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A polynomial time algorithm for computing the nucleolus for a class of disjunctive games with a permission structure

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Abstract Recently, applications of cooperative game theory to economic allocation problems have gained popularity. In many such allocation problems there is some hierarchical ordering of the players. In this paper we consider a class of *games with a permission structure* describing situations in which players in a cooperative TU-game are hierarchically ordered in the sense that there are players that need permission from other players before they are allowed to cooperate. The corresponding restricted game takes account of the limited cooperation possibilities by assigning to every coalition the worth of its largest feasible subset. In this paper we provide a polynomial time algorithm for computing the *nucleolus* of the restricted games corresponding to a class of games with a permission structure which economic applications include auction games, dual airport games, dual polluted river games and information market games.

Keywords TU-game · Nucleolus · Game with permission structure · Peer group game · Information market game · Algorithm · Complexity

JEL Classification C71

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1 Introduction

A *cooperative game with transferable utility*, or simply a TU-game, is a finite set of players and for any subset (coalition) of players a worth representing the total payoff that the coalition can obtain by cooperating. A payoff vector is a vector which gives a payoff to each of the players, i.e., each component corresponds to precisely one of the players. A payoff vector is efficient if the sum of the payoffs is equal to the worth of the grand coalition of all players. A set-valued solution for TU-games assigns a set of payoff vectors (possibly empty) to every TU-game. A single-valued solution assigns precisely one payoff vector to every TU-game. A solution is said to be efficient if for every game any payoff vector assigned by the solution is efficient. The best known efficient set-valued solution is the *Core* (Gillies 1953). The two best known efficient single-valued solutions are the *Shapley value* (Shapley 1953) and the nucleolus (Schmeidler 1969).

In this paper we assume that the players in a TU-game are part of some hierarchical structure that is represented by a directed graph such that some players need permission from other players before they are allowed to cooperate within a coalition. In the literature two approaches to these *games with a permission structure* can be found. In the *conjunctive* approach, as considered in Gilles et al. (1992) and van den Brink and Gilles (1996), it is assumed that each player needs permission from all its predecessors in the directed graph before it is allowed to cooperate. Alternatively, in the *disjunctive* approach as developed in Gilles and Owen (1994) and van den Brink (1997), a player needs permission to cooperate of at least one of its direct predecessors (if it has any). So, according to the conjunctive approach a coalition is feasible if and only if for any player in the coalition it holds that all its predecessors are also in the coalition, whereas according to the disjunctive approach a coalition is feasible if and only if for any player in the coalition at least one of its predecessors (if it has any) is also in the coalition. Following an approach similar to that of Myerson (1977) for games with limited communication (graph) structure, in both the conjunctive and disjunctive approach to games with a permission structure a restricted game is derived. In games with a permission structure the restricted game assigns to every coalition the worth of its largest feasible subset. Applying well-known solutions as the Shapley value, Core or nucleolus to such restricted games yields solutions for games with a permission structure.

A special subclass of games with a permission structure arises from peer group situations, as introduced in Brânzei et al. (2002). A *peer group situation* is a triple consisting of a set of players, a hierarchical structure represented by a rooted directed tree, and for each player a real number representing its potential individual (economic) contribution to the society of all players. This yields an associated TU-game being the additive game in which the worth of any coalition is equal to the sum of the individual potentials of its members. In a rooted directed tree there is one top node (not having a predecessor), while any other node has precisely one predecessor. So, in case the hierarchical structure on the player set is a rooted directed tree, the conjunctive approach and the disjunctive approach as described above, coincide. The restricted game of the associated TU-game with respect to such a permission structure is called a *peer group game*. These peer group games have many interesting applications, such as auction games, dual airport games (see Brânzei et al. 2002) and dual polluted river games (see

Ni and Wang 2007). The larger class of games that we consider here also contains information market games (see Muto et al. 1989).

Clearly, in a peer group game the worth of a coalition is the sum of the individual potentials of the members of the largest feasible subset of the coalition. Since the top player is always in this set when it belongs to the coalition, and the largest feasible set is the empty set for any coalition not containing the unique top player, it follows that the top player is a veto player, i.e., any coalition not containing the (veto) top player has zero worth in the restricted game. In Arin and Feltkamp (1997) an exponential time algorithm has been given to compute the nucleolus for veto-rich games. In Brânzei et al. (2005) a polynomial time algorithm is given to compute the nucleolus of a peer group game. In this paper we modify the Arin-Feltkamp algorithm to compute the nucleolus of the restricted game induced by more general situations, including the examples mentioned above as special cases. The generalization concerns both the hierarchical graph structure and the class of unrestricted TU-games by allowing for any digraph having one top node and no directed cycles and any game satisfying a so-called weak digraph monotonicity condition and a weak digraph concavity condition. The algorithm finds the nucleolus in polynomial time.

The paper is organized as follows. Section 2 is a preliminary section on cooperative TU-games (with special attention for the nucleolus) and directed graphs. In Sect. 3 games with a permission structure are defined as considered in this paper and we introduce the properties of weak digraph monotonicity and weak digraph concavity for such games. In Sect. 4 we present some properties of essential and feasible coalitions. These properties are crucial for the algorithm given in Sect. 5. Section 5 contains three subsections. In the first subsection the algorithm is given and the second one contains the proof that the algorithm indeed finds the nucleolus. In the third subsection the algorithm is illustrated by an example. In Sect. 6 we discuss the complexity of the algorithm. Finally, Sect. 7 contains concluding remarks.

2 Preliminaries

2.1 TU-games

A situation in which a finite set of players can obtain certain payoffs by cooperation can be described by a *cooperative game with transferable utility*, or simply a TU-game, being a pair (N, v), where $N \subset \mathbb{N}$ is a finite set of n players and $v\colon 2^N \to \mathbb{R}$ is a characteristic function on N such that $v(\emptyset) = 0$. For any coalition $S \subseteq N$, $v(S)$ is the worth of coalition S, i.e., the members of coalition S can obtain a total payoff of $v(S)$ by agreeing to cooperate. For simplicity, for a single player i we denote its worth $v(\{i\})$ by $v(i)$. We denote the collection of all characteristic functions on N by $\mathcal{G}^N$. A TU-game (N, v) is *monotone* if $v(S) \leq v(T)$ for all $S \subseteq T \subseteq N$. It is *convex (concave)* if $v(S) + v(T) \leq (\geq) v(S \cap T) + v(S \cup T)$ for all $S, T \subseteq N$.

A collection $B = \{S_1, \ldots, S_m\}$ of subsets of N is said to be a *balanced collection* when the system of equations

$$\sum_{j=1}^{m} \lambda_j e^{S_j} = e^N \tag{1}$$

has a positive solution. A balanced collection B is called a *minimal* balanced collection if the system of Eq. 1 has a unique positive solution. In that case we denote it by λ_j^B, $j = 1, \ldots, m$, where, for $S \subseteq N$, the n-vector e^S is given by $e_i^S = 1$ when $i \in S$ and $e_i^S = 0$ otherwise. By $\mathcal{B}$ we denote the set of all minimal balanced collections, excluding the balanced collection $\{N\}$ having the grand coalition N as its single element. In the following we will only use minimal balanced collections and, since there is no confusion, just refer to a minimal balanced collection $B \in \mathcal{B}$ as a balanced collection. A game (N, v) is *balanced* if

$$\sum_{j=1}^{m} \lambda_j^B \, v(S_j) \leq v(N)$$

for any balanced collection $B = \{S_1, \ldots, S_m\} \in \mathcal{B}$.

A payoff vector is a vector $x \in \mathbb{R}^n$ assigning a payoff x_i to every $i \in N$. In the sequel, for $S \subseteq N$ we denote $x(S) = \sum_{i \in S} x_i$. A payoff vector is efficient if $x(N) = v(N)$ and it is individually rational if $x_i \geq v(i)$ for every $i \in N$. The imputation set $I(N, v)$ of game v is given by

$$I(N, v) = \left\{x \in \mathbb{R}^n | x(N) = v(N) \text{ and } x_i \geq v(i) \text{ for every } i \in N\right\},$$

i.e., $I(N, v)$ is the set of all efficient and individually rational payoff vectors.

A (set-valued) *solution* F on $\mathcal{G}^N$ assigns a set $F(N, v) \subset \mathbb{R}^n$ of payoff vectors to every characteristic function $v \in \mathcal{G}^N$. The best known set-valued solution is the Core assigning to every $v \in \mathcal{G}^N$ the set

$$C(N, v) = \{x \in I(N, v) | x(S) \geq v(S) \quad \text{for all } S \subset N\},$$

i.e., it is the set of all imputations that are stable in the sense that no coalition can do better by separating from the grand coalition. The Core of (N, v) is non-empty if and only if the game is balanced, see e.g. Bondareva (1962) or Shapley (1967).

A solution F is said to be single-valued if it assigns to any $v \in \mathcal{G}^N$ a unique payoff vector. The two best known single-valued solutions are the *Shapley value* (Shapley 1953) and the *nucleolus* (Schmeidler 1969). Since the aim of this paper is to give an algorithm for computing the nucleolus for a special class of characteristic functions, we devote the next subsection to this solution.

2.2 Nucleolus

Consider a given characteristic function $v \in \mathcal{G}^N$, and payoff vector $x \in \mathbb{R}^n$. Then the excess $e(S, x)$ of a coalition $S \subseteq N$ is defined by

$$e(S, x) = v(S) - x(S).$$

Further, let $E(x)$ be the $(2^n - 2)$-component vector that is composed of the excesses of all coalitions $S \subset N$, $S \neq \emptyset$, in a non-increasing order, so $E_1(x) \geq E_2(x) \geq$

$\cdots \geq E_{2^n-2}(x)$. Then the nucleolus $Nuc(N, v)$ of the game (N, v) is the unique imputation which lexicographically minimizes the vector-valued function $E(\cdot)$ over the imputation set. Formally,

$$Nuc(N, v) = x \in I(N, v) \text{ such that } E(x) \preceq_L E(y) \quad \text{for all } y \in I(N, v),$$

where $\preceq_L$ denotes the lexicographic order of vectors. It is well-known that $Nuc(N, v) \in C(N, v)$ when $C(N, v) \neq \emptyset$.

