

LEBESGUE NUMBERS AND ATSUJI SPACES IN SUBSYSTEMS OF SECOND ORDER ARITHMETIC

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ABSTRACT. We study Lebesgue and Atsugi spaces within subsystems of second order arithmetic. The former spaces are those such that every open covering has a Lebesgue number, while the latter are those such that every continuous function defined on them is uniformly continuous. The main results we obtain are the following: the statement “every compact space is Lebesgue” is equivalent to WKL_0 ; the statements “every perfect Lebesgue space is compact” and “every perfect Atsugi space is compact” are equivalent to ACA_0 ; the statement “every Lebesgue space is Atsugi” is provable in RCA_0 ; the statement “every Atsugi space is Lebesgue” is provable in ACA_0 . We also prove that the statement “the distance from a closed set is a continuous function” is equivalent to $\Pi_1^1\text{-}CA_0$.

1. INTRODUCTION

This paper is part of the program started by Harvey Friedman and Steve Simpson and known as *reverse mathematics*: the aim of this program is to understand the role of set existence axioms in the development of ordinary mathematics and its present stage consists of establishing the weakest subsystem of second order arithmetic in which a theorem of ordinary mathematics can be proved. The basic reference for this program is Simpson’s monograph ([15]) while an overview can be found in [14].

We are interested in the theory of complete separable metric spaces (also called Polish spaces) and therefore we need to develop this theory within weak subsystems of second order arithmetic. Such a development is possible by using an appropriate coding of these spaces, their subsets and the continuous functions among them. This coding is now standard (see e.g. [6, 4, 5, 13, 7, 15]) and has been used to study — among other things — various notions of open, closed and compact sets, properties of continuous functions and of sequential convergence, and basic theorems such as the category theorem, various fixed point theorems and the Hahn-Banach theorem for separable Banach spaces.

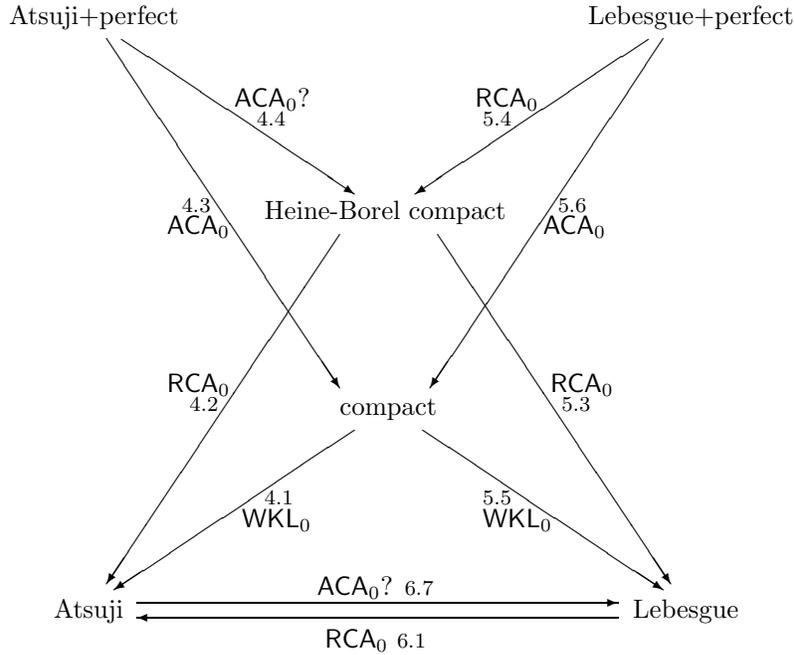
In this paper we concentrate on properties regarding Lebesgue numbers of open coverings: these are introduced in basic topology (see e.g. [10]). They are a tool for proving that every continuous function with compact domain is uniformly continuous (see e.g. [9]): by combining lemma 5.1 and theorem 6.1 in this paper we obtain that this proof can be carried out in the subsystem WKL_0 . Lebesgue numbers are also used to prove more advanced results in various areas of geometry: see e.g. [8] and [11].

We call Lebesgue spaces the spaces such that every open covering has a Lebesgue number: they turn out to be the spaces X such that every continuous function on X is uniformly continuous. Spaces having the latter property have been studied

for their own sake (see [1, 2, 3] and the references quoted therein) and are usually called Atsuji spaces.

Let us notice that since every Lebesgue (or Atsuji) metric space is complete the restriction imposed by the expressive power of the language of second order arithmetic on the spaces we study consists solely of forsaking non separable spaces.

The following picture shows the results we obtain by indicating the systems needed to prove the various implications between the notions of Atsuji and Lebesgue space and these and two notions of compactness. The numbers refer to the lemmas or theorems where the results are established and the question marks appear beside the two implications where we do not know whether our result is optimal.



The plan of the paper is as follows. The last part of this introductory section gives a brief presentation (for a more detailed presentation see [15]) of the subsystems we will deal with. Section 2 reviews the basic techniques and results about the coding of complete separable metric spaces in second order arithmetic. In section 3 we deal with various notions of compact space and introduce Lebesgue and Atsuji spaces. In sections 4 and 5 we study the relationships between the various notions of compactness on one side and Atsuji and Lebesgue spaces on the other side: we prove the results expressed by the diagonal arrows of our picture (the reversals of the arrows pointing towards “compact” are proved in WKL_0 and we show that this is necessary). In section 6 we consider the equivalence of Lebesgue and Atsuji spaces and we prove the results expressed by the horizontal arrows: the “easy” direction (i.e. Lebesgue implies Atsuji) can be obtained in RCA_0 , while the opposite direction is provable in ACA_0 (we do not know whether it is equivalent to ACA_0). Section 7 considers a problem arisen during the investigations of section 6, namely showing that the function distance from a closed set is continuous: this requires the even stronger subsystem $\Pi_1^1-CA_0$.

The formal systems we will consider are, in order of increasing strength, RCA_0 , WKL_0 , ACA_0 and $\Pi_1^1\text{-CA}_0$. These are all theories which use classical first order logic and the language of second order arithmetic, which consists of number variables m, n, \dots , set variables X, Y, \dots , primitives $+$, \cdot , 0 , 1 , $=$, $<$ and \in , logical connectives and quantifiers on both sorts of variables. Formulas of this language are classified according to the number of alternating quantifiers: Σ_1^0 formulas have one existential number quantifier in front of a matrix containing only bounded number quantifiers; arithmetical formulas contain no set quantifiers; Σ_1^1 formulas have one existential set quantifier in front of an arithmetical matrix; Π_n^i formulas are negations of Σ_n^i formulas.

All systems share a set of basic arithmetical axioms, an induction axiom

$$0 \in X \wedge \forall n(n \in X \longrightarrow n + 1 \in X) \longrightarrow \forall n(n \in X)$$

and differ by the formulas φ allowed in the comprehension scheme

$$\exists X \forall n(n \in X \longleftrightarrow \varphi(n))$$

or by the presence of other additional axioms.

RCA_0 has comprehension only for Δ_1^0 formulas, i.e. formulas which are provably equivalent both to a Σ_1^0 and to a Π_1^0 formula (and, for technical reasons, it has also an induction scheme for Σ_1^0 formulas): this is the base theory for most reverse mathematics investigations. WKL_0 extends RCA_0 by adding to it weak König's lemma (i.e. König's lemma for trees consisting of sequences of 0's and 1's): this allows for a good theory of compactness and continuity. ACA_0 has comprehension for arbitrary arithmetical formulas and allows for a good theory of sequential convergence. $\Pi_1^1\text{-CA}_0$ is the strongest system that turns out to be needed to prove theorems of ordinary mathematics: Π_1^1 formulas are allowed in the comprehension scheme.

A typical reverse mathematics result is the statement that, within a weaker base theory (typically RCA_0 , but see theorems 3.9, 4.6 and 5.8), one of these systems is equivalent to some theorem of ordinary mathematics.

In the next sections whenever we begin a definition, lemma or theorem by the name of one of these subsystems between parenthesis we mean that the definition is given, or the statement provable, within that subsystem.

