\boldsymbol{a}) = \widetilde{\mathrm{app}}(\boldsymbol{a}, \imath(F)) \text{ and } e(\jmath(F))(\boldsymbol{a}) = \widetilde{\mathrm{app}}(\boldsymbol{a}, \jmath(F))$$

(that holds for n-tuples of non-principal ultrafilters as well). This completes the proof. □

Lemma 3.21. *Both operations e and E are surjective and non-injective. More precisely,*

(i) e (and E) on $\boldsymbol{\beta}\big(Y^{X_1 \times \ldots \times X_n}\big)$ is a surjection onto $RC_{X_1,\ldots,X_{n-1}}(\boldsymbol{\beta}X_1, \ldots, \boldsymbol{\beta}X_n, Y)$,

(ii) e on $\boldsymbol{\beta}\mathcal{P}(X_1 \times \ldots \times X_n)$ is a surjection onto $\big\{\widetilde{R} \in \mathcal{P}(\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n) : R \subseteq X_1 \times \ldots \times X_n\big\}$ $= \big\{Q \in \mathcal{P}(\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n) : Q$ is right clopen w.r.t. $X_1, \ldots, X_n\big\}$,

(iii) E on $\boldsymbol{\beta}\mathcal{P}(X_1 \times \ldots \times X_n)$ is a surjection onto $\big\{R^* \in \mathcal{P}(\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n) : R \subseteq X_1 \times \ldots \times X_n\big\}$ $= \big\{Q \in \mathcal{P}(\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n) : Q$ is regular closed$\big\}$,

and each of the three maps is not an injection whenever at least one of the X_i is infinite.

Proof. Item (i) is Lemma 2.10; items (ii) and (iii) are immediate from Lemma 3.17; the non-injectivity is easy from the cardinality argument since both maps $R \mapsto \widetilde{R}$ and $R \mapsto R^*$ are bijections. (Alternatively, (ii) can be obtained from (i) by replacing relations with their characteristic functions.) The equalities in (ii) and (iii) were stated in Theorem 1.14(i),(ii). □

By Lemma 3.17, relations of the model $e(\mathfrak{B})$ are the $^\sim$-extensions of some relations on X, while relations of the model $E(\mathfrak{B})$ are the *-extensions of the same relations. Whether the whole models $e(\mathfrak{B})$ and $E(\mathfrak{B})$ are the ultrafilter extensions of some models depends only on the ultrafilter interpretation of functional symbols in $\mathfrak{B}$:

Theorem 3.22. *Let $\mathfrak{B}$ be an ultrafilter model with the universe $\boldsymbol{\beta} X$. The following are equivalent:*

(i) $e(\mathfrak{B}) = \widetilde{\mathfrak{A}}$ for an ordinary model $\mathfrak{A}$ with the universe X,

(ii) $E(\mathfrak{B}) = \mathfrak{A}^*$ for an ordinary model $\mathfrak{A}$ with the universe X,

(iii) the interpretation in $\mathfrak{B}$ is pseudo-principal on functional symbols.

Moreover, the model $\mathfrak{A}$ in (i) and (ii) is the same.

$$\begin{tikzcd} & \mathfrak{B} \arrow[dl, "e"'] \arrow[dr, "E"] & \\ \widetilde{\mathfrak{A}} & & \mathfrak{A}^* \\ & \mathfrak{A} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Proof. The implications from each of (i) and (ii) to (iii) are obvious: if the interpretation $\jmath$ in $\mathfrak{B}$ is not pseudo-principal, then there are a functional symbol F and a sequence $\boldsymbol{a}$ of principal ultrafilters over $\boldsymbol{\beta} X$ such that the operation $G = e(\jmath(F))$ on $\boldsymbol{\beta} X$ takes $\boldsymbol{a}$ to a non-principal ultrafilter $G(\boldsymbol{a})$ over X. Therefore, G is not of form $\widetilde{f}$ for any operation f on X. Since G is the interpretation of F in both models $e(\mathfrak{B})$ and $E(\mathfrak{B})$, it follows that these models are not of form $\widetilde{\mathfrak{A}}$ and $\mathfrak{A}^*$ for any ordinary model $\mathfrak{A}$.

Let us show now that, conversely, (iii) implies each of (i) and (ii). By Lemma 3.20, it suffices to handle the case when the pseudo-principal interpretation $\jmath$ in $\mathfrak{B}$ is principal. So suppose this is the case and define an ordinary interpretation $\imath$ of the same language by letting, for all functional symbols F and predicate symbols R,

$$\begin{aligned} &\imath(F) = G \text{ if the principal ultrafilter } \jmath(F) \text{ over } X^{X \times \ldots \times X} \text{ is given by } G, \\ &\imath(R) = Q \text{ if } Q = e(\jmath(R)) \cap (X \times \ldots \times X). \end{aligned}$$

We have:

$$e(\jmath(F)) = E(\jmath(F)) = \widetilde{\imath(F)}$$

since $\jmath(F)$ is principal and e (and E) on principal ultrafilters is ext, and

$$e(\jmath(R)) = \widetilde{\imath(R)} \text{ and } E(\jmath(R)) = (\imath(R))^*$$

by Lemma 3.17. Thus if $\mathfrak{A}$ is the ordinary model given by $\imath$, we obtain $e(\mathfrak{B}) = \widetilde{\mathfrak{A}}$ and $E(\mathfrak{B}) = \mathfrak{A}^*$, as required. □

Finally, we point out that the fact whether an ordinary model with the universe $\boldsymbol{\beta}X$ is of form $e(\mathfrak{B})$, and whether it is of form $E(\mathfrak{B})$, for some ultrafilter model $\mathfrak{B}$ (clearly, with the same universe $\boldsymbol{\beta}X$) depends only on its topological properties:

Theorem 3.23. *Let $\mathfrak{A}$ be an ordinary model with the universe $\boldsymbol{\beta}X$. Then:*

(i) $\mathfrak{A} = e(\mathfrak{B})$ for an ultrafilter model $\mathfrak{B}$ iff in $\mathfrak{A}$ all operations are right continuous w.r.t. X and all relations are right clopen w.r.t. X,

(ii) $\mathfrak{A} = E(\mathfrak{B})$ for an ultrafilter model $\mathfrak{B}$ iff in $\mathfrak{A}$ all operations are right continuous w.r.t. X and all relations are regular closed.

Proof. Lemma 3.21. □

Since by Theorem 3.22, e and E applied to ultrafilter models with pseudo-principal interpretations give the $^{\sim}$- and *-extensions of ordinary models, Theorem 3.23 can be considered as a generalization of Theorems 1.10 and 1.12.

First Extension Theorems. Here we discuss a possible generalization of the First Extension Theorems (Theorems 1.8 and 1.7) to ultrafilter models. To start, let us restate both them in a single way as follows.

Theorem 3.24. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two (ordinary) models of the same signature, and let $h : X \to Y$ be a map between their universes. The following are equivalent:*

(i) h is a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$,

(ii) $\widetilde{h}$ is a homomorphism of $\widetilde{\mathfrak{A}}$ into $\widetilde{\mathfrak{B}}$,

(iii) $\widetilde{h}$ is a homomorphism of $\mathfrak{A}^*$ into $\mathfrak{B}^*$:

$$\begin{tikzcd}
& & \mathfrak{A}^* \arrow[rr, dashed, "\widetilde{h}"] & & \mathfrak{B}^* \\
\widetilde{\mathfrak{A}} \arrow[rr, dashed, "\widetilde{h}"] & & \widetilde{\mathfrak{B}} & & \\
& \mathfrak{A} \arrow[ul] \arrow[uur] \arrow[rr, "h"] & & \mathfrak{B} \arrow[ul] \arrow[uur] &
\end{tikzcd}$$

Proof. Theorems 1.7 and 1.8. □

This leads to a conclusion for our ultrafilter models:

Lemma 3.25. *Let $\mathfrak{U}$ and $\mathfrak{V}$ be two ultrafilter models of the same signature, and let $h : \boldsymbol{\beta}X \to \boldsymbol{\beta}Y$ be a map between their universes. The following are equivalent:*

(i) h is a homomorphism of $e(\mathfrak{U})$ into $e(\mathfrak{V})$,

(ii) h is a homomorphism of $E(\mathfrak{U})$ into $E(\mathfrak{V})$.

Proof. If $\mathfrak{f}$ is an ultrafilter over operations, we have $e(\mathfrak{f}) = E(\mathfrak{f})$ by definition of e and E, hence the claim for homomorphisms w.r.t. operations holds trivially. If $\mathfrak{r}$ is an ultrafilter over relations, we have $e(\mathfrak{r}) = \widetilde{R}$ and $E(\mathfrak{r}) = R^*$ by Lemma 3.17, hence the claim for homomorphisms w.r.t. relations holds by Theorem 3.24. □

This observation leads to the following definition:

Definition 3.26. If $\mathfrak{U}$ and $\mathfrak{V}$ are two ultrafilter models of the same signature, we say that a map $h : \boldsymbol{\beta} X \to \boldsymbol{\beta} Y$ between their universes is a *homomorphism* (*of ultrafilter models*) iff it is a homomorphism of $e(\mathfrak{U})$ into $e(\mathfrak{V})$ (or a homomorphism of $E(\mathfrak{U})$ into $E(\mathfrak{V})$, which is equivalent by Lemma 3.25).

The concepts of *epimorphisms*, *quotients*, *isomorphic embeddings*, *submodels*, *elementary embeddings*, *elementary submodels*, etc., for ultrafilter models are defined likewise.

The following can be considered as the First Extension Theorem for ultrafilter models:

Theorem 3.27. *Let $\mathfrak{U}$ and $\mathfrak{V}$ be two ultrafilter models of the same signature with the universes $\boldsymbol{\beta} X$ and $\boldsymbol{\beta} Y$, both having pseudo-principal interpretations on functional symbols, let $\mathfrak{A}$ and $\mathfrak{B}$ denote the models such that $\widetilde{\mathfrak{A}} = e(\mathfrak{U})$ and $\widetilde{\mathfrak{B}} = e(\mathfrak{V})$, and so $\mathfrak{A}^* = E(\mathfrak{U})$ and $\mathfrak{B}^* = E(\mathfrak{V})$, and let $h : X \to Y$. The following are equivalent:*

(i) h is a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$,

(ii) $\widetilde{h}$ is a homomorphism of $\mathfrak{U}$ into $\mathfrak{V}$,

(iii) $\widetilde{h}$ is a homomorphism of $\widetilde{\mathfrak{A}}$ into $\widetilde{\mathfrak{B}}$,

(iv) $\widetilde{h}$ is a homomorphism of $\mathfrak{A}^*$ into $\mathfrak{B}^*$:

$$\begin{tikzcd}
 & \mathfrak{U} \arrow[rr, dashed, "\widetilde{h}"] \arrow[dr, "E"] \arrow[ddl, "e"'] & & \mathfrak{V} \arrow[dr, "E"] \arrow[ddl, "e"'] & \\
 & & \mathfrak{A}^* \arrow[rr, dashed, "\widetilde{h}"] & & \mathfrak{B}^* \\
\widetilde{\mathfrak{A}} \arrow[rr, dashed, "\widetilde{h}"] & & \widetilde{\mathfrak{B}} & & \\
 & \mathfrak{A} \arrow[ul] \arrow[uur] \arrow[rr, "h"] & & \mathfrak{B} \arrow[ul] \arrow[uur] &
\end{tikzcd}$$

Proof. The equivalence of items (i) and (ii) follows from Theorem 3.22, Lemma 3.25, and the definition of homomorphisms of ultrafilter models. The equivalence of items (i), (iii), and (iv) repeats Theorem 3.24. □

For an ultrafilter model $\mathfrak{U}$ with the universe $\boldsymbol{\beta} X$, the set X of principal ultrafilters forms an ultrafilter submodel (and also ordinary submodels of $e(\mathfrak{U})$ and $E(\mathfrak{U})$) iff the interpretation in $\mathfrak{U}$ is pseudo-principal on functional symbols; this can be added as item (iv) to Theorem 3.22. We shall call the submodel consisting of principal ultrafilters the *principal submodel*. Thus Theorem 3.27 can be reformulated by replacing "both having pseudo-principal interpretations" with "both having principal submodels".

