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Cut elimination for coherent theories in negation normal form

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Abstract
We present a cut-free sequent calculus for a class of first-order theories in negation normal form which include coherent and co-coherent theories alike. All structural rules, including cut, are admissible.

Keywords Cut elimination · Negation normal form · Classical first-order logic

Mathematics Subject Classification 03F03

1 Introduction

At least since [7] one of the main problem in proof theory has been to find for a given first-order theory T an equivalent sequent calculus G satisfying Gentzen's celebrated cut-elimination theorem.[1] When T is just classical or intuitionistic first-order logic, then we can see that the issue has already been settled in the early days of proof theory by Gentzen himself with the sequent calculi LK and LJ, respectively. However, if T consists of non-logical axioms it is not entirely clear whether such a G can be found at all. For G cannot simply be LK or LJ extended with one new 'axiomatic sequent' $\Rightarrow A$ for each non-logical axiom A of T, since in this case there is no guarantee that G would still satisfy cut elimination.

In [6] Negri introduced cut-free sequent calculi for coherent theories, i.e., first-order theories in which the non-logical axioms are (closed) formulas of the form $\forall \bar{x}(P_1 \wedge \cdots \wedge P_n \supset \exists \bar{y}(C_1 \vee \cdots \vee C_m))$, where each P_i is an atom and each C_j is a conjunction of atoms.[2] The methodology of [6] were later generalized in [3] by Negri and Dyckhoff, who showed that any first-order theory admits a cut-free systematization

[1] Here a theory is just as set of axioms in the language of first-order logic.
[2] In [6] coherent theories were called 'geometric'.

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in sequent calculus. To attain such a greater level of generality, the methodology of [3] dictates that for theories not falling under the coherent fragment, the language is to be extended with new symbols and consequently more axioms need to be considered. For example, the axiom of strict seriality, namely $\forall x \exists y(x \leq y \wedge \neg y \leq x)$, is not coherent, nor there exists a coherent formula *in the same language* which is equivalent to it. The basic idea of [3] is that such an axiom is nevertheless equivalent to coherent one *of another, richer, language*. Specifically, if the original language is extended with a new binary predicate constant $\not\leq$ and a new atom $x \not\leq y$ is defined as $\neg x \leq y$, then strict seriality can be written in such extended language as $\forall x \exists y(x \leq y \wedge y \not\leq x)$, which is entirely coherent. However, we need consider also an extra axiom corresponding (at least partly) to the definition of $x \not\leq y$. In general, when the axioms of a theory are of certain syntactic complexity, extending the language may become exceedingly involved and in many ways undesirable. For example, it almost entirely prevents one to prove interesting results such as interpolation theorem which are notoriously sensitive to the choice of the non-logical constants in the language.[3]

In this paper we suggest a much simpler way to extend the methodology of [6] beyond the coherent fragment without resorting to the linguistic extensions of [3]. In particular, we show that if the language is in negation normal form, then the class of theories admitting a cut-free systematization in sequent calculus is significantly larger than the class of coherent ones, although it is still smaller than the class of all first-order theories. Interestingly such a class includes all co-coherent theories investigated in [9], for which so far the methodology of [6] was not applicable, as well as well-known theories such as classical mereology [2] and Tarski's formalization of geometry [12].

2 The calculus G_n

The language $\mathcal{L}_n$ is a first-order language (without identity) in negation normal form, i.e., $\mathcal{L}_n$ contains two sorts of atoms, the positive atoms and the negative ones. A positive (negative) atom is indicated as P ($\bar{P}$, respectively). Compound formulas are built up from atoms, $\perp$ and $\top$ in the usual way using conjunction, disjunction and the quantifiers. The negation $\neg A$ of a formula A is defined inductively as follows: $\neg P := \bar{P}, \neg \bar{P} := P, \neg(B \wedge C) := \neg B \vee \neg C, \neg(B \vee C) := \neg B \wedge \neg C, \neg \forall x B := \exists x \neg B$ and $\neg \exists x B := \forall x \neg B$. The depth $d(A)$ of a formula A is defined inductively as follows: $d(\perp) = d(\top) = d(P) = d(\bar{P}) = 0, d(B \wedge C) = d(B \vee C) = d(B) + d(C) + 1$, and $d(\forall x B) = d(\exists x B) = d(B) + 1$. It follows that $d(\neg A) = d(A)$. We agree that $A \supset B$ and $A \equiv B$ are abbreviations for $\neg A \vee B$ and $(A \supset B) \wedge (B \supset A)$, respectively. A sequent is a pair $\langle \Gamma, \Delta \rangle$ of (finite, possibly empty) multisets Γ, Δ of formulas and will be indicated as $\Gamma \Rightarrow \Delta$. The sequent calculus G_n consists of the following initial sequents and inference rules, where z is the proper variable of $R\forall$ and $L\exists$, namely it does not occur free in their conclusion.

[3] For interpolation in extensions of first-order logic the reader is referred to [4].

$$P, \Gamma \Rightarrow \Delta, P \qquad \bar{P}, \Gamma \Rightarrow \Delta, \bar{P}$$

$$\frac{}{P, \bar{P}, \Gamma \Rightarrow \Delta}\ LNC \qquad \frac{}{\Gamma \Rightarrow \Delta, P, \bar{P}}\ LEM$$

$$\frac{}{\bot, \Gamma \Rightarrow \Delta}\ L\bot \qquad \frac{}{\Gamma \Rightarrow \Delta, \top}\ R\top$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ R\wedge$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ R\vee$$

$$\frac{A(y/x), \forall x A, \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta}\ L\forall \qquad \frac{\Gamma \Rightarrow \Delta, A(z/x)}{\Gamma \Rightarrow \Delta, \forall x A}\ R\forall$$

$$\frac{A(z/x), \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta}\ L\exists \qquad \frac{\Gamma \Rightarrow \Delta, \exists x A, A(y/x)}{\Gamma \Rightarrow \Delta, \exists x A}\ R\exists$$

A logical axiom is an initial sequent or the conclusion of a rule with no premise.[4] The notions of derivation of a sequent and derivable sequent are defined as usual. We assume that if $\Gamma \Rightarrow \Delta$ is derivable and $\Gamma' \Rightarrow \Delta'$ differs from $\Gamma \Rightarrow \Delta$ only in the names of bound variables, then $\Gamma' \Rightarrow \Delta'$ is also derivable. The height $h(\mathcal{D})$ of a derivation $\mathcal{D}$ of $\Gamma \Rightarrow \Delta$ is defined inductively as follows: if $\Gamma \Rightarrow \Delta$ is a logical axiom, then $h(\mathcal{D}) = 0$; if $\Gamma \Rightarrow \Delta$ is the conclusion of a rule with two premises $\Gamma' \Rightarrow \Delta'$ and $\Gamma'' \Rightarrow \Delta''$ whose derivations are $\mathcal{D}'$ and $\mathcal{D}''$, respectively, then $h(\mathcal{D}) = \max(h(\mathcal{D}'), h(\mathcal{D}'')) + 1$; if $\Gamma \Rightarrow \Delta$ is the conclusion of a rule with one premise $\Gamma' \Rightarrow \Delta'$ whose derivation is $\mathcal{D}'$, then $h(\mathcal{D}) = h(\mathcal{D}') + 1$. If $\mathcal{D}$ is a derivation of $\Gamma \Rightarrow \Delta$ and $n \in \mathbb{N}$, then we say that $\Gamma \Rightarrow \Delta$ is n-derivable, if $h(\mathcal{D}) \leq n$. A rule is admissible if its conclusion is derivable, whenever its premises are derivable and it is height-preserving admissible if its conclusion is n-derivable, whenever its premises are n-derivable.

We shall also consider the standard structural rules of weakening, contraction and cut.

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ RW$$

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ LC \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ RC$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ cut$$

We do not assume any of these rules as primitive since they are all admissible. In fact, we shall see that weakening and contraction are also height-preserving admissible.