In a game (N, v), a coalition S is called *inessential* if it has a partition $\{S_1, \ldots, S_r\}$ with $r \geq 2$, such that $v(S) \leq \sum_{j=1}^{r} v(S_j)$. Coalitions which are not inessential are called *essential*. Notice that single player coalitions are always essential. It is straightforward to observe that for an inessential coalition S it holds that

$$e(S, x) \leq \sum_{j=1}^{r} e(S_j, x), \quad \text{for all } x \in \mathbb{R}^n.$$

Therefore the Core, and thus also the nucleolus, is independent of inessential coalitions, as was noticed by Huberman (1980). In fact, in any n player game there are at most $(2n - 2)$ coalitions which actually determine the nucleolus, see Brune (1983) and Reynierse and Potters (1998). Although, as noticed by Brânzei et al. (2005), identifying these coalitions is no less laborious as computing the nucleolus itself, in the following we state some facts for games with non-empty Core which will appear to be useful later on. We denote

$$e^*(N, v) = \min_{\{S \subset N | S \neq \emptyset\}} - e(S, x) \text{ at } x = Nuc(N, v),$$

i.e., $e^*(N, v)$ is the minimal negative excess at the nucleolus of game (N, v). Clearly, $e^*(N, v) \geq 0$ if and only if $Core(N, v) \neq \emptyset$.

Lemma 2.1 *If $e^*(N, v) > 0$, then every coalition $S \subset N$ with $-e(S, x) = e^*(N, v)$ at $x = Nuc(N, v)$ is essential.*

Proof Suppose $S \subset N$ with $-e(S, x) = e^*(N, v)$ is inessential. Then there is a partition $\{S_1, \ldots, S_m\}$ such that $e^*(N, v) = -e(S, x) \geq \sum_{j=1}^{m} - e(S_j, x)$. Since $e^*(N, v) > 0$ there must be at least one $j \in \{1, \ldots, m\}$ such that $-e(S_j, x) < -e(S, x)$, which contradicts that $e^*(N, v) = \min_{\{S \subset N | S \neq \emptyset\}} - e(S, x)$. □

Let $B = \{S_1, \ldots, S_m\} \in \mathcal{B}$ be a balanced collection of coalitions. The next lemma follows from Arin and Inarra (1998, Theorem 3.2)

Lemma 2.2 (Arin and Inarra 1998) *If $e^*(N, v) \geq 0$ then*

$$e^*(N, v) = \min_{B \in \mathcal{B}} \frac{v(N) - \sum_{j=1}^{m} \lambda_j^B v(S_j)}{\sum_{j=1}^{m} \lambda_j^B},$$

with λ_j^B, $j = 1, \ldots, m$, the solution of the system (1) for the balanced collection $B = \{S_1, \ldots, S_m\} \in \mathcal{B}$.

The next two corollaries follow immediately.

Corollary 2.3 *Let $B = \{S_1, \ldots, S_m\} \in \mathcal{B}$ be a balanced collection with weights λ_j^B, $j = 1, \ldots, m$, satisfying*

$$e^*(N, v) = \frac{v(N) - \sum_{j=1}^m \lambda_j^B v(S_j)}{\sum_{j=1}^m \lambda_j^B}. \tag{2}$$

Then at $x = Nuc(N, v)$ we have that $-e(S_j, x) = e^*(N, v)$, $j = 1, \ldots, m$.

Proof Since $\sum_{\{j|i \in S_j\}} \lambda_j^B = 1$ for every $i \in N$, it holds that $x(S) = \sum_{i \in S} x_i = \sum_{i \in S} \sum_{\{j|i \in S_j\}} \lambda_j^B x_i$ for every $x \in \mathbb{R}^n$ and $S \subset N$. Hence,

$$\sum_{j=1}^m \lambda_j^B x(S_j) = \sum_{j=1}^m \sum_{i \in S_j} \lambda_j^B x_i = \sum_{i \in N} \sum_{\{j|i \in S_j\}} \lambda_j^B x_i = x(N)$$

and thus at $x = Nuc(N, v)$ we have that

$$\begin{aligned} \frac{v(N) - \sum_{j=1}^m \lambda_j^B \, v(S_j)}{\sum_h \lambda_h^B} &= \sum_{j=1}^m \frac{\lambda_j^B}{\sum_h \lambda_h^B} \cdot (x(S_j) - v(S_j)) \\ &= \sum_{j=1}^m \frac{\lambda_j^B}{\sum_h \lambda_h^B} \cdot (-e(S_j, x)). \end{aligned}$$

Thus, the right-hand side of Eq. 2 is a convex combination of the numbers $-e(S_j, x)$. Therefore, for each j, $e^*(N, v) \leq -e(S_j, x)$ must hold with equality. □

Corollary 2.4 *If $e^*(N, v) > 0$, then for any balanced collection $B = \{S_1, \ldots, S_m\} \in \mathcal{B}$ satisfying $e^*(N, v) = \frac{v(N) - \sum_{j=1}^m \lambda_j^B v(S_j)}{\sum_{j=1}^m \lambda_j^B}$, it holds that any set S_j is essential.*

Proof This follows immediately from Lemma 2.1 and Corollary 2.3. □

Arin and Feltkamp (1997) propose an exponential time algorithm to find the nucleolus of a *veto-rich* game, i.e., a game (N, v) such that there exists (at least one) veto player being a player i such that $v(S) = 0$ when $i \notin S$. In this paper we modify this algorithm to find, in polynomial time, the nucleolus of restricted games arising from games with a permission structure in which players in a cooperative TU-game belong to a hierarchical structure that is represented by a directed graph.

2.3 Directed graphs

A directed graph or *digraph* is a pair (N, D) where $N \subset \mathbb{N}$ is a finite set of nodes (representing the players) and $D \subseteq N \times N$ is a binary relation on N. Given (N, D) and $S \subseteq N$, the digraph $(S, D(S))$ is the induced subgraph on S, that is given by $D(S) =$

$\{(i, j) \in D | i, j \in S\}$. In the sequel we simply refer to D for a digraph (N, D) and to $D(S)$ for the subgraph $(S, D(S))$. For $i \in N$ the nodes in $S_D(i) := \{j \in N \mid (i, j) \in D\}$ are called the *successors* of i, and the nodes in $P_D(i) := \{j \in N \mid (j, i) \in D\}$ are called the *predecessors* of i.

For given D on N, a *path* between i and j in N is a sequence of distinct nodes $(i_1, \ldots, i_m)$ such that $i_1 = i, i_m = j$, and $\{(i_k, i_{k+1}), (i_{k+1}, i_k)\} \cap D \neq \emptyset$ for $k = 1, \ldots, m-1$. A set of nodes $T \subseteq N$ is *connected* in digraph D if there is a path between any two nodes in T that only uses arcs between nodes in T, i.e., if for every $i, j \in T$ there is a path $(i_1, \ldots, i_m)$ between i and j such that $\{i_1, \ldots, i_m\} \subseteq T$. A *component* in D is a maximally connected set T of nodes, i.e., T is connected and $T \cup \{i\}$ is not connected for every $i \in N \backslash T$. A path $(i_1, \ldots, i_m)$ between i and j in D is a *directed path* if $(i_k, i_{k+1}) \in D$ for $k = 1, \ldots, m-1$. A directed path $(i_1, \ldots, i_m)$, $m \geq 1$, in D is a *cycle* if $(i_m, i_1) \in D$. We call digraph D *acyclic* if it does not contain any cycle. Note that acyclicity of a digraph D implies that D is irreflexive, i.e., $(i, i) \notin D$ for all $i \in N$.

A digraph is called *quasi-strongly connected* if there exists a node $i_0 \in N$, such that for every $j \neq i_0$ there is a directed path from i_0 to j. Note that this implies that N is connected. When D is acyclic then i_0 is the unique node in N having no predecessors and i_0 is called the *top-node* of the digraph. The collection of all acyclic, quasi-strongly connected digraphs on N is denoted by $\mathcal{D}^N$. A digraph $D \in \mathcal{D}^N$ is a rooted directed tree with root i_0 if there is precisely one path from the top-node i_0 to every other node. Node $j \in N$ is a *complete subordinate* of node $i \in N$ in $D \in \mathcal{D}^N$ if every directed path from the top-node i_0 to node j contains node i. We denote the set of complete subordinates of node i by $\overline{S}_D(i)$, i.e.,

$$\overline{S}_D(i) = \left\{ j \in N \;\middle|\; \begin{array}{c} i \in \{h_1, \ldots, h_{t-1}\} \text{ for every sequence of nodes } h_1, \ldots, h_t \\ \text{such that } h_1 = i_0, h_{k+1} \in S_D(h_k),\ k \in \{1, \ldots, t-1\}, \text{ and } h_t = j \end{array} \right\}.$$

3 Games with a permission structure

In this paper we assume that the players in a TU-game are part of a hierarchical structure that is represented by a directed graph, referred to as a *permission structure*, such that some players need permission from other players before they are allowed to cooperate within a coalition. A triple (N, v, D) with (N, v) a TU-game and (N, D) a digraph with the player set N as the set of nodes is called a *game with a permission structure*. In the sequel we assume that $D \in \mathcal{D}^N$ and, without loss of generality, we assume that $i_0 = 1$ as its unique top-node[1] and that N is essential in the restricted game (N, r).

Assumption 3.1 (i) (N, D) is acyclic and quasi-strongly connected with $P_D(1) = \emptyset$.
(ii) N is essential in the restricted game (N, r).

[1] This implies that $1 \in N$. Later we consider reduced games on proper subsets of $N' \subset N$, but the top-player 1 always belongs to N'.

The first part of Assumption 3.1 implies that $P_D(i) \neq \emptyset$ for every $i \neq 1$.

As noticed in the introduction, we can distinguish between the conjunctive and disjunctive approach. In this paper we take the *disjunctive* approach as developed in Gilles and Owen (1994) and van den Brink (1997), where a player $i \neq 1$ needs permission to cooperate of at least one of its direct predecessors. Therefore a coalition is feasible if and only if it contains the top-player 1 and for every other player in the coalition at least one of its predecessors is also in the coalition. So, for digraph (N, D), the set of disjunctive feasible coalitions is given by[2]

$$\Phi^D = \{S \subseteq N \mid P_D(i) \cap S \neq \emptyset \text{ for all } i \in S \backslash \{1\}\}.$$

For any $S \subseteq N$, let $\sigma(S) = \cup\{T \in \Phi^D \mid T \subseteq S\}$ be the largest disjunctive feasible subset of S in D.[3] Since, by quasi-strongly connectedness, for every $i \neq 1$ there is at least one directed path from 1 to i, it follows that for every $S \subseteq N$ with $\sigma(S) \neq \emptyset$, the subgraph $(\sigma(S), D(\sigma(S))$ is acyclic and quasi-strongly connected with node $1 \in \sigma(S)$ as its unique top-node.