In fact, we can omit here the assumption about the pseudo-principality in the ultrafilter model $\mathfrak{V}$ by applying the Second Extension Theorems (Theorems 1.11 and 1.13):

Theorem 3.28. *Let $\mathfrak{U}$ and $\mathfrak{V}$ be two ultrafilter models of the same signature with the universes $\boldsymbol{\beta} X$ and $\boldsymbol{\beta} Y$, let the interpretation of $\mathfrak{U}$ be pseudo-principal on functional symbols with $\mathfrak{A}$ the principal submodel (having the universe X), so $\widetilde{\mathfrak{A}} = e(\mathfrak{U})$ and $\mathfrak{A}^* = E(\mathfrak{U})$, and let $h : X \to Y$. The following are equivalent:*

(i) h is a homomorphism of $\mathfrak{A}$ into $e(\mathfrak{V})$,

(ii) h is a homomorphism of $\mathfrak{A}$ into $E(\mathfrak{V})$,

(iii) $\widetilde{h}$ is a homomorphism of $\mathfrak{U}$ into $\mathfrak{V}$,

(iv) $\widetilde{h}$ is a homomorphism of $\widetilde{\mathfrak{A}}$ into $e(\mathfrak{V})$,

(v) $\widetilde{h}$ is a homomorphism of $\mathfrak{A}^*$ into $E(\mathfrak{V})$:

$$
\begin{tikzcd}
 & \mathfrak{U} \arrow[rr, dashed, "\widetilde{h}"] \arrow[dr, "E"] \arrow[ddl, "e"'] & & \mathfrak{V} \arrow[dr, "E"] \arrow[ddl, "e"'] & \\
 & & \mathfrak{A}^* \arrow[rr, dashed, "\widetilde{h}"] & & E(\mathfrak{V}) \\
\widetilde{\mathfrak{A}} \arrow[rr, dashed, "\widetilde{h}"] & & e(\mathfrak{V}) & & \\
 & \mathfrak{A} \arrow[ul] \arrow[uur] \arrow[ur, "h"] \arrow[uurrr, "h"'] & & &
\end{tikzcd}
$$

Proof. The equivalence of items (i) and (iii) follows from Theorem 1.11 and Theorem 3.23(i), while the equivalence of items (ii) and (iii) follows from Theorem 1.13 and Theorem 3.23(ii). Finally, (ii) is equivalent to (iv) by Theorem 1.11 and to (v) by Theorem 1.13. □

Observe that, by Theorem 3.23, whenever the interpretation of $\mathfrak{V}$ is *not* pseudo-principal on functional symbols, then the models $e(\mathfrak{V})$ and $E(\mathfrak{V})$ are *not* of form $\widetilde{\mathfrak{A}}$ and $\mathfrak{A}^*$ for any ordinary model $\mathfrak{A}$; nevertheless, these models still satisfy the conditions of Theorems 1.11 and 1.13 (playing the role of the model $\mathfrak{C}$ there). Therefore, Theorem 3.28 is indeed more general than Theorem 3.27; it has a character intermediate between the First and the Second Extension Theorems. To have a reasonable generalization of Second Extension Theorems in the full form, we need to have a more general concept of ultrafilter models; this is the subject of the next, last section of our article.

Remark 3.29. Theorems 3.24–3.27 remain true for epimorphisms and isomorphic embeddings, and Theorem 3.28 for epimorphisms. Also they can be stated for so-called homotopies and isotopies; these concepts (generalizing homomorphisms and isomorphisms) for ordinary models, together with both extension theorems, were introduced in [27] (and [28]). For ultrafilter models they can be defined in the same way as this was done for homomorphisms and embeddings. Finally, versions for multi-sorted models (having rather many universes $X_1, \ldots, X_n$ than one universe X) can be also easily stated.

4 Wider ultrafilter interpretations

Here we discuss a possible generalization of the Second Extension Theorems (Theorems 1.11 and 1.13) to ultrafilter models. For this, we should have a wider concept of ultrafilter models which, on the one hand, would replace compact Hausdorff right topological models in these theorems, and on the other hand, would turn into our previous concept of ultrafilter models whenever the universe is of form $\boldsymbol{\beta}Y$ to include our versions of the First Extension Theorem for ultrafilter models (Theorems 3.27 and 3.28). Also we should have a concept of satisfiability in these models which would turn into the satisfiability in our previous ultrafilter model; recall that the latter can be redefined in terms of the map e (Theorem 3.16). Actually, our new concepts of ultrafilter models and satisfiability will be wide enough to cover all ordinary models, not only ultrafilter extensions.

Ultrafilter models in the wide sense. The new definition of ultrafilter models requires only a minor modification of the former one. By an *ultrafilter interpretation* we still mean a map which takes functional and relational symbols to ultrafilters over operations and relations on a set X. But valuations of variables now will be in the set X itself, not in $\boldsymbol{\beta}X$.

Definition 4.1. Given a signature τ, an *ultrafilter model of τ in the wide sense* is a structure $\mathfrak{U}$ of the form

$$\mathfrak{U} = (X, \mathfrak{f}, \ldots, \mathfrak{r}, \ldots),$$

where X is the universe of $\mathfrak{U}$ (so individual variables are valuated by elements of X), each n-ary functional symbol in τ is interpreted by some $\mathfrak{f} \in \boldsymbol{\beta}(X^{X^n})$, and each n-ary predicate symbol in τ is interpreted by some $\mathfrak{r} \in \boldsymbol{\beta}\mathcal{P}(X^n)$.

Ultrafilter models in the sense of Definition 3.1 will be referred to as *ultrafilter models in the narrow sense*. Henceforth we shall never omit the words “in the narrow sense”.

To revise the concept of satisfiability making it adequate for ultrafilter models in the wide sense, we use the notion of convergence of ultrafilters. Recall that a filter $\mathfrak{d}$ over a topological space X *converges* to a point $x \in X$ iff any neighborhood of x belongs to $\mathfrak{d}$. If a filter $\mathfrak{d}$ converges to a unique point x then x is called the *limit* of $\mathfrak{d}$, in which case we shall write $\lim \mathfrak{d} = x$. As well-known, any filter over X converges to at most one point iff X is Hausdorff, and any *ultra*filter over X converges to at least one point iff X is compact (see e.g. [12]; for the concept of $\mathfrak{u}$-limit with ultrafilters $\mathfrak{u}$ see [17]). Note also that, even if X is compact Hausdorff, some filter over X that is not an ultrafilter may converge to *no* single point (consider e.g. any discrete X with $1 < |X| < \omega$ and the trivial filter consisting of X as its single element).

For an ultrafilter model in the wide sense $\mathfrak{U} = (X, \mathfrak{f}, \ldots, \mathfrak{r}, \ldots)$, fix some topologies on the sets X^{X^n} and $\mathcal{P}(X^n)$ for all $n < \omega$ such that the signature has n-ary operations, respectively, relations.

Definition 4.2. The ultrafilter model $\mathfrak{U}$ *converges* (w.r.t the specified family of topologies) to an ordinary model $\mathfrak{A} = (X, F, \ldots, R, \ldots)$ of the same signature iff the interpretation of each symbol in $\mathfrak{U}$ converges to one in $\mathfrak{A}$. Moreover, $\mathfrak{A}$ is the *limit* of $\mathfrak{U}$ iff the interpretation of $\mathfrak{A}$ is the pointwise limit of one of $\mathfrak{U}$, in which case we write $\mathfrak{A} = \lim \mathfrak{U}$.

Thus whenever the limits of all the ultrafilters $\mathfrak{f}, \ldots, \mathfrak{r}, \ldots$ exist then the ultrafilter model $\mathfrak{U}$ converges to its limit:

$$\lim \mathfrak{U} = (X, \lim \mathfrak{f}, \ldots, \lim \mathfrak{r}, \ldots),$$

which is an ordinary model of the same signature.

Definition 4.3. The ultrafilter model $\mathfrak{U}$ (endowed with the specified family of topologies) has a *well-defined satisfiability* iff there exists the limit of $\mathfrak{U}$, in which case we define it as the ordinary satisfiability in the limit:

$$\mathfrak{U} \Vdash_{\lim} \varphi\,[v] \;\text{ iff }\; \lim \mathfrak{U} \vDash \varphi\,[v]$$

for all formulas φ and valuations v.

We use the symbol $\Vdash_{\lim}$ for the renewed concept of satisfiability temporarily; after Theorem 4.5, which states that on ultrafilter models in the narrow sense this concept coincides with the former one, we shall continue to use the former symbol $\Vdash$.

Let us firstly show that all ordinary models and the satisfiability in them can be regarded as ultrafilter models in the wide sense and the satisfiability defined via limits.

Theorem 4.4. *Any ordinary model $\mathfrak{A}$ with the usual satisfaction relation $\vDash$ is (up to a natural identification) an ultrafilter model $\mathfrak{U}$ with the satisfaction relation $\Vdash_{\lim}$, so we have*

$$\mathfrak{U} \Vdash_{\lim} \varphi\,[v] \ \text{ iff } \ \mathfrak{A} \vDash \varphi\,[v]$$

for all formulas φ and valuations v. Moreover, the same is true for ordinary models endowed with arbitrary topologies.

Proof. Define $\mathfrak{U}$ as follows: let the universe of $\mathfrak{U}$ coincide with one of $\mathfrak{A}$, which we denote by X, and let the interpretation in $\mathfrak{U}$ be the principal interpretation giving with one in $\mathfrak{A}$, i.e. if an n-ary functional symbol is interpreted in $\mathfrak{A}$ by $F \in X^{X^n}$ then it is interpreted in $\mathfrak{U}$ by the principal ultrafilter over X^{X^n} given by F, and likewise for predicate symbols. We can suppose that all topologies on X^{X^n} and $\mathcal{P}(X^n)$ are discrete. Since any principal ultrafilter given by a point x has the limit x, we conclude that $\Vdash_{\lim}$ in $\mathfrak{U}$ coincides with $\vDash$ in $\mathfrak{A}$. Moreover, the same fact is true for every topologies on X^{X^n} and $\mathcal{P}(X^n)$, which proves the last claim of the theorem. $\square$

Now we are going to show that the wider concepts of ultrafilter models and the satisfiability in them cover the former, narrow concepts. (Let us also note that the new concept is not exhausted by the two cases of ordinary models and ultrafilter models in the narrow sense.)

Theorem 4.5. *Any ultrafilter model $\mathfrak{A}$ in the narrow sense with the satisfaction relation is, up to a natural identification, an ultrafilter model $\mathfrak{U}$ in the wide sense with the satisfaction relation defined via limits in certain appropriate topologies, so we have*

$$\mathfrak{U} \Vdash_{\lim} \varphi\,[v] \ \text{ iff } \ \mathfrak{A} \Vdash \varphi\,[v]$$

for all formulas φ and valuations v.

Proof. We start by describing how to represent an ultrafilter model $\mathfrak{A}$ in the narrow sense by a certain ultrafilter model $\mathfrak{U}$ in the wide sense. Let $\boldsymbol{\beta} X$ be the universe of $\mathfrak{A}$, and suppose that the universe of $\mathfrak{U}$ coincides with it. Now we must identify ultrafilters over the sets X^{X^n} and $\mathcal{P}(X^n)$ with certain ultrafilters over the sets $(\boldsymbol{\beta} X)^{(\boldsymbol{\beta} X)^n}$ and $\mathcal{P}((\boldsymbol{\beta} X)^n)$, respectively. Let us provide a more general procedure, which will be referred to as the *identification map* and denoted by i.

Identification map $\boldsymbol{i}$**.** For any positive $n < \omega$, discrete spaces $X_1, \ldots, X_n$, compact Hausdorff space Y, and $S \subseteq Y$, we construct the map i taking ultrafilters over $S^{X_1 \times \ldots \times X_n}$ to ultrafilters over $Y^{\boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n}$:

$$i``\boldsymbol{\beta}\big(S^{X_1 \times \ldots \times X_n}\big) \subseteq \boldsymbol{\beta}\big(Y^{\boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n}\big).$$

The construction is going in two steps.

First, recall that the map ext provides a canonical one-to-one correspondence between the set $S^{X_1 \times \ldots \times X_n}$ and its image

$$\operatorname{ext}``\, S^{X_1 \times \ldots \times X_n} = \big\{\widetilde{f} : f \in S^{X_1 \times \ldots \times X_n}\big\} \subseteq RC_{X_1, \ldots, X_{n-1}}(\boldsymbol{\beta} X_1, \ldots, \boldsymbol{\beta} X_n, Y) \subseteq Y^{\boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n}.$$

This induces the bijection $^+$ of $\boldsymbol{\beta}(S^{X_1 \times \ldots \times X_n})$ onto $\boldsymbol{\beta}(\mathrm{ext}``S^{X_1 \times \ldots \times X_n})$ taking each ultrafilter $\mathfrak{f}$ over $S^{X_1 \times \ldots \times X_n}$ to an ultrafilter $\mathfrak{f}^+$ over $\mathrm{ext}``S^{X_1 \times \ldots \times X_n}$ by letting

$$\mathfrak{f}^+ = \{\mathrm{ext}``A : A \in \mathfrak{f}\}.$$

Second, for any $S \subseteq T$ we define the *lifting* map of $\boldsymbol{\beta}S$ into $\boldsymbol{\beta}T$, by letting for all $\mathfrak{u} \in \boldsymbol{\beta}S$,

$$\mathfrak{u}^T = \{B \subseteq T : A \in \mathfrak{u} \text{ and } B \supseteq A\}.$$

Define also the *projection* map of $\{\mathfrak{v} \in \boldsymbol{\beta}T : S \in \mathfrak{v}\}$ into $\boldsymbol{\beta}S$, by letting for all such $\mathfrak{v}$,

$$\mathfrak{v}_S = \{A \cap S : A \in \mathfrak{v}\}.$$

Clearly, the domain of the projection is the range of the lifting, and moreover, $(\mathfrak{v}_S)^T = \mathfrak{v}$ and $(\mathfrak{u}^T)_S = \mathfrak{u}$, thus the lifting and the projection maps are two mutually inverse bijections. (Often one identifies these ultrafilters, thus considering $\boldsymbol{\beta}S$ as the closed subset of $\boldsymbol{\beta}T$ consisting of those ultrafilters over T that are concentrated on S, see e.g. [17], Section 3.3).