[4] The labels LNC and LEM are abbreviations for 'law of non-contradiction' and 'law of excluded middle', respectively.

Finally, the rules for negations, i.e.,

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}\, L_\neg \qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}\, R_\neg$$

will also be proved admissible. However, since we are primarily interested in the extensions of G_n we shall postpone the proof of the admissibility of the structural rules and of the negation rules once such extensions have been introduced.

3 The extensions of G_n

We now consider how to extend the calculus G_n with inference rules. We first introduce the notion of coherent formula and coherent theory in $\mathcal{L}_n$. To ease the notation let a literal ℓ be either a positive atom P or a negative one $\bar{P}$. We shall use $\ell, \jmath, \imath, \ldots$ to indicate literals. A (closed) formula of $\mathcal{L}_n$ is *coherent* when it is equivalent to:

$$\forall \bar{x}\Big(\ell_1 \wedge \cdots \wedge \ell_n \supset \exists \bar{y}\big((\jmath 1_1 \wedge \cdots \wedge \jmath 1_p) \vee \cdots \vee (\jmath m_1 \wedge \cdots \wedge \jmath m_q)\big)\Big)$$

Without loss of generality we assume that none of the variables in $\bar{y}$ occurs free in $\ell_1 \wedge \cdots \wedge \ell_n$. We shall also agree that if $n = 0$ ($m = 0$), then the consequent (antecedent) of $\supset$ is $\top$ ($\bot$, respectively). A theory is coherent when all its non-logical axioms are coherent formulas. Given a coherent formula, the corresponding coherent rule C_n is:

$$\frac{\jmath 1_1, \ldots, \jmath 1_p, \ell_1, \ldots, \ell_n, \Gamma \Rightarrow \Delta \quad \cdots \quad \jmath m_1, \ldots, \jmath m_q, \ell_1, \ldots, \ell_n, \Gamma \Rightarrow \Delta}{\ell_1, \ldots, \ell_n, \Gamma \Rightarrow \Delta}\, C_n$$

where in each premise of C_n each variable in $\bar{y}$ of the coherent formula has been replaced by a variable in $\bar{z}$ not occurring free in the conclusion of C_n. Let G_n^* be any extension of G_n with finitely many coherent rules. It is easy to see that a coherent formula A and the corresponding coherent rule C_n are equivalent in the that sense that the sequent $\Rightarrow A$ is derivable in G_n^* and C_n is admissible in the calculus G_n plus the cut rule and the initial sequent $\Rightarrow A$.

To prove the admissibility of the contraction rules, we shall assume that coherent rules satisfy the following closure condition for contraction. Let C_n be a coherent rule of G_n^* such that:

$$\frac{\jmath 1_1, \ldots, \jmath 1_p, \ell, \ell, \ell_1, \ldots, \ell_{n-2}, \Gamma \Rightarrow \Delta \quad \cdots \quad \jmath m_1, \ldots, \jmath m_q, \ell, \ell, \ell_1, \ldots, \ell_{n-2}, \Gamma \Rightarrow \Delta}{\ell, \ell, \ell_1, \ldots, \ell_{n-2}, \Gamma \Rightarrow \Delta}\, C_n$$

Then, the following rule cC_n is also a rule of G_n^*:

$$\frac{J1_1, \ldots, J1_p, \ell, \ell_1, \ldots, \ell_{n-2}, \Gamma \Rightarrow \Delta \quad \cdots \quad Jm_1, \ldots, Jm_q, \ell, \ell_1, \ldots, \ell_{n-2}, \Gamma \Rightarrow \Delta}{\ell, \ell_1, \ldots, \ell_{n-2}, \Gamma \Rightarrow \Delta}\ cC_n$$

We say that the rule cC_n is the contracted version of C_n.

Perhaps the most well-known example of a coherent theory is first-order logic with identity. If the predicates $=$ and $\neq$ are added to $\mathcal{L}_n$, then a sequent calculus for first-order logic with identity can be obtained by extending G_n with the following two coherent rules $Ref_=$ and $Repl_=$, corresponding each to the two coherent axioms of reflexivity and replacement of identicals, $\forall x(\top \supset x = x)$ and $\forall x \forall y(\ell(x) \wedge x = y \supset \ell(y))$.

$$\frac{x = x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ Ref_= \qquad \frac{\ell(y), \ell(x), x = y, \Gamma \Rightarrow \Delta}{\ell(x), x = y, \Gamma \Rightarrow \Delta}\ Repl_=$$

In general, adding a coherent rule to G_n does *not* suffices to obtain a cut-free calculus for a given first-order theory. For example, consider a simple first-order theory axiomatized by the coherent formula $\forall x(x < x \supset \bot)$ and let $\mathsf{G}_n^<$ be the extension of G_n by the following coherent rule with no premise:

$$\frac{}{x < x, \Gamma \Rightarrow \Delta}\ Irref_<$$

As a counter-example to cut elimination in $\mathsf{G}_n^<$ consider the sequent $\Rightarrow x \nless x$. Although such a sequent is derivable by *cut*, i.e.,

$$\frac{\dfrac{}{\Rightarrow x < x, x \nless x}\ LEM \qquad \dfrac{}{x < x \Rightarrow}\ Irref_<}{\Rightarrow x \nless x}\ cut$$

it is clear that a cut-free derivation of it can hardly be found, unless we avail ourselves with the rules of negation. Certainly, using the rules of negation, especially $R_\neg$, the sequent can easily be proved to be cut-free derivable.

$$\dfrac{\dfrac{\dfrac{}{x < x \Rightarrow}\ Irref_<}{\Rightarrow \neg x < x}\ R_\neg}{\Rightarrow x \nless x}\ df_\neg$$

However, since no rule of negation is primitive in G_n, we should show that at least $R_\neg$ is admissible. Alas, it appears that in the presence of $Irref_<$ the admissibility of $R_\neg$ essentially requires the admissibility of *cut*. To see this, let the premise of $R_\neg$ be an instance of $Irref_<$ with A principal formula. Thus, A is $x < x$ and the derivation of the conclusion of $R_\neg$ is:

$$\dfrac{\dfrac{}{x < x, \Gamma \Rightarrow \Delta}\ Irref_<}{\Gamma \Rightarrow \Delta, \neg x < x}\ R_\neg$$

However, it should be clear that the only way to derive the conclusion of $R_\neg$ is to use *cut*.

$$\dfrac{\dfrac{}{\Rightarrow x < x, x \nless x}\,LEM \qquad \dfrac{}{x < x, \Gamma \Rightarrow \Delta}\,Irref_<}{\dfrac{\Gamma \Rightarrow \Delta, x \nless x}{\Gamma \Rightarrow \Delta, \neg x < x}\,df_\neg}\,cut$$

Thus, it seems that there is a circularity in the admissibility of *cut* and the rules of negation: the former presupposes the latter and *vice versa*.

To overcome this obstacle we shall impose on G_n^* an additional closure condition. Let C_n be a coherent rule of G_n^* except $Repl_=$ and Λ the multi-set of its principal atoms. Moreover, let $\mathsf{Rel}(\Lambda) = \bigcup_{\ell \in \Lambda} \mathsf{Rel}(\ell)$, where $\mathsf{Rel}(\ell)$ is the set of predicates of ℓ. In $\mathsf{G}_n^<$, for example, we have $\mathsf{Rel}(\Lambda) = \{<\}$, since $Irref_<$ is the only coherent rule. Now, we shall assume that G_n^* contains all instances of the following rule:

$$\frac{P, \Gamma \Rightarrow \Delta \quad \bar{P}, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,LEM^*$$

where $\mathsf{Rel}(P) \subseteq \mathsf{Rel}(\Lambda)$ or $\mathsf{Rel}(\bar{P}) \subseteq \mathsf{Rel}(\Lambda)$. Moreover, if C_n is $Repl_=$ and $\Gamma \Rightarrow \Delta$ a sequent to be derived in G_n^* such that Δ contains either P or $\bar{P}$ and as well as an identity atom Q or $\bar{Q}$ in Λ, then we shall also add all instances of LEM^* such that $\mathsf{FV}(P, \bar{P}) \cap \mathsf{FV}(Q, \bar{Q}) \neq \emptyset$, where $\mathsf{FV}(A)$ is the set of the free variables of a formula A.