2. CODING COMPLETE SEPARABLE METRIC SPACES

Definition 2.1 (RCA_0). A (code for a) *complete separable metric space* \widehat{A} is a set $A \subseteq \mathbb{N}$ together with a function $d : A \times A \rightarrow \mathbb{R}$ such that for all $a, b, c \in A$ we have $d(a, a) = 0$, $d(a, b) = d(b, a) \geq 0$ and $d(a, b) \leq d(a, c) + d(c, b)$.

A (code for a) *point of \widehat{A}* is a sequence $\langle a_n : n \in \mathbb{N} \rangle$ of elements of A such that for every n we have $d(a_n, a_{n+1}) < 2^{-n}$.

Within RCA_0 (or any other subsystem of second order arithmetic) \widehat{A} does not formally exist as a set: notations as $x \in \widehat{A}$ are just abbreviations for “ x is a point of \widehat{A} ”. Similar considerations can be made for the various notions of subsets of \widehat{A} we will introduce.

The metric d can be extended to $\widehat{A} \times \widehat{A}$ in an obvious way: this extension will still be denoted by d (or by $d_{\widehat{A}}$ when there is danger of confusion) and represents the metric of the complete separable metric space. If $x, y \in \widehat{A}$ are such that $d(x, y) = 0$ we identify them and write $x = y$.

A standard example of a complete separable metric space is obtained by taking $A = \mathbb{Q}$ with d the usual metric and denoting $\widehat{\mathbb{Q}}$ by \mathbb{R} . By restricting the code to the rationals between 0 and 1 we get a code for the closed interval $[0, 1]$.

Other important complete separable metric spaces are the Cantor space $2^{\mathbb{N}}$ and the Baire space $\mathbb{N}^{\mathbb{N}}$ of infinite sequences respectively of 0's and 1's and of natural numbers with the product topology originated by the discrete topologies respectively on $\{0, 1\}$ and \mathbb{N} . Complete metrics compatible with these topologies are obtained by setting, whenever $x \neq y$, $d(x, y) = 2^{-n}$ where n is least such that $x(n) \neq y(n)$. Explicit codings of $2^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$ within RCA_0 are provided in [15] and [4].

Definition 2.2 (RCA_0). For every $x \in \widehat{A}$ and $q \in \mathbb{R}^+$ let $B(x, q)$ denote the *open ball* of center x and radius q in \widehat{A} . This means that for every $y \in \widehat{A}$ we have that $y \in B(x, q)$ if and only if $d(x, y) < q$.

A (code for an) *open set* in \widehat{A} is a sequence $U = \langle (a_n, q_n) : n \in \mathbb{N} \rangle$ of elements of $A \times \mathbb{Q}^+$. The meaning of this coding is that $U = \bigcup_{n \in \mathbb{N}} B(a_n, q_n)$ and hence $x \in U$ if and only if $\exists n d(x, a_n) < q_n$. A *closed set* in \widehat{A} is the complement of an open set, and thus is represented by the same code.

A basic fact about open sets in complete separable metric spaces is the following lemma proved in [15].

Lemma 2.3 (RCA_0). Let $\varphi(x)$ be a Σ_1^0 formula such that $x, y \in \widehat{A}$ and $x = y$ imply $\varphi(x) \leftrightarrow \varphi(y)$. Then there exists an open set U in \widehat{A} such that $x \in U$ if and only if $\varphi(x)$ holds.

Definition 2.4 (RCA_0). For every $x \in \widehat{A}$ and $q \in \mathbb{R}^+$ let

$$P(x, q) = \{y \in \widehat{A} \mid 0 < d(x, y) < q\} = B(x, q) \setminus \{x\}$$

be the *punctured ball* of center x and radius q .

The formula defining $P(x, q)$ is Σ_1^0 : hence by lemma 2.3 within RCA_0 $P(x, q)$ is an open set.

Definition 2.5 (RCA_0). A point $x \in \widehat{A}$ is *isolated* if for some $q \in \mathbb{R}^+$ we have $P(x, q) = \emptyset$. A complete separable metric space is *perfect* if it does not have isolated points.

We defined closed sets to be complements of open sets; another natural definition can be obtained by viewing a closed set as the closure of a countable set.

Definition 2.6 (RCA_0). A code for a *separably closed set* in \widehat{A} is a sequence $C = \langle x_n : n \in \mathbb{N} \rangle$ of points of \widehat{A} . The separably closed set is then denoted by \overline{C} and $x \in \overline{C}$ if and only if $\forall q \in \mathbb{Q}^+ \exists \times (\curvearrowright, \curvearrowleft) < \mathbb{1}$.

The two notions of closed set we introduced are not equivalent within RCA_0 : their relationship has been studied in depth by Brown ([4, 5]), who obtained the following results.

Theorem 2.7 (RCA_0). *The following are equivalent:*

1. ACA_0 .
2. Every separably closed set in a complete separable metric space is closed.

Theorem 2.8 (RCA₀). *The following are equivalent:*

1. Π_1^1 -CA₀.
2. *Every closed set in a complete separable metric space is separably closed.*

If W and Z are open, closed or separably closed sets of \widehat{A} we write $W \subseteq Z$ to mean $\forall x(x \in W \rightarrow x \in Z)$. $W = Z$ and $W \not\subseteq Z$ have the obvious meanings.

Continuous functions are coded in second order arithmetic in the following way (see [6, 15]).

Definition 2.9 (RCA₀). Let \widehat{A} and \widehat{B} be two complete separable metric spaces. A (code for a) *continuous function from \widehat{A} to \widehat{B}* is a set $\Phi \subseteq \mathbb{N} \times \mathbb{A} \times \mathbb{Q}^+ \times \mathbb{B} \times \mathbb{Q}^+$ such that, if we denote by $(a, r)\Phi(b, s)$ the formula $\exists n (n, a, r, b, s) \in \Phi$, the following properties hold:

- $(a, r)\Phi(b, s) \wedge (a, r)\Phi(b', s') \rightarrow d(b, b') < s + s'$;
- $(a, r)\Phi(b, s) \wedge d(b, b') + s \leq s' \rightarrow (a, r)\Phi(b', s')$;
- $(a, r)\Phi(b, s) \wedge d(a, a') + r' \leq r \rightarrow (a', r')\Phi(b, s)$;
- $\forall x \in \widehat{A} \forall q \in \mathbb{Q}^+ \exists (\varrho, \searrow, \sim)((\varrho, \searrow)\Phi(\sim) \wedge (\swarrow, \varrho) < \searrow \wedge \sim < \ulcorner)$.

In this situation for every $x \in \widehat{A}$ there exists a unique $y \in \widehat{B}$ such that $d(y, b) < s$ whenever $d(x, a) < r$ and $(a, r)\Phi(b, s)$. This y is denoted by $f(x)$ and is the image of x under the function f coded by Φ .

Sometimes we will need to use continuous functions which are defined only on a subset of \widehat{A} . These can be coded omitting the last clause in the above definition: their domain consists precisely of those $x \in \widehat{A}$ for which

$$\forall q \in \mathbb{Q}^+ \exists (\varrho, \searrow, \sim)((\varrho, \searrow)\Phi(\sim) \wedge (\swarrow, \varrho) < \searrow \wedge \sim < \ulcorner).$$

RCA₀ proves (see [15]) that d is a continuous function, that the class of continuous function contains the constant functions and is closed under the basic arithmetical operations, max, min and composition and that the preimage by a continuous function of an open set in \widehat{B} is an open set in \widehat{A} . We will use these facts without explicit mention.

In section 7 we will show that not all continuous functions between complete separable metric spaces which are commonly used exist within RCA₀: indeed constructing for every closed set a code for the continuous function that associates to every point its distance from the closed set requires Π_1^1 -CA₀, while the same construction for separably closed sets requires ACA₀.

The following versions of Urysohn's lemma and Tietze extension theorem are proved in [4] and [15].

Theorem 2.10 (RCA₀). *If C_0 and C_1 are closed sets in a complete separable metric space \widehat{A} and $C_0 \cap C_1 = \emptyset$ then there exists a continuous function $f : \widehat{A} \rightarrow \mathbb{R}$ such that for every $i < 2$ and $x \in C_i$ we have $f(x) = i$.*

Theorem 2.11 (RCA₀). *If C is a closed set in a complete separable metric space \widehat{A} and $f : C \rightarrow \mathbb{R}$ is a continuous function there exists a continuous function $g : \widehat{A} \rightarrow \mathbb{R}$ such that $g \upharpoonright C = f$, i.e. $g(x) = f(x)$ for every $x \in C$.*

We will consider also uniformly continuous functions. In the context of subsystems of second order arithmetic sometimes (e.g. in [15]) functions which admit a modulus of uniform continuity have been most useful. Here we consider the usual

(and weaker, from the point of view of subsystems of second order arithmetic) notion of uniformly continuous function because we want to be as close as possible to standard mathematical practice.