Given the triple (N, v, D) with $v \in \mathcal{G}^N$ and $D \in \mathcal{D}^N$, under the disjunctive approach the induced *restricted* game $r\colon 2^N \to \mathbb{R}$ is given by

$$r(S) = v\,(\sigma(S)) \quad \text{for all } S \subseteq N. \tag{3}$$

Since player 1 is the top-node it holds that $r(S) = 0$ when $1 \notin S$, i.e., the restricted game is a veto-rich game with respect to the top-player 1. If D is a rooted directed tree (with node 1 as its root), then $|P_D(i)| = 1$ for all $i \neq 1$, and the conjunctive and disjunctive approach coincide. When D is a rooted tree and (N, v) is a non-negative additive game, i.e., there exist real numbers $a_i, i \in N$, such that $v(S) = \sum_{i \in S} a_i, S \subseteq N$, then the triple (N, v, D) is a *peer group situation* (see Brânzei et al. 2002). For a peer group situation the restricted game (N, r) as defined in (3) is a so-called *peer group game* and is given by

$$r(S) = v(\sigma(S)) = \sum_{\{i \in S \mid \widehat{P}_D(i) \subseteq S\}} a_i,$$

where $j \in \widehat{P}_D(i)$ if and only if there exists a sequence of players $(h_1, \ldots, h_t)$ such that $h_1 = j, h_{k+1} \in S(h_k)$ for all $1 \leq k \leq t-1$, and $h_t = i$. A peer group game (N, r) is a monotone veto-rich game and has a non-empty Core. In particular (with 1 the veto player) the payoff vector $x \in \mathbb{R}^n_+$ given by $x_1 = v(N)$ and $x_i = 0, i \neq 1$, belongs to the Core.

In Sect. 5 we will present an algorithm for computing the nucleolus of the restricted game (N, r) of a game with a permission structure (N, v, D) that satisfies *weak digraph monotonicity* and *weak digraph concavity*.

[2] Since we allow for cycles in the underlying undirected graph, the set of feasible coalitions Φ^D is not necessarily the set of connected coalitions in some cycle-free undirected graph as considered in Kuipers et al. (2000).

[3] Every coalition having a unique largest feasible subset follows from Φ^D being closed under union.

Definition 3.2 (a) A game with permission structure (N, v, D) satisfies *weak digraph monotonicity* if

$$S \in \Phi^D \Rightarrow v(S) \leq v(N).$$

(b) A game with permission structure (N, v, D) satisfies *weak digraph concavity* if

$$\left[S \cup T = N \text{ and } S, T \in \Phi^D\right] \Rightarrow v(S) + v(T) \geq v(S \cap T) + v(N).$$

Part (a) states that the triple (N, v, D) satisfies weak digraph monotonicity if the worth of every feasible coalition is at most equal to the worth of the grand coalition N. Weak digraph monotonicity weakens monotonicity in two respects, namely (i) the monotocity condition $v(S) \leq v(T)$ if $S \subseteq T$ only has to hold for $T = N$ and (ii) for sets S that are feasible given the disjunctive permission structure on the digraph D.[4] Part (b) states that (N, v, D) satisfies weak digraph concavity if the usual inequalities for concavity are only required for feasible coalitions which union equals the grand coalition. This property weakens concavity in two respects, namely that the concavity condition only has to hold for sets S and T satisfying (i) $S \cup T = N$ and (ii) S and T are feasible given the disjunctive permission structure on D. In fact, for both properties the adjunctive *'weak'* means that the inequality conditions are only required for $T = N$, respectively $S \cup T = N$, and the adjunctive *'digraph'* means that the inequality conditions are only required for feasible sets with respect to the permission structure. Monotonicity is a condition satisfied by most of the games that arise from economic or social situations, so this is certainly the case for weak digraph monotonicity. Although concavity is a strong condition for profit games[5], weak digraph concavity is considerably weaker and is also satisfied by several interesting classes of profit games with a permission structure. We give some examples.

Example 3.3 Generalised peer group situations It is obvious that a peer group situation (N, v, D) satisfies weak digraph monotonicity. Further, for any feasible S and T such that $S \cup T = N$ we have that $S \cap T$ is feasible (since D is a rooted tree) and

$$v(S) + v(T) = \sum_{i \in S} a_i + \sum_{i \in T} a_i = \sum_{i \in S \cap T} a_i + \sum_{i \in N} a_i = v(S \cap T) + v(N).$$

So, (N, v, D) also satisfies weak digraph concavity. A triple (N, v, D) is a *generalised peer group situation* when $D \in \mathcal{D}^N$ is an acyclic and quasi-strongly connected

[4] For (N, v, D) weak digraph monotone, part (ii) of Assumption 3.1 is without loss of generality. When N is inessential, then there exists a partition $\{S_1, \ldots, S_m\}$ such that $r(N) \leq \sum_{j=1}^{m} r(S_j)$, S_1 is essential, and $1 \in S_1$. Because of the latter we have that $S_2, \ldots, S_m$ are not feasible and thus $r(S_j) = 0$ for $j = 2, \ldots, m$. Together with weak digraph monotonicity this implies that $r(N) = r(S_1)$. According to Arin and Feltkamp (1997), the nucleolus then assigns zero payoff to every player not in S_1 and we can restrict ourselves to the subgame and subgraph on the essential coalition S_1 containing player 1.

[5] Given our nucleolus concept in which the maximum excess $v(S) - x(S)$ is minimized, in this paper we deal with profit games.

digraph and v is again a nonnegative additive game. Clearly, also every generalised peer group situation satisfies weak digraph monotonicity and weak digraph concavity.

Example 3.4 Generalised information market situations Let $\mathcal{S} = \{S_1, \ldots, S_K\}$ be a collection of K (nonempty) subsets of N, and α_k, $k = 1, \ldots, K$, be positive numbers. Then we consider the game (N, v) given by

$$v(S) = \sum_{\{k \mid S_k \cap S \neq \emptyset\}} \alpha_k, \quad S \subseteq N. \tag{4}$$

Further, let $D \in \mathcal{D}^N$ be any digraph satisfying $(1, j) \in D$ for all $j \in \{2, \ldots, n\}$. So, $j = 1$ is the top-player and $S \subseteq N$ is feasible if and only if $1 \in S$. Now, the restricted game (N, r) is given by $r(S) = 0$ if $1 \notin S$ and

$$r(S) = \sum_{\{k \mid S_k \cap S \neq \emptyset\}} \alpha_k, \text{ if } 1 \in S.$$

The game (N, r) is an information game as introduced in Muto et al. (1989). Obviously, (N, v, D) satisfies weak digraph monotonicity. Further, for any feasible S and T such that $S \cup T = N$ we have that $S \cap T$ is feasible and

$$\begin{aligned} v(S) + v(T) &= \sum_{\{k \mid S_k \cap S \neq \emptyset\}} \alpha_k + \sum_{\{k \mid S_k \cap T \neq \emptyset\}} \alpha_k \\ &= \sum_{\{k \mid S_k \cap (S \cap T) \neq \emptyset\}} \alpha_k + \sum_{\{k \mid S_k \cap N \neq \emptyset\}} \alpha_k = v(S \cap T) + v(N) \end{aligned}$$

where the last but one equality follows since $S \cup T = N$. Thus (N, v, D) also satisfies weak digraph concavity. In fact, this condition is satisfied for any $D \in \mathcal{D}^N$. Also any game with a permission structure (N, v, D) where v is the sum of an additive game and a game as given in Eq. 4, satisfies the conditions of weak digraph monotonicity and weak digraph concavity.

4 Essential and feasible coalitions

In this section we prove several results of essential and feasible coalitions for games with a permission structure (N, v, D) satisfying Assumption 3.1. These results will be used later on to prove that the algorithm of Sect. 5 will indeed find the nucleolus of the induced restricted game (N, r) under the disjunctive approach. The first lemma states that any essential coalition with at least two elements is feasible.

Lemma 4.1 *Let (N, v, D) be a game with a permission structure. If $S \subseteq N$ with $|S| \geq 2$ is essential in the restricted game (N, r), then S is feasible.*

Proof Suppose that S is not feasible. Then $r(S) = r(\sigma(S))$ with $\sigma(S) \subset S$. Since $r(j) = 0$ for all $j \in S \backslash \sigma(S)$, it holds that $r(S) = r(\sigma(S)) + \sum_{j \in S \backslash \sigma(S)} r(j)$, implying that S is not essential. $\square$

When (N, v, D) satisfies weak digraph monotonicity, then the restricted game (N, r) is a weak monotone ($r(S) \leq r(N)$ for all $S \subseteq N$) veto-rich game (with veto player 1) and therefore the Core contains the payoff vector $(r(N), 0, \ldots, 0)^\top \in \mathbb{R}^n$ and thus is not empty. (Observe that $r(N) = v(N)$.) Hence, $Nuc(N, r)$ is in the Core of (N, r) and independent of inessential coalitions. Thus Lemma 4.1 implies that every non-feasible coalition S with $|S| \geq 2$ can be ignored.

According to Arin and Feltkamp (1997) the nucleolus assigns positive payoff to every player in N when N is essential. Notice also that $r(N) > r(S)$ for every $S \subset N$ when N is essential. This yields the next lemma.

Lemma 4.2 *Let (N, v, D) be a game with a permission structure. If (N, v, D) is weak digraph monotone, then $e^*(N, r) > 0$.*

Proof By weak digraph monotonicity we have that $C(N, r) \neq \emptyset$ and thus $e^*(N, r) \geq 0$. Hence, according to Lemma 2.2,

$$e^*(N, r) = \min_{B \in \mathcal{B}} \frac{r(N) - \sum_{j=1}^{m} \lambda_j^B r(S_j)}{\sum_{j=1}^{m} \lambda_j^B},$$

with λ_j^B, $j = 1, \ldots, m$, the solution of the system (1) for the balanced collection $B \in \mathcal{B}$. Since $r(S_j) = 0$ when $1 \notin S_j$, we obtain that

$$e^*(N, r) = \min_{B \in \mathcal{B}} \frac{r(N) - \sum_{\{j|1 \in S_j\}} \lambda_j^B r(S_j)}{\sum_{j=1}^{m} \lambda_j^B}.$$

Since the collection $\{N\}$ does not belong to $\mathcal{B}$, any S_j in a balanced collection $B \in \mathcal{B}$ is a real subset of N and thus $r(S_j) < r(N)$ for every S_j. Since $\sum_{\{j|1 \in S_j\}} \lambda_j^B = 1$ by the definition of balancedness, it follows that $r(N) - \sum_{\{j|1 \in S_j\}} \lambda_j^B r(S_j) > 0$ for any $B \in \mathcal{B}$, which proves the lemma. □

Similar as in Arin and Feltkamp (1997), in the sequel we denote for $S \subset N$ and the restricted game (N, r),

$$\tau(S, r) = \frac{r(N) - r(S)}{|N \backslash S| + 1}.$$

In the following, $\Omega^D = \Phi^D \backslash \{N\}$ denotes the collection of all feasible coalitions not equal to N. We now have the following lemmas.