Some remarks on LEM^* are in order. Firstly, LEM^* is a multi-succedent version of the rule *Gem-at* of [8], introduced to provide a single-succedent calculus for classical propositional logic; the main difference is that in *Gem-at* the principal formulas are an atom and its negation, whereas here we have two atoms.

Secondly, from a semantic point of view, LEM^* is just another way to express in sequent calculus the law of excluded middle; in this respect, it does not differ substantially from LEM. The motivation for having another rule for the law of excluded middle is that with LEM^* we can address satisfactorily the issue of circularity in the admissibility of negation rule and *cut*. In the case of $\mathsf{G}_n^<$ the sequent $\Rightarrow x \nless x$ becomes cut-free derivable:

$$\dfrac{\dfrac{}{x < x \Rightarrow x \nless x}\,Irref_< \qquad x \nless x \Rightarrow x \nless x}{\Rightarrow x \nless x}\,LEM^*$$

(The case of the admissibility of $R_\neg$ in $\mathsf{G}_n^<$ can be dealt with similarly). Much more generally, we shall see that in any extension G_n^* of G_n, although $R_\neg$ is still needed in the proof of admissibility of *cut* (Theorem 2), *cut* is not needed at all in the proof of the admissibility of $R_\neg$ (Lemma 2).

Thirdly, LEM^* is manifestly a non-analytic rule; inasmuch as it makes P and $\bar{P}$ disappear, LEM^* is similar to a cut rule. While the presence of a rule similar to cut in the extensions of first-order logic is arguably acceptable from a conceptual point of view, such a rule in a sequent calculus for first-order logic would certainly be a disturbing upshot for anyone subscribing to the idea that logic is analytic. Here we can retain the idea that at least G_n is analytic by imposing that LEM^* is only present in G_n^*. Moreover, the side conditions circumscribe significantly the non-analytic character of G_n^* brought about by LEM^*. Indeed, we need to consider the instances of LEM^* where all predicates occurring in the disappearing formulas occur also in the coherent rules, provided that these rule are not the identity rules. And since we only consider finitely many rules at a time, it is clear that the number of instances to be added is finite. Moreover, if we also consider the identity rules, especially *Repl*$_=$ the disappearing formulas of LEM^* can be taken to be atoms whose variables are the variables occurring in the identity atoms principal in *Repl*$_=$. Also in this case we shall add only finitely many instances of LEM^*.

4 Admissibility results

In this section we shall prove the main admissibility results concerning the structural rules of G_n^* as well as the admissibility of the rules of negation. We proceed by showing first the height-preserving admissibility of weakening and contraction rules. These results will be then used in the proof of the admissibility of the negation rules which, in turn, will be applied in the proof of the admissibility of cut rule.

We need some preliminary results regarding the substitution of variables. Let the variable y be free for x in Γ, Δ. The substitution rule *Sb* is:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma(y/x) \Rightarrow \Delta(y/x)}\ Sb$$

Lemma 1 *The substitution rule is height-preserving admissible in* G_n^*.

Proof Let $\mathcal{D}$ be the derivation of the premise $\Gamma \Rightarrow \Delta$. We proceed by induction on $h(\mathcal{D})$. If $h(\mathcal{D}) = 0$, then $\Gamma \Rightarrow \Delta$ is a logical axiom or the conclusion of a coherent rule with no premise. Then, also the conclusion of *Sb* is a logical axiom or the conclusion of a coherent rule with no premise. If $h(\mathcal{D}) > 0$, then we reason by cases on the last rule R applied in the derivation of $\Gamma \Rightarrow \Delta$. If R is a propositional rule or a quantifier rule, then the proof is as in Lemma 4.1.2 of [8]. If R is LEM^*, then the premises of LEM^* are $P, \Gamma \Rightarrow \Delta$ and $\bar{P}, \Gamma \Rightarrow \Delta$ and the conclusion of *Sb* can be derived as follows:

$$\frac{\dfrac{P, \Gamma \Rightarrow \Delta}{P(y/x), \Gamma(y/x) \Rightarrow \Delta(y/x)}\ Sb \qquad \dfrac{\bar{P}, \Gamma \Rightarrow \Delta}{\bar{P}(y/x), \Gamma(y/x) \Rightarrow \Delta(y/x)}\ Sb}{\Gamma(y/x) \Rightarrow \Delta(y/x)}\ LEM^*$$

Notice that the two applications of *Sb* are height-preserving admissible by the inductive hypothesis and that the application of *Sb* is legitimate since if y is free for x in Γ, Δ, then so is in P, Γ, Δ and $\bar{P}, \Gamma \Rightarrow \Delta$ (because P and $\bar{P}$ are atoms). Finally, if R is a coherent rule C_n with at least one premise, then the proof does not differ substantially from that given in Lemma 1 of [6]. In this case Γ contains the literals $\ell_1, \ldots, \ell_n$. Since

y may be one the proper variables of C_n, we first need replace each premise of C_n all such variables with new ones; this can be done via several applications of Sb, all of which are height-preserving admissible by the inductive hypothesis. Next we apply again Sb to replace x with y and then C_n again. □

Now we can prove that the structural rules, including cut, are admissible in G_n^*. In fact, we shall prove something stronger, namely the height-preserving admissibility of weakening and contraction as well as the height-preserving invertibility of all rules.

Theorem 1 *The rules of weakening and contraction are height-preserving admissible in* G_n^*. *Moreover, all rules are height-preserving invertibile.*

Proof We start with the height-preserving admissibility of weakening. We only consider LW and leave RW to the reader. We proceed by induction on $h(\mathcal{D})$, where $\mathcal{D}$ is the derivation of the premise $\Gamma \Rightarrow \Delta$ of LW. If $h(\mathcal{D}) = 0$, then $\Gamma \Rightarrow \Delta$ is a logical axiom or the conclusion of a coherent rule with no premise. Then, also the conclusion of LW is a logical axiom or the conclusion of a coherent rule with no premise. If $h(\mathcal{D}) > 0$, then $\Gamma \Rightarrow \Delta$ is the conclusion of some rule R and we proceed by cases on R. If R is a propositional or quantifier rule, then the proof is as in Theorem 2 of [6]. If R is LEM^*, then from the premises $P, \Gamma \Rightarrow \Delta$ and $\bar{P}, \Gamma \Rightarrow \Delta$ of LEM^* the conclusion of LW is obtained as follows:

$$\dfrac{\dfrac{P, \Gamma \Rightarrow \Delta}{A, P, \Gamma \Rightarrow \Delta}\,LW \qquad \dfrac{\bar{P}, \Gamma \Rightarrow \Delta}{A, \bar{P}, \Gamma \Rightarrow \Delta}\,LW}{A, \Gamma \Rightarrow \Delta}\,LEM^*$$

Notice that the two applications of LW in the derivation above are height-preserving admissible by the inductive hypothesis. If R is a coherent rule C_n with at least one premise, then Γ contains $\ell_1, \ldots, \ell_n$. Since A may contain some of the proper variables of C_n, we first apply Sb on the premises of C_n to replace all such variables with new ones. Notice that Sb is height-preserving admissible by Lemma 1 and the substitution does not affect $\ell_1, \ldots, \ell_n, \Gamma', \Delta$ in virtue of the variable condition. Then, the conclusion of LW is obtained from here by applying LW (which is height-preserving admissible by the inductive hypothesis) and C_n again.

