Auctions with Synergy and Resale

Xiaoshu Xu, Dan Levin, and Lixin Ye*

May 2010

Abstract

We study a sequential auction of two objects with two bidders, where the winner of the package obtains a synergy from the second object. If reselling after the two auctions occurs, it proceeds as either monopoly or monopsony take-it-or-leave-it offer. We find that a post-auction resale has a significant impact on bidding strategies in the auctions: Under the monopoly offer, there does not exist an equilibrium (symmetric or asymmetric) where bidders reveal their types with positive probability. Under the monopsony offer, however, we can identify symmetric increasing equilibrium strategies in auctions for both items. While allowing resale always improves efficiency, we demonstrate that the effect of resale on expected revenue and the probability of exposure are both ambiguous.

Keywords: Auctions, auctions with synergy, resale

JEL Classification: D44, D80, D82, D40

1 Introduction

In auctions with multiple objects, obtaining more than one object often makes the value of a package larger than the sum of the stand-alone values. This superadditivity is referred to as *synergy* in the auction literature. Synergies can arise for a variety of reasons: they can be due to the complementarity derived from a subset of objects auctioned, the geographic advantage in cost saving among

*Department of Economics, The Ohio State University, 410 Arps Hall, 1945 North High Street, Columbus, OH 43220. Email addresses: xiaoshu.xu7@gmail.com (X. Xu), levin.36@osu.edu (D. Levin), ye.45@osu.edu (L. Ye).

nationwide companies, or skills learning from previous objects, etc.[1]

Mainly motivated by the spectrum auctions conducted all over the world in the past decade, there is a growing literature on auctions with synergies. Rosenthal and Wang (1996) study a simultaneous-auction model with synergies and common values. They construct strategies in which bidders of different types randomize over different bid intervals and provide conditions under which such strategies form symmetric equilibria. Krishna and Rosenthal (1996) find that global bidders (e.g. nationwide companies) bid less aggressively than local bidders when there are more competitors. However, Rothkopf et al. (1998) argue that, though attractive, package bidding may create new problems such as the huge complexity in determining the optimal bids for the most interested package when the objects auctioned are numerous. Ausubel and Milgrom (2002) propose proxy bidding in ascending package auctions and show that even when goods are complementary, ascending proxy auction equilibria lie in the core with respect to the true preferences. Kagel and Levin (2005) experimentally compare open outcry and sealed-bid uniform-price auctions with synergies. They identify two forces behind the bidding behavior, one due to demand reduction and the other due to the "exposure problem" (bidders may win only part of a package and earn negative profits). Their results show that bidding outcomes are closer to equilibrium in clock compared to sealed-bid auctions. Cantillon and Pesendorfer (2006) conduct an empirical analysis on London bus route auctions and conclude that the benefits of allowing package bidding is ambiguous.

Synergies also arise in sequential auctions. Branco (1997) modified Krishna and Rosenthal (1996) into a sequential auction of two identical items to a global bidder and two local bidders, where one local bidder only wants the first item, while the other only wants the second item. In this setting he identifies a decline in the price trend. Menezes and Monteiro (2003, 2004) modify Branco (1997) in that local bidders (with a single demand) have equal valuations for both items. They show that with this modification, bidding price may not necessarily decline because local bidders bid less aggressively in the first auction. Jeitshko and Wolfstetter (2002) analyze a model with two bidders and two non-identical items in a sequential setting. Evaluations for both items are drawn from a Bernoulli distribution with the same support. However, they introduce the synergy with the assumption that the probability of getting a higher valuation for the second item is higher for the winner of the first

[1] For example, Ausubel, Cramton and McAfee (1997) provide some evidence for synergies in wireless telephone from the broadband PCS auctions.

item. They show that the positive synergy intensifies competition in the first auction and causes a decline in the price trend.

While a common approach to modeling synergy is to add a constant value to the sum of the stand-alone values of the package, Menezes and Monteiro (2003, 2004) argue that it is unrealistic to assume that a package of relatively small value will yield the same synergy as a package with a relatively high value. Leufkens et al. (2006, 2007) consider both theoretically and experimentally a sequential auction of two projects where the synergy factor appears as a multiplier on the stand-alone value of the second project. They model the sequential auctions as second-price sealed bid and show that the presence of synergies induces more competitive bidding, leading to lower expected profit to the bidders and higher expected revenue for the seller. The presence of synergies also leads to a decreasing price trend.

The contribution of this paper is to introduce resale into auctions with synergy. More specifically, we model resale in the framework adopted by Leufkens et al. (2006, 2007) described above. As is well known, one undesirable feature caused by synergies (in both simultaneous and sequential auctions) is that the auction outcome is typically inefficient. In the sequential auction setting, Leufkens et al. (2006) identify three types of inefficient outcomes: 1) the first-item winner has the highest valuation of the package but does not win the second item, 2) the second-item winner has the highest valuation of the package but does not win the first-item, and 3) a third bidder has the highest valuation of the package but does not win any item. A natural way to restore efficiency is to allow for post-auction resale. However, as the previous literature on resale has pointed out, the possibility of resale changes bidder behavior substantially, and the existence of a monotonic equilibrium is nontrivial. Haile (2003) considers a setup in which bidders only have noisy signals at the auction stage, where the motive for resale arises when the true value of the auction winner turns out to be low. Garratt and Tröger (2006) consider a model with a speculator who can only benefit from participating in the auction when she can resell the item to the other bidder. Garratt et al. (2009) prove the existence of a seemingly collusive equilibrium in a standard English auction with resale, and this equilibrium Pareto dominates the "bid your value" equilibrium. Hafalir and Krishna (2008) analyze auctions with resale in an asymmetric independent private value auction environment and show that the expected revenue is higher under a first-price auction than under a second-price auction. Hafalir and Krishna (2009) further demonstrate that for three distribution families, allowing resale in the first-price asymmetric auction environment can lead to higher revenue. However, there exist some cases where allowing resale may decrease efficiency.