Lemma 4.3 *Let (N, v, D) be a game with a permission structure. If (N, v, D) is weak digraph monotone, then*

$$e^*(N, r) = \min_{S \in \Omega^D} \tau(S, r).$$

Proof According to Kohlberg's theorem (Kohlberg 1971) there exists a balanced collection $\{S_1, \ldots, S_m\}$ such that $-e(S_k, x) = e^*(N, r)$ for all $k = 1, \ldots, m$. Since $e^*(N, r) > 0$ by Lemma 4.2, according to Corollary 2.4 we have that any S_j is essential. Without loss of generality, let $1 \in S_1$. Then, we have that either $S_1 = \{1\}$ and thus feasible, or $|S_1| > 1$ and thus feasible according to Lemma 4.1. Denote $U = S_1$. Now, consider $j \notin U$. Since the collection is balanced, there must be a coalition $S_k \neq S_1 = U$ containing j, but not 1. Then S_k is essential, but not feasible. Hence it follows with Lemma 4.1 that $|S_k| = 1$ and thus $S_k = \{j\}$. Now, let λ_U^B and λ_j^B, $j \notin U$, be the corresponding weights. Then $\lambda_U^B = \lambda_j^B = 1$, $j \notin U$. Further $r(j) = 0$ for all $j \notin U$ since $\{j\}$ is not feasible. Substituting these values in (2) gives $e^*(N, r) = \frac{r(N) - \lambda_U^B r(U)}{|N \backslash U| + 1} = \tau(U, r)$, showing that there exists a coalition $U \in \Omega^D$ satisfying $e^*(N, r) = \tau(U, r)$. Next, consider any $S \in \Omega^D$. Then $B = \{S\} \cup \{\{j\} \mid j \notin S\}$ is a balanced collection with corresponding weights $\lambda_S^B = \lambda_j^B = 1$, $j \notin S$. Since $1 \in S$ (because S is feasible), it follows that $r(j) = 0$ for all $j \notin S$. Hence with Lemma 2.2 we obtain that $e^*(N, r) \leq \frac{r(N) - \lambda_S^B r(S) - \sum_{j \notin S} \lambda_j^B r(j)}{|N \backslash S| + 1} = \frac{r(N) - r(S)}{|N \backslash S| + 1} = \tau(S, r)$. □

Lemma 4.4 *Let (N, v, D) be a game with a permission structure satisfying weak digraph monotonicity, let $U \in \Omega^D$ be such that $\tau(U, r) = e^*(N, r)$, and let $y \in \mathbb{R}^n$ be such that $y(U) = r(U) + \tau(U, r)$ and $y_j = \tau(U, r)$ for all $j \notin U$. Then $x = Nuc(N, r)$ satisfies $x(U) = y(U)$ and $x_j = y_j$ for all $j \notin U$.*

Proof First, observe that

$$y(N) = y(U) + \sum_{j \notin U} y_j = r(U) + (|N \backslash U| + 1)\tau(U, r) = r(N),$$

so y is efficient. Next, observe that U is feasible and thus $1 \in U$. Hence for any $j \notin U$, the singleton coalition $\{j\}$ is not feasible and thus $r(j) = 0$. Therefore the excesses for the coalitions $U \in \Phi^D$ and the singletons $\{j\}$, $j \notin U$, at y are equal to $e(U, y) = -\tau(U, r) = e(\{j\}, y)$, $j \notin U$. Now, suppose that $x = Nuc(N, r)$ does not satisfy $x(U) = y(U)$ and $x_j = y_j$. Then

$$\min[-e(U, x), \ \min_{j \notin U} -e(\{j\}, x)] < \tau(U, r),$$

contradicting that $\tau(U, r) = e^*(N, r) = \min_{\{S \subset N, S \neq \emptyset\}} \ -e(S, x)$. □

The two lemmas above show that as soon as a coalition $U \in \Omega^D$ has been found with $\tau(U, r) = \min_{S \in \Omega^D} \ \tau(S, r)$, the nucleolus values of all players $j \notin U$ have been found and that these values are equal to $\tau(U, r)$. This gives us the basic idea for the algorithm in the next section. In the sequel, denote $\tau^*(r) = \min_{S \in \Omega^D} \ \tau(S, r)$. In the first step the algorithm searches for a coalition $U_1 \in \Omega^D$ satisfying

$$\tau(U_1, r) = \tau^*(r) \text{ and } |U_1| = \max_{\{U \in \Omega^D \mid \tau(U, r) = \tau^*(r)\}} |U|, \tag{5}$$

i.e., any other feasible set $U \neq N$ satisfying $\tau(U, r) = \tau^*(r)$ contains at most the same number of players as U_1. This gives nucleolus payoffs $\tau^*(r) = \tau(U_1, r)$ to any player $j \notin U_1$ and in the next step the algorithm continues with a search on a reduced set of players U_1. The details of the algorithm will be given in the next section.

In the rest of this section we give several results with respect to a set U_1 satisfying condition (5). Notice that in the previous three lemmas only weak digraph monotonicity is required. The next results require both weak digraph monotonicity and weak digraph concavity.

Lemma 4.5 *Let game with permission structure (N, v, D) satisfy weak digraph monotonicity and weak digraph concavity and, for a coalition U_1 satisfying condition (5), let $\{T_1,\ T_2\}$ be a partition of $N\backslash U_1$. Then at least one of the two coalitions $U_1 \cup T_1, U_1 \cup T_2$ is not feasible.*

Proof Suppose that both sets $U_1 \cup T_1$ and $U_1 \cup T_2$ are feasible. Then we have that

$$
\begin{aligned}
&\frac{|T_2|+1}{|T_1|+|T_2|+2}\tau(U_1 \cup T_1, r) + \frac{|T_1|+1}{|T_1|+|T_2|+2}\tau(U_1 \cup T_2, r) \\
&= \frac{r(N) - r(N\backslash T_2)}{|T_1|+|T_2|+2} + \frac{r(N) - r(N\backslash T_1)}{|T_1|+|T_2|+2} \\
&= \frac{2r(N) - r(N\backslash T_1) - r(N\backslash T_2)}{|T_1|+|T_2|+2} \leq \frac{r(N) - r(U_1)}{|T_1|+|T_2|+2},
\end{aligned}
$$

where the last inequality follows from weak digraph concavity for the sets $N\backslash T_j$, $j = 1, 2$, since $r(N) = v(N), r(U_1) = v(U_1)$ by the feasibility of U_1, and for $i \in \{1, 2\}, i \neq j$, we have that $r(N\backslash T_j) = r(U_1 \cup T_i) = v(U_1 \cup T_i)$ because of the feasibility of $U_1 \cup T_i$. Further since $r(U_1) = v(U_1) \leq v(N) = r(N)$ because of weak digraph monotonicity, we have that

$$
\frac{r(N) - r(U_1)}{|T_1|+|T_2|+2} \leq \frac{r(N) - r(U_1)}{|T_1|+|T_2|+1} = \tau(U_1, r).
$$

So, $\tau(U_1, r)$ is at least equal to the given convex combination of $\tau(U_1 \cup T_1, r)$ and $\tau(U_1 \cup T_2, r)$, implying that for at least one i, $i = 1, 2$, it holds that

$$
\tau(U_1 \cup T_i, r) \leq \tau(U_1, r).
$$

This contradicts condition (5). □

The next proposition says that for a set U_1 satisfying condition (5) the complement $N\backslash U_1$ is connected and that the collection of all successors of players in U_1 contains precisely one player not in U_1. For $T \subseteq N$, let $S_D(T) = \cup_{i \in T} S_D(i)$ denote the union of all successors of at least one player of T in the digraph (N, D).

Proposition 4.6 *Let game with permission structure (N, v, D) satisfy weak digraph monotonicity and weak digraph concavity and let U_1 be a coalition satisfying condition (5). Then:*

1. *The set $N\backslash U_1$ is connected, and*
2. $|S_D(U_1) \cap (N\backslash U_1)| = 1$.

Proof To prove **1**, suppose $N\backslash U_1$ consists of at least two components. Let T_1 be one of the components and denote $T_2 = N\backslash(U_1 \cup T_1)$. We show that both $U_1 \cup T_i$, $i = 1, 2$ are feasible. To do so, let i be any player in T_1. By quasi-strongly connectedness of (N, D), there exists a directed path $(i_1, i_2, \ldots, i_m)$ from $i_1 = 1$ to $i_m = i$. Let i_k, $1 \leq k < m$, be the last player in the path not in T_1, thus $i_k \in U_1 \cup T_2$ and $i_{k+1}, \ldots, i_m \in T_1$. Since $(i_k, i_{k+1}) \in D$, $i_k \in T_2$ contradicts that T_1 is a component of $N\backslash U_1$. Hence $i_k \in U_1$. Since U_1 is feasible, $1 \in U_1$ and there is a path $(j_1, \ldots, j_\ell)$ from $j_1 = 1$ to $j_\ell = i_k$ with $j_r \in U_1$ for all $r = 1, \ldots, \ell$. Hence for any $i \in T_1$ there is a path $(j_1, \ldots, j_\ell, i_{k+1}, \ldots, i_m)$ from 1 to i only containing nodes in $U_1 \cup T_1$. This shows that $U_1 \cup T_1$ is feasible. Similarly it follows that $U_1 \cup T_2$ is feasible. This contradicts Lemma 4.5, which proves the first statement.