The proof of height-preserving invertibility of logical rules is similar to the proof of Theorem 3.1.1 and Lemma 4.2.8 in [8]. Moreover, coherent rules as well as LEM^* are height-preserving invertible via the height-preserving admissibility of LW.

To prove height-preserving admissibility of contraction, we proceed by induction on the premise $A, A, \Gamma \Rightarrow \Delta$ of LC (the case of RC is left to the reader). Let $\mathcal{D}$ be the derivation of $A, A, \Gamma \Rightarrow \Delta$. If $h(\mathcal{D}) = 0$, then $A, A, \Gamma \Rightarrow \Delta$ is a logical axiom or the conclusion of a coherent rule with no premise. If it is a logical axiom, then we have the following cases.

(i) $P \in \Gamma \cap \Delta$. The conclusion of LC is also a logical axiom.
(ii) $\bar{P} \in \Gamma \cap \Delta$. The conclusion of LC is also a logical axiom.
(iii) $P, \bar{P} \in \Gamma$. The conclusion of LC is also a logical axiom.
(iv) $P, \bar{P} \in \Delta$. The conclusion of LC is also a logical axiom.

(v) A is P and $\bar{P} \in \Gamma$. The conclusion of LC is also a logical axiom.
(vi) A is $\bar{P}$ and $P \in \Gamma$. The conclusion of LC is also a logical axiom.
(vii) $\bot \in \Gamma$. The conclusion of LC is also a logical axiom.
($viii$) $\top \in \Delta$. The conclusion of LC is also a logical axiom.

If the premise of LC is the conclusion of a coherent rule C_n with no premise, then we distinguish the following three cases:

(ix) none of the occurrences of A is principal in C_n. In this case, $\ell_1, \ldots, \ell_n \in \Gamma$. Then, the conclusion of LC is a conclusion of C_n.
(x) exactly one occurrence of A is principal in C_n. In this case, A is ℓ and $\ell_1, \ldots, \ell_{n-1} \in \Gamma$, namely the premise of LC is $\ell, \ell, \ell_1, \ldots, \ell_{n-1}, \Gamma' \Rightarrow \Delta$. Then, the conclusion $\ell, , \ell_1, \ldots, \ell_{n-1}, \Gamma' \Rightarrow \Delta$ of LC is a conclusion of C_n.
(xi) both occurrences of A are principal of C_n. Thus, A is ℓ and $\ell_1, \ldots, \ell_{n-2} \in \Gamma$, i.e., the premise of LC is $\ell, \ell, \ell_1, \ldots, \ell_{n-2}, \Gamma' \Rightarrow \Delta$. Notice that the conclusion $\ell, \ell_1, \ldots, \ell_{n-2}, \Gamma' \Rightarrow \Delta$ of LC is *not* a conclusion of C_n (since C_n needs n literals to be applied). However the sequent $\ell, \ell_1, \ldots, \ell_{n-2}, \Gamma' \Rightarrow \Delta$ is a conclusion of the contracted version cC_n, which is a rule of G_n^* in virtue of the closure condition.

If $h(\mathcal{D}) > 0$, then we need to reason by cases on the last rule R applied in the derivation of the premise of LC. If R is a propositional or quantifier rule, then see Theorem 4 of [6]. If R is LEM^*, then none of the occurrences of the contracted formula A can be principal. Then, from the premises $P, A, A, \Gamma \Rightarrow \Delta$ and $\bar{P}, A, A, \Gamma \Rightarrow \Delta$ we derive the conclusion of LC as follows:

$$\dfrac{\dfrac{A, A, P, \Gamma \Rightarrow \Delta}{A, P, \Gamma \Rightarrow \Delta}\,LC \qquad \dfrac{A, A, \bar{P}, \Gamma \Rightarrow \Delta}{A, \bar{P}, \Gamma \Rightarrow \Delta}\,LC}{A, \Gamma \Rightarrow \Delta}\,LEM^*$$

Clearly, the two applications of LC are height-preserving admissible by the inductive hypothesis. If R is coherent rule C_n with at least one premise, then have three cases, all similar to (ix)-(xi). Firstly, none of the occurrences of A is principal in C_n. Then, we take the premises of C_n and via several applications of LC (all height-preserving admissible by the inductive hypothesis) and C_n we obtain the conclusion of LC as in (ix). Secondly, exactly one occurrence of A is principal of C_n. The reasoning is as in (x). Thirdly, both occurrences of A are principal. In this case we the conclusion of LC is obtained via several height-preserving admissible applications of LC and the contracted version of cC_n, like in (xi). □

These preliminary results will be now used to prove the admissibility of the negation rules as well as the admissibility of the cut rule. We start with the admissibility of the rules of negation.

Lemma 2 *The rules $L_\neg$ and $R_\neg$ are admissible in G_n^*. Moreover, in each rule if the premise is n-derivable, then the conclusion is $n+1$-derivable.*

Proof We consider $L_\neg$ first. Let $\mathcal{D}$ be the derivation of the premise $\Gamma \Rightarrow \Delta, A$. We proceed by induction on $h(\mathcal{D})$. If $h(\mathcal{D}) = 0$, then $\Gamma \Rightarrow \Delta, A$ is either a logical axiom or the conclusion of a coherent rule with no premise. If it is a logical axiom, then we consider the following cases.

(i) $P \in \Gamma \cap \Delta$. The conclusion of $L_\neg$ is a logical axiom.
(ii) $\bar{P} \in \Gamma \cap \Delta$. The conclusion of $L_\neg$ is a logical axiom.
(iii) $P, \bar{P} \in \Gamma$. The conclusion of $L_\neg$ is a logical axiom.
(iv) $P, \bar{P} \in \Delta$. The conclusion of $L_\neg$ is a logical axiom.
(v) A is P and $P \in \Gamma$. The conclusion of $L_\neg$ is $\neg P, P, \Gamma' \Rightarrow \Delta$ and this is a logical axiom since $\neg P := \bar{P}$.
(vi) A is $\bar{P}$ and $\bar{P} \in \Gamma$. The conclusion of $L_\neg$ is $\neg \bar{P}, \bar{P}, \Gamma' \Rightarrow \Delta$ and this is a logical axiom since $\neg \bar{P} := P$.
(vii) A is P and $\bar{P} \in \Delta$. The conclusion of $L_\neg$ is $\neg P, \Gamma \Rightarrow \Delta', \bar{P}$ and this is a logical axiom since $\neg P := \bar{P}$.
($viii$) A is $\bar{P}$ and $P \in \Delta$. The conclusion of $L_\neg$ is $\neg \bar{P}, \Gamma \Rightarrow \Delta', P$ and this is a logical axiom since $\neg \bar{P} := P$.
(ix) $\bot \in \Gamma$. The conclusion of $L_\neg$ is a logical axiom.
(x) $\top \in \Delta$. The conclusion of $L_\neg$ is a logical axiom.

If $\Gamma \Rightarrow \Delta, A$ is the conclusion of a coherent rule C with no premise, then Γ contains $\ell_1, \ldots, \ell_n$. Also in this case, the conclusion of $L_\neg$ is also a conclusion of C. If $h(\mathcal{D}) > 0$, then we distinguish according to whether A is principal or not principal of R. In the latter case, then the conclusion of $L_\neg$ is obtained from the premise of R by an application of $L_\neg$ and then by one of R. Clearly, such an application of $L_\neg$ is admissible by the inductive hypothesis. (Here we tacitly assumed that R is a rule with one premise, but if it has has two or more premises the reasoning is the same). If A is principal of R, then R is either $R_\wedge$ or $R_\vee$ or $R_\forall$ or else $R_\exists$. We consider only the first case. If R is $R_\wedge$, then A is $B \wedge C$ and $\mathcal{D}$ is:

$$\dfrac{\Gamma \Rightarrow \Delta, B \quad \Gamma \Rightarrow \Delta, C}{\Gamma \Rightarrow \Delta, B \wedge C}\ R_\wedge$$

Considering that $\neg(B \wedge C) := \neg B \vee \neg C$, the conclusion of $L_\neg$ is thus obtained as follows:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, B}{\neg B, \Gamma \Rightarrow \Delta}\ L_\neg \quad \dfrac{\Gamma \Rightarrow \Delta, C}{\neg C, \Gamma \Rightarrow \Delta}\ L_\neg}{\neg B \vee \neg C, \Gamma \Rightarrow \Delta}\ R_\wedge$$

Once again the two applications of $L_\neg$ are admissible by the inductive hypothesis.