Definition 2.12 (RCA_0). A continuous function $f : \widehat{A} \rightarrow \widehat{B}$ is *uniformly continuous* if

$$\forall \varepsilon \in \mathbb{R}^+ \exists \delta \in \mathbb{R}^+ \forall \curvearrowright, \curvearrowleft \in \widehat{A} (\widehat{\curvearrowright}(\curvearrowright, \curvearrowleft) < \delta \rightarrow \widehat{B}(\mathcal{U}(\curvearrowright), \mathcal{U}(\curvearrowleft)) < \varepsilon).$$

3. COMPACT, LEBESGUE AND ATSUJI SPACES

Another important concept is that of compact space. As is the case for the notions of closed sets also in this case the usual equivalent notions of compact spaces are not equivalent in weak subsystems of second order arithmetic. We will consider the following notions.

Definition 3.1 (RCA_0). A complete separable metric space \widehat{A} is *compact* if there exists an infinite sequence of finite sequences of points of \widehat{A} $\langle \langle x_{n,m} : m < i_n \rangle : n \in \mathbb{N} \rangle$ such that

$$\forall x \in \widehat{A} \forall n \in \mathbb{N} \exists \triangleright < \beth_{\times} (\curvearrowright, \curvearrowleft_{\times}, \triangleright) < \aleph^{-\times}.$$

Notice that our definition of compact space requires more than the existence for every $\varepsilon \in \mathbb{R}^+$ of a finite set $B \subseteq \widehat{A}$ such that $\forall x \in \widehat{A} \exists y \in B d(x, y) < \varepsilon$. RCA_0 does not prove that the latter condition implies compactness but ACA_0 does and indeed proves the following equivalence.

Lemma 3.2 (ACA_0). *A complete separable metric space \widehat{A} is compact if and only if for every $\varepsilon \in \mathbb{R}^+$ there exists a finite set $B \subseteq A$ such that $\forall a \in A \exists b \in B d(a, b) \leq \varepsilon$.*

Proof. Suppose $\langle \langle x_{n,m} : m < i_n \rangle : n \in \mathbb{N} \rangle$ witnesses the compactness of \widehat{A} . For any $\varepsilon \in \mathbb{R}^+$ let n be such that $2^{-n+1} \leq \varepsilon$. For every $m < i_n$ let $b_m \in A$ be such that $d(b_m, x_{n,m}) < 2^{-n}$. Then $B = \{b_m \mid m < i_n\}$ satisfies $\forall x \in \widehat{A} \exists b \in B d(x, b) \leq \varepsilon$ and a fortiori the desired property.

For the other direction of the equivalence recall that, within RCA_0 , every finite set can be coded as a natural number. Hence if for every $n \in \mathbb{N}$ there exists a finite set $B \subseteq A$ such that $\forall a \in A \exists b \in B d(a, b) \leq 2^{-(n+1)}$, within ACA_0 , there exists a function f that to each n associates the least finite set B satisfying this arithmetical condition. If $\langle \langle x_{n,m} : m < i_n \rangle : n \in \mathbb{N} \rangle$ is the sequence of the sequences of the elements of the various $f(n)$'s it is immediate to check that it witnesses the compactness of \widehat{A} . \square

Definition 3.3 (RCA_0). A sequence $\mathfrak{U} = \langle \mathfrak{U}_n : n \in \mathbb{N} \rangle$ of open sets in \widehat{A} is an *open covering* if for every $x \in \widehat{A}$ there exists n such that $x \in U_n$.

A complete separable metric space \widehat{A} is *Heine-Borel compact* if for every open covering \mathfrak{U} of \widehat{A} there exists a finite covering $\mathfrak{U}' \subseteq \mathfrak{U}$.

The results summarized in the next theorem are contained in [4] and [15].

Theorem 3.4 (RCA_0). *The following are equivalent:*

1. WKL_0 .
2. Every compact complete separable metric space is Heine-Borel compact.
3. The closed interval $[0, 1]$ is Heine-Borel compact.

4. If \widehat{A} is a compact complete separable metric space and $f : \widehat{A} \rightarrow \mathbb{R}$ a continuous function then f attains a minimum.

We also have the following result which is a corollary of the proof of one of the implications of the above theorem.

Lemma 3.5 (RCA_0). *If \widehat{A} is a Heine-Borel compact complete separable metric space and $f : \widehat{A} \rightarrow \mathbb{R}$ a continuous function then f attains a minimum.*

Proof. It suffices to inspect the proof of (1) \implies (4) of the previous theorem in [15] and notice that WKL_0 is only used to deduce Heine-Borel compactness from compactness. \square

The following equivalence is essentially due to Brown ([4]), but we prove a slightly different result that will be useful in the proof of theorem 4.3.

Theorem 3.6 (WKL_0). *The following are equivalent:*

1. ACA_0 .
2. Every Heine-Borel compact complete separable metric space is compact.
3. Every perfect Heine-Borel compact complete separable metric space is compact.

Proof. The equivalence between (1) and (2) is proved in [4] within RCA_0 . Since (2) implies (3) is obvious it suffices to prove that (3) implies (1): to this end we modify slightly Brown's proof of (2) implies (1).

It is well-known that ACA_0 is equivalent over RCA_0 (and, a fortiori, over WKL_0) to the statement that the range of every one-to-one function from \mathbb{N} to \mathbb{N} exists. Fix $f : \mathbb{N} \rightarrow \mathbb{N}$ one-to-one. We want to define a code for a complete separable metric space \widehat{A} homeomorphic to

$$\{(0, 0)\} \cup \bigcup_{k \in \mathbb{N}} \left(\{2^{-f(k)}\} \times [0, 2^{-f(k)}] \right) \subseteq \mathbb{R}^{\neq}.$$

Since the range of f is not available we cannot use $\{(2^{-f(k)}, q) \mid 0 \leq q \leq 2^{-f(k)}\}$ as a code. This problem can be overcome by defining

$$A = \{(k, q) \in \mathbb{N} \times \mathbb{Q}^+ \mid q \leq 2^{-f(k)}\}$$

and letting

$$d((k, q), (k', q')) = \max(|2^{-f(k)} - 2^{-f(k')}|, |q - q'|)$$

(we are using a metric different from, but equivalent to, the usual metric on \mathbb{R}^{\neq}).

\widehat{A} is clearly perfect and we claim that it is also Heine-Borel compact. To see this suppose $\mathfrak{U} = \langle U_n : n \in \mathbb{N} \rangle$ is an open covering of \widehat{A} . For some n_0 we have that $(0, 0) \in U_{n_0}$ and hence there exists m such that if $f(k) > m$ then $(k, y) \in U_{n_0}$ for every $y \in [0, 2^{-f(k)}]$. There are only finitely many k 's such that $f(k) \leq m$ and we can define $n_1 = \max(\{n_0\} \cup \{g(k) : f(k) \leq m\})$, where $g : \mathbb{N} \rightarrow \mathbb{N}$ is such that for every k we have that $\langle U_n : n < g(k) \rangle$ is a covering of $\{2^{-f(k)}\} \times [0, 2^{-f(k)}]$ (g exists within WKL_0 by the uniform version of the Heine-Borel compactness of $[0, 1]$ proved in [15]). We have that $\langle U_n : n < n_1 \rangle$ is a finite subcovering of \mathfrak{U} .

Therefore (3) implies that \widehat{A} is compact: let $\langle \langle x_{n,m} : m < i_n \rangle : n \in \mathbb{N} \rangle$ be such that

$$\forall x \in \widehat{A} \forall n \in \mathbb{N} \exists \triangleright < \sqsupset_x (\curvearrowright, \curvearrowleft_x, \triangleright) < \neq^{-\kappa}.$$

Every $x_{n,m}$ represents some $(2^{-f(k_{n,m})}, y_{n,m})$, whose actual code is $(k_{n,m}, y_{n,m})$. It is easy to check that for every n we have

$$\exists k f(k) = n \quad \text{iff} \quad \exists m < i_{n+1} f(k_{n+1,m}) = n.$$

By recursive comprehension the range of f exists. \square

The reader may notice that the preceding theorem has been proved within WKL_0 , while most reverse mathematics results are proved within RCA_0 . The use of a stronger base theory is indeed necessary to prove that statement (3) implies ACA_0 in this theorem, as we are now going to show, and the same situation will occur also for other results we will obtain later.