Now we define i as the composition of $^+$ and lifting, thus for all $\mathfrak{f} \in \boldsymbol{\beta}(S^{X_1 \times \ldots \times X_n})$ we let

$$i(\mathfrak{f}) = (\mathfrak{f}^+)^{Y^{\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n}}.$$

In result, for any ultrafilter $\mathfrak{f}$ over $S^{X_1 \times \ldots \times X_n}$, its image $i(\mathfrak{f})$ is an ultrafilter over $Y^{\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n}$ which is concentrated on $\mathrm{ext}``S^{X_1 \times \ldots \times X_n} = \{\widetilde{f} : f \in S^{X_1 \times \ldots \times X_n}\}$.

Let us now expand the domain of the map i to ultrafilters over relations. We want to get i taking ultrafilters over $\mathcal{P}(X_1 \times \ldots \times X_n)$ to ultrafilters over $\mathcal{P}(\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n)$:

$$i``\boldsymbol{\beta}\mathcal{P}(X_1 \times \ldots \times X_n) \subseteq \boldsymbol{\beta}\mathcal{P}(\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n).$$

For this, we may identify n-ary relations with their characteristic functions, i.e. n-ary maps into $2 = \{0, 1\}$, where 2 is endowed with the discrete topology, and use the definition of i for ultrafilters over maps (with $Y = S = 2$). Equivalently, we might imitate the above construction: for each $\mathfrak{r} \in \boldsymbol{\beta}\mathcal{P}(X_1 \times \ldots \times X_n)$, we might turn it firstly to $\mathfrak{r}^+ \in \boldsymbol{\beta}(\mathrm{ext}``\mathcal{P}(X_1 \times \ldots \times X_n))$ where $\mathrm{ext}(R) = \widetilde{R}$, so by Theorem 1.14,

$$\begin{aligned}\mathrm{ext}``\mathcal{P}(X_1 \times \ldots \times X_n) &= \{\widetilde{R} : R \subseteq X_1 \times \ldots \times X_n\} \\ &= \{Q \subseteq \boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n : Q \text{ is right clopen w.r.t. } X_1, \ldots, X_n\},\end{aligned}$$

by letting

$$\mathfrak{r}^+ = \{\mathrm{ext}``A : A \in \mathfrak{r}\},$$

and secondly, by lifting the obtaining ultrafilter to an ultrafilter over $\mathcal{P}(\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n)$, thus letting

$$i(\mathfrak{r}) = (\mathfrak{r}^+)^{\mathcal{P}(\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n)}.$$

In result, for any ultrafilter $\mathfrak{r}$ over $\mathcal{P}(X_1 \times \ldots \times X_n)$, its image $i(\mathfrak{r})$ is an ultrafilter over $\mathcal{P}(\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n)$ which is concentrated on $\{Q \subseteq \boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n : Q \text{ is right clopen w.r.t. } X_1, \ldots, X_n\}$.

Remark 4.6. In fact, the map $^+$ is $\widetilde{\mathrm{ext}}$ for ext considered as a bijection between two discrete spaces, and thus a homeomorphism between the spaces of ultrafilters over them.

For maps these discrete spaces are $S^{X_1\times\ldots\times X_n}$ and $\mathrm{ext}``S^{X_1\times\ldots\times X_n}$, so $\widetilde{\mathrm{ext}}$ is a homeomorphism between $\boldsymbol{\beta}(S^{X_1\times\ldots\times X_n})$ and $\boldsymbol{\beta}(\mathrm{ext}``S^{X_1\times\ldots\times X_n})$, and $\mathfrak{f}^+ = \widetilde{\mathrm{ext}}(\mathfrak{f})$:

$$\begin{tikzcd}
\boldsymbol{\beta}\big(S^{X_1\times\ldots\times X_n}\big) \arrow[r, dashed, "+"] & \boldsymbol{\beta}\big(\mathrm{ext}``S^{X_1\times\ldots\times X_n}\big) \\
S^{X_1\times\ldots\times X_n} \arrow[u] \arrow[r, "\mathrm{ext}"] & \mathrm{ext}``S^{X_1\times\ldots\times X_n} \arrow[u]
\end{tikzcd}$$

and analogously, for relations the discrete spaces are $\mathcal{P}(X_1\times\ldots\times X_n)$ and $\mathrm{ext}``\mathcal{P}(X_1\times\ldots\times X_n)$, so $\widetilde{\mathrm{ext}}$ is a homeomorphism between $\boldsymbol{\beta}\mathcal{P}(X_1\times\ldots\times X_n)$ and $\boldsymbol{\beta}(\mathrm{ext}``\mathcal{P}(X_1\times\ldots\times X_n))$, and $\mathfrak{r}^+ = \widetilde{\mathrm{ext}}(\mathfrak{r})$:

$$\begin{tikzcd}
\boldsymbol{\beta}\mathcal{P}(X_1\times\ldots\times X_n) \arrow[r, dashed, "+"] & \boldsymbol{\beta}(\mathrm{ext}``\mathcal{P}(X_1\times\ldots\times X_n)) \\
\mathcal{P}(X_1\times\ldots\times X_n) \arrow[u] \arrow[r, "\mathrm{ext}"] & \mathrm{ext}``\mathcal{P}(X_1\times\ldots\times X_n) \arrow[u]
\end{tikzcd}$$

(cf. Remark 2.8 explaining a similar situation with currying). Nevertheless, we use the symbol $^+$ to avoid confusing with $\widetilde{\mathrm{ext}}$ for the map ext into a compact Hausdorff space Y, which will be also used in our arguments below.

Lemma 4.7. *The map i is a bijection between:*

(i) the set of all ultrafilters over $S^{X_1\times\ldots\times X_n}$ and the set of the ultrafilters over $Y^{\boldsymbol{\beta}X_1\times\ldots\times\boldsymbol{\beta}X_n}$ that are concentrated on $\mathrm{ext}``S^{X_1\times\ldots\times X_n}$*,*

(ii) the set of all ultrafilters over $\mathcal{P}(X_1\times\ldots\times X_n)$ and the set of the ultrafilters over $\mathcal{P}(\boldsymbol{\beta}X_1\times\ldots\times\boldsymbol{\beta}X_n)$ that are concentrated on $\mathrm{ext}``\mathcal{P}(X_1\times\ldots\times X_n)$.

Proof. For brevity, we let:

$$A = \boldsymbol{\beta}\big(S^{X_1\times\ldots\times X_n}\big),\;\; B = \boldsymbol{\beta}\big(\mathrm{ext}``S^{X_1\times\ldots\times X_n}\big),\;\; C = \big\{\mathfrak{g}\in\boldsymbol{\beta}\big(Y^{\boldsymbol{\beta}X_1\times\ldots\times\boldsymbol{\beta}X_n}\big) : \mathrm{ext}``S^{X_1\times\ldots\times X_n}\in\mathfrak{g}\big\}.$$

As we have already pointed out, the map $^+$ is a bijection of A onto B, and the lifting map is a bijection of B onto C. Therefore, i, as the composition of the two maps, is a bijection of A onto C, which proves item (i). Item (ii) is either proved similarly or obtained from (i) by replacing relations with their characteristic functions. (We may also note that these maps are homeomorphic embeddings.) ☐

Let us now turn back to Theorem 4.5 and the discussed there situation with ultrafilter models in the former, narrow sense. In this case, all the discrete spaces $X_1,\ldots,X_n$ are equal to X, while the compact Hausdorff space Y is $\boldsymbol{\beta}X$ and its subset S is X or 2 in the cases of operations and relations of the model, respectively. (Recall that we identify elements of X with the principal ultrafilters given by them, so any n-ary operation on X is identified with a map of X^n into $\boldsymbol{\beta}X$.) We expand i to ultrafilter models in the narrow sense by defining it pointwise:

Definition 4.8. Given an ultrafilter model $\mathfrak{A} = (\boldsymbol{\beta}X,\mathfrak{f},\ldots,\mathfrak{r},\ldots)$ in the narrow sense, we let

$$i(\mathfrak{A}) = (\boldsymbol{\beta}X, i(\mathfrak{f}),\ldots,i(\mathfrak{r}),\ldots).$$

Continuing the proof of Theorem 4.5, we define $\mathfrak{U}$, the ultrafilter model in the wide sense corresponding to $\mathfrak{A}$, the given ultrafilter model in the narrow sense, by letting

$$\mathfrak{U} = i(\mathfrak{A}).$$

It remains to verify that the new satisfiability, defined via limits, coincides with the old one. By Theorem 3.16, which states that for all formulas φ and valuations v, we have $\mathfrak{A} \Vdash \varphi\,[v]$ iff $e(\mathfrak{A}) \vDash \varphi\,[v]$, the latter can be redefined via the map e. Therefore, it suffices to check that for all formulas φ and valuations v, we have

$$\lim \mathfrak{U} \vDash \varphi\,[v] \;\text{ iff }\; e(\mathfrak{A}) \vDash \varphi\,[v].$$

But actually, a stronger fact is true:

$$\lim \mathfrak{U} = e(\mathfrak{A}),$$

thus leading to the following result.

Theorem 4.9. *If $\mathfrak{A}$ is an ultrafiler model in the narrow sense, then $\lim i(\mathfrak{A}) = e(\mathfrak{A})$:*

$$\begin{tikzcd}
 & i(\mathfrak{A}) \arrow[dd, "\lim"] \\
\mathfrak{A} \arrow[ur, "i"] \arrow[dr, "e"'] & \\
 & e(\mathfrak{A})
\end{tikzcd}$$

Proof. We must verify the equalities $\lim i(\mathfrak{f}) = e(\mathfrak{f})$ and $\lim i(\mathfrak{r}) = e(\mathfrak{r})$ for all $\mathfrak{f} \in \boldsymbol{\beta}(X^{X\times\ldots\times X})$ and $\mathfrak{r} \in \boldsymbol{\beta}\,\mathcal{P}(X \times \ldots \times X)$. This will be stated in the next, more general lemma.

Recall once more that the map e on ultrafilters over n-ary maps is $\widetilde{\mathrm{ext}}$, the continuous extension of the map ext, where the latter, in turn, takes n-ary maps $f : X_1 \times \ldots \times X_n \to Y$ of discrete spaces $X_1, \ldots, X_n$ into a compact Hausdorff space Y, to their extensions $\widetilde{f} = \mathrm{ext}(f)$ that are right continuous w.r.t. $X_1, \ldots, X_{n-1}$, and that these extensions form a compact Hausdorff space w.r.t. the $(X_1, \ldots, X_n)$-pointwise convergence topology:

$$\begin{tikzcd}
\boldsymbol{\beta}\big(Y^{X_1\times\ldots\times X_n}\big) \arrow[dr, dashed, "\widetilde{\mathrm{ext}}"] & \\
Y^{X_1\times\ldots\times X_n} \arrow[u] \arrow[r, "\mathrm{ext}"] & RC_{X_1,\ldots,X_{n-1}}(\boldsymbol{\beta}X_1, \ldots, \boldsymbol{\beta}X_n, Y)
\end{tikzcd}$$

The next lemma states that the map e is the composition of the identification map i and taking the limit.