We now prove the admissibility of $R_\neg$ which is slightly more challenging. We proceed by induction on $h(\mathcal{D})$, where $\mathcal{D}$ is the derivation of the premise $A, \Gamma \Rightarrow \Delta$. If $h(\mathcal{D}) = 0$, then $A, \Gamma \Rightarrow \Delta$ is a logical axiom or the conclusion of coherent rule with no premise. In the first case, we reason by cases.

(i) $P \in \Gamma \cap \Delta$. The conclusion of $L_\neg$ is a logical axiom.
(ii) $\bar{P} \in \Gamma \cap \Delta$. The conclusion of $L_\neg$ is a logical axiom.
(iii) $P, \bar{P} \in \Gamma$. The conclusion of $L_\neg$ is a logical axiom.
(iv) $P, \bar{P} \in \Delta$. The conclusion of $L_\neg$ is a logical axiom.

(*v*) A is P and $P \in \Delta$. The conclusion of $L_\neg$ is $\Gamma \Rightarrow \Delta', P, \neg P$ and this is a logical axiom since $\neg P := \bar{P}$.
(*vi*) A is $\bar{P}$ and $\bar{P} \in \Delta$. The conclusion of $L_\neg$ is $\Gamma \Rightarrow \Delta', \bar{P}, \neg\bar{P}$ and this is a logical axiom since $\neg\bar{P} := P$.
(*vii*) A is P and $\bar{P} \in \Gamma$. The conclusion of $L_\neg$ is $\bar{P}, \Gamma' \Rightarrow \Delta\neg P$ and this is a logical axiom since $\neg P := \bar{P}$.
(*viii*) A is $\bar{P}$ and $P \in \Gamma$. The conclusion of $L_\neg$ is $P, \Gamma' \Rightarrow \Delta\neg\bar{P}$ and this is a logical axiom since $\neg\bar{P} := P$.
(*ix*) $\bot \in \Gamma$. The conclusion of $L_\neg$ is a logical axiom.
(*x*) $\top \in \Delta$. The conclusion of $L_\neg$ is a logical axiom.

If $A, \Gamma \Rightarrow \Delta$ is the conclusion of a coherent rule C_n with no premise, then we need to distinguish two cases. Either A is principal of C_n or it is not. If it is not, then the conclusion of $R_\neg$ is a conclusion of C_n. Else, A is ℓ_n and $\ell_1, \ldots, \ell_{n-1} \in \Gamma$. In this case we use LEM^*. In particular, we need to find a derivation $\mathcal{D}'$ of the the sequent $\ell_1, \ldots, \ell_{n-1}, \Gamma' \Rightarrow \Delta, \neg\ell_n$ starting from the conclusion of C_n, namely $\ell_1, \ldots, \ell_{n-1}, \ell_n, \Gamma' \Rightarrow \Delta$. Let $\mathcal{D}'$ be the following derivation.

$$\dfrac{\dfrac{\dfrac{}{\ell_1, \ldots, \ell_{n-1}, \ell_n, \Gamma' \Rightarrow \Delta}\,C_n}{\ell_1, \ldots, \ell_{n-1}, \ell_n, \Gamma' \Rightarrow \Delta, \bar{\ell}_n}\,RW \qquad \ell_1, \ldots, \ell_{n-1}, \bar{\ell}_n, \Gamma' \Rightarrow \Delta, \bar{\ell}_n}{\ell_1, \ldots, \ell_{n-1}, \Gamma' \Rightarrow \Delta, \bar{\ell}_n}\,LEM^*$$

Since $\neg\ell_n := \bar{\ell}_n$ we have derived $\ell_1, \ldots, \ell_{n-1}, \Gamma' \Rightarrow \Delta, \neg\ell_n$. If $h(\mathcal{D}) > 0$, then we distinguish according to whether A is principal or not principal of the last rule R applied in the derivation of the premise. In the latter case, we reason as in the corresponding case in the proof of the admissibility of $L_\neg$. If A is principal of R, then we need to consider, besides the cases already considered in the proof of the admissibility of $L_\neg$, also the possibility that R is a coherent rule C_n. Let $\mathcal{D}$ be the derivation of the premise of $R_\neg$ with A principal of C_n. To prevent the derivation trees to spread we assume without loss of generality that C_n has just one premise. Thus, premise of $R_\neg$ has the following derivation:

$$\dfrac{J1_1, \ldots, J1_p, \ell_1, \ldots, \ell_{n-1}, \ell_n, \Gamma' \Rightarrow \Delta}{\ell_1, \ldots, \ell_{n-1}, \ell_n, \Gamma' \Rightarrow \Delta}\,C_n$$

We can find a derivation $\mathcal{D}'$ of the conclusion of $R_\neg$, namely the sequent $\ell_1, \ldots, \ell_{n-1}, \Gamma' \Rightarrow \Delta, \neg\ell_n$, by considering the following derivation:

$$\dfrac{\dfrac{\dfrac{J1_1, \ldots, J1_p, \ell_1, \ldots, \ell_{n-1}, \ell_n, \Gamma' \Rightarrow \Delta}{J1_1, \ldots, J1_p, \ell_1, \ldots, \ell_{n-1}, \ell_n, \Gamma' \Rightarrow \Delta, \bar{\ell}_n}\,RW}{\ell_1, \ldots, \ell_{n-1}, \ell_n, \Gamma' \Rightarrow \Delta, \bar{\ell}_n}\,C_n \qquad \ell_1, \ldots, \ell_{n-1}, \bar{\ell}_n, \Gamma' \Rightarrow \Delta, \bar{\ell}_n}{\ell_1, \ldots, \ell_{n-1}, \Gamma' \Rightarrow \Delta, \bar{\ell}_n}\,LEM^*$$

Notice that the applicability of C_n is legitimate since no proper variable of C_n may occur in $\bar{\ell}_n$ introduced in the application of RW. From here we immediately see that $\ell_1, \ldots, \ell_{n-1}, \Gamma' \Rightarrow \Delta, \neg\ell_n$ is derivable since $\neg\ell_n := \bar{\ell}_n$. □

We can now prove the central admissibility result for G_n^*, namely the admissibility of the cut rule.

Theorem 2 *The rule of cut is admissible in* G_n^*.

Proof We proceed by induction on the depth $d(A)$ of the cut formula A with a sub-induction on the height of a cut, defined as the sum of the heights of the two premises of cut. The proof follows the patter of the proof of Theorem 4.2.10 of [8]. We have the following two main cases. (A) One of the two premises of cut is a logical axiom or the conclusion of a coherent rule with no premise; (B) none of the two premise of cut is a logical axiom or the conclusion of a coherent rule with no premise. In case (A), we need to consider the following sub-cases: (i) the left premise is a logical axiom or the conclusion of a coherent rules with no premise; (ii) the right premise is a logical axiom or the conclusion of a coherent rules with no premise. In case (B), there are three sub-cases: (iii) A is not principal in the left premise; (iv) A is principal only in the left premise; (v) A is principal in both premises. We start from (i) and consider first the case in which the left premise is a logical axiom.