To prove **2**, assume that there are two players $i_1, i_2 \in S_D(U_1) \cap (N\backslash U_1)$, $i_1 \neq i_2$. For any player $i \in N\backslash U_1$, let $\widetilde{S}_D(i)$ be defined as the subset of $N\backslash U_1$ such that node $j \in N\backslash U_1$ belongs to $\widetilde{S}_D(i)$ if and only if $j = i$ or there is a directed path from node i to node j that only consists of nodes in $N\backslash U_1$. Since (N, D) is acyclic by Assumption 3.1, we have that $i_1 \notin \widetilde{S}_D(i_2)$ or $i_2 \notin \widetilde{S}_D(i_1)$ (or both). Suppose $i_2 \notin \widetilde{S}_D(i_1)$. We now consider the partition of $N\backslash U_1$ into two non-empty sets $T_1 = \widetilde{S}_D(i_1)$ and $T_2 = (N\backslash U_1)\backslash T_1$ and obtain a contradiction by using Lemma 4.5. Since there is a directed path from node 1 to $i_1 \in T_1$ consisting of nodes in $U_1 \cup \{i_1\}$, and from $i_1 \in T_1$ to any other node in T_1 consisting of nodes in T_1, for each $j \in U_1 \cup T_1$ there is a path from 1 to j in $U_1 \cup T_1$, and thus $U_1 \cup T_1$ is feasible.

Next consider $U_1 \cup T_2$. For a node $j \in T_2$, let $(i_1, i_2, \ldots, i_m)$ be a path from $i_1 = 1$ to $i_m = j$ and let i_k, $1 \leq k < m$, be the last player in the path not in T_2, thus $i_k \in N\backslash T_2 = U_1 \cup T_1$. Then $i_k \in U_1$, because $i_k \in T_1 = \widetilde{S}_D(i_1)$ contradicts that $j \notin T_1$. Since U_1 is feasible, there is a path $(j_1, \ldots, j_\ell)$ from $j_1 = 1$ to $j_\ell = i_k$ with $j_r \in U_1$ for all $r = 1, \ldots, \ell$. Hence for any $j \in T_2$ there is a path $(j_1, \ldots, j_\ell, i_{k+1}, \ldots, i_m)$ from 1 to j only containing nodes in $U_1 \cup T_2$. This shows that $U_1 \cup T_2$ is feasible. Hence the existence of two players in $S_D(U_1) \cap (N\backslash U_1)$ contradicts Lemma 4.5, which proves the second statement. □

For U_1 satisfying condition (5), let i_1 be the unique node in $S_D(U_1) \cap (N\backslash U_1)$ i.e., i_1 is the unique successor of U_1 in $N\backslash U_1$. Since $1 \in U_1$, this implies that any path from node 1 to a player $j \in N\backslash U_1$ has node i_1 as the first player on the path not in U_1. Together with the connectedness of $N\backslash U_1$ (see Proposition 4.6) this gives the following corollary.

Corollary 4.7 *Let game with permission structure (N, v, D) satisfy weak digraph monotonicity and weak digraph concavity and let U_1 be a coalition satisfying condition (5). Then the subgraph $(N\backslash U_1, D(N\backslash U_1))$ of (N, D) on $N\backslash U_1$ is also a quasi-strongly connected, acyclic directed graph with one top-node (node i_1).*

5 An algorithm for computing the nucleolus

In this section we modify the exponential time algorithm of Arin and Feltkamp (1997) to find, in polynomial time, the nucleolus of the restricted game (N, r) of a game with a permission structure (N, v, D) satisfying weak digraph monotonicity and weak digraph concavity. It will appear that Proposition 4.6 is needed to prove that the algorithm indeed finds the nucleolus of the restricted game (N, r). Since this proposition holds for games (N, v, D) that are weak digraph monotone and weak digraph concave, both properties are required. Further recall from Assumption 3.1 that node 1 is the unique top node in (N, D) (thus 1 is a veto player in the restricted game (N, r)) and that N is essential in (N, r).

This section contains three subsections. In the first subsection the algorithm is given and the second one contains the proof that the algorithm indeed finds the nucleolus. In the third subsection the algorithm is illustrated by an example.

5.1 The algorithm

For the reduced game with permission structure (U_k, v_k, D_k) defined in iteration $k-1$ at Step 3 of the algorithm given below, the set Ω^{D_k} denotes the set of all feasible coalitions not equal to U_k in the digraph (U_k, D_k). Also, for $i \in U_k$, we denote by $S_{D_k}(i)$ and $P_{D_k}(i)$ the set of successors, respectively predecessors in (U_k, D_k). Then the algorithm proceeds as follows.

Algorithm

Step 1 Set $k = 0$, $U_0 = N$, $v_0 = v$, $D_0 = D$ and $r_0 = r$. Go to Step 2.

Step 2 Find $U_{k+1} \subset U_k$ satisfying condition (5) with respect to game with permission structure (U_k, v_k, D_k), i.e.,

$$\tau(U_{k+1}, r_k) = \tau^*(r_k) \quad \text{and} \quad |U_{k+1}| = \max_{\{U \in \Omega^{D_k} | \tau(U, r_k) = \tau^*(r_k)\}} |U|,$$

where $\tau^*(r_k) = \min_{U \in \Omega^{D_k}} \tau(U, r_k)$ with $\tau(U, r_k) = \frac{r_k(U_k) - r_k(U)}{|U_k \backslash U| + 1}$. Assign $y_j = \tau^*(r_k)$ to every player $j \in U_k \backslash U_{k+1}$. Go to Step 3.

Step 3 If $U_{k+1} = \{1\}$ then Go to Step 4. If $U_{k+1} \neq \{1\}$, let i_{k+1} be the unique top-player of the subgraph $(U_k \backslash U_{k+1}, D_k(U_k \backslash U_{k+1})$ of the digraph (U_k, D_k) restricted to $U_k \backslash U_{k+1}$. Define game (U_{k+1}, v_{k+1}) by setting for every $U \subseteq U_{k+1}$,

$$v_{k+1}(U) = \begin{cases} v_k(U) & \text{if } P_{D_k}(i_{k+1}) \cap U = \emptyset \\ v_k(U \cup (U_k \backslash U_{k+1})) - \tau(U_{k+1}, r_k)|U_k \backslash U_{k+1}| & \text{else,} \end{cases} \tag{6}$$

digraph (U_{k+1}, D_{k+1}) given by

$$(i, j) \in D_{k+1} \quad \text{if} \begin{cases} (i, j) \in D_k \text{ or} \\ i \in P_{D_k}(i_{k+1}) \quad \text{and} \quad j \in S_{D_k}(U_k \backslash U_{k+1}) \cap U_{k+1}, \end{cases} \tag{7}$$

and let r_{k+1} be the restricted game of $(U_{k+1}, v_{k+1}, D_{k+1})$. Set $k = k + 1$. Go to Step 2.

Step 4 Assign $y_1 = v(N) - \sum_{j \in N \backslash \{1\}} y_j$. Stop.

In every step of the algorithm, for $U_{k+1} \subset U_k$ satisfying condition (5) with respect to (U_k, v_k, D_k), any player in $U_k \backslash U_{k+1}$ receives payoff $\tau(U_{k+1}, r_k)$. Observe that at any iteration the new found set U_{k+1} is essential in (U_{k+1}, r_{k+1}). If not, there exists an essential subset S of U_{k+1} with $r_{k+1}(S) = r_{k+1}(U_k)$, yielding payoff $y_j = 0$ for all $j \in U_{k+1} \backslash S$. This contradicts that all players get positive payoff (because N is essential). Since in any iteration the payoff of at least one player is determined, in at most $k = n - 1$ iterations the algorithm stops with $U_{k+1} = \{1\}$ and player 1 getting what is left from $v(N)$ after all other players received their payoffs as determined by the algorithm. (Note that player 1 belongs to the player set of every game (U_k, D_k) that appears in the algorithm.)

5.2 The algorithm finds the nucleolus

Let K be such that $U_{K+1} = \{1\}$. To show that the algorithm is well-defined, it is needed that the results of Sect. 4 hold for every game (U_k, r_k), $k = 1, \dots, K$. This is shown in the next two lemmas. The first lemma states that for any $k = 0, 1, \dots, K - 1$ the digraph (U_{k+1}, D_{k+1}) is acyclic and quasi-strongly connected with $i = 1$ as its unique top-node.

Lemma 5.1 *For every $k = 0, 1, \dots, K - 1$, the digraph (U_{k+1}, D_{k+1}) satisfies property (i) of Assumption 3.1.*

Proof Since (N, D) satisfies Assumption 3.1.(i), the statement is true for $k = 0$. We now proceed by induction and suppose that the statement is true for $j = 0, \dots, k$, $k < K - 1$. Then it remains to show that the statement is true for $j = k + 1$. By the induction hypothesis we have that (U_k, D_k) is acyclic and quasi-strongly connected and has $i = 1$ as its unique top node. So, for any $j \neq 1$ in U_{k+1} there is a directed path $(i_1, \dots, i_m)$ in (U_k, D_k) with $i_1 = 1$ and $i_m = j$. If any node i_k, $k = 2, \dots, m - 1$ in this path is in U_{k+1}, then this path also exists in (U_{k+1}, D_{k+1}). Otherwise, for any node i_h on the path not in U_{k+1}, there exist two (not necessarily different) nodes i_r, i_s on the path with $r \leq h \leq s$ such that $i_{r-1}, i_{s+1} \in U_{k+1}$ and $i_r, i_s \notin U_{k+1}$. Then by (7) we have that $(i_{r-1}, i_{s+1}) \in D_{k+1}$. Hence there is a directed path from $i = 1$ to $i = j$ in (U_{k+1}, D_{k+1}), showing (U_{k+1}, D_{k+1}) is quasi-strongly connected with node 1 as top node. Because in (U_{k+1}, D_{k+1}) there can only be a directed path from node i to node j if there is a directed path from i to j in (U_k, D_k), the acyclicity of (U_{k+1}, D_{k+1}) follows immediately from the fact that (U_k, D_k) is acyclic. □

The next lemma shows that every game (U_k, v_k, D_k), $k = 0, 1, \dots, K$, is weak digraph monotone and weak digraph concave. Again the proof is by induction, where Proposition 4.6 is used to show the weak digraph monotonicity.

Lemma 5.2 *Let game with permission structure (N, v, D) satisfy weak digraph monotonicity and weak digraph concavity. Then the game with permission structure (U_k, v_k, D_k) satisfies these conditions on the player set U_k for every $k = 0, \dots, K$.*

Proof We prove the proposition by induction on k. For $k = 0$ both weak digraph monotonicity and weak digraph concavity are satisfied by assumption. Proceeding by induction, assume that these conditions are satisfied for $j = 0, \dots, k, k < K - 1$. By Lemma 5.1 the digraph (U_k, D_k) satisfies Assumption 3.1.(i). So, the game (U_k, v_k, D_k) satisfies all conditions of Proposition 4.6.