We need to formalize within RCA_0 the fact that every perfect complete separable metric space has a closed subset which is homeomorphic to $2^{\mathbb{N}}$ (with the metric described in section 2). This amounts essentially to Exercise 3D.15 in [12] (which does not mention RCA_0).

Theorem 3.7 (RCA_0). *For every perfect complete separable metric space \widehat{A} there exists a sequence $\langle B(a_s, q_s) : s \in 2^{<\mathbb{N}} \rangle$ of open balls with $a_s \in A$ and $q_s \in \mathbb{R}^+$ such that:*

1. $\forall s \in 2^{<\mathbb{N}} \forall i < 2 d(a_s, a_{s \smallfrown \langle i \rangle}) + q_{s \smallfrown \langle i \rangle} < q_s$;
2. $\forall s \in 2^{<\mathbb{N}} d(a_{s \smallfrown \langle 0 \rangle}, a_{s \smallfrown \langle 1 \rangle}) < q_{s \smallfrown \langle 0 \rangle} + q_{s \smallfrown \langle 1 \rangle}$;
3. $\forall s \in 2^{<\mathbb{N}} q_s \leq 2^{-\text{lh}(s)}$.

Using $\langle B(a_s, q_s) : s \in 2^{<\mathbb{N}} \rangle$ we can define an embedding (i.e. a continuous function which is injective and has continuous inverse) $\varphi : 2^{\mathbb{N}} \rightarrow \widehat{A}$ such that the range of φ is closed in \widehat{A} . Moreover φ is Lipschitz with constant 1, i.e. $d_{\widehat{A}}(\varphi(x), \varphi(y)) \leq d_{2^{\mathbb{N}}}(x, y)$ for every $x, y \in 2^{\mathbb{N}}$.

Proof. Define a_s and q_s by recursion on $\text{lh}(s)$. Let $a_{\langle \rangle}$ be an element of A and $q_{\langle \rangle} = 1$. Assuming we have defined a_s and q_s , since \widehat{A} is perfect $B(a_s, q_s)$ contains at least two distinct elements $a_{s \smallfrown \langle 0 \rangle}$ and $a_{s \smallfrown \langle 1 \rangle}$. For every $i < 2$ let

$$q_{s \smallfrown \langle i \rangle} = \min \left\{ \frac{1}{2}(q_s - d(a_s, a_{s \smallfrown \langle i \rangle})), \frac{1}{3}d(a_{s \smallfrown \langle 0 \rangle}, a_{s \smallfrown \langle 1 \rangle}) \right\}.$$

φ is defined by letting, for every $x \in 2^{\mathbb{N}}$, $\varphi(x) = \langle a_{x \upharpoonright n} : n \in \mathbb{N} \rangle$, where $x \upharpoonright n$ is the initial segment of x of length n . The properties of $\langle B(a_s, q_s) : s \in 2^{<\mathbb{N}} \rangle$ imply that $\varphi(x)$ is a point of \widehat{A} and that φ is injective and Lipschitz. The complement of the range of φ is $U = \{ x \in \widehat{A} \mid \exists n \forall s \in 2^n d(x, a_s) > q_s \}$ which is open by lemma 2.3. We leave to the reader the routine details (which involve the details of the coding of $2^{\mathbb{N}}$) of the definition of codes for φ and its inverse. \square

Lemma 3.8 (RCA_0). *The following are equivalent:*

1. WKL_0 .
2. *There exists a complete separable metric space which is perfect and Heine-Borel compact.*

Proof. (1) implies (2) follows immediately from theorem 3.4 and the fact that RCA_0 proves that $[0, 1]$ is perfect.

To prove that (2) implies (1) let \widehat{A} be a perfect Heine-Borel compact complete separable metric space. Let $\langle B(a_s, q_s) : s \in 2^{<\mathbb{N}} \rangle$ and φ be given by theorem 3.7 and denote by U be the complement of the range of φ .

Now let $T \subseteq 2^{<\mathbb{N}}$ be a binary tree with no paths. Consider the collection of open sets $\mathfrak{U} = \{\mathfrak{U}\} \cup \{\mathfrak{B}(a_s, q_s) \mid s \notin \mathfrak{T}\}$. \mathfrak{U} is a covering of \widehat{A} because T has no paths. Let $\mathfrak{U}' \subseteq \mathfrak{U}$ be a finite subcovering: only for finitely many s we have $B(a_s, q_s) \in \mathfrak{U}'$. Since \mathfrak{U}' is a covering every $t \in T$ has an extension s such that $B(a_s, q_s) \in \mathfrak{U}'$ and for each s there are only finitely many such t , this entails that T is finite. \square

Theorem 3.9. *Statement (3) of theorem 3.6 does not imply ACA_0 in any theory stronger than RCA_0 and properly weaker than WKL_0 .*

Proof. Let \mathbb{T} be a theory stronger than RCA_0 and properly weaker than WKL_0 and let \mathfrak{M} be a model of \mathbb{T} in which WKL_0 fails. By lemma 3.8 in \mathfrak{M} there are no perfect complete separable metric spaces which are Heine-Borel compact and hence statement (3) of theorem 3.6 is vacuously true. Since \mathfrak{M} is not a model of ACA_0 \mathbb{T} does not prove that (3) implies ACA_0 . \square

Our main goal is to explore the relationships among the following notions and between them and the notions of compactness we just introduced.

Definition 3.10 (RCA_0). A complete separable metric space \widehat{A} is *Atsuji* if every continuous function $f : \widehat{A} \rightarrow \widehat{B}$ (where \widehat{B} is an arbitrary complete separable metric space) is uniformly continuous.

Definition 3.11 (RCA_0). A complete separable metric space \widehat{A} is *Lebesgue* if for every open covering $\mathfrak{U} = \langle \mathfrak{U}_n \mid n \in \mathbb{N} \rangle$ of \widehat{A} there exists $q \in \mathbb{R}^+$ such that

$$\forall x \in \widehat{A} \exists n \in \mathbb{N} \mathbb{B}(x, q) \subseteq \mathfrak{U}_n.$$

q is called a *Lebesgue number* for \mathfrak{U} .

Remark 3.12. The set of natural numbers \mathbb{N} with the usual metric is a complete separable metric space which is Atsuji (for every $\varepsilon > 0$, $\delta = 1$ suffices in the definition of uniform continuity) and Lebesgue (1 is a Lebesgue number for every covering of \mathbb{N}) but not compact.

Another example of a Lebesgue and Atsuji non compact complete separable metric space is obtained by taking $\{e_n \mid n \in \mathbb{N}\}$ to be an orthonormal basis for an infinite dimensional separable real Hilbert space and considering $\{0\} \cup \{2^{-m}e_n \mid m, n \in \mathbb{N}\}$.

4. COMPACT AND ATSUJI

One direction of the relationship between compact spaces and uniform continuity has been already explored by Brown and Simpson. The following is the statement in our terminology of the main results they obtained.

Theorem 4.1 (RCA_0). *The following are equivalent:*

1. WKL_0 .
2. *Every complete separable metric space which is compact is Atsuji.*
3. *The closed interval $[0, 1]$ is Atsuji.*

Proof. (1) implies (2) is proved in [4] and [15]. (2) implies (3) holds because in RCA_0 it is easy to show that $[0, 1]$ is compact. (3) implies (1) is proved in [15]. \square

We also have the following result which is a corollary of the proof of one of the implications of the above theorem.

Lemma 4.2 (RCA₀). *Every complete separable metric space which is Heine-Borel compact is Atsuji.*

Proof. It suffices to inspect the proof of implication (1) \implies (2) of the previous theorem and notice that WKL₀ is only used to deduce Heine-Borel compactness from compactness. \square

Remark 3.12 shows that not all Atsuji spaces are compact. However a perfect space which is Atsuji is compact. ACA₀ is needed to prove this result.