($i.1$) $P \in \Gamma \cap \Delta$. The conclusion of cut is a logical axiom.
($i.2$) $\bar{P} \in \Gamma \cap \Delta$. The conclusion of cut is a logical axiom.
($i.3$) A is $P \in \Gamma$. The original cut is:

$$\dfrac{P, \Gamma' \Rightarrow \Delta, P \quad P, \Pi \Rightarrow \Sigma}{P, \Gamma', \Pi \Rightarrow \Delta, \Sigma}\ cut$$

The conclusion of cut is obtained by several application of LW and RW, jointly indicated as W, from the right premise as follows:

$$\dfrac{P, \Pi \Rightarrow \Sigma}{P, \Gamma', \Pi \Rightarrow \Delta, \Sigma}\ W$$

($i.4$) A is $\bar{P} \in \Gamma$. Similar to case ($i.3$).
($i.5$) $P, \bar{P} \in \Gamma$. The conclusion of cut is a logical axiom.
($i.6$) $P, \bar{P} \in \Delta$. The conclusion of cut is a logical axiom.
($i.7$) A is P and $\bar{P} \in \Delta$. The original cut is:

$$\dfrac{\dfrac{}{\Gamma \Rightarrow \Delta', \bar{P}, P}\ LEM \qquad P, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta', \bar{P}, \Sigma}\ cut$$

The conclusion of *cut* is obtained by W and $R_\neg$ from the right premise as follows:

$$\cfrac{\cfrac{\cfrac{P, \Pi \Rightarrow \Sigma}{\Gamma, P, \Pi \Rightarrow \Delta', \Sigma}\, W}{\Gamma, \Pi \Rightarrow \Delta', \neg P, \Sigma}\, R_\neg}{\Gamma, \Pi \Rightarrow \Delta', \bar{P}, \Sigma}\, df_\neg$$

(*i.8*) A is $\bar{P}$ and $P \in \Delta$. Similar to (*i.7*).
(*i.9*) $\bot \in \Gamma$. The conclusion of *cut* is a logical axiom.
(*i.10*) $\top \in \Delta$. The conclusion of *cut* is a logical axiom.

If the left premise of *cut* is the conclusion of a coherent C_n rule with no premise, then the cut formula A is not principal in C_n. Thus, we have only one case, namely:

(*i.11*) $\ell_1, \dots, \ell_n \in \Gamma$. The conclusion of *cut*, i.e., $\ell_1, \dots, \ell_n, \Gamma', \Pi \Rightarrow \Delta, \Sigma$, is a conclusion of C_n.

We now consider the case (*ii*). If the right premise of *cut* is a logical axiom, then there are several cases:

(*ii.1*) $P \in \Pi \cap \Sigma$. The conclusion of *cut* is a logical axiom.
(*ii.2*) $\bar{P} \in \Pi \cap \Sigma$. The conclusion of *cut* is a logical axiom.
(*ii.3*) A is $P \in \Sigma$. Similar to (*i.3*).
(*ii.4*) A is $\bar{P} \in \Sigma$. Similar to case (*i.3*).
(*ii.5*) $P, \bar{P} \in \Pi$. The conclusion of *cut* is a logical axiom.
(*ii.6*) A is P and $\bar{P} \in \Pi$. Similar to (*i.7*).
(*ii.7*) A is $\bar{P}$ and $P \in \Pi$.. Similar to (*i.7*).
(*ii.8*) $P, \bar{P} \in \Sigma$. The conclusion of *cut* is a logical axiom.
(*ii.9*) $\bot \in \Pi$. The conclusion of *cut* is a logical axiom.
(*ii.10*) $\top \in \Sigma$. The conclusion of *cut* is a logical axiom.

If the right premise is the conclusion of a coherent rule C_n with no premise, then we need to distinguish according to whether the cut formula is principal or not principal in C_n:

(*ii.11*) A is not principal in C_n. Then, $\ell_1, \dots, \ell_n \in \Pi$. The conclusion of *cut* $\Gamma, \ell_1, \dots, \ell_n, \Pi' \Rightarrow \Delta, \Sigma$ is also a conclusion of C_n.
(*ii.12*) A is principal in C_n. In this case, A is ℓ_n and $\ell_1, \dots, \ell_{n-1} \in \Pi$. Thus, the original *cut* is:

$$\cfrac{\Gamma \Rightarrow \Delta, \ell_n \qquad \cfrac{}{\ell_1, \dots, \ell_{n-1}, \ell_n, \Pi' \Rightarrow \Sigma}\, C_n}{\Gamma, \ell_1, \dots, \ell_{n-1}, \Pi' \Rightarrow \Delta, \Sigma}\, cut$$

Here we need to reason on the left premise. Let ℓ_n be P (the case of $\bar{P}$ is similar). If the left premise is a logical axiom and P is principal, then either $P \in \Gamma$ or $\bar{P} \in \Delta$. In the first case, the conclusion of *cut* is obtained by W as in (*i.3*). In the second case, the conclusion of *cut* is obtained by W and $R_\neg$ as in (*i.7*). If the left premise is a logical axiom and P is not principal, then the conclusion of cut is a logical axiom, too. If the left premise is the conclusion of a coherent rule C'_n with no premise, then Γ contains $\ell'_1, \ldots, \ell'_m$ and also the conclusion of cut is a conclusion of C'_n. If the left premise is the conclusion of a rule R, then the cut formula ℓ_n, being atomic, cannot be principal in R. If R is a propositional rule or LEM^* or else a quantifier rule without variable condition, then we take the premise of R, apply an instance of cut on it and the right premise of cut and then R again. If R is a quantifier rule with variable condition or a coherent rule, we take again the premise(s) of R, apply the rule Sb in order to replace all the proper variable with new ones, apply an instance of cut on the right premise of cut and finally R again. This concludes the proof of case (*ii*).

Regarding case (*iii*) we proceed by cases on the last rule R applied in the derivation of the left premise. Here we permute cut and R as in case (*ii.11*). Regarding case (*iv*) we proceed by cases on the last rule R applied in the derivation of the right premise. Here, too, cut is permuted with R as in (*ii.11*). Finally, the case (*v*) does not differ from the same case in the proof of Theorem 4.2.10 of [8] since it involves only logical rules. □

5 Beyond coherent theories

In this section we shall compare using some examples the present method with that of [6]. We first recall some definitions from [6]. Let $\mathcal{L}$ be the standard first-order language (possibly with identity). A (closed) formula of $\mathcal{L}$ is coherent when it is equivalent to:

$$\forall \bar{x}\Big(P_1 \wedge \cdots \wedge P_n \supset \exists \bar{y}\big((Q_{1_1} \wedge \cdots \wedge Q_{1_p}) \vee \cdots \vee (Q_{m_1} \wedge \cdots \wedge Q_{m_q})\big)\Big)$$

We assume that no variable in $\bar{y}$ occurs free in $P_1 \wedge \cdots \wedge P_n$. Given a coherent formula of $\mathcal{L}$, a coherent rule C is:

$$\frac{Q_{1_1}, \ldots, Q_{1_p}, P_1, \ldots, P_n, \Gamma \Rightarrow \Delta \quad \cdots \quad Q_{m_1}, \ldots, Q_{m_q}, P_1, \ldots, P_n, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_n, \Gamma \Rightarrow \Delta}\, C$$

where in each premise of C each variable in $\bar{y}$ has been replaced by a variable in $\bar{z}$ not occurring in the conclusion of C. Let G^* be any extension of the calculus $\mathsf{G3c}$ for first-order logic with finitely many coherent rules. As shown in [6], the admissibility results for structural rules hold for G^*.