To show that weak digraph monotonicity holds for $(U_{k+1}, v_{k+1}, D_{k+1})$, we have to show that $[U \subseteq U_{k+1}$ and U feasible in $(U_{k+1}, D_{k+1})] \Rightarrow v_{k+1}(U) \leq v_{k+1}(U_{k+1})$. Since $P_{D_k}(i_{k+1}) \cap U_{k+1} \neq \emptyset$, we have that

$$\begin{aligned} v_{k+1}(U_{k+1}) &= v_k(U_{k+1} \cup (U_k \backslash U_{k+1})) - \tau(U_{k+1}, r_k)|U_k \backslash U_{k+1}| \\ &= v_k(U_k) - \tau(U_{k+1}, r_k)|U_k \backslash U_{k+1}|. \end{aligned}$$

Next, let $U \subseteq U_{k+1}$ be a feasible subset of U_{k+1} in (U_{k+1}, D_{k+1}). We consider two cases, either $P_{D_k}(i_{k+1}) \cap U \neq \emptyset$ or $P_{D_k}(i_{k+1}) \cap U = \emptyset$. In the latter case we have that (i) $v_{k+1}(U) = v_k(U)$ and (ii) there is an arc between two nodes i and j of U in the digraph (U_{k+1}, D_{k+1}) if and only if there is also an arc between i and j in (U_k, D_k). Hence U is also feasible in (U_k, D_k) and thus $v_{k+1}(U) = v_k(U) = r_k(U)$. Moreover, $\tau(U, r_k) = \frac{r_k(U_k) - r_k(U)}{|U_k \backslash U| + 1} \geq \tau(U_{k+1}, r_k)$ and thus $r_k(U_k) - r_k(U) \geq (|U_k \backslash U| + 1)\tau(U_{k+1}, r_k)$. Hence

$$\begin{aligned} v_{k+1}(U) = r_k(U) &\leq r_k(U_k) - (|U_k \backslash U| + 1)\tau(U_{k+1}, r_k) \\ &< v_k(U_k) - |U_k \backslash U_{k+1}|\tau(U_{k+1}, r_k) = v_{k+1}(U_{k+1}). \end{aligned}$$

In case $P_{D_k}(i_{k+1}) \cap U \neq \emptyset$, we obtain from applying Proposition 4.6 to (U_k, v_k, D_k), that $U \cup (U_k \backslash U_{k+1})$ is feasible in (U_k, D_k). From this it follows that

$$\begin{aligned} v_{k+1}(U) &= v_k(U \cup (U_k \backslash U_{k+1})) - \tau(U_{k+1}, r_k)|U_k \backslash U_{k+1}| \\ &\leq v_k(U_k) - \tau(U_{k+1}, r_k)|U_k \backslash U_{k+1}| = v_{k+1}(U_{k+1}) \end{aligned}$$

because weak digraph monotonicity holds for (U_k, v_k, D_k).

Next we consider weak digraph concavity, i.e., we have to show that $[S \cup T = U_{k+1}$ and S, T feasible in $(U_{k+1}, D_{k+1})] \Rightarrow v_{k+1}(S) + v_{k+1}(T) \geq v_{k+1}(S \cap T) + v_{k+1}(U_{k+1})$. Since $S \cup T = U_{k+1}$ we have that $P_{D_k}(i_{k+1}) \cap S \neq \emptyset$ or $P_{D_k}(i_{k+1}) \cap T \neq \emptyset$ (or both). We first consider the case that both intersections are nonempty and thus also $P_{D_k}(i_{k+1}) \cap (S \cap T) \neq \emptyset$. Then $S' = S \cup (U_k \backslash U_{k+1})$, $T' = T \cup (U_k \backslash U_{k+1})$ are feasible in (U_k, D_k) and $S' \cup T' = U_k$, and thus it follows from weak digraph concavity for (U_k, v_k, D_k) that

$$\begin{aligned} v_{k+1}(S) + v_{k+1}(T) &= v_k(S') + v_k(T') - 2\tau(U_{k+1}, r_k)|U_k \backslash U_{k+1}| \\ &\geq v_k(S' \cap T') + v_k(U_k) - 2\tau(U_{k+1}, r_k)|U_k \backslash U_{k+1}| \\ &= v_k((S \cap T) \cup (U_k \backslash U_{k+1})) + v_k(U_k) - 2\tau(U_{k+1}, r_k)|U_k \backslash U_{k+1}| \\ &= v_{k+1}(S \cap T) + v_{k+1}(U_{k+1}), \end{aligned}$$

where the last equality follows from the fact that $v_{k+1}(S \cap T) = v_k((S \cap T) \cup (U_k \backslash U_{k+1})) - \tau(U_{k+1}, r_k)|U_k \backslash U_{k+1}|$ and $v_{k+1}(U_{k+1}) = v_k(U_k) - \tau(U_{k+1}, r_k)|U_k \backslash U_{k+1}|$. In case only one of the sets S and T has a nonempty intersection with $P_{D_k}(i_{k+1})$

and thus $P_{D_k}(i_{k+1}) \cap (S \cap T) = \emptyset$, suppose without loss of generality that $T \cap P_{D_k}(i_{k+1}) = \emptyset$. Then $S' = S \cup (U_k \backslash U_{k+1})$ and T are feasible in (U_k, D_k), $S' \cup T = N$ and thus it follows from weak digraph concavity for (U_k, v_k, D_k) that

$$\begin{aligned} v_{k+1}(S) + v_{k+1}(T) &= v_k(S') + v_k(T) - \tau(U_{k+1}, r_k)|U_k \backslash U_{k+1}| \\ &\geq v_k(S' \cap T) + v_k(U_k) - \tau(U_{k+1}, r_k)|U_k \backslash U_{k+1}| \\ &= v_{k+1}(S \cap T) + v_{k+1}(U_{k+1}), \end{aligned}$$

where the last equality follows from the fact that $v_k(S' \cap T) = v_k(S \cap T) = v_{k+1}(S \cap T)$ and $v_k(U_k) - \tau(U_{k+1}, r_k)|U_k \backslash U_{k+1}| = v_{k+1}(U_{k+1})$. $\square$

We now show that for $k = 1, \ldots, K$, the game (U_{k+1}, r_{k+1}) is the Davis-Maschler reduced game (Davis and Maschler 1965) of the game (U_k, r_k) with respect to the nucleolus. For a game (N, v), let $T \subset N$ be a nonempty coalition and $y \in \mathbb{R}^n$ a payoff vector. Then the Davis-Maschler reduced game on T at y is the game (T, v_T^y) given by $v_T^y(T) = v(N) - x(N \backslash T)$ and $v_T^y(S) = \max_{Q \subseteq N \backslash T} (v(S \cup Q) - y(Q))$, $S \subset T$, $S \neq N$. Observe that in the definition of the reduced game only the values y_j of the players $j \in N \backslash T$ appear.

Property 5.3 (Davis-Maschler reduced game property) *For a game* (N, v)*, let* $x = Nuc(N, v)$*. Then for any nonempty* $T \subset N$ *it holds that*

$$Nuc_i(N, v) = Nuc_i(T, v_T^x), \quad \textit{for all } i \in T.$$

The Davis-Maschler reduced game property holds for the prenucleolus as shown by Sobolev (1975). If we consider a game with nonempty core then the nucleolus and the prenucleolus coincide and we can postulate property 5.3 for the nucleolus. Since the restricted games (N, r) in this paper have a nonempty core, we can formulate and use property 5.3.

In the following, let (U_{k+1}, r'_k) denote the Davis-Maschler reduced game of the game (U_k, r_k) on the set U_{k+1} at y with $y_j = \tau^*(r_k) = \tau(U_{k+1}, r_k)$ for $j \in U_k \backslash U_{k+1}$. We first show the following lemma on the largest disjunctive feasible subset of a coalition U in the digraph (U_k, D_k). In the sequel we denote this set by $\sigma_k(U)$. Observe that for $U \subseteq N$ we have that $\sigma_0(U) = \sigma(U)$.

Lemma 5.4 *For the game with permission structure* (U_k, v_k, D_k)*, let* $U_{k+1} \subset U_k$ *and* $i_{k+1} \notin U_{k+1}$ *be the set and node as obtained in the iteration k of the algorithm. Then for each* $U \subseteq U_{k+1}$ *we have that*

(i) $\sigma_{k+1}(U) = \sigma_k(U)$ *if* $S_{D_k}(\sigma_k(U)) \subset U_{k+1}$*;*
(ii) $\sigma_{k+1}(U) = \sigma_k(U \cup (U_k \backslash U_{k+1})) \backslash (U_k \backslash U_{k+1})$ *if* $i_{k+1} \in S_{D_k}(\sigma_k(U))$.

Proof 1. Consider $U \subseteq U_{k+1}$ with $S_{D_k}(\sigma_k(U)) \subset U_{k+1}$. Clearly, then $\sigma_k(U)$ is feasible in (U_{k+1}, D_{k+1}) and thus $\sigma_k(U) \subseteq \sigma_{k+1}(U)$. Next, suppose that there exists some player $i \in \sigma_{k+1}(U) \backslash \sigma_k(U)$. Then there is path $(a_0, a_1, \ldots, a_l)$ such that (i) $a_0 = 1$, (ii) $a_l = i$, (iii) $a_t \in U$ for all $t = 1, \ldots, l-1$, and (iv) $(a_t, a_{t+1}) \in D_{k+1}$ for all $t = 0, \ldots, l-1$. If $(a_t, a_{t+1}) \in D_k$ for all $t =$

$0, \ldots, l-1$, then $i \in \sigma_k(U)$ and we get a contradiction with our assumption that $i \in \sigma_{k+1}(U) \backslash \sigma_k(U)$. So, there must exist a $t \in \{0, \ldots, l-1\}$ such that $(a_t, a_{t+1}) \notin D_k$. By definition of digraph D_{k+1} it holds that $a_t \in P_{D_k}(i_{k+1})$, which contradicts $S_{D_k}(\sigma_k(U)) \subset U_{k+1}$. Hence $\sigma_{k+1}(U) = \sigma_k(U)$.