Lemma 4.10. *Let $X_1, \ldots, X_n$ be discrete spaces, Y a compact Hausdorff space, and let the spaces $RC_{X_1,\ldots,X_{n-1}}(\boldsymbol{\beta}X_1, \ldots, \boldsymbol{\beta}X_n, Y)$ and $\{Q \subseteq \boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n : Q$ is right clopen w.r.t. $X_1, \ldots, X_{n-1}\}$ be endowed with the $(X_1, \ldots, X_n)$-pointwise convergence topologies. Then we have*

$$\lim i(\mathfrak{f}) = e(\mathfrak{f}) \;\; \textit{and} \;\; \lim i(\mathfrak{r}) = e(\mathfrak{r})$$

for every ultrafilters $\mathfrak{f} \in \boldsymbol{\beta}(Y^{X_1\times\ldots\times X_n})$ and $\mathfrak{r} \in \boldsymbol{\beta}\,\mathcal{P}(X_1 \times \ldots \times X_n)$.

Proof. For brevity, let RC denote the space $RC_{X_1,\dots,X_{n-1}}(\boldsymbol{\beta}X_1,\dots,\boldsymbol{\beta}X_n,Y)$. By Lemma 2.4, the space RC is compact Hausdorff. Hence, $i(\mathfrak{f})$, which is an ultrafilter over the set $Y^{\boldsymbol{\beta}X_1\times\ldots\times\boldsymbol{\beta}X_n}$ concentrated on its subset RC and thus can be identified with its projection to RC, converges to a unique point of RC, i.e. has a limit in RC. We need to show that the limit is exactly the map $e(\mathfrak{f})$.

Denote $e(\mathfrak{f})$ by F. Then, since $e=\widetilde{\mathrm{ext}}$ and $\mathrm{ext}=\widetilde{\;\;}$, we get:

$$\{F\}=\bigcap_{A\in\mathfrak{f}}\mathrm{cl}_{\mathrm{RC}}\,\mathrm{ext}``A=\bigcap_{A\in\mathfrak{f}}\mathrm{cl}_{\mathrm{RC}}\{\widetilde{f}:f\in A\}.$$

Therefore, for every $A\in\mathfrak{f}$ and any neighborhood O of the point F in the space, there exists $f\in A$ such that $\widetilde{f}\in O$; here we use that the set $\mathrm{ext}``Y^{X_1\times\ldots\times X_n}=\{\widetilde{f}:f\in Y^{X_1\times\ldots\times X_n}\}$ is dense in RC by Lemma 2.5.

Let us verify that, moreover, for any neighborhood O of $F\in\mathrm{RC}$ the set $\{f\in Y^{X_1\times\ldots\times X_n}:\widetilde{f}\in O\}$ is in $\mathfrak{f}$. Assume the converse: there exists a neighborhood O of F such that the set $\{f\in Y^{X_1\times\ldots\times X_n}:\widetilde{f}\in O\}$ is not in $\mathfrak{f}$. Then, as $\mathfrak{f}$ is an ultrafilter, the complement

$$A=\big\{f\in Y^{X_1\times\ldots\times X_n}:\widetilde{f}\notin O\big\}$$

is in $\mathfrak{f}$. However, this contradicts to the above stated fact.

The case of ultrafilters over relations reduces to the case of ultrafilters over maps with $Y=2$. The lemma is proved. □

This proves Theorem 4.9. □

Now the proof of Theorem 4.5 is complete. □

Theorem 4.5 permits us to eliminate our temporary symbol $\Vdash_{\lim}$ and use the former symbol $\Vdash$ also to denote the satisfaction in ultrafilter models in the wide sense. Moreover, by Theorem 4.4 we might use the only ordinary symbol $\models$ to denote the satisfaction in both ordinary and ultrafilter models; we however prefer to retain the symbol $\Vdash$ for a convenience of reading.

Finally, we refine the first part of Theorem 4.5 (concerning rather models than the satisfaction relation) by characterizing the ultrafilter models in the wide sense that correspond to those in the narrow sense:

Theorem 4.11. *Let $\mathfrak{U}$ be an ultrafilter model in the wide sense. Then:*

(i) $\mathfrak{U}=i(\mathfrak{A})$ *for some ultrafilter model $\mathfrak{A}$ in the narrow sense iff the universe of $\mathfrak{U}$ is $\boldsymbol{\beta}X$ for some X and the interpretation takes all functional symbols to ultrafilters concentrated on* $\mathrm{ext}``X^{X^n}$*, and all relational symbols to ultrafilters concentrated on $\{Q\subseteq(\boldsymbol{\beta}X)^n:Q$ is right clopen w.r.t. $X\}$;*

(ii) $\lim\mathfrak{U}=\widetilde{\mathfrak{A}}$ *for some ordinary model $\mathfrak{A}$ iff the universe of $\mathfrak{U}$ is $\boldsymbol{\beta}X$ for some X and the interpretation takes all functional symbols to ultrafilters in $\{i(\mathfrak{f}):\mathfrak{f}\in\boldsymbol{\beta}(X^{X^n})$ is pseudo-principal$\}$, and all relational symbols to ultrafilters concentrated on $\{Q\subseteq(\boldsymbol{\beta}X)^n:Q$ is right clopen w.r.t. $X\}$.*

Proof. Item (i) is immediate from Lemma 4.7; we recall only that the images of ultrafilters over $\mathcal{P}(X^n)$ under i are exactly ultrafilters over $\mathcal{P}((\boldsymbol{\beta}X)^n)$ that are concentrated on $\mathrm{ext}``\mathcal{P}(X^n)$:

$$\begin{aligned}
i``\boldsymbol{\beta}\mathcal{P}(X^n)&=\big\{\mathfrak{s}\in\boldsymbol{\beta}\mathcal{P}((\boldsymbol{\beta}X)^n):\mathrm{ext}``\mathcal{P}(X^n)\in\mathfrak{s}\big\}\\
&=\big\{\mathfrak{s}\in\boldsymbol{\beta}\mathcal{P}((\boldsymbol{\beta}X)^n):\{\widetilde{R}:R\subseteq X^n\}\in\mathfrak{s}\big\}\\
&=\big\{\mathfrak{s}\in\boldsymbol{\beta}\mathcal{P}((\boldsymbol{\beta}X)^n):\{Q\subseteq(\boldsymbol{\beta}X)^n:Q\text{ is right clopen w.r.t. }X\}\in\mathfrak{s}\big\}.
\end{aligned}$$

Item (ii) follows from item (i) and Theorem 3.22. □

Map I**.** Here we consider a variant of the map i, which we denote by I. This map relates to the operation E in the same way as the map i to the operation e does (which explains our choosing of the symbol I).

The map I has the same domain and range that the map i does:

$$I``\boldsymbol{\beta}\big(S^{X_1\times\ldots\times X_n}\big)\subseteq\boldsymbol{\beta}\big(Y^{\boldsymbol{\beta}X_1\times\ldots\times\boldsymbol{\beta}X_n}\big),$$
$$I``\boldsymbol{\beta}\mathcal{P}(X_1\times\ldots\times X_n)\subseteq\boldsymbol{\beta}\mathcal{P}(\boldsymbol{\beta}X_1\times\ldots\times\boldsymbol{\beta}X_n).$$

and is defined as follows: on ultrafilters over $S^{X_1\times\ldots\times X_n}$ it coincides with i, and on ultrafilters over $\mathcal{P}(X_1\times\ldots\times X_n)$ it is defined likewise i except for taking $^\times$ instead of $^+$, where $\mathfrak{r}^\times$ uses rather cl than ext, i.e. turning $R\subseteq X_1\times\ldots\times X_n$ not to $\widetilde{R}$ but to R^* (recall that, by Theorem 1.14, cl is a bijection between all subsets of $X_1\times\ldots\times X_n$ and regular closed subsets of $\boldsymbol{\beta}X_1\times\ldots\times\boldsymbol{\beta}X_n$).

Thus for each $\mathfrak{r}\in\boldsymbol{\beta}\mathcal{P}(X_1\times\ldots\times X_n)$, we might turn it firstly to $\mathfrak{r}^\times\in\boldsymbol{\beta}(\mathrm{cl}``\mathcal{P}(X_1\times\ldots\times X_n))$ where $\mathrm{cl}(R)=R^*$, so by Theorem 1.14,

$$\begin{aligned}\mathfrak{r}^\times&\in\boldsymbol{\beta}\big\{R^*:R\subseteq X_1\times\ldots\times X_n\big\}\\&=\boldsymbol{\beta}\big\{Q\subseteq\boldsymbol{\beta}X_1\times\ldots\times\boldsymbol{\beta}X_n:Q\text{ is regular closed}\big\},\end{aligned}$$

by letting

$$\mathfrak{r}^\times=\{\mathrm{cl}``A:A\in\mathfrak{r}\},$$

and secondly, by lifting the obtained ultrafilter to an ultrafilter over $\mathcal{P}(\boldsymbol{\beta}X_1\times\ldots\times\boldsymbol{\beta}X_n)$, thus letting

$$I(\mathfrak{r})=(\mathfrak{r}^\times)^{\mathcal{P}(\boldsymbol{\beta}X_1\times\ldots\times\boldsymbol{\beta}X_n)}.$$

In result, for any ultrafilter $\mathfrak{r}$ over $\mathcal{P}(X_1\times\ldots\times X_n)$, its image $I(\mathfrak{r})$ is an ultrafilter over $\mathcal{P}(\boldsymbol{\beta}X_1\times\ldots\times\boldsymbol{\beta}X_n)$ which is concentrated on $\{Q\subseteq\boldsymbol{\beta}X_1\times\ldots\times\boldsymbol{\beta}X_n:Q\text{ is regular closed}\}$. (To make an analogy between $^+$ and $^\times$ more complete, we can also let that $^\times$ is defined on ultrafilters over $S^{X_1\times\ldots\times X_n}$ and coincides there with $^+$; but in fact we do not need this.)

Remark 4.12. Again, the map $^\times$ is $\widetilde{\mathrm{cl}}$ for cl considered as a bijection between two discrete spaces $\mathcal{P}(X_1\times\ldots\times X_n)$ and $\mathrm{cl}``\mathcal{P}(X_1\times\ldots\times X_n)$, so $\widetilde{\mathrm{cl}}$ is a homeomorphism between $\boldsymbol{\beta}\mathcal{P}(X_1\times\ldots\times X_n)$ and $\boldsymbol{\beta}(\mathrm{cl}``\mathcal{P}(X_1\times\ldots\times X_n))$, and $\mathfrak{r}^\times=\widetilde{\mathrm{cl}}(\mathfrak{r})$:

$$\begin{tikzcd}\boldsymbol{\beta}\mathcal{P}(X_1\times\ldots\times X_n) \arrow[r, dashed, "\times"] & \boldsymbol{\beta}(\mathrm{cl}``\mathcal{P}(X_1\times\ldots\times X_n))\\ \mathcal{P}(X_1\times\ldots\times X_n) \arrow[u] \arrow[r, "\mathrm{cl}"] & \mathrm{cl}``\mathcal{P}(X_1\times\ldots\times X_n) \arrow[u]\end{tikzcd}$$

Nevertheless, we use the symbol $^\times$ to keep the analogy with $^+$.

Two next lemmas and the subsequent theorem are counterparts of Lemmas 4.7 and 4.10 and Theorem 4.9, respectively.

Lemma 4.13. *The map I is a bijection between:*

(i) the set of all ultrafilters over $S^{X_1\times\ldots\times X_n}$ and the set of the ultrafilters over $Y^{\boldsymbol{\beta}X_1\times\ldots\times\boldsymbol{\beta}X_n}$ that are concentrated on $\mathrm{ext}``S^{X_1\times\ldots\times X_n}$*,*

(ii) the set of all ultrafilters over $\mathcal{P}(X_1 \times \ldots \times X_n)$ *and the set of the ultrafilters over* $\mathcal{P}(\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n)$ *that are concentrated on* $\mathrm{cl}_{\boldsymbol{\beta}X_1\times\ldots\times\boldsymbol{\beta}X_n}``\,\mathcal{P}(X_1 \times \ldots \times X_n)$.

Proof. Item (i) just repeats Lemma 4.7(i) since I coincides with i on ultrafilters over maps. For item (ii), let

$$A = \boldsymbol{\beta}\,\mathcal{P}(X_1 \times \ldots \times X_n), \quad B = \boldsymbol{\beta}(\mathrm{cl}_{\boldsymbol{\beta}X_1\times\ldots\times\boldsymbol{\beta}X_n}``\,\mathcal{P}(X_1 \times \ldots \times X_n)),$$
$$C = \big\{\mathfrak{s} \in \boldsymbol{\beta}\,\mathcal{P}(\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n) : \mathrm{cl}_{\boldsymbol{\beta}X_1\times\ldots\times\boldsymbol{\beta}X_n}``\,\mathcal{P}(X_1 \times \ldots \times X_n) \in \mathfrak{s}\big\}.$$

(Here the closure $\mathrm{cl}_{\boldsymbol{\beta}X_1\times\ldots\times\boldsymbol{\beta}X_n}$ refers to the product topology where the spaces $\boldsymbol{\beta}X_i$ are endowed with their standard topologies, and the set $\mathrm{cl}_{\boldsymbol{\beta}X_1\times\ldots\times\boldsymbol{\beta}X_n}``\,\mathcal{P}(X_1 \times \ldots \times X_n)$ in the definition of B is considered as a discrete space.) As the map $^{\times}$ is a bijection of A onto B and the lifting map is a bijection of B onto C, the map I, which the composition of the two maps, is a bijection of A onto C, thus proving (ii). ☐

In what follows we consider the space $\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n$ endowed with the usual product topology of the spaces $\boldsymbol{\beta}X_i$ and the set of regular closed sets in this space endowed with a compact Hausdorff topology. This topology is induced from the compact Hausdorff space $\mathcal{P}(X_1 \times \ldots \times X_n)$, which we identify with the space $2^{X_1\times\ldots\times X_n}$ (where 2 is discrete and the space carries the usual product topology) by the natural bijection taking $R \subseteq X_1 \times \ldots \times X_n$ to $R^* \subseteq \boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n$.