Theorem 4.3 (WKL₀). *The following are equivalent:*

1. ACA₀.
2. *Every complete separable metric space which is perfect and Atsuji is compact.*

Proof. (1) \implies (2). We reason in ACA₀ and suppose that \widehat{A} is a perfect complete separable metric space which is not compact. We will show that \widehat{A} is not Atsuji.

Since \widehat{A} is not compact by lemma 3.2 there exists $\varepsilon \in \mathbb{R}^+$ such that for no finite $B \subseteq A$ we have that for all $a \in A$ there exists $b \in B$ such that $d(a, b) \leq \varepsilon$. Using this fact we can define by recursion a sequence $\langle a_n^0 : n \in \mathbb{N} \rangle$ of elements of A such that $n \neq m \implies d(a_n^0, a_m^0) > \varepsilon$. Since \widehat{A} is perfect for every n there exists $a_n^1 \in A$ such that $a_n^1 \in P(a_n^0, 2^{-n-1}\varepsilon)$. Using the triangle inequality we get that $n \neq m \implies d(a_n^1, a_m^1) > \frac{\varepsilon}{4}$. Setting $C_i = \langle a_n^i : n \in \mathbb{N} \rangle$ (for $i = 0, 1$) these facts entail that $C_0 \cap C_1 = \emptyset$ and $\overline{C_i} = C_i$. In other words, C_0 and C_1 code two disjoint separably closed sets. By theorem 2.7 each C_i is closed and we can apply theorem 2.10 to get a continuous function $f : \widehat{A} \rightarrow \mathbb{R}$ such that $f(C_i) = \{i\}$. To see that f is not uniformly continuous fix $\delta \in \mathbb{R}^+$: if $2^{-n-1}\varepsilon \leq \delta$ we have $d(a_n^0, a_n^1) < \delta$ but $|f(a_n^0) - f(a_n^1)| = 1$.

(2) \implies (1). We will use theorem 3.6: it suffices to prove that if \widehat{A} is Heine-Borel compact and perfect then \widehat{A} is compact. This follows immediately from lemma 4.2 and (2). \square

Remark 4.4. Combining theorems 4.3 and 3.4 we have a proof within ACA₀ that every complete separable metric space which is perfect and Atsuji is Heine-Borel compact. We do not know whether ACA₀ is necessary to prove this statement.

We will prove that WKL₀ is necessary to obtain the equivalence of theorem 4.3 by the same argument we used to prove theorem 3.9.

Lemma 4.5 (RCA₀). *The following are equivalent:*

1. WKL₀.
2. *There exists a complete separable metric space which is perfect and Atsuji.*

Proof. (1) implies (2) follows immediately from theorem 4.1.

To prove that (2) implies (1) let \widehat{A} be a perfect Atsuji complete separable metric space and φ be given by theorem 3.7. Denote by C and ψ respectively the range and the inverse of φ . We will show that every continuous function $f : 2^{\mathbb{N}} \rightarrow \mathbb{R}$ is uniformly continuous. This implies WKL₀ by a simplified version of the argument used in [15] to prove (3) \implies (1) of theorem 4.1 (that argument uses only functions from $[0, 1]$ to \mathbb{R} and the explicit embedding of $2^{\mathbb{N}}$ into $[0, 1]$ given by Cantor middle-third set).

Let $f : 2^{\mathbb{N}} \rightarrow \mathbb{R}$ be continuous and define $g : C \rightarrow \mathbb{R}$ by setting $g = f \circ \psi$. By theorem 2.11 let $h : \widehat{A} \rightarrow \mathbb{R}$ be continuous such that $h \upharpoonright C = g$. Since \widehat{A} is Atsujii h is uniformly continuous. To show that f is uniformly continuous fix $\varepsilon > 0$ and let $\delta > 0$ be such that for all $y, y' \in \widehat{A}$ if $d_{\widehat{A}}(y, y') < \delta$ then $|h(y) - h(y')| < \varepsilon$. If $x, x' \in 2^{\mathbb{N}}$ are such that $d_{2^{\mathbb{N}}}(x, x') < \delta$ then, since φ is Lipschitz with constant 1, $d_{\widehat{A}}(\varphi(x), \varphi(x')) < \delta$ and hence $|h(\varphi(x)) - h(\varphi(x'))| < \varepsilon$. But $h(\varphi(x)) = f(x)$ and $h(\varphi(x')) = f(x')$, so that the uniform continuity of f is established. \square

Theorem 4.6. *Statement (2) of theorem 4.3 does not imply ACA_0 in any theory stronger than RCA_0 and properly weaker than WKL_0 .*

Proof. Repeat the argument of the proof of theorem 3.9 using lemma 4.5. \square

5. COMPACT AND LEBESGUE

We now explore the relationship between compact spaces and Lebesgue numbers.

Lemma 5.1 (WKL_0). *Every complete separable metric space which is compact is Lebesgue.*

Proof. We reason in WKL_0 . Let \widehat{A} be a compact complete separable metric space and \mathfrak{U} an open covering of \widehat{A} . Each element of \mathfrak{U} is union of open balls with center in A and rational radius: since a Lebesgue number for the covering consisting of these open balls is also a Lebesgue number for the original covering we can assume that each element of \mathfrak{U} is actually such an open ball. By theorem 3.4 \widehat{A} is Heine-Borel compact and there exists a finite subcovering of \mathfrak{U} . Since a Lebesgue number for any subcovering is a Lebesgue number also for the original covering we can assume that \mathfrak{U} is finite and has the form $\langle B(a_n, r_n) : n < k \rangle$ with $a_n \in A$ and $r_n \in \mathbb{Q}^+$ for every $n < k$.

For every $n < k$ let $f_n : \widehat{A} \rightarrow \mathbb{R}$ be the continuous function defined by $f_n(x) = \max(0, r_n - d(a_n, x))$. Let $f : \widehat{A} \rightarrow \mathbb{R}$ be the continuous function defined by $f(x) = \max\{f_n(x) \mid n < k\}$. Since \mathfrak{U} is a covering for every $x \in \widehat{A}$ there exists $n < k$ such that $f_n(x) > 0$ and hence $f(x) > 0$. By 3.4 since \widehat{A} is compact f attains a minimum $q \in \mathbb{R}^+$. We claim that q is a Lebesgue number for \mathfrak{U} .

To prove the claim let $x \in \widehat{A}$: since $f(x) \geq q$ for some $n < k$ we have $r_n - d(a_n, x) = f_n(x) \geq q$ which implies $B(x, q) \subseteq B(a_n, r_n)$ completing the proof of the claim and of the lemma. \square

Remark 5.2. The functions f_n used in the proof of the preceding lemma compute a lower bound for the distance from the complement of $B(a_n, r_n)$ (the latter is the function used in textbook proofs of this result). This suffices to prove that if $f_n(x) \geq q$ then $B(x, q) \subseteq B(a_n, r_n)$, which is all is needed to complete the argument. In theorem 7.1 we will show that WKL_0 does not suffice to prove that the function computing the actual distance from the complement of $B(a_n, r_n)$ exists.

The following results will be used in the proofs of theorems 5.5 and 5.6 but are also interesting in their own right.

Lemma 5.3 (RCA_0). *Every complete separable metric space which is Heine-Borel compact is Lebesgue.*

Proof. It suffices to repeat the proof of the above lemma using lemma 3.5. \square

Theorem 5.4 (RCA₀). *Every complete separable metric space which is perfect and Lebesgue is Heine-Borel compact.*

Proof. Let \mathfrak{U} be an open covering of \widehat{A} . Since every open subset of \widehat{A} is union of open balls with center in A and rational radius we may assume that \mathfrak{U} has the form $\langle B(a_n, r_n) : n \in \mathbb{N} \rangle$ with $a_n \in A$ and $r_n \in \mathbb{Q}^+$ for every $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$ and $b \in A$ let $q_{n,b} = \min(r_n - d(a_n, b), 2^{-n}) \in \mathbb{R}$. Let $V_{n,b} = P(b, q_{n,b})$. Notice that the definition of $q_{n,b}$ implies that $V_{n,b} \subseteq B(a_n, r_n)$ and that if $b \notin B(a_n, r_n)$ we have $q_{n,b} \leq 0$ and hence $V_{n,b} = \emptyset$.