2. Consider $U \subseteq U_{k+1}$ with $i_{k+1} \in S_{D_k}(\sigma_k(U))$. If there is a player $i \in \sigma_{k+1}(U)$ then there is a path $(a_0, a_1, \ldots, a_l)$ such that (i) $a_0 = 1$, (ii) $a_l = i$, (iii), $a_t \in U$ for all $t = 1, \ldots, l-1$, and (iv) $(a_t, a_{t+1}) \in D_{k+1}$ for all $t = 0, \ldots, l-1$. We show that these four conditions also describe all elements of
$$\sigma_k(U \cup (U_k \backslash U_{k+1})) \backslash (U_k \backslash U_{k+1})$$
If $(a_t, a_{t+1}) \in D_k$ for all $t = 0, \ldots, l-1$, then $i \in \sigma_k(U)$. Since $U \subseteq U_{k+1}$, it follows that $i \in \sigma_k(U \cup (U_k \backslash U_{k+1})) \backslash (U_k \backslash U_{k+1})$. Otherwise, if $(a_t, a_{t+1}) \in D_{k+1} \backslash D_k$ for some t, then $a_t \in P_{D_k}(i_{k+1})$ and $a_{t+1} \in S_{D_k}(U_k \backslash U_{k+1})$. So there is a path from a_t to a_{t+1} which contains only elements from $U_k \backslash U_{k+1}$. In the path $(a_0, a_1, \ldots, a_l)$, replace the arc (a_t, a_{t+1}) by this path from a_t to a_{t+1}.
Continuing in this way, we can change each arc in the path $(a_0, a_1, \ldots, a_l)$ that belongs to $D_{k+1} \backslash D_k$ by a path which consists only of elements from $U_k \backslash U_{k+1}$. So, we have a path from 1 to i which consists only of elements from $U \cup (U_k \backslash U_{k+1})$, implying that $i \in \sigma_k(U \cup (U_k \backslash U_{k+1}))$. Since $i \notin U_k \backslash U_{k+1}$, we conclude that $i \in \sigma_k(U \cup (U_k \backslash U_{k+1})) \backslash (U_k \backslash U_{k+1})$. So, in both cases we have that $i \in \sigma_k(U \cup (U_k \backslash U_{k+1})) \backslash (U_k \backslash U_{k+1})$ and therefore we get
$$\sigma_{k+1}(U) = \sigma_k(U \cup (U_k \backslash U_{k+1})) \backslash (U_k \backslash U_{k+1}).$$
□

The next lemma shows that the game (U_{k+1}, r_{k+1}) is the Davis-Maschler reduced game of the game (U_k, r_k) with respect to the nucleolus.

Lemma 5.5 *Let game with permission structure (N, v, D) satisfy weak digraph monotonicity and weak digraph concavity. Then, for $k = 0, \ldots, K$, the game (U_{k+1}, r_{k+1}) is equal to the Davis-Maschler reduced game (U_{k+1}, r'_k) of the game (U_k, r_k) on U_{k+1} at y with $y_j = \tau^*(r_k)$ for $j \in U_k \backslash U_{k+1}$.*

Proof For coalition $T \subseteq U_{k+1}$, we consider two cases, namely whether or not $S_{D_k}(\sigma_k(T)) \subset U_{k+1}$.

In case $S_{D_k}(\sigma_k(T)) \subset U_{k+1}$, assertion 1 of Lemma 5.4 implies that $\sigma_{k+1}(T) = \sigma_k(T)$. Further, since $P_{D_k}(i_{k+1}) \cap \sigma_k(T) = \emptyset$ we have by Eq. 6 in Step 3 of the algorithm that $v_{k+1}(T) = v_k(T)$ and thus $r_{k+1}(T) = r_k(T)$ because $\sigma_{k+1}(T) = \sigma_k(T)$. On the other hand, for the Davis-Mashler reduced game (U_{k+1}, r'_k) it holds for any $T \subset U_{k+1}$ that
$$r'_k(T) = \max_{Q \subseteq U_k \backslash U_{k+1}} (r_k(T \cup Q) - y(Q)) = r_k(T),$$
because for any $Q \subseteq U_k \backslash U_{k+1}$ we have that

$$r_k(T \cup Q) = v_k(\sigma_k(T \cup Q)) = v_k(\sigma_k(T)) = r_k(T),$$

where the second equality follows since for any pair $j \in (T \backslash \sigma_k(T) \cup Q)$ and $i \in \sigma_k(T)$, it holds that $(i, j) \notin D_k$ and thus $\sigma_k(T \cup Q) = \sigma_k(T)$. Hence $r'_k(T) = r_k(T) = r_{k+1}(T)$.

In case $S_{D_k}(\sigma_k(T))$ is not a subset of U_{k+1} we have that $P_{D_k}(i_{k+1}) \cap \sigma_k(T) \neq \emptyset$, because i_{k+1} is the unique successor of U_{k+1} in $U_k \backslash U_{k+1}$. So, by Eq. 6 in Step 3 of the algorithm we have that

$$r_{k+1}(T) = v_{k+1}(\sigma_{k+1}(T)) = v_k(\sigma_{k+1}(T) \cup (U_k \backslash U_{k+1})) - \tau(U_{k+1}, r_k)|U_k \backslash U_{k+1}|.$$

From Lemma 5.4 we have that $\sigma_{k+1}(T) \cup (U_k \backslash U_{k+1}) = \sigma_k(T \cup (U_k \backslash U_{k+1}))$ and so

$$\begin{aligned} r_{k+1}(T) &= v_k(\sigma_k(T \cup (U_k \backslash U_{k+1}))) - \tau(U_{k+1}, r_k)|U_k \backslash U_{k+1}| \\ &= r_k(T \cup (U_k \backslash U_{k+1})) - \tau(U_{k+1}, r_k)|U_k \backslash U_{k+1}|. \end{aligned}$$

To show that $r_{k+1}(T) = r'_k(T)$ it remains to prove that the right-hand term in the equation

$$r'_k(T) = \max_{Q \subseteq U_k \backslash U_{k+1}} (r_k(T \cup Q) - \tau(U_{k+1}, r_k)|Q|)$$

obtains its maximum for $U_k \backslash U_{k+1}$. To do so, denote $\overline{Q} = U_k \backslash U_{k+1}$, $V = T \cup \overline{Q}$ and, for $Q \subseteq \overline{Q}$, denote $W = U_{k+1} \cup Q$. Then (because of Lemma 5.4) the sets $\sigma_k(V) = \sigma_{k+1}(T) \cup \overline{Q}$ and $\sigma_k(W) = \sigma_k(U_{k+1} \cup Q) \supseteq U_{k+1}$ are feasible and satisfy $\sigma_k(V) \cup \sigma_k(W) = U_k$. By Lemma 5.2 the game with permission structure (U_k, v_k, D_k) satisfies weak digraph concavity and thus

$$\begin{aligned} r_k(V) + r(W) &= v_k(\sigma_k(V)) + v_k(\sigma_k(W)) \geq v_k(U_k) + v_k(\sigma_k(V) \cap \sigma_k(W)) \\ &= v_k(U_k) + v_k(\sigma_k(V \cap W)) = r_k(U_k) + r_k(V \cap W), \end{aligned}$$

where the second equality follows from the fact that $\sigma_k(V \cap W) = \sigma_k(V) \cap \sigma_k(W)$ because of the graph structure. With $V \cap W = (T \cup \overline{Q}) \cap (U_{k+1} \cup Q) = T \cup Q$ this yields

$$\begin{aligned} r_k(T \cup \overline{Q}) - r_k(T \cup Q) &\geq r_k(U_k) - r_k(U_{k+1} \cup Q) \\ &> \frac{r_k(U_k) - r(U_{k+1} \cup Q)}{|\overline{Q}| - |Q| + 1}(|\overline{Q}| - |Q|) \\ &= \tau(U_{k+1} \cup Q)(|\overline{Q}| - |Q|) \geq \tau(U_{k+1}, r_k)(|\overline{Q}| - |Q|) \end{aligned}$$

by definition of U_{k+1}. Hence

$$r_k(T \cup \overline{Q}) - \tau(U_{k+1}, r_k)|\overline{Q}| > r_k(T \cup Q) - \tau(U_{k+1}, r_k)|Q|,$$

for all $Q \subseteq \overline{Q}$, which shows that indeed

$$r_k(T \cup \overline{Q}) - \tau(U_{k+1}, r_k)|\overline{Q}| = \max_{Q \subseteq U_k \setminus U_{k+1}} (r_k(T \cup Q) - \tau(U_{k+1}, r_k)|Q|)\,.$$

□

We now have the following proposition.

Proposition 5.6 *Given game with permission structure* (N, v, D) *satisfying the weak digraph monotonicity and weak digraph concavity, the algorithm described in Sect.* 5.1 *yields the nucleolus of* (N, r).

Proof In iteration $k = 0$ the algorithm assigns in Step 2 the value $\tau^*(r_0) = \tau(U_1, r_0) = \tau(U_1, r)$ to any player $j \in U_0 \setminus U_1 = N \setminus U_1$. According to Lemma 4.4, $\tau^*(r_0)$ is the nucleolus value of the players in $N \setminus U_1$. Applying Lemma 5.5 for $k = 0$, the game (U_1, r_1) is the Davis-Maschler reduced game of the game (N, r) with respect to the nucleolus values $y_j = \tau^*(r_0)$ of the players not in U_1. Since the nucleolus satisfies the Davis-Maschler reduced game consistency property, the nucleolus values of the reduced game (U_1, r_1) are equal to the nucleolus values of the players of U_1 in the game (N, r). In iteration $k = 1$ the algorithm assigns in Step 2 the value $\tau^*(r_1)$ to any player $j \in U_1 \setminus U_2$. According to Lemma 4.4, $\tau^*(r_1)$ is the nucleolus value of the players in $U_1 \setminus U_2$ in the game (U_1, r_1), and hence it is also the nucleolus value of these players in the game (N, r). Continuing this reasoning we have that in any iteration k, the algorithm assigns in Step 2 the value $\tau^*(r_k)$ to any player $j \in U_k \setminus U_{k+1}$, which is the nucleolus value of the players in $U_k \setminus U_{k+1}$ in the game (N, r). At the final iteration K we have that $U_{K+1} = \{1\}$ and player 1 gets its nucleolus value in Step 4 of the algorithm. □

5.3 Example

In this subsection we illustrate the algorithm by an example of a game with a permission structure on a set of five players. The player set is $N = \{A, B, C, D, E\}$ and (N, v) is an additive with the weights of players given by 1, 2, 0, 4 and 6 respectively. The set $D \subset N \times N$ is given by

$$D = \{(A, B),\ (A, C),\ (B, D),\ (C, D),\ (C, E)\}.$$

Notice that (N, v, D) satisfies Assumption 3.1 (with player A the unique top player) and is weak digraph monotone and weak digraph concave. The game is given in Fig. 1, in which the numbers near the players are there weights.