Recall also that by Lemma 2.4 (and its proof), the $(X_1, \ldots, X_n)$-pointwise convergence topology on $RC_{X_1,\ldots,X_{n-1}}(\boldsymbol{\beta}X_1, \ldots, \boldsymbol{\beta}X_n, Y)$ can be induced from the product topology on $Y^{X_1\times\ldots\times X_n}$ by the bijection ext, which takes each $f \in Y^{X_1\times\ldots\times X_n}$ to $\widetilde{f} \in Y^{\boldsymbol{\beta}X_1\times\ldots\times\boldsymbol{\beta}X_n}$. In particular, if $Y = 2$ then ext on relations (identified with their characteristic functions), which takes each $R \subseteq X_1 \times \ldots \times X_n$ to $\widetilde{R} \subseteq \boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n$, induces the above considered compact Hausdorff topology on $\mathcal{P}(\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n)$. Therefore, we have three homeomorphic spaces: the space of subsets R of $X_1 \times \ldots \times X_n$ and its images under the homeomorphisms ext and cl taking $R, \widetilde{R}, R^*$ into each others:

$$\begin{tikzcd}
\mathrm{RClop} \arrow[rr, leftrightarrow] & & \mathrm{RegCl} \\
& \mathcal{P}(X_1 \times \ldots \times X_n) \arrow[ul, "\mathrm{ext}"] \arrow[ur, "\mathrm{cl}"'] &
\end{tikzcd}$$

where

$$\begin{aligned}
\mathrm{RClop} &= \ \mathrm{ext}``\,\mathcal{P}(X_1 \times \ldots \times X_n) \\
&= \big\{\widetilde{R} : R \in \mathcal{P}(X_1 \times \ldots \times X_n)\big\} \\
&= \big\{Q \subseteq \boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n : Q \ \text{ is right clopen w.r.t. } X_1, \ldots, X_{n-1}\big\}, \\
\mathrm{RegCl} &= \ \mathrm{cl}``\,\mathcal{P}(X_1 \times \ldots \times X_n) \\
&= \big\{R^* : R \in \mathcal{P}(X_1 \times \ldots \times X_n)\big\} \\
&= \big\{Q \subseteq \boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n : Q \ \text{ is regular closed }\big\}.
\end{aligned}$$

Question 4.14. Redefine the topology on the set RegCl as a restricted version of the Vietoris topology (in an analogy with the restricted version of pointwise convergence topology turning out RC into a compact Hausdorff space homeomorphic to the product space $Y^{X_1\times\ldots\times X_n}$). Note that we cannot use the usual (unrestricted) Vietoris topology since in it, RegCl is not a closed subset of the space $\{Q \subseteq \boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n : Q \text{ is closed}\}$. (Problem 5.7.)

Lemma 4.15. *Let f be the homeomorphism between* $\mathrm{RClop} = \{\widetilde{R} : R \in \mathcal{P}(X_1 \times \ldots \times X_n)\}$ *and* $\mathrm{RegCl} = \{R^* : R \in \mathcal{P}(X_1 \times \ldots \times X_n)\}$ *taking $\widetilde{R}$ to R^*. Then we have*

$$f \circ e = E \;\; \textit{and} \;\; \widetilde{f} \circ i = I:$$

$$\begin{tikzcd}
\boldsymbol{\beta}\,\mathrm{RClop} \arrow[rr, dashed, "\widetilde{f}"] & & \boldsymbol{\beta}\,\mathrm{RegCl} \\
\mathrm{RClop} \arrow[u] \arrow[rr, "f"] & & \mathrm{RegCl} \arrow[u] \\
& \boldsymbol{\beta}\,\mathcal{P}(X_1 \times \ldots \times X_n) \arrow[ul, "e"] \arrow[ur, "E"'] \arrow[uul, "i"] \arrow[uur, "I"'] &
\end{tikzcd}$$

Proof. The equality $f \circ e = E$ follows from Theorem 3.17 and, in turn, implies the equality $\widetilde{f} \circ i = I$. □

Lemma 4.16. *Let $X_1, \ldots, X_n$ be discrete spaces, Y a compact Hausdorff space, and let the spaces* $\mathrm{RC} = RC_{X_1,\ldots,X_{n-1}}(\boldsymbol{\beta}X_1, \ldots, \boldsymbol{\beta}X_n, Y)$ *and* $\mathrm{RegCl} = \{Q \subseteq \boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n : Q$ *is regular closed*$\}$ *be endowed with the topology induced from $Y^{X_1 \times \ldots \times X_n}$ and $\mathcal{P}(X_1 \times \ldots \times X_n)$, respectively. Then we have*

$$\lim I(\mathfrak{f}) = E(\mathfrak{f}) \;\; \textit{and} \;\; \lim I(\mathfrak{r}) = E(\mathfrak{r})$$

for every ultrafilters $\mathfrak{f} \in \boldsymbol{\beta}(Y^{X_1 \times \ldots \times X_n})$ and $\mathfrak{r} \in \boldsymbol{\beta}\,\mathcal{P}(X_1 \times \ldots \times X_n)$.

Proof. The first equality repeats the first equality in Lemma 4.10 as the topology on RC induced from $Y^{X_1 \times \ldots \times X_n}$ coincides with the $(X_1, \ldots, X_n)$-pointwise convergence topology by Theorem 2.4.

For the second equality, recall first a general fact: if a map $g : A \to B$ is continuous and $\mathfrak{u}$ is an ultrafilter over A, then $g(\lim \mathfrak{u}) = \lim \widetilde{g}(\mathfrak{u})$ whenever both limits exist. It easily follows that if g is a homeomorphism, then $\lim \mathfrak{u} = g^{-1}(\lim \widetilde{g}(\mathfrak{u}))$ and $\lim \widetilde{g}(\mathfrak{u}) = g(\lim \widetilde{g}^{-1}(\mathfrak{u}))$. In our situation, we have:

$$\lim I(\mathfrak{r}) = f(\lim \widetilde{f}^{-1}(I(\mathfrak{r}))) = f(\lim i(\mathfrak{r})) = f(e)(\mathfrak{r}) = E(\mathfrak{r})$$

where the first equality holds by this general fact, the second follows from Lemma 4.15, the third holds by Lemma 4.10, and the last again by Lemma 4.15. □

Likewise i, we expand I to ultrafilter models in the narrow sense pointwise:

Definition 4.17. For all ultrafilter models $\mathfrak{A} = (\boldsymbol{\beta}X, \mathfrak{f}, \ldots, \mathfrak{r}, \ldots)$ in the narrow sense, let

$$I(\mathfrak{A}) = (\boldsymbol{\beta}X, I(\mathfrak{f}), \ldots, I(\mathfrak{r}), \ldots).$$

Theorem 4.18. *If $\mathfrak{A}$ is an ultrafilter model in the narrow sense, then* $\lim I(\mathfrak{A}) = E(\mathfrak{A})$*:*

$$\begin{tikzcd}
& I(\mathfrak{A}) \arrow[dd, "\lim"] \\
\mathfrak{A} \arrow[ur, "I"] \arrow[dr, "E"'] & \\
& E(\mathfrak{A})
\end{tikzcd}$$

Proof. Immediate from Lemma 4.16. □

We summarize the interplay between ultrafilter models $\mathfrak{A}$ in the narrow sense, their limits, and the operations i, I, e, E, and f (where f is defined on models of form $e(\mathfrak{A})$ as expected) in the following diagram:

$$\begin{tikzcd}
i(\mathfrak{A}) \arrow[rr, dashed, "\tilde{f}"] \arrow[d, "\lim"'] & & I(\mathfrak{A}) \arrow[d, "\lim"] \\
e(\mathfrak{A}) \arrow[rr, "f"] & & E(\mathfrak{A}) \\
& \mathfrak{A} \arrow[uul, "i"'] \arrow[uur, "I"] \arrow[ul, "e"] \arrow[ur, "E"'] &
\end{tikzcd}$$

Despite the fact that Theorem 4.18 is an E-analog of Theorem 4.9 used to get Theorem 4.5, the latter theorem has no such analog. This is due to an asymmetry between the operations e and E w.r.t. the satisfiability in ultrafilter models in the narrow sense, as it has been defined: there is no E-analog of Theorem 3.16, which was also used in proving Theorem 4.5 (cf., however, Problem 5.5). Nonetheless, we are still able to get a counterpart of Theorem 4.11, which does not involve satisfiability:

Theorem 4.19. *Let $\mathfrak{U}$ be an ultrafilter model in the wide sense. Then:*

(i) $\mathfrak{U} = I(\mathfrak{A})$ *for some ultrafilter model $\mathfrak{A}$ in the narrow sense iff the universe of $\mathfrak{U}$ is $\boldsymbol{\beta} X$ for some X and the interpretation takes all functional symbols to ultrafilters concentrated on* $\operatorname{ext}``X^{X^n}$, *and all relational symbols to ultrafilters concentrated on* $\{Q \subseteq (\boldsymbol{\beta} X)^n : Q$ *is regular closed*$\}$*;*

(ii) $\lim \mathfrak{U} = \mathfrak{A}^*$ *for some ordinary model $\mathfrak{A}$ iff the universe of $\mathfrak{U}$ is $\boldsymbol{\beta} X$ for some X and the interpretation takes all functional symbols to ultrafilters in* $\{I(\mathfrak{f}) : \mathfrak{f} \in \boldsymbol{\beta}(X^{X^n})$ *is pseudo-principal*$\}$, *and all relational symbols to ultrafilters concentrated on* $\{Q \subseteq (\boldsymbol{\beta} X)^n : Q$ *is regular closed*$\}$.

Proof. Item (i) is immediate from Lemma 4.13; recall that the images of ultrafilters over $\mathcal{P}(X^n)$ under I are exactly ultrafilters over $\mathcal{P}((\boldsymbol{\beta} X)^n)$ that are concentrated on $\operatorname{cl}_{\boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n}`` \mathcal{P}(X^n)$:

$$\begin{aligned}
i``\boldsymbol{\beta}\mathcal{P}(X^n) &= \big\{\mathfrak{s} \in \boldsymbol{\beta}\mathcal{P}((\boldsymbol{\beta} X)^n) : \operatorname{cl}_{\boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n}`` \mathcal{P}(X^n) \in \mathfrak{s}\big\} \\
&= \big\{\mathfrak{s} \in \boldsymbol{\beta}\mathcal{P}((\boldsymbol{\beta} X)^n) : \{R^* : R \subseteq X^n\} \in \mathfrak{s}\big\} \\
&= \big\{\mathfrak{s} \in \boldsymbol{\beta}\mathcal{P}((\boldsymbol{\beta} X)^n) : \{Q \subseteq (\boldsymbol{\beta} X)^n : Q \text{ is regular closed}\} \in \mathfrak{s}\big\}.
\end{aligned}$$

Item (ii) follows from item (i) and Theorem 3.22. □

Second Extension Theorems. Now we define homomorphisms between ultrafilter models in the wide sense as homomorphisms of their limits:

Definition 4.20. Let $\mathfrak{U}$ and $\mathfrak{V}$ be two ultrafilter models in the wide sense, of the same signature, with the universes X and Y, respectively. A map $h : X \to Y$ is a *homomorphism* (*of ultrafilter models in the wide sense*) iff it is a homomorphism of $\lim \mathfrak{U}$ into $\lim \mathfrak{V}$.

Theorem 4.5 guaranties that for ultrafilter models in the narrow sense, Definition 4.20 gives the same, up to the identification map i, that Definition 3.26. Similar concepts (*epimorphisms*, *quotients*, *isomorphic embeddings*, *submodels*, *elementary embeddings*, *elementary submodels*, etc. of ultrafilter models in the wide sense) are defined likewise and also coincide with the corresponding concepts for ultrafilter models in the narrow sense.