We claim that $\mathfrak{V} = \langle \mathfrak{V}_{n,b} : n \in \mathbb{N}, b \in A \rangle$ is a covering of \widehat{A} . To see this let $x \in \widehat{A}$: since \mathfrak{U} is a covering, for some $n \in \mathbb{N}$ we have $d(a_n, x) < r_n$ and hence there exists $\varepsilon \in \mathbb{R}^+$ such that $d(a_n, x) \leq r_n - 2\varepsilon$. Since \widehat{A} is perfect and hence x is not isolated there exists $b \in A$ such that $0 < d(b, x) < \min(\varepsilon, 2^{-n})$: this implies $d(b, a_n) < r_n - \varepsilon$. Since $d(b, x) < \min(\varepsilon, 2^{-n}) < \min(r_n - d(a_n, b), 2^{-n}) = q_{n,b}$ we have $x \in V_{n,b}$. This completes the proof of the claim.

Since \widehat{A} is Lebesgue there exists a Lebesgue number $q \in \mathbb{R}^+$ for \mathfrak{V} . Let $k \in \mathbb{N}$ be such that $2^{-k} < q$. We now prove that $\langle B(a_n, r_n) : n < k \rangle$ is a finite subcovering of \mathfrak{U} , thereby establishing the lemma.

To see that $\langle B(a_n, r_n) : n < k \rangle$ is a covering of \widehat{A} let $x \in \widehat{A}$: by definition of Lebesgue number there exist $n \in \mathbb{N}$ and $b \in A$ such that $B(x, q) \subseteq V_{n,b}$. Thus $b \notin B(x, q)$ and $x \in V_{n,b}$ which imply $q \leq d(b, x) < q_{n,b}$. Therefore $2^{-k} < q < q_{n,b} \leq 2^{-n}$ which entails $n < k$. Since $x \in V_{n,b} \subseteq B(a_n, r_n)$ the proof is complete. \square

The following are our reverse mathematics results on the relationship between Lebesgue spaces and compactness.

Theorem 5.5 (RCA₀). *The following are equivalent:*

1. WKL₀.
2. *Every complete separable metric space which is compact is Lebesgue.*
3. *The closed interval $[0, 1]$ is Lebesgue.*

Proof. (1) implies (2) is lemma 5.1. (2) implies (3) is immediate. To prove (3) implies (1) use theorem 3.4 and notice that $[0, 1]$ is perfect: it follows from theorem 5.4 that it is Heine-Borel compact. \square

Theorem 5.6 (WKL₀). *The following are equivalent:*

1. ACA₀.
2. *Every complete separable metric space which is perfect and Lebesgue is compact.*

Proof. (1) implies (2) follows by theorem 5.4 and (1) \implies (2) of theorem 3.6.

To prove that (2) implies (1) we use theorem 3.6: we suppose that \widehat{A} is perfect and Heine-Borel compact and show, using (2), that it is compact. This is immediate using lemma 5.3. \square

Also in this case we are able to prove that WKL₀ is necessary to obtain the equivalence of theorem 5.6.

Lemma 5.7 (RCA₀). *The following are equivalent:*

1. WKL₀.
2. *There exists a complete separable metric space which is perfect and Lebesgue.*

Proof. (1) implies (2) follows immediately from lemma 5.1.

To prove that (2) implies (1) we could give a proof similar to the proofs of the corresponding statement in lemmas 3.8 and 4.5, but this is not necessary: combining theorem 5.4 and lemma 3.8 we obtain an immediate proof. \square

Theorem 5.8. *Statement (2) of theorem 5.6 does not imply ACA_0 in any theory stronger than RCA_0 and properly weaker than WKL_0 .*

Proof. Repeat the argument of the proof of theorem 3.9 using lemma 5.7. \square

6. ATSUJI AND LEBESGUE

In [2] and [3] Beer remarks that the notions of Atsuji space and Lebesgue space are equivalent. The simmetries of theorems 4.1 and 5.5, which show that both notions can be derived from compactness in WKL_0 , and of theorems 4.3 and 5.6, which show that both notions imply compactness for perfect spaces in ACA_0 , may suggest that this equivalence should be provable in a rather weak subsystems. This, as the next theorem shows, is indeed the case for one direction of the equivalence.

Theorem 6.1 (RCA_0). *Every complete separable metric space which is Lebesgue is Atsuji.*

Proof. Let \widehat{A} be a Lebesgue complete separable metric space, \widehat{B} a complete separable metric space and $f : \widehat{A} \rightarrow \widehat{B}$ a continuous function. Fix $\varepsilon \in \mathbb{R}^+$. For every $b \in B$ let $U_b = f^{-1}(B(b, \frac{\varepsilon}{2}))$: the continuity of f implies that $\mathfrak{U} = \langle U_b : b \in B \rangle$ is an open covering of \widehat{A} . Let $\delta \in \mathbb{R}^+$ be a Lebesgue number for \mathfrak{U} .

Suppose that $x, y \in \widehat{A}$ are such that $d_{\widehat{A}}(x, y) < \delta$: this means that $x, y \in B(x, \delta)$. Since δ is a Lebesgue number for \mathfrak{U} there exists $b \in B$ such that $B(x, \delta) \subseteq U_b$. Therefore $f(x), f(y) \in B(b, \frac{\varepsilon}{2})$ and hence $d_{\widehat{B}}(f(x), f(y)) < \varepsilon$. This completes the proof of the uniform continuity of f . \square

The reverse implication appears to be harder to prove and we present a proof of it within ACA_0 . We do not know whether it is provable in a weaker system. Another clue of the difficulties involved in proving this implication (and an earlier asymmetry between Atsuji and Lebesgue) derives from the fact that we are unable to prove the analogue of theorem 5.4 with Atsuji in place of Lebesgue (see remark 4.4).

The basic tool for our proof that Atsuji spaces are Lebesgue is the notion of ε -witness.

Definition 6.2 (RCA_0). Let $\mathfrak{U} = \langle U_n : n \in \mathbb{N} \rangle$ be an open covering of the complete separable metric space \widehat{A} and $\varepsilon \in \mathbb{R}^+$. We say that $x \in \widehat{A}$ is an ε -witness for \mathfrak{U} if for every n we have $B(x, \varepsilon) \not\subseteq U_n$, i.e. if for every n there exists $y \in \widehat{A}$ such that $y \in B(x, \varepsilon)$ and $y \notin U_n$.

Lemma 6.3 (RCA_0). *Let $\mathfrak{U} = \langle U_n : n \in \mathbb{N} \rangle$ be an open covering of the complete separable metric space \widehat{A} . The following properties are equivalent:*

1. \mathfrak{U} has no Lebesgue number.
2. For every $\varepsilon \in \mathbb{R}^+$ there exists $x \in \widehat{A}$ which is an ε -witness for \mathfrak{U} .
3. For every $\varepsilon \in \mathbb{R}^+$ there exists $a \in A$ which is an ε -witness for \mathfrak{U} .

Proof. The equivalence between (1) and (2) follows immediately from the definitions. (3) implies (2) is trivial.

To prove that (2) implies (3) suppose that (2) holds and let $\varepsilon \in \mathbb{R}^+$. Let $x \in \widehat{A}$ be an $\frac{\varepsilon}{2}$ -witness for \mathfrak{U} and let $a \in A$ be such that $d(x, a) < \frac{\varepsilon}{2}$. Then $B(a, \varepsilon) \supseteq B(x, \frac{\varepsilon}{2})$ and hence a is an ε -witness for \mathfrak{U} . \square

Remark 6.4. The formula asserting that x is an ε -witness for \mathfrak{U} is of the form $\forall n \exists y \psi(n, y, \varepsilon, x)$ with ψ arithmetical, i.e. it is an essentially Σ_1^1 formula (which in ACA_0 is not even provably equivalent to a Σ_1^1 formula).

Actually it is easy to see that the set of ε -witnesses is a G_δ (countable intersection of open sets) in \widehat{A} and hence is definable by a Π_2^0 formula. The obvious way of doing this requires the set $\{x \mid d(x, \widehat{A} \setminus U_n) < \varepsilon\}$ to be open, which is a consequence of the continuity of the map $x \mapsto d(x, \widehat{A} \setminus U_n)$: in theorem 7.1 we will show that these two statements are equivalent to $\Pi_1^1\text{-CA}_0$ and hence not available in ACA_0 .