In Step 2 at the first iteration with $U_0 = N$ and $r_0 = r$ we find that $\tau^*(r_0) = 1$ and that $\{B\}$ is the unique set with $\tau(\{B\}, r_0) = \tau^*(r_0)$. So in the first iteration of the algorithm we set $U_1 = (\{B\})$ and $Nuc_B(N, r) = 1$. Next the Davis-Maschler reduced game on $N \setminus U_1 = \{A, C, D, E\}$ is constructed. This is again an additive game, given in Fig. 2.

In Step 2 of the second iteration we now have $\tau(\{E\}, r_1) = \tau(\{C, E\}, r_1) = \tau^*(r_1)$=2 and $U_2 = \{C, E\}$ is chosen, because it is the unique maximal set with minimal

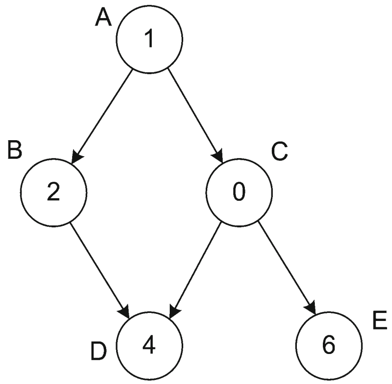

Fig. 1 Example 5.3

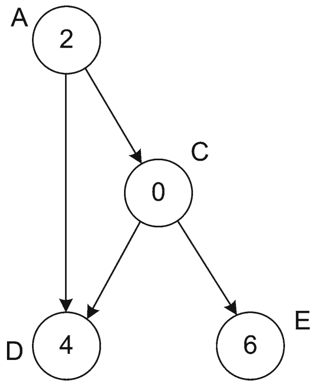

Fig. 2 Example 5.3: first iteration

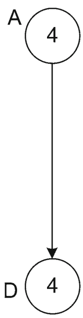

Fig. 3 Example 5.3: second iteration

value $\tau^*(r_1)$. So, $Nuc_C(U_1, r_1) = Nuc_E(U_1, r_1) = 2$. Next the reduced game on $U_1 \backslash U_2 = \{A, D\}$ is again an additive game given in Fig. 3. From this figure it follows straightforwardly that $Nuc_A(U_2, r_2) = 6$ and $Nuc_D(U_2, r_2) = 2$. As final result we get $Nuc(N, r) = (6, 1, 2, 2, 2)$.

6 Complexity of the algorithm

For arbitrary veto-rich games the algorithm of Arin and Feltkamp (1997) to compute the nucleolus is an exponential time algorithm of the order $\mathcal{O}(n.2^{n-1})$. Brânzei et al. (2005) argue that applying the algorithm to the specific case of a peer group game the complexity reduces to a polynomial time algorithm of order $\mathcal{O}(n^3)$. They show that the algorithm given in their paper to find the nucleolus of a peer group game is a polynomial time algorithm of order $\mathcal{O}(n^2)$. In this section we show that the algorithm given in the previous section to find the nucleolus of the more general restricted game of a game with disjunctive permission structure is a polynomial time algorithm of order $\mathcal{O}(n^4)$.

6.1 Good sets

To show the complexity of the algorithm, we first define the concept of a *good set* in a digraph.

Definition 6.1 For a digraph (N, D) with $D \in \mathcal{D}^N$, a set $T \subset N$ is a good set, when

(i) there is a unique top node in the subgraph $(T, D(T))$ of (N, D) and for any other node i in T there is a path from this unique top node to node i that only contains nodes in T,
(ii) the set $N \backslash T$ is connected, and
(iii) the top node in $(T, D(T))$ is the only node in T that has predecessors in $N \backslash T$.

We now have the following lemma.

Lemma 6.2 *In any iteration k of the algorithm, the set $U_k \backslash U_{k+1}$ is a good set.*

Proof Applying Corollary 4.7 to (U_k, D_k) we have that the subgraph of (U_k, D_k) restricted to $U_k \backslash U_{k+1}$ is a quasi-strongly connected, acyclic directed graph with one top node, so condition (i) holds. Next, denote $T_k = U_k \backslash U_{k+1}$. Then $U_k \backslash T_k = U_{k+1}$. Therefore condition (ii) holds, because U_{k+1} is feasible in (U_k, D_k) and thus connected in (U_k, D_k). Further, by applying the second statement of Proposition 4.6 to (U_k, D_k) we have that U_{k+1} has only one successor in $T_k = U_k \backslash U_{k+1}$. Let this only successor be node j in T_k. Since the digraph (U_k, D_k) is acyclic and quasi-strongly connected, there is a path from top node 1 in (U_k, D_k) to any other node in U_k, so also to any node in T_k. Since j is the only successor of U_{k+1} in T_k, any path from 1 to some node $h \in T_k$ must contain the node j. Moreover, the path from j to h can not contain nodes not in T_k, otherwise U_{k+1} has more than one successor in T_k. Hence j is also a top node in T_k such that for any other node in T_k there is a path from j to this node that only contains nodes in T_k. □

Lemma 6.2 implies that in Step 2 of the algorithm the set U_{k+1} that we must find is such that its complement $U_k \backslash U_{k+1}$ is a good set. Conversely, when $\mathcal{T}_k$ is the collection of all good sets in (U_k, D_k), then the search for U_{k+1} can be restricted to sets in the collection $U_k \backslash T_k$, $T_k \in \mathcal{T}_k$. The next lemma says that in a game with permission structure (N, v, D) there is precisely one good set for any player $j \in N$. Applying this to (U_k, D_k) this means that at iteration k of the algorithm the number of good sets is equal to $|U_k|$. Observe that j itself is a singleton good set if j has no successors.

Lemma 6.3 *Let (N, D) be a digraph with $D \in \mathcal{D}^N$. Then for any node $j \in N$ there is exactly one good set T such that j is the unique top node in T.*

Proof Recall from Sect. 2.3 that the set $\overline{S}_D(j)$ of all complete subordinates of j is the set of nodes i such that any path from top node 1 in (N, D) to node i contains node j. It is straightforward to verify that $\overline{S}_D(j)$ is a good set having node j as its unique top node. Next, suppose that there are two good sets with j as their unique top node, say T_1 and T_2 and, w.l.o.g., suppose that $T_1 \backslash T_2$ is non-empty. Consider some node $h \in T_1 \backslash T_2$. By definition of a good set we know that any path from top player 1 to the player h contains the node j. However, $N \backslash T_2$ does not contain j and so there is no path from top node 1 to h in $N \backslash T_2$, contradicting condition (ii) of Definition 6.1. $\square$

6.2 Complexity

We are now ready to consider the complexity of the algorithm.

Proposition 6.4 *The complexity of the algorithm is of order $\mathcal{O}(n^4)$.*

Proof First, in iteration k we have to find all good sets in U_k. To find the good set with some player j in U_k as its unique top node, delete player j from U_k. Then the good set consists of player j and all nodes in U_k that are no longer connected to player 1 when player j is deleted. Since U_k contains at most $n - 1$ nodes not equal to 1, this requires at most $\mathcal{O}(n^2)$ actions to find the good set of node j. So, it requires at most $\mathcal{O}(n^3)$ actions to find all $n - 1$ good sets of all players $j \neq 1$. Next, at each iteration k we need to calculate the number $\tau(U_k \backslash T, r_k)$ for any good set T. For this we need at most $O((n-1)m_k)$ actions, where m_k is the number of actions to find all values $v_k(U)$, $U \subseteq U_k$ in Step 3 of iteration $k - 1$. Clearly $m_0 = 1$. Further, from Eq. 6 in Step 3 of the algorithm it follows that we need m_{k-1} actions to find $v_k(U)$ if $P_{D_k}(i_{k+1}) \cap U = \emptyset$. Otherwise m_{k-1} actions are needed to calculate $v_k(U) = v_{k-1}(U \cup (U_{k-1} \backslash U_k))$ and $\mathcal{O}(1)$ actions are needed for calculating $\tau(U_k, r_{k-1})|U_{k-1} \backslash U_k|$ and for substraction, because $\tau(U_k, r_{k-1})$ was already found before. Hence $m_k = m_{k-1} + \mathcal{O}(1)$. Together with $m_0 = 1$ this yields that $m_k \leq \mathcal{O}(n)$. Since the number of iterations is at most equal to n, it follows that the complexity of the algorithm is given by $n \cdot (O(n^3) + O((n-1)m_k)) = O(n^4)$. $\square$

7 Concluding remarks

In this paper we presented a polynomial time algorithm to compute the nucleolus of restricted games induced by a class of games with a permission structure which generalizes the class of peer group games. Whereas the hierarchical structure in peer group games is a rooted tree and the game is additive, we allow for any acyclic quasi-strongly connected digraph (so players are allowed to have more than one predecessor) and games that satisfy the so-called weak digraph monotonicity and weak digraph concavity properties.

To show that the algorithm indeed yields the nucleolus of the restricted game in polynomial time, we obtained several other interesting results that give insight into the nucleolus for this class of games with restricted cooperation. In particular, we showed that the modified game with permission structure in each iteration of the algorithm satisfies the conditions of weak digraph monotonicity and weak digraph concavity and, moreover, the digraph is acyclic and quasi-strongly connected. Also, we saw that in every iteration of the algorithm, the set of players for which the nucleolus payoffs are determined form a 'good set' meaning that (i) the restricted digraph on that set of players is quasi-strongly connected, (ii) its complement is connected in the digraph, and (iii) the top node is the only node in the 'good set' that has predecessors outside T.

We showed that the algorithm yields the nucleolus of the restricted game by showing that the restricted game of the modified game with permission structure in each iteration of the algorithm is equal to the Davis-Maschler reduced game of the restricted game from the previous iteration. This then also implies that the Davis-Maschler reduced game of a restricted game in the class that we consider satisfies weak digraph monotonicity and weak digraph concavity. Whereas, the Davis-Maschler reduced game of a disjunctive restricted game is always a disjunctive restricted game, the properties of weak digraph monotonicity and weak digraph concavity might be lost. However, from our results it follows that this does not happen for the reduced games that appear in the algorithm.

Although our purpose is not to give an axiomatization of the nucleolus on this class of restricted games, the properties that follow from the algorithm might be useful in characterizing the nucleolus for the class of games considered here using a reduced game property similar to that of Davis and Maschler. Although the kernel coincides with the prekernel and with the nucleolus for every balanced game with a veto-player, the converse consistency property does not hold for the prekernel on the class of restricted games that is considered in this paper since we cannot delete an arbitrary player.

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