The proofs of the Second Extension Theorems (Theorems 1.11 and 1.13) are based on Theorems 1.10 and 1.12, which describe the topological properties of the $\tilde{}$- and *-extensions, respectively, and a result called the "abstract extension theorem" in [28]. This result is rather about restrictions of continuous maps than about continuous extensions of maps, but it also states that such a map is a homomorphism of the whole models whenever it is a homomorphism of certain submodels in them; we restate it in the next theorem:

Theorem 4.21. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two (ordinary) models of the same signature whose universes X and Y, respectively, both carry topologies, the topology on Y is Hausdorff, and let $D \subseteq X$ be a dense subset of X which forms a submodel $\mathfrak{D}$ of $\mathfrak{A}$. Let, moreover, $h : X \to Y$ be a continuous map, and suppose that*

- *(a) all operations in $\mathfrak{A}$ are right continuous w.r.t. D, and in $\mathfrak{B}$ right continuous w.r.t. $h``D$,*
- *(b) one of two following items holds:*
  - *(α) all relations in $\mathfrak{A}$ are right open w.r.t. D, and in $\mathfrak{B}$ right closed w.r.t. $h``D$,*
  - *(β) all relations in $\mathfrak{A}$ are regular closed in the product topology on X^n, in $\mathfrak{B}$ closed in the product topology on Y^n (where n is the arity of a given relation), and h is a closed map.*

Then the following are equivalent:

- *(i) $h \restriction D$ is a homomorphism of $\mathfrak{D}$ into $\mathfrak{B}$,*
- *(ii) h is a homomorphism of $\mathfrak{A}$*

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, dashed, "h"] & \mathfrak{B} \\
\mathfrak{D} \arrow[u] \arrow[r, "h \restriction D"] & \mathfrak{E} \arrow[u]
\end{tikzcd}$$

where $\mathfrak{E}$ denotes the submodel of $\mathfrak{B}$ with the universe $h``D$.

Proof. That (ii) implies (i) is trivial since $\mathfrak{D}$ is a submodel of $\mathfrak{A}$. For the converse implication in the case of (α), see [27] or [28], Theorem 4.1. The case of (β) is obtained from the case of (α) as follows.

If R is an n-ary relation on X belonging to the model $\mathfrak{A}$, consider R as a unary relation on X^n and note that under (i), the restriction $h \restriction D$ is also a homomorphism between the model $(D^n, (\mathrm{int}_{X^n} R) \cap D^n)$ and the model (Y^n, S), where $\mathrm{int}_{X^n} R$ is the interior of R in the product topology on X^n, so the set $(\mathrm{int}_{X^n} R) \cap D^n = \mathrm{int}_{D^n}(R \cap D^n)$ is an open unary relation on D^n, and S is the relation on Y interpreting the same predicate symbol that R doing and also considered as a unary relation on Y^n. By (α) we conclude that h is a homomorphism between $(X^n, \mathrm{int}_{X^n} R)$ and (Y^n, S). But as in the case of (β) the map h is closed, we have: $h``\mathrm{cl}_{X^n}\mathrm{int}_{X^n} R \subseteq \mathrm{cl}_{Y^n} S$, thus h is a homomorphism between $(X^n, \mathrm{cl}_{X^n}\mathrm{int}_{X^n} R)$ and $(Y^n, \mathrm{cl}_{Y^n} S)$. Finally, as under ($\beta$), R is regular closed in X^n and S is closed in Y^n, we have $\mathrm{cl}_{X^n}\mathrm{int}_{X^n} R = R$ and $\mathrm{cl}_{Y^n} S = S$, thus showing that h is a homomorphism between (X^n, R) and (Y^n, S), and hence, between (X, R) and (Y, S) where the relations R and S are considered as n-ary. This gives (ii), completing the proof of the theorem. $\square$

Remark 4.22. The argument for proving (β) from (α) allows to obtain stronger statements. Instead of the assumption of (β), it suffices to suppose that the interior of R is dense in the closure of R, i.e. the closure of R is regular closed: $\mathrm{cl}_{X^n}\mathrm{int}_{X^n}R = \mathrm{cl}_{X^n}R$. This includes the cases of open as well as of regular closed R. Also instead of the assumption of (α), it suffices to suppose that R "has right dense interior w.r.t. D", i.e. that for each $i, 1 \leqslant i \leqslant n$, and every $a_1, \ldots, a_{i-1} \in D$ and $x_{i+1}, \ldots, x_n \in X$, the set $\{x \in X : (a_1, \ldots, a_{i-1}, x, x_{i+1}, \ldots, x_n) \in R\}$ has the interior dense in the closure of this set. As easy to see, in both cases the same proof works as well.

The same is applied to the following theorem.

The next result is an immediate analog of Theorem 4.21 for ultrafilter models in the wide sense.

Theorem 4.23. *Let $\mathfrak{U}$ and $\mathfrak{V}$ be two ultrafilter models in the wide sense, of the same signature, whose universes X and Y, respectively, both carry topologies, the topology on Y is Hausdorff, and let $D \subseteq X$ be a dense subset of X which forms an ultrafilter submodel $\mathfrak{D}$ of $\mathfrak{U}$. Let, moreover, $h : X \to Y$ be a continuous map, and suppose that for any $n < \omega$,*

(a) n-ary functional symbols are interpreted: in $\mathfrak{U}$ by ultrafilters having limits in $RC_D(X^n, X)$, and in $\mathfrak{V}$ by ultrafilters having limits in $RC_{h``D}(Y^n, Y)$,

(b) one of two following items holds:

(α) n-ary predicate symbols are interpreted: in $\mathfrak{U}$ by ultrafilters having limits in $\{R \subseteq X^n : R$ is right open w.r.t. $D\}$, and in $\mathfrak{V}$ by ultrafilters having limits in $\{S \subseteq Y^n : S$ is right closed w.r.t. $h``D\}$,

(β) n-ary predicate symbols are interpreted: in $\mathfrak{U}$ by ultrafilters having limits in $\{R \subseteq X^n : R$ is regular closed$\}$, and in $\mathfrak{V}$ by ultrafilters having limits in $\{S \subseteq Y^n : S$ is closed$\}$, in the product topologies on X^n and Y^n, respectively, and h is a closed map.

Then the following are equivalent:

(i) $h \upharpoonright D$ is a homomorphism of $\mathfrak{D}$ into $\mathfrak{V}$,

(ii) h is a homomorphism of $\mathfrak{U}$ into $\mathfrak{V}$:

$$\begin{tikzcd}
\mathfrak{U} \arrow[rr, dashed, "h"] \arrow[ddr, "\lim"] & & \mathfrak{V} \arrow[ddr, "\lim"] & \\
\mathfrak{D} \arrow[u] \arrow[rr, "h\upharpoonright D"] \arrow[ddr, "\lim"'] & & \mathfrak{E} \arrow[u] \arrow[ddr, "\lim"] & \\
& \lim \mathfrak{U} \arrow[rr, dashed, "h"] & & \lim \mathfrak{V} \\
& \lim \mathfrak{D} \arrow[u] \arrow[rr, "h\upharpoonright D"] & & \lim \mathfrak{E} \arrow[u]
\end{tikzcd}$$

where $\mathfrak{E}$ denotes the ultrafilter submodel of $\mathfrak{V}$ with the universe $h``D$.

Proof. By definition, homomorphisms of ultrafilter models $\mathfrak{D}$, $\mathfrak{U}$, and $\mathfrak{V}$ are precisely homomorphisms of the ordinary models $\lim \mathfrak{D}$, $\lim \mathfrak{U}$, and $\lim \mathfrak{V}$. Now apply Theorem 4.21 to the latter three models. □

Note that Theorem 4.23 includes Theorem 4.21 as a particular case by identifying operations and relations with principal ultrafilters given by them as in Theorem 4.4.

Before formulating an extension theorem for ultrafilter models in the wide sense, let us state one more auxiliary result:

Lemma 4.24. *Let $\mathfrak{U}$ and $\mathfrak{V}$ be two ultrafilter models in the wide sense, of the same signature, whose universes $\boldsymbol{\beta}X$ and Y, respectively, both carry topologies where the topology on $\boldsymbol{\beta}X$ is standard, let $\mathfrak{U}$ coincide, up to the identification map i, with an ultrafilter model $\mathfrak{B}$ in the narrow sense (also having the universe $\boldsymbol{\beta}X$), and let $g : \boldsymbol{\beta}X \to Y$. Then the following are equivalent:*

(i) g is a homomorphism of $\mathfrak{U}$ into $\mathfrak{V}$,

(ii) g is a homomorphism of $e(\mathfrak{B})$ into $\mathfrak{V}$.

If moreover, the interpretation in $\mathfrak{V}$ takes n-ary predicate symbols to ultrafilters having limits in $\{S \subseteq Y^n : S \text{ is closed}\}$ where Y^n carries the product topology, and the map g is closed, then the following item:

(iii) g is a homomorphism of $E(\mathfrak{B})$ into $\mathfrak{V}$,

also is equivalent to each of items (i) and (ii):

$$\begin{tikzcd}
 & \mathfrak{U} \arrow[dd, "\lim"] \arrow[ddrr, "g"] & & \\
\mathfrak{B} \arrow[ur, "i"] \arrow[rr, "E"] \arrow[dr, "e"'] & & E(\mathfrak{B}) \arrow[dr, "g"] & \\
 & e(\mathfrak{B}) \arrow[rr, "g"] & & \mathfrak{V}
\end{tikzcd}$$

Proof. Before proving the equivalences, recall that by Theorem 4.4, the ordinary models $e(\mathfrak{B})$ and $E(\mathfrak{B})$ are identified with ultrafilter models in the wide sense having the principal interpretations, and the limits of the principal ultrafilters over the sets of operations and relations are the operations and relations that generate them. The formulations of (i) and (ii) imply such an identification.

The equivalence of items (i) and (ii) requires no special assumptions about $\mathfrak{B}$, $\mathfrak{V}$, and g; it is immediate from the following: our definition of homomorphisms of ultrafilter models via their limits, the assumption $\mathfrak{U} = i(\mathfrak{B})$, and the equality $e(\mathfrak{B}) = \lim i(\mathfrak{B})$ stated in Theorem 4.9.

To prove that item (iii) under the additional assumption is also equivalent to (i) and (ii), we repeat the part of the proof of Theorem 4.21 that deduces the case of (β) from the case of (α), taking into account Theorem 3.23 stating that all relations in $E(\mathfrak{B})$ are regular closed in the product topology on $(\boldsymbol{\beta}X)^n$.

The proof is complete. □

Now we are ready to formulate a version of the Second Extension Theorem for ultrafilter models in the wide sense.

Theorem 4.25. *Let $\mathfrak{U} = (\boldsymbol{\beta} X, \mathfrak{f}, \ldots, \mathfrak{r}, \ldots)$ and $\mathfrak{V} = (Y, \mathfrak{g}, \ldots, \mathfrak{s}, \ldots)$ be two ultrafilter models in the wide sense, of the same signature, let $\boldsymbol{\beta} X$ carry its standard topology and Y a compact Hausdorff topology, let $h : X \to Y$, and suppose that*

(a) $\mathfrak{U}$ coincides, up to the identification map i, with an ultrafilter model $\mathfrak{B}$ in the narrow sense, and the interpretation in $\mathfrak{B}$ is pseudo-principal on functional symbols with $\mathfrak{A}$ the principal submodel (having the universe X),

(b) the interpretation in $\mathfrak{V}$ takes all n-ary functional symbols to ultrafilters having limits in $RC_{h``X}(Y^n, Y)$, and all $n$-ary predicate symbols to ultrafilters having limits in $\{S \subseteq Y^n : S$ is right closed w.r.t. $h``X\}$, for any $n < \omega$.

Then the following are equivalent:

(i) h is a homomorphism of $\mathfrak{A}$ into $\mathfrak{V}$,

(ii) $\widetilde{h}$ is a homomorphism of $\mathfrak{U}$ into $\mathfrak{V}$,

(iii) $\widetilde{h}$ is a homomorphism of $\widetilde{\mathfrak{A}}$ into $\mathfrak{V}$.