In view of the preceding remark the notion of ε -witness appears inadequate for a proof in ACA_0 : we need to modify it by using an arithmetical definition, much more manageable within ACA_0 .

Definition 6.5 (RCA_0). Let $\mathfrak{U} = \langle U_n : n \in \mathbb{N} \rangle$ be an open covering of the complete separable metric space \widehat{A} and $\varepsilon \in \mathbb{R}^+$. We say that $x \in \widehat{A}$ is a *strong ε -witness* for \mathfrak{U} if for every n there exists $b \in A$ such that $b \in B(x, \varepsilon)$ and $b \notin U_n$.

Remark 6.6. Notice that not every ε -witness for \mathfrak{U} is a strong ε -witness. Indeed there exist complete separable metric spaces \widehat{A} and open coverings \mathfrak{U} of \widehat{A} such that for every $\varepsilon \in \mathbb{R}^+$ small enough there exist ε -witnesses but no strong ε -witnesses for \mathfrak{U} . An example consists, for $\alpha \in \mathbb{R}^+ \setminus \mathbb{Q}$ with $\alpha < 1/2$, of the space $\widehat{A} = \bigcup_{n \in \mathbb{N}} [n - \alpha 2^{-n}, n + \alpha 2^{-n}] \subset \mathbb{R}$ coded by $A = \widehat{A} \cap \mathbb{Q}$ with the covering $\mathfrak{U} = \{ (n - \alpha 2^{-n}, n + \alpha 2^{-n}), [n - \alpha 2^{-n}, n + \alpha 2^{-n}] \mid n \in \mathbb{N} \}$.

This shows that lack of Lebesgue number does not imply existence of strong ε -witnesses and we cannot replace strong ε -witness in place of ε -witness in the statement of lemma 6.3. Nevertheless in the proof of the next theorem we will change the open covering we deal with so that we can use strong ε -witnesses.

Theorem 6.7 (ACA_0). *Every Atsugi complete separable metric space is Lebesgue.*

Proof. Suppose \widehat{A} is a complete separable metric space which is not Lebesgue and let \mathfrak{U} be an open covering of \widehat{A} which has no Lebesgue number. Starting from \mathfrak{U} we will construct a continuous function $f : \widehat{A} \rightarrow \mathbb{R}$ which is not uniformly continuous, thereby showing that \widehat{A} is not Atsugi.

The first step in our construction is to replace \mathfrak{U} by a finer open covering \mathfrak{V} which not only has no Lebesgue number but for every $\varepsilon \in \mathbb{R}^+$ has a strong ε -witness. First of all we may assume that \mathfrak{U} consists of open balls with center in A and rational radius: let $\mathfrak{U} = \langle \mathfrak{B}(a_n, r_n) : n \in \mathbb{N} \rangle$. Define $\mathfrak{V} = \{ \mathfrak{B}(a_n, s) \mid n \in \mathbb{N} \wedge s \in \mathbb{Q}^+ \wedge s < r_n \}$. It is straightforward to check that \mathfrak{V} is an open covering of \widehat{A} .

We claim that for every $\varepsilon \in \mathbb{R}^+$ there exists $a \in A$ which is a strong ε -witness for \mathfrak{V} . To prove the claim fix ε and let, by lemma 6.3, $a \in A$ be an ε -witness for \mathfrak{U} . Now fix $B(a_n, s) \in \mathfrak{V}$: since a is an ε -witness for \mathfrak{U} there exists $y \in B(a, \varepsilon) \setminus B(a_n, r_n)$.

Let $b \in A$ be such that $d(b, y) < \min\{\varepsilon - d(a, y), r_n - s\}$. Then it is immediate to check that $b \in B(a, \varepsilon)$ and $b \notin B(a_n, s)$, thereby showing that a is a strong ε -witness for \mathfrak{V} and establishing the claim.

Observe that if a is a (strong) ε -witness then $P(a, \varepsilon) \neq \emptyset$ and hence there exists $b \in A$ such that $b \in P(a, \varepsilon)$. Therefore, using the fact that being a strong ε -witness for \mathfrak{V} is an arithmetical property, within ACA_0 we can construct a sequence $\langle (b_m^0, b_m^1) : m \in \mathbb{N} \rangle$ of pairs of elements of A such that for every m b_m^0 is a strong 2^{-m} -witness for \mathfrak{V} and $b_m^1 \in P(b_m^0, 2^{-m})$.

The following fact about sequences of b_m^i 's will be useful in the remainder of the proof.

Sublemma 6.7.1 (RCA_0). *Suppose $\langle x_k : k \in \mathbb{N} \rangle$ is a sequence of elements of A such that for some unbounded function $g : \mathbb{N} \rightarrow \mathbb{N}$ we have that for every k x_k is $b_{g(k)}^i$ for some $i < 2$. Then $\lim_{k \rightarrow \infty} x_k$ does not exist.*

Proof. Suppose $x = \lim_{k \rightarrow \infty} x_k$. Since \mathfrak{V} is an open covering of \widehat{A} there exist $B(a_n, s) \in \mathfrak{V}$ and $\varepsilon \in \mathbb{R}^+$ such that $B(x, \varepsilon) \subseteq B(a_n, s)$. Since the x_k 's converge to x and g is unbounded there exists k such that $d(x_k, x) < \frac{\varepsilon}{3}$ and $2^{-g(k)} < \frac{\varepsilon}{3}$. Now it is easy to check that $B(b_{g(k)}^0, 2^{-g(k)}) \subseteq B(x, \varepsilon) \subseteq B(a_n, s)$, contradicting the fact that $b_{g(k)}^0$ is a $2^{-g(k)}$ -witness for \mathfrak{V} . \square

The sublemma implies that for every $i < 2$ the sequence $\langle b_m^i : m \in \mathbb{N} \rangle$ does not contain infinitely many repetitions of the same element of A . Hence we can define by recursion a strictly increasing function $h : \mathbb{N} \rightarrow \mathbb{N}$ by setting $h(0) = 0$ and $h(n+1) =$ the least k such that for all $m \geq k$ and all $i, j < 2$ $b_m^i \neq b_{h(n)}^j$. The definition of h implies that if we let $C_i = \{b_{h(n)}^i \mid n \in \mathbb{N}\}$ then $C_0 \cap C_1 = \emptyset$.

Another consequence of the sublemma is that if a sequence of elements of C_i converges it is eventually constant. This means $\overline{C_i} = C_i$, i.e. that C_0 and C_1 are separably closed. Exactly as in the proof of (1) \implies (2) in theorem 4.3 we use theorems 2.7 and 2.10 to construct a continuous function $f : \widehat{A} \rightarrow \mathbb{R}$ such that $f(C_i) = \{i\}$. To see that f is not uniformly continuous fix $\delta \in \mathbb{R}^+$: let n be such that $2^{-h(n)} \leq \delta$: then $d(b_{h(n)}^0, b_{h(n)}^1) < \delta$ but $|f(b_{h(n)}^1) - f(b_{h(n)}^0)| = 1$. \square

7. THE CONTINUITY OF THE DISTANCE FROM A CLOSED SET

In this section we study the function that computes the distance of points of a complete separable metric space from a fixed closed set: the definition of this function involves a greatest lower bound and it is well-known that the existence of inf's and sup's is equivalent to ACA_0 (see [15]: this is indeed one of the very first reverse mathematics results obtained by Friedman). However we show that ACA_0 does not suffice to prove the continuity of the distance from a closed set: this continuity is equivalent to $\Pi_1^1\text{-CA}_0$. We also show that if instead of a closed set we consider a separably closed set the continuity of the function is equivalent to ACA_0 (and hence to the existence of the inf needed for its definition).

The last equivalent condition of the next theorem asserts that the sets needed in the straightforward definition of the set of ε -witnesses as a G_δ are indeed open.

Theorem 7.1 (RCA_0). *The following are equivalent:*

1. $\Pi_1^1\text{-CA}_0$.