It is clear that a coherent formula of $\mathcal{L}$ is a special case of a coherent formula of $\mathcal{L}_n$, the one in which all literals are positive atoms. On the other hand, there are coherent formulas of $\mathcal{L}_n$ that are not coherent in $\mathcal{L}$. Consider once again the axiom of strict seriality $\forall x \exists y (x \leq y \wedge \neg y \leq x)$. In $\mathcal{L}$ strict seriality is *not* coherent since $\neg y \leq x$ is not atomic. However, in $\mathcal{L}_n$ we have since the beginning that $\neg y \leq x := y \nleq x$ and consequently strict seriality is equivalent to the coherent formula $\forall x \exists y (\top \supset x \leq$

$y \wedge y \not\leq x)$. Thus, it corresponds to the following coherent rule *StrSer*, with z proper variable:

$$\frac{x \leq z, z \not\leq x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \textit{StrSer}$$

One may object that the example of strict seriality is too artificial. After all, such a principle can easily be treated by the standard coherent rules once it is formalized in $\mathcal{L}$ as the coherent formula $\forall x \exists y (x < y)$. While this certainly the case, for certain axioms there does not seem to be a similar solution. Consider, for example, the axiom of extensionality in set theory, namely the formula $\forall x \forall y (\forall z (z \in x \equiv z \in y) \supset x = y)$. In $\mathcal{L}_n$ such a formula is coherent, since it is equivalent to $\forall x \forall y (x \neq y \supset \exists z ((z \in x \wedge z \notin y) \vee (z \notin x \wedge z \in y)))$. Thus, in G_n^* extensionality corresponds to the following coherent rule Ext, where u is the proper variable:

$$\frac{u \in x, u \notin y, x \neq y, \Gamma \Rightarrow \Delta \quad u \notin x, u \in y, x \neq y, \Gamma \Rightarrow \Delta}{x \neq y, \Gamma \Rightarrow \Delta}\ \textit{Ext}$$

However, extensionality is not coherent in $\mathcal{L}$. In $\mathcal{L}$ the closest we can get to a coherent formulation of extensionality is $\forall x \forall y (x = y \vee \exists z ((z \in x \wedge \neg z \in y) \vee (\neg z \in x \wedge z \in y)))$, which still fails to be coherent precisely because both $\neg z \in y$ and $\neg z \in x$ are compound formulas.

Other examples of axioms that are coherent in $\mathcal{L}_n$ but not in $\mathcal{L}$ include the supplementation principles in mereology.[5] Consider the so-called strong supplementation principle: $\neg x\mathrm{P}y \rightarrow \exists z (z\mathrm{P}x \wedge \neg z\mathrm{O}y)$, where P and O are the relations of part and overlap, respectively.[6] Strong supplementation is clearly not coherent in $\mathcal{L}$. However, in $\mathcal{L}_n$ the formulas $\neg x\mathrm{P}y$ and $\neg z\mathrm{O}y$ are the atoms $x\bar{\mathrm{P}}y$ and $z\bar{\mathrm{O}}y$ and strong supplementation can be converted into the following coherent rule of G_n, where u is the proper variable:

$$\frac{x\bar{\mathrm{P}}y, u\mathrm{P}x, u\bar{\mathrm{O}}y, \Gamma \Rightarrow \Delta}{x\bar{\mathrm{P}}y, \Gamma \Rightarrow \Delta}\ \textit{StrSupp}$$

A final example is from Tarski's classical formalization of geometry in [12] based on a ternary predicate B of betweennes among points. As noted in [1], the theory is not coherent since the so-called lower 2-dimensional axiom $\exists x \exists y \exists z (\neg B(x, y, z) \wedge \neg B(y, z, x) \wedge \neg B(z, x, y))$ asserting the existence of three non-collinear points on the same plane is not coherent. Thus, in order to provide a cut-free sequent calculus equivalent to Tarski's system, in [1] the language $\mathcal{L}$ is extended with predicates corresponding to the negations of the primitive predicates, in particular of B. In $\mathcal{L}_n$ this is not necessary since the predicate $\bar{B}$ is already present. The lower 2-dimensional axiom can be thus reformulated in $\mathcal{L}_n$ as the coherent formula

[5] See [2] for an overview on mereology and [5] for a proof-theoretic approach.

[6] Intuitively, the principle says that if an individual x fails to be part of an individual y, then there must be a part of x disjoint from y—such a part being helpfully understood as a remainder.

$\top \supset \exists x \exists y \exists z(\bar{B}(x, y, z) \wedge \bar{B}(y, z, x) \wedge \bar{B}(z, x, y))$, which corresponds to the following coherent rule of G_n^* with x, y and z not occurring free in the conclusion:

$$\frac{\bar{B}(x, y, z), \bar{B}(y, z, x), \bar{B}(z, x, y), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \; L2D$$

These examples suffice to show that the class of axioms to which the present method applies is larger that the class of ordinary coherent axioms. How much larger? We provide a partial answer to this question. For a start we notice that not any (first-order) axiom can be treated in G_n^*. Consider, for example, the axiom of minimal posets from [3], according to which every element is less or equal than some minimal element, namely $\forall x \exists y(\top \supset x \leq y \wedge \forall z(y \leq z \supset y = z))$. Even in $\mathcal{L}_n$ such a formula is not coherent. Following the method of semindefinitional extensions developed in [3], we extend $\mathcal{L}$ with a new symbol unary predicate M and define $M(x)$ as $\forall z(x \leq z \supset x = z)$. Then, we take the axiom of minimal posets and replace $\forall z(x \leq z \supset x = z)$ with $M(x)$. In this way the new axiom of minimal posets becomes the coherent formula $\forall x \exists y(\top \supset x \leq y \wedge M(y))$. Clearly this is not enough since there are no axiom governing $M(x)$. However, as shown in [3], the only axiom for $M(x)$ that needs to be considered is the coherent formula $M(x) \wedge x \leq z \supset x = z$, whereas the conditional $(\forall z(x \leq z \supset x = z)) \supset M(x)$ is not needed since in the new axiom of minimal posets the atom $M(y)$ occurs in the consequent of $\supset$ (or positively, in the terminology of [3]). A sequent calculus for the theory of minimal posets is thus obtained by adding on top of $\mathsf{G3c}^=$ the following coherent rules, where u is proper variable in PM_1.

$$\frac{x \leq u, M(u), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \; PM_1 \qquad \frac{x = z, M(x), x \leq z, \Gamma \Rightarrow \Delta}{M(x), x \leq z, \Gamma \Rightarrow \Delta} \; PM_2$$

The method of semidefinitional extensions is certainly very general: any first-order formulas can be converted into a rule of sequent calculus without jeopardizing cut elimination. In other words, the class of axioms to which the method of semidefinitional extensions applies is the class of all first-order formulas. The drawback of the semidefinitional approach is that the language $\mathcal{L}$ needs to be extended with new symbols; and for certain theories this procedure, though viable in theory, is in practice exceeding complex. The examples above show that certain axioms can easily be treated with the method of semidefinitional extensions, but they do not need to. Indeed, those axioms already correspond to coherent rules of G_n, with no need to extend the underlying language $\mathcal{L}_n$. Thus, it appears that the class of axioms that can adequately dealt with in G_n is intermediate between the class of coherent axioms and the class of all axioms expressible in some extension of a standard first-order language.

6 Co-coherent formulas in negation normal form

In this section we shall see that the axioms that the present method can treat include all the so-called co-coherent formulas. Recall from [9] that a formula of $\mathcal{L}$ is co-coherent when it is equivalent to:

$$\forall \bar{x}\Big(\forall \bar{y}\big((Q_{1_1} \vee \cdots \vee Q_{1_p}) \wedge \cdots \wedge (Q_{m_1} \vee \cdots \vee Q_{1_q})\big) \supset P_1 \vee \cdots \vee P_n\Big)$$

where we assume that none of the variables in $\bar{y}$ occurs free in $P_1 \vee \cdots \vee P_n$. A theory is co-coherent when all its proper axioms are co-coherent formulas. Given a co-coherent formula, the corresponding rule *co-C* is:

$$\frac{\Gamma \Rightarrow \Delta, P_1, \ldots, P_n, Q_{1_1}, \ldots, Q_{1_p} \quad \cdots \quad \Gamma \Rightarrow \Delta, P_1, \ldots, P_n, Q_{m_1}, \ldots, Q_{m_q}}{\Gamma \Rightarrow \Delta, P_1, \ldots, P_n}\ \textit{co-C}$$

where in each premise each variable in $\bar{y}$ has been replaced by a variable in $\bar{z}$ not occurring free in the conclusion. Let $\mathsf{co\text{-}G}^*$ be any extension of $\mathsf{G3c}^=$ with finitely many co-coherent rules. Also $\mathsf{co\text{-}G}^*$ satisfies the admissibility results of the previous section. In $\mathcal{L}$ coherent rules fall short of treating co-coherent formulas, which require co-coherent rules. This is clearly disadvantageous since there might be theories in which some axioms are coherent and others are co-coherent. Alas, for such ‘mixed’ theories the standard method of [6] to preserve the admissibility results is not applicable. In fact, evidence suggests that when we work with such ‘combined calculi’, then cut elimination is jeopardized. Consider, for example, in $\mathcal{L}$ a sequent calculus GE resulting from $\mathsf{G3c}^=$ by adding a zero-premise co-coherent rule E_1 corresponding to the fact that a binary relation $\sim$ is reflexive as well as a one-premise coherent rule E_2 corresponding to the fact that $\sim$ is Euclidean.