If moreover, the interpretation in $\mathfrak{V}$ takes n-ary predicate symbols to ultrafilters having limits in $\{S \subseteq Y^n : S$ is closed$\}$ where Y^n carries the product topology, then the following item:

(iv) $\widetilde{h}$ is a homomorphism of $\mathfrak{A}^*$ into $\mathfrak{V}$,

also is equivalent to each of items (i)–(iii):

$$\begin{tikzcd}
 & \mathfrak{U} \arrow[dd, "\lim"'] \arrow[dddrrr, dashed, "\widetilde{h}"] & & & \\
\mathfrak{B} \arrow[ur, "i"] \arrow[rrr, "E"] \arrow[dr, "e"'] & & & \mathfrak{A}^* \arrow[ddr, dashed, "\widetilde{h}"] & \\
 & \widetilde{\mathfrak{A}} \arrow[drrr, dashed, "\widetilde{h}"] & & & \\
 & & \mathfrak{A} \arrow[ul] \arrow[uur] \arrow[rr, "h"] & & \mathfrak{V}
\end{tikzcd}$$

Proof. The assumptions about $\mathfrak{U}$, $\mathfrak{V}$, and $\widetilde{h}$ of this theorem repeat the assumptions about $\mathfrak{U}$, $\mathfrak{V}$, and g of Lemma 4.24 with an extra requirement stating that the interpretation in $\mathfrak{B}$ is pseudo-principal on functional symbols. So as the principal submodel of $\mathfrak{B}$ is $\mathfrak{A}$, we have $e(\mathfrak{B}) = \widetilde{\mathfrak{A}}$ and $E(\mathfrak{B}) = \mathfrak{A}^*$ by Theorem 3.22. Thus by Lemma 4.24, items (ii) and (iii) are equivalent, and under the additional assumption about $\mathfrak{V}$, item (iv) is also equivalent to each of them.

Let us now prove that (i) and (ii) are equivalent. It suffices to show that the models $\mathfrak{U}$ and $\mathfrak{V}$ satisfy the conditions of Theorem 4.23(α). For $\mathfrak{V}$, this is true by the assumption of (b). As for $\mathfrak{U}$, by the assumption of (a) we have $\mathfrak{U} = i(\mathfrak{B})$ with $\mathfrak{B}$ an ultrafilter model in the narrow sense. Furthermore, $i(\mathfrak{B})$ is an ultrafilter model in the wide sense, and by Lemma 4.7, the interpretation of $i(\mathfrak{B})$ takes functional symbols to ultrafilters concentrated on $RC_X((\boldsymbol{\beta} X)^n, \boldsymbol{\beta} X)$, and relational symbols to ultrafilters concentrated on $\{Q \subseteq (\boldsymbol{\beta} X)^n : Q$ is right clopen w.r.t. $X\}$. Therefore, the ultrafilters have limits in these sets endowed with the X-pointwise convergence topologies as the latter are compact Hausdorff by Lemma 2.4. Thus $\mathfrak{U}$ also satisfies the conditions of Theorem 4.23(α), with the principal submodel $\mathfrak{A}$ here as the submodel $\mathfrak{D}$ from that theorem (again by identifying ordinary models with ultrafilter models having the principal interpretations). This shows the equivalence of (i) and (ii), thus completing the proof. $\square$

Finally, by changing i with I, we obtain the counterparts of Lemma 4.24 and Theorem 4.25:

Lemma 4.26. *Let $\mathfrak{U}$ and $\mathfrak{V}$ be two ultrafilter models in the wide sense, of the same signature, whose universes $\boldsymbol{\beta} X$ and Y, respectively, both carry topologies where the topology on $\boldsymbol{\beta} X$ is standard, let $g : \boldsymbol{\beta} X \to Y$, and suppose that*

(a) $\mathfrak{U} = I(\mathfrak{B})$ for some ultrafilter model $\mathfrak{B}$ in the narrow sense (also having the universe $\boldsymbol{\beta} X$),

(b) the interpretation in $\mathfrak{V}$ takes all n-ary functional symbols to ultrafilters having limits in $RC_{g``X}(Y^n, Y)$, and all n-ary predicate symbols to ultrafilters having limits in $\{S \subseteq S^n : R$ is closed$\}$ where Y^n carries the product topology.

Then the following are equivalent:

(i) g is a homomorphism of $\mathfrak{U}$ into $\mathfrak{V}$,

(ii) g is a homomorphism of $e(\mathfrak{B})$ into $\mathfrak{V}$,

(iii) g is a homomorphism of $E(\mathfrak{B})$ into $\mathfrak{V}$:

$$\begin{tikzcd}
& \mathfrak{U} \arrow[dr, "\lim"] \arrow[dddrr, "g"] & & \\
\mathfrak{B} \arrow[ur, "I"] \arrow[rr, "E"'] \arrow[dr, "e"'] & & E(\mathfrak{B}) \arrow[ddr, "g"] & \\
& e(\mathfrak{B}) \arrow[drr, "g"] & & \\
& & & \mathfrak{V}
\end{tikzcd}$$

Proof. Again, by using Theorem 4.4, we identify the ordinary models $e(\mathfrak{B})$ and $E(\mathfrak{B})$ in (ii) and (iii) with the corresponding ultrafilter models in the wide sense having the principal interpretations (and thus having $e(\mathfrak{B})$ and $E(\mathfrak{B})$ as their limits). The equivalence of items (i) and (iii) is immediate from the following: our definition of homomorphisms of ultrafilter models via their limits, the assumption $\mathfrak{U} = I(\mathfrak{B})$, and the equality $e(\mathfrak{B}) = \lim I(\mathfrak{B})$ stated in Theorem 4.18. But then item (ii) is also equivalent to each of items (i) and (iii) since the identity map on $\boldsymbol{\beta} X$ is a homomorphism of $e(\mathfrak{B})$ onto $E(\mathfrak{B})$ by Theorem 3.19. The proof is complete. □

Theorem 4.27. *Let $\mathfrak{U} = (\boldsymbol{\beta} X, \mathfrak{f}, \ldots, \mathfrak{r}, \ldots)$ and $\mathfrak{V} = (Y, \mathfrak{g}, \ldots, \mathfrak{s}, \ldots)$ be two ultrafilter models in the wide sense, of the same signature, let $\boldsymbol{\beta} X$ carry its standard topology and Y a compact Hausdorff topology, let $h : X \to Y$, and suppose that*

(a) $\mathfrak{U} = I(\mathfrak{B})$ for some ultrafilter model $\mathfrak{B}$ in the narrow sense, and the interpretation in $\mathfrak{B}$ is pseudo-principal on functional symbols with $\mathfrak{A}$ the principal submodel (having the universe X),

(b) the interpretation in $\mathfrak{V}$ takes all n-ary functional symbols to ultrafilters having limits in $RC_{h``X}(Y^n, Y)$, and all n-ary predicate symbols to ultrafilters having limits in $\{S \subseteq Y^n : S$ is closed$\}$ where Y^n carries the product topology.

Then the following are equivalent:

(i) h is a homomorphism of $\mathfrak{A}$ into $\mathfrak{V}$,

(ii) $\widetilde{h}$ is a homomorphism of $\mathfrak{U}$ into $\mathfrak{V}$,

(iii) $\widetilde{h}$ is a homomorphism of $\widetilde{\mathfrak{A}}$ into $\mathfrak{V}$,

(iv) $\widetilde{h}$ is a homomorphism of $\mathfrak{A}^*$ into $\mathfrak{V}$:

$$\begin{tikzcd}
{} & \mathfrak{U} \arrow[dr, "\lim"] \arrow[dddrr, dashed, "\widetilde{h}"] & {} & {} \\
\mathfrak{B} \arrow[ur, "I"] \arrow[rr, "E"'] \arrow[dr, "e"'] & {} & \mathfrak{A}^* \arrow[ddr, dashed, "\widetilde{h}"] & {} \\
{} & \widetilde{\mathfrak{A}} \arrow[drr, dashed, "\widetilde{h}"] & {} & {} \\
{} & {} & \mathfrak{A} \arrow[ul] \arrow[uu] \arrow[r, "h"] & \mathfrak{V}
\end{tikzcd}$$

Proof. The assumptions about $\mathfrak{U}$, $\mathfrak{V}$, and $\widetilde{h}$ of this theorem repeat the assumptions about $\mathfrak{U}$, $\mathfrak{V}$, and g of Lemma 4.26 with an extra requirement stating that the interpretation in $\mathfrak{B}$ is pseudo-principal on functional symbols. So as the principal submodel of $\mathfrak{B}$ is $\mathfrak{A}$, we have $e(\mathfrak{B}) = \widetilde{\mathfrak{A}}$ and $E(\mathfrak{B}) = \mathfrak{A}^*$ by Theorem 3.22. Thus by Lemma 4.26, items (ii), (iii), and (iv) all are equivalent.

Let us now prove that (i) and (ii) are equivalent. It suffices to show that the models $\mathfrak{U}$ and $\mathfrak{V}$ satisfy the conditions of Theorem 4.23(β). For $\mathfrak{V}$, this is true by the assumption of (b). As for $\mathfrak{U}$, by the assumption of (a) we have $\mathfrak{U} = I(\mathfrak{B})$ with $\mathfrak{B}$ an ultrafilter model in the narrow sense. Furthermore, $I(\mathfrak{B})$ is an ultrafilter model in the wide sense, and by Lemma 4.13, the interpretation of $I(\mathfrak{B})$ takes functional symbols to ultrafilters concentrated on $RC_X((\boldsymbol{\beta} X)^n, \boldsymbol{\beta} X)$, and relational symbols to ultrafilters concentrated on $\{Q \subseteq (\boldsymbol{\beta} X)^n : Q \text{ is regular closed}\}$. Therefore, the ultrafilters have limits in these sets endowed with the corresponding compact Hausdorff topologies described above. Thus $\mathfrak{U}$ also satisfies the conditions of Theorem 4.23(β), with the principal submodel $\mathfrak{A}$ here as the submodel $\mathfrak{D}$ from that theorem (again by identifying ordinary models with ultrafilter models having the principal interpretations). This shows the equivalence of (i) and (ii), thus completing the proof. □

Remark 4.28. Theorems 4.21–4.27 admits some variants and generalizations. E.g. they remain true for epimorphisms (since for any compact Hausdorff Y, if $h : X \to Y$ is such that $h``X$ is dense in Y, then $\widetilde{h} : \boldsymbol{\beta} X \to Y$ is surjective), as well as for homotopies and isotopies (in sense of [27], [28]), which can be defined for ultrafilter models in the wide sense in the same way as this was done for homomorphisms and embeddings. Also versions for multi-sorted models (having rather many universes $X_1, \ldots, X_n$ than one universe X) can be easily stated.

5 Problems

This section contains a list of questions and tasks, including all ones posed in the text above. Some of them are rather technical (Problems 5.1, 5.3, 5.4, 5.7) while others are more program.

Problem 5.1. Does Lemma 2.5 remain true for the space $RC_{X_1,\ldots,X_{n-1}}(\boldsymbol{\beta} X_1, \ldots, \boldsymbol{\beta} X_n, Y)$ (or moreover, the space $Y^{\boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n}$) endowed with the full pointwise convergence topology? i.e.

given discrete spaces $X_1, \dots, X_n$, a compact Hausdorff space Y, and a dense subset S of Y, is the set

$$\mathrm{ext}``\, S^{X_1\times\ldots\times X_n} = \{\widetilde{f} : f \in S^{X_1\times\ldots\times X_n}\}$$

dense in this space? It can be seen that the answer is affirmative for unary maps, i.e. the set $\{\widetilde{f} : f \in S^X\}$ is dense in $C(\boldsymbol{\beta} X, Y)$. What happens for binary maps?

Problem 5.2. Given discrete $X_1, \dots, X_n$ and compact Hausdorff Y, let ext be a map of $Y^{X_1\times\ldots\times X_n}$ endowed with the discrete topology into $Y^{\boldsymbol{\beta} X_1\times\ldots\times\boldsymbol{\beta} X_n}$ endowed with the usual product topology (or equivalently, the usual pointwise convergence topology). As the range is a compact Hausdorff space, the map ext continuously extends to $\widetilde{\mathrm{ext}}$:

$$\begin{tikzcd}
\boldsymbol{\beta}\big(Y^{X_1\times\ldots\times X_n}\big) \arrow[dr, dashed, "\widetilde{\mathrm{ext}}"] & \\
Y^{X_1\times\ldots\times X_n} \arrow[u] \arrow[r, "\mathrm{ext}"] & Y^{\boldsymbol{\beta} X_1\times\ldots\times\boldsymbol{\beta} X_n}
\end{tikzcd}$$

Can this alternative version of self-applying of the map ext lead to some interesting possibilities, including variants of the theory of ultrafilter models?