2. For every complete separable metric space \widehat{A} and every closed set C in \widehat{A} there exists a continuous function $f_C : \widehat{A} \rightarrow \mathbb{R}$ such that for every $x \in \widehat{A}$ we have $f_C(x) = \inf\{d(x, y) \mid y \in C\}$.
3. For every complete separable metric space \widehat{A} and every open set U in \widehat{A} the set $\{(x, \varepsilon) \in \widehat{A} \times \mathbb{R} \mid B(x, \varepsilon) \not\subseteq U\}$ is open.
4. For every complete separable metric space \widehat{A} , every open set U in \widehat{A} and every $\varepsilon \in \mathbb{R}$ the set $\{x \in \widehat{A} \mid B(x, \varepsilon) \not\subseteq U\}$ is open.

Proof. (1) \implies (2). Let \widehat{A} and C be given. Let Φ be a set which enumerates all quadruples $(a, r, c, s) \in A \times \mathbb{Q}^+ \times \mathbb{Q} \times \mathbb{Q}^+$ such that

$$\forall b \in A (d(a, b) < r \implies \exists x \in C (d(b, x) < c + s) \wedge \forall y \in C (d(b, y) > c - s)).$$

The preceding formula is equivalent to a Boolean combination of Π_1^1 formulas and hence Φ exists within $\Pi_1^1\text{-CA}_0$. Moreover $\Pi_1^1\text{-CA}_0$ shows that Φ is a code for the function f_C .

(2) \implies (3). If C is the complement of U we have that (x, ε) is such that $B(x, \varepsilon) \not\subseteq U$ if and only if $f_C(x) < \varepsilon$. If Φ is a code for the continuous function f_C then $\{(x, \varepsilon) \mid B(x, \varepsilon) \not\subseteq U\} = \{(x, \varepsilon) \mid \exists(n, a, r, c, s) \in \Phi (d(a, x) < r \wedge c + s < \varepsilon)\}$ is open by lemma 2.3.

(3) \implies (4) is trivial.

(4) \implies (1). We reason within RCA_0 and begin by showing that (4) implies ACA_0 . To this end let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a one-to-one function: we need to show that the range of f exists. \mathbb{N} with the usual metric can be viewed as a complete separable metric space \widehat{A} and we can consider the open set $U = \{n \in \widehat{A} \mid \exists k n = f(k)\}$: by lemma 2.3 U can be coded within RCA_0 . By (4) let U' be the open set $\{n \in \widehat{A} \mid B(n, 1) \not\subseteq U\}$. It is immediate to check that for every n we have

$$\exists k f(k) = n \quad \text{iff} \quad n \notin U'.$$

The right-hand side of the above equivalence gives a Π_1^0 definition of the range of f , that therefore exists within RCA_0 by Δ_1^0 -comprehension.

Now we can prove within ACA_0 that (4) implies $\Pi_1^1\text{-CA}_0$. It is well-known that $\Pi_1^1\text{-CA}_0$ is equivalent over RCA_0 (and, a fortiori, over ACA_0) to the statement that if $\langle T_n : n \in \mathbb{N} \rangle$ is an infinite sequence of trees of finite sequences of natural numbers then the set $X = \{n \mid T_n \text{ is well-founded}\}$ exists. Fix a sequence of trees $\langle T_n : n \in \mathbb{N} \rangle$ and let $T = \{\langle \rangle\} \cup \{s \in \mathbb{N}^{<\mathbb{N}} \mid \langle s(1), \dots, s(\text{lh}(s) - 1) \rangle \in T_{s(0)}\}$. We work in the Baire space $\mathbb{N}^{\mathbb{N}}$ (with the metric described in section 2) and consider the open set $U = \{x \in \mathbb{N}^{\mathbb{N}} \mid \exists n x[n] \notin T\}$, whose elements are all infinite sequences which are not paths through T . Once more lemma 2.3 insures that we can find a code for U . By (4) let U' be the open set $\{x \in \mathbb{N}^{\mathbb{N}} \mid B(x, 1) \not\subseteq U\}$. Let $x_n \in \mathbb{N}^{\mathbb{N}}$ be the infinite sequence consisting of n followed by infinitely many 0's. It is immediate to check that for every n we have

$$T_n \text{ is well-founded} \quad \text{iff} \quad x_n \notin U'.$$

The right-hand side of the above equivalence gives an arithmetical definition of X , that therefore exists within ACA_0 . \square

Remark 7.2. The open set U used in the second part of the preceding proof is the same used by Brown in his proof of theorem 2.8. The same open set yields an immediate proof of (2) \implies (1) without the need of obtaining ACA_0 first.

Theorem 7.3 (RCA₀). *The following are equivalent:*

1. ACA₀.
2. *For every complete separable metric space \hat{A} and every separably closed set \overline{C} in \hat{A} there exists a continuous function $f_{\overline{C}} : \hat{A} \rightarrow \mathbb{R}$ such that for every $x \in \hat{A}$ we have $f_{\overline{C}}(x) = \inf\{d(x, y) \mid y \in \overline{C}\}$.*

Proof. (1) \implies (2). Let \hat{A} and $C = \langle x_n : n \in \mathbb{N} \rangle$ (the code for \overline{C}) be given. A code for the function $f_{\overline{C}}$ can be obtained by setting $(a, r)\Phi(c, s)$ if and only if

$$\exists q \in \mathbb{Q}^+ \forall \epsilon \in \mathbb{A} ((\exists \delta, \epsilon) < \setminus \rightarrow - \sim + \parallel < \inf\{(\cdot, \curvearrowright_{\kappa}) \mid \kappa \in \mathbb{N}\} < + \sim).$$

Since ACA₀ proves that every sequence of reals which has a lower bound (in this case 0) has a greatest lower bound and the formula is arithmetical we have that the code exists within ACA₀.

(2) \implies (1). We use theorem 2.7 and prove that (2) implies that every separably closed set is closed. This is immediate because the complement of \overline{C} is the preimage by $f_{\overline{C}}$ of the open interval $(0, +\infty)$ and hence it is open. \square

Remark 7.4. Another proof of (2) \implies (1) of the last theorem consists of deducing from (2) the existence of the least upper bound for any bounded sequence of real numbers.

REFERENCES

- [1] M. ATSUJI, Uniform continuity of continuous functions of metric spaces, *Pacific J. Math.* **8** (1958) 11–16.
- [2] G. BEER, Metric spaces on which continuous functions are uniformly continuous and Hausdorff distance, *Proc. Amer. Math. Soc.* **95** (1985) 653–658.
- [3] G. BEER, More about metric spaces on which continuous functions are uniformly continuous, *Bull. Austral. Math. Soc.* **33** (1986) 397–406.
- [4] D. K. BROWN, *Functional Analysis in Weak Subsystems of Second Order Arithmetic*, Ph.d. thesis, The Pennsylvania State University, 1987.
- [5] D. K. BROWN, Notions of closed subsets of a complete separable metric space in weak subsystems of second order arithmetic, in *Logic and Computation* (W. Sieg, ed.), *Contemporary Mathematics* 106, American Mathematical Society, 1990, pp. 39–50.
- [6] D. K. BROWN AND S. G. SIMPSON, Which set existence axioms are needed to prove the separable Hahn-Banach theorem?, *Ann. Pure Appl. Logic* **31** (1986) 123–144.
- [7] D. K. BROWN AND S. G. SIMPSON, The Baire category theorem in weak subsystems of second order arithmetic, *J. Symb. Logic* **58** (1993) 557–578.
- [8] M. DO CARMO, *Differential Geometry of Curves and Surfaces*, Prentice Hall, 1976.
- [9] J. G. HOCKING AND G. S. YOUNG, *Topology*, Addison Wesley, 1961.
- [10] J. L. KELLEY, *General Topology*, Van Nostrand, 1955.
- [11] C. KOSNIOWSKI, *A First Course in Algebraic Topology*, Cambridge University Press, 1980.
- [12] Y.N. MOSCHOVAKIS, *Descriptive Set Theory*, North-Holland, 1980.
- [13] N. SHIOJI AND K. TANAKA, Fixed point theory in weak second-order arithmetic, *Ann. Pure Appl. Logic* **47** (1990) 167–188.
- [14] S. G. SIMPSON, Subsystems of Z_2 and reverse mathematics, appendix to G. Takeuti, *Proof Theory*, 2nd edition, North-Holland, 1986.
- [15] S. G. SIMPSON, *Subsystems of Second Order Arithmetic*, in preparation.

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