$$\frac{}{\Gamma \Rightarrow \Delta, x \sim x}\ E_1 \qquad \frac{y \sim z, x \sim y, x \sim z, \Gamma \Rightarrow \Delta}{x \sim y, x \sim z, \Gamma \Rightarrow \Delta}\ E_2$$

In such a calculus it is easy to find a counter-example to cut elimination. Indeed, the sequent corresponding to the symmetry of $\sim$, namely $x \sim y \Rightarrow y \sim x$, is not derivable without *cut*, though is derivable with it, as the following derivation shows:

$$\dfrac{\dfrac{}{\Rightarrow x \sim x}\ E_2 \qquad \dfrac{y \sim x, x \sim y, x \sim x \Rightarrow y \sim x}{x \sim y, x \sim x \Rightarrow y \sim x}\ E_2}{x \sim y \Rightarrow y \sim x}\ \textit{cut}$$

While in $\mathcal{L}$ the combination of coherent and co-coherent rules may impair cut elimination, this issue does not arise in $\mathcal{L}_n$ for the simple reason that in $\mathcal{L}_n$ co-coherent formulas are, in fact, coherent. More precisely, given a co-coherent formula of $\mathcal{L}$:

$$\forall \bar{x}\Big(\forall \bar{y}\big((Q_{1_1} \vee \cdots \vee Q_{1_p}) \wedge \cdots \wedge (Q_{m_1} \vee \cdots \vee Q_{1_q})\big) \supset P_1 \vee \cdots \vee P_n\Big)$$

we can use classically valid equivalences to obtain the following

$$\forall \bar{x}\Big(\neg P_1 \wedge \cdots \wedge \neg P_n \supset \exists \bar{y}\big((\neg Q_{1_1} \wedge \cdots \wedge \neg Q_{1_p}) \vee \cdots \vee (\neg Q_{m_1} \wedge \cdots \wedge \neg Q_{m_q})\big)\Big)$$

However, given the definition of negation in $\mathcal{L}_n$, the latter is a coherent formula:

$$\forall \bar{x}\Big(\bar{P}_1 \wedge \cdots \wedge \bar{P}_n \supset \exists \bar{y}\big((\bar{Q}_{1_1} \wedge \cdots \wedge \bar{Q}_{1_p}) \vee \cdots \vee (\bar{Q}_{m_1} \wedge \cdots \wedge \bar{Q}_{m_q})\big)\Big)$$

In other words, co-coherent formulas of $\mathcal{L}$ are special cases of coherent formulas of $\mathcal{L}_n$. This entails that a co-coherent formula of $\mathcal{L}$ can be expressed by a coherent rule of G^*_n.

$$\frac{\bar{Q}_{1_1}, \ldots, \bar{Q}_{1_p}, \bar{P}_1, \ldots, \bar{P}_n, \Gamma \Rightarrow \Delta \quad \cdots \quad \bar{Q}_{m_1}, \ldots, \bar{Q}_{m_q}, \bar{P}_1, \ldots, \bar{P}_n, \Gamma \Rightarrow \Delta}{\bar{P}_1, \ldots, \bar{P}_n, \Gamma \Rightarrow \Delta}\ C_n$$

Thus, in $\mathcal{L}_n$ there is no need of special, co-coherent rules for co-coherent axioms as coherent rules in negation normal form suffice to treat both coherent and co-coherent formulas. This means that in G^*_n we can treat coherent and co-coherent theories alike.

7 Conclusions

Since Negri's work [6] there has been a growing interest in methods for obtaining cut-free sequent calculi for first-order theories falling outside the coherent fragment. Recent works such as [3] introduced the method of semidefinitional extensions and showed any first-order theory admits a cut-free systematization in sequent calculus. While such a method, in virtue of its wide applicability, has superseded all previous ones, including Negri's own method of systems of rules of [3], alternative approaches are still worth to be investigated, if only because semidefinitional extensions are arguably quite artificial and, though unavoidable in certain cases, they are not needed at all for certain others. To what extent are semidefinitional extensions dispensable is not clear and one of the main contribution of the present work was to take some steps towards providing an answer to this question. In particular, using languages in negation normal form we showed that without resorting to semidefinitional extensions we can still provide cut-free sequent calculi for a class of theories which include, among others, also co-coherent theories investigated in [9]. While the present method is inevitably a special case of the method of semidefinitional extensions, in the sense that the latter is applicable to all first-order theories while the former only to a proper subclass thereof, it has the advantage of being entirely standard in proof theory. Indeed, languages in negation normal form have been extensively investigated in proof theory and were the primary source of inspiration for Gentzen-Schütte sequent calculi for classical and intuitionistic first-order logic.[7] From this perspective, one can see that among the contributions of this paper there is also the development of extensions of Gentzen-Schütte sequent calculi which have so far been hardly considered at all.

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[7] Such calculi are named after K. Schütte who introduced them in [10], although his original calculi are still in a standard first-order language. Gentzen-Schütte calculi in a language in negation normal form came only with Tait's classical paper [11].

References

1. Avigad, J., Dean, E., Mumma, J.: A formal system for Euclid's *Elements*. Rev. Symb. Log. **2**(4), 700–768 (2009)
2. Cotnoir, A., Varzi, A.: Mereology. Oxford University Press, Oxford (2021)
3. Dyckhoff, R., Negri, S.: Geometrisation of first-order logic. Bull. Symb. Log. **21**(2), 123–163 (2015)
4. Gherardi, G., Maffezioli, P., Orlandelli, E.: Interpolation in extensions of first-order logic. Stud. Logica. **108**, 619–648 (2020)
5. Maffezioli, P.: Analytic rules for mereology. Stud. Logica. **104**(1), 79–114 (2016)
6. Negri, S.: Contraction-free sequent calculi for geometric theories with an application to Barr's theorem. Arch. Math. Logic **42**(4), 389–401 (2003)
7. Negri, S., von Plato, J.: Cut elimination in the presence of axioms. Bull. Symb. Log. **4**(4), 418–435 (1998)
8. Negri, S., von Plato, J.: Structural Proof Theory. Cambridge University Press, Cambridge (2001)
9. Negri, S., von Plato, J.: The duality of classical and constructive notions and proofs. In: Schuster, P., Crosilla, L. (eds.) From Sets and Types to Topology and Analysis: Towards Practicable Foundations for Constructive Mathematics, pp. 149–161. Oxford University Press, Oxford (2005)
10. Schütte, K.: Schlussweisen-Kalküle der Prädicatenlogik. Mathematischen Annalen **122**, 47–65 (1950)
11. Tait, W.: Normal derivability in classical logic. In: Barwise, K. (ed.) The Syntax and Semantics of Infinitary Languages, pp. 240–251. Springer, Berlin (1968)
12. Tarski, A., Givant, S.: Tarski's system of geometry. Bull. Symb. Log. **5**(2), 175–214 (1999)

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