Note that now $\widetilde{\mathrm{ext}}``\boldsymbol{\beta}(Y^{X_1\times\ldots\times X_n})$ does not coincide with $\mathrm{ext}``\,Y^{X_1\times\ldots\times X_n}$ (unlike our previous situation); however, the latter set is still dense in the former:

$$\mathrm{ext}``\; Y^{X_1\times\ldots\times X_n} \subset \widetilde{\mathrm{ext}}``\; \boldsymbol{\beta}\big(Y^{X_1\times\ldots\times X_n}\big) = \mathrm{cl}_{Y^{\boldsymbol{\beta} X_1\times\ldots\times\boldsymbol{\beta} X_n}}\big(\mathrm{ext}``\; Y^{X_1\times\ldots\times X_n}\big).$$

Also, is this version of $\widetilde{\mathrm{ext}}$ surjective? This would be the case if the previous question in its stronger form, i.e. for the space $Y^{\boldsymbol{\beta} X_1\times\ldots\times\boldsymbol{\beta} X_n}$, had the affirmative answer.

Problem 5.3. For which compact Hausdorff spaces Y, instead of $\boldsymbol{\beta} Y$ with a discrete Y, does Lemma 3.10 remain true, i.e. for any discrete $X_1, \dots, X_n$ and the map $\widetilde{\mathrm{app}}$ defined as in the remark in the beginning of Section 3:

$$\begin{tikzcd}
\boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n \times \boldsymbol{\beta}(Y^{X_1\times\ldots\times X_n}) \arrow[dr, dashed, "\widetilde{\mathrm{app}}"] & \\
X_1 \times \ldots \times X_n \times Y^{X_1\times\ldots\times X_n} \arrow[u] \arrow[r, "\mathrm{app}"] & Y
\end{tikzcd}$$

the statements

$$\begin{aligned}
&e(\mathfrak{f})(\mathfrak{u}_1, \dots, \mathfrak{u}_n) = \widetilde{\mathrm{app}}(\mathfrak{u}_1, \dots, \mathfrak{u}_n, \mathfrak{f}),\\
&e(\mathfrak{r})(\mathfrak{u}_1, \dots, \mathfrak{u}_n) \ \text{ iff } \ \widetilde{\mathrm{in}}\,(\mathfrak{u}_1, \dots, \mathfrak{u}_n, \mathfrak{r})
\end{aligned}$$

hold for all $\mathfrak{f} \in \boldsymbol{\beta}(Y^{X_1\times\ldots\times X_n})$, $\mathfrak{r} \in \boldsymbol{\beta}\mathcal{P}(X_1 \times \ldots \times X_n)$, and $\mathfrak{u}_1 \in \boldsymbol{\beta} X_1, \dots, \mathfrak{u}_n \in \boldsymbol{\beta} X_n$? Does this hold at least for all compact Hausdorff spaces Y that are zero-dimensional, or extremally disconnected?

Problem 5.4. What are topological properties of the subset of the space $\boldsymbol{\beta}(Y^{X_1\times\ldots\times X_n})$ consisting of pseudo-principal ultrafilters? Of the preimage of this set under e, i.e. the set $\{\widetilde{f} : f \in Y^{X_1\times\ldots\times X_n}\}$, in the space $RC_{X_1,\ldots,X_{n-1}}(\boldsymbol{\beta} X_1, \dots, \boldsymbol{\beta} X_n, \boldsymbol{\beta} Y)$ with the $(X_1, \dots, X_n)$-pointwise convergence topology (except for the fact that it is dense there, as stated in Lemma 2.5), or with the (usual) pointwise convergence topology? in the space $(\boldsymbol{\beta} Y)^{\boldsymbol{\beta} X_1\times\ldots\times\boldsymbol{\beta} X_n}$ with the pointwise convergence topology?

Often objects naturally defined in terms of ultrafilter extensions have rather hardly definable topological properties, as shown in [18, 19].

In the next two problems, we wonder about variants of the definition of the satisfiability in ultrafilter models in the narrow sense.

Problem 5.5. Define an alternative satisfaction relation $\Vdash$ by using rather in^* than $\widetilde{\mathrm{in}}$; i.e. if $R(t_1, \dots, t_n)$ is an atomic formula in which R is not the equality predicate, let

$$\mathfrak{A} \Vdash R(t_1, \dots, t_n)\ [v] \ \text{ iff } \ \mathrm{in}^*(v_\imath(t_1), \dots, v_\imath(t_n), \imath(P)).$$

Does this give a E-counterpart of the semantics of ultrafilter models in the narrow sense? More precisely, is the following E-counterpart of Theorem 3.16 true: If $\mathfrak{A}$ is an ultrafilter model in the narrow sense, then for all formulas φ and elements $\mathfrak{u}_1, \dots, \mathfrak{u}_n$ of the universe of $\mathfrak{A}$,

$$\mathfrak{A} \Vdash \varphi\,[\mathfrak{u}_1, \dots, \mathfrak{u}_n] \ \text{ iff } \ E(\mathfrak{A}) \vDash \varphi\,[\mathfrak{u}_1, \dots, \mathfrak{u}_n]$$

(where $\Vdash$ has this new meaning)?

Problem 5.6. Another way to vary the definition $\Vdash$ is by letting

$$\mathfrak{A} \Vdash R(t_1, \dots, t_n)\ [v] \ \text{ iff } \ \left(\forall^{\imath(R)} Q\right)\, \widetilde{Q}(v_\imath(t_1), \dots, v_\imath(t_n)).$$

This version looks less smooth. Does this, nevertheless, give something interesting?

Problem 5.7. To define the map I, we considered the set RegCl of regular closed subsets of the space $\boldsymbol{\beta} X_1 \times \dots \times \boldsymbol{\beta} X_n$ with a topology turning it into a space homeomorphic to the usual product space $2^{X_1 \times \dots \times X_n}$ with the discrete space 2. Redefine this topology on RegCl as a restricted version of Vietoris topology (in an analogy with the restricted version of pointwise convergence topology turning out RC into a compact Hausdorff space homeomorphic to the product space $Y^{X_1 \times \dots \times X_n}$). Note that in the usual Vietoris topology, the space $\{Q \subseteq \boldsymbol{\beta} X_1 \times \dots \times \boldsymbol{\beta} X_n : Q \text{ is closed}\}$ is compact Hausdorff but RegCl is not a closed subspace of it.

Problem 5.8. Investigate filter extensions of first-order models (as was started in [16, 31]) and the corresponding concepts of filter interpretations and filter models.

Problem 5.9. Isolate and investigate other possible types of ultrafilter extensions (in the sense of Definition 1.1), besides the $\widetilde{\ }$- and *-extensions, establish special features of the two canonical extensions among others (as was proposed at the end of [31]).

Problem 5.10. Investigate ultrafilter extensions of syntax (including those of languages, of valuation and interpretation maps, of the satisfaction relation).

Problem 5.11. Investigate iterations of ultrafilter extensions (taking unions at limit steps).

Problem 5.12. Investigate higher-order and infinitary analogs of ultrafilter extensions and ultrafilter interpretations, more generally, analogs for model-theoretic languages (in the sense of [1]).

Problem 5.13. The *-extensions play a special role in modal propositional logic; if $\mathfrak{A}$ is a model of a relational language, all canonical modal formulas are preserved under passing from $\mathfrak{A}$ to $\mathfrak{A}^*$, provided both first-order models are considered as Kripke frames (see [2, 5]). What is a (nonclassical) propositional logic with a similar property w.r.t. the $\widetilde{\ }$-extensions? (Perhaps, this connects to Shelah's theorem on fragments of second-order logic, see [33, 1].)

Problem 5.14. Do the concepts of ultrafilter interpretations and ultrafilter models have any interesting applications? e.g. combinatorial (Ramsey-theoretic) applications in model theory?

Also a list of problems related to ultrafilter extensions can be found in Section 5 of [28].

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References

[1] J. Barwise, S. Feferman (eds.). *Model-theoretic logics*. Perspectives in Mathem. Logic 8, Springer-Verlag, 1985.

[2] J. F. A. K. van Benthem. *Canonical modal logics and ultrafilter extensions.* J. Symbolic Logic 44:1 (1979), 1–8.

[3] J. F. A. K. van Benthem. *Notes on modal definability.* Notre Dame J. Formal Logic, 30:1 (1988), 20–35.

[4] J. Berglund, H. Junghenn, P. Milnes. *Analysis on semigroups*. Wiley, New York, 1989.

[5] P. Blackburn, M. de Rijke, Y. Venema. *Modal logic*. Cambridge Univ. Press, Cambridge, 2004.

[6] E. Čech. *On bicompact spaces.* Ann. of Math. 38:2 (1937), 823–844.

[7] C. C. Chang, H. J. Keisler. *Model theory.* North-Holland, Amsterdam–London–N.Y., 1973.

[8] W. W. Comfort, S. Negrepontis. *The theory of ultrafilters.* Springer, Berlin, 1974.

[9] M. Di Nasso, L. Luperi Baglini. *Ramsey properties of nonlinear Diophantine equations,* Advances in Mathematics, 324 (2018), 84–117.

[10] R. Ellis. *Distal transformation groups,* Pacific J. Math. 8 (1958), 401–405.

[11] R. Ellis. *Lectures on topological dynamics,* Benjamin, N.Y., 1969.

[12] R. Engelking. *General topology*. Monogr. Matem. 60, Warszawa, 1977.

[13] T. Frayne, A. C. Morel, D. S. Scott. *Reduced direct products.* Fund. Math. 51:3 (1962), 195–228; Correction: 53:1 (1963), 117.

[14] R. I. Goldblatt, S. K. Thomason. *Axiomatic classes in propositional modal logic.* In: J. N. Crossley (ed.). Algebra and Logic. Lecture Notes in Math. 450 (1975), 163–173.

[15] R. I. Goldblatt. *Varieties of complex algebras.* Annals of Pure and Applied Logic, 44 (1989), 173–242.

[16] V. Goranko. *Filter and ultrafilter extensions of structures: universal-algebraic aspects.* Preprint, 2007.

[17] N. Hindman, D. Strauss. *Algebra in the Stone–Čech compactification.* 2nd ed., revised and expanded, W. de Gruyter, Berlin–N.Y., 2012.

[18] N. Hindman, D. Strauss. *Topological properties of some algebraically defined subsets of* $\boldsymbol{\beta}\,\mathbb{N}$. Topology and its Application, 220 (2017), 43–49.

[19] N. Hindman, D. Strauss. *Sets and mapping in* $\boldsymbol{\beta} S$ *which are not Borel.* New York Journal of Mathematics, 24 (2018), 689–701.

[20] B. Jónsson, A. Tarski. *Boolean algebras with operators.* Part I: Amer. J. Math. 73:4 (1951), 891–939. Part II: ibid. 74:1 (1952), 127–162.

[21] A. Kanamori. *The higher infinite: large cardinals in set theory from their beginnings.* 2nd ed., Springer, Berlin, 2005.

[22] S. Kochen. *Ultraproducts in the theory of models.* Ann. of Math. 74:2 (1961), 221–261.

[23] E. J. Lemmon. *Algebraic semantics for modal logic.* Part II: J. Symbolic Logic 31:2 (1966), 191–218.

[24] E. J. Lemmon, D. S. Scott. *An introduction to modal logic.* Oxford, Blackwell, 1977.

[25] N. L. Poliakov, D. I. Saveliev. *On two concepts of ultrafilter extensions of first-order models and their generalizations.* LLIC, Lecture Notes in Computer Science 10388, J. Kennedy, R. J. G. B. de Queiroz (eds.), Springer, Berlin–Heidelberg, 2017, 336–348.

[26] D. I. Saveliev. *On Hindman sets.* Preprint, 2008.

[27] D. I. Saveliev. *Ultrafilter extensions of models.* Lecture Notes in Computer Science 6521 (2011), 162–177.

[28] D. I. Saveliev. *On ultrafilter extensions of models.* The Infinity Project Proc., S.-D. Friedman, M. Koerwien, M. M. Müller (eds.), CRM Documents 11, Barcelona, 2012, 599–616.

[29] D. I. Saveliev. *On idempotents in compact left topological universal algebras*. Topology Proc. 43 (2014), 37–46.

[30] D. I. Saveliev. *Ultrafilter extensions of linearly ordered sets.* Order 32:1 (2015), 29–41.

[31] D. I. Saveliev. *On two types of ultrafilter extensions of binary relations.* arXiv: 2001.02456.

[32] D. I. Saveliev, S. Shelah. *Ultrafilter extensions do not preserve elementary equivalence*. Mathematical Logic Quarterly, 65:4 (2019), 511–516.

[33] S. Shelah. *There are just four second-order quantifiers*. Israel J. Math., 15 (1973), 282–300.

[34] M. H. Stone. *Applications of the theory of Boolean rings to general topology.* Trans. Amer. Math. Soc., 41 (1937), 375–481.

[35] Y. Venema. *Model definability, purely modal.* In: J. Gerbrandy et al. (eds). *JFAK. Essays Dedicated to Johan van Benthem on the Occasion on his 50th Birthday*. Amsterdam, 1999.

[36] H. Wallman. *Lattices and topological spaces.* Ann. of Math. 39 (1938), 112–126.