By explicitly taking into account post-auction resale, we demonstrate that the specific resale mechanism matters for the existence of an equilibrium in the bidding stage. More specifically, if the post-auction resale takes the form of a monopoly offering (i.e., the seller makes the take-it-or-leave-it offer), we fail to identify any equilibrium (symmetric or asymmetric) in which a bidder has to reveal her type over some range of values with positive probability. When the resale takes the form of a monopsony offering (i.e., the buyer makes the take-it-or-leave-it offer), however, we can construct a perfect Bayesian equilibrium (PBE) in which bidders follow a symmetric strictly increasing bid function in the first auction. In this PBE, we show that while bidders bid higher for the second item when resale is available, the effect of resale on the bidding in the first auction is ambiguous. Thus, while allocative efficiency always improves with resale, the expected revenue to the seller can either increase or decrease with resale, and we identify some sufficient conditions for either case to happen. Moreover, we demonstrate that the "exposure" problem (i.e., over-paying for one item without winning both items) can be either more or less severe when resale is allowed; if the synergy effect is sufficiently large, however, allowing resale will unambiguously lead to a lower probability of exposure.

The rest of the paper is organized as follows. Section 2 introduces the model, Section 3 considers the monopoly offering mechanism in resale, Section 4 considers the monopsony offering mechanism in resale, Section 5 discusses our modeling restrictions and the robustness of our results, and Section 6 concludes.

2 The Model

We consider a private and independent value auction with two risk neutral bidders. Two items are auctioned sequentially using the second-price sealed bid format. The stand-alone value of the first item (x_i) is distributed according to the differentiable CDF $F(\cdot)$ with a positive density function $f(\cdot)$ over $[0, 1]$. The stand-alone value of the second item (y_i) follows a Bernoulli distribution, taking value 1 (High) with probability $p \in (0, 1)$, and 0 (Low) with probability $1 - p$. If a bidder only obtains a single item, the value is the stand-alone value; if a bidder obtains both items, the value is

$x_i + \theta y_i$, where the synergy factor $\theta \in (1, +\infty)$.

We consider post-auction resale so that after the completion of both auctions, resale may take place in the form of either a monopoly offer or a monopsony offer. In a monopoly mechanism, the seller, who won at least one item from the auctions, makes a take-it-or-leave-it offer to the other bidder; in a monopsony mechanism, the buyer, who won at most one item from the auctions and wants to buy one or two items from the other bidder, makes a take-it-or-leave-it offer to the other bidder. Without loss of generality, we assume that offers are accepted whenever the other bidder is weakly better off by taking the offer.[2]

As is standard in auction analysis, we assume that ties are announced and are broken at random.

When resale is absent or banned,[3] it is easily verified that the symmetric equilibrium for the first auction is given by

$$\beta(x_i) = x_i + p(\theta - 1).$$

Note that bidders only bid their values when there is no synergy between the two items for sale (i.e., when $\theta = 1$).

In the second auction, the bidding strategies of the first-item winner and loser are different because of the synergy: in equilibrium, the winner bids θy_i and the loser bids y_i.

In the symmetric equilibrium described above, it can be easily seen that when the first-item loser obtains signal 1, while the winner obtains signal 0, the outcome is not efficient when resale is banned. Both bidders can be better off if the first-item winner can resell the first item to the loser. However, when resale is allowed, the option value and additional profit induced by resale will affect bidding strategies in the initial auctions. Moreover, the resale mechanism matters for the final allocation. In this paper, we analyze two mechanisms in the resale stage: the monopoly case, where the seller of the item makes a take-it-or-leave-it offer, and the monopsony case, where the buyer makes a take-it-or-leave-it offer.

[2] A bidder is allowed to buy or to sell simultaneously. However, in the equilibrium we are going to construct, buying and selling simultaneously will not occur.

[3] This is the case analyzed by Leufkens et al. (2006), except that they assume the first and second item value distributions are the same.

3 Resale with Monopoly Offer

When resale takes the form of monopoly offers, we can rule out any equilibrium in which at least one bidder reveals her first-auction value over some non-degenerate interval.

PROPOSITION 1 *There does not exist any equilibrium in which at least one bidder reveals her first-auction value with positive probability.*

Proof. See the appendix. ■

In particular, Proposition 1 implies that there does not exist an equilibrium in which both bidders bid according to a common symmetric strictly increasing function in the first auction.

The proof is tedious, but the idea is straightforward. We start with the assumption, contrary to the claim of the proposition, that there exists an equilibrium in which at least one bidder (say, bidder 1) bids according to a strictly increasing function over a non-degenerate interval in the first auction. We then show that in such an equilibrium, there must exist a non-degenerate value interval over which some bidder (bidder 1 or bidder 2) follows a strictly increasing bid function; moreover, when this bidder loses over this interval, resale will occur with positive probability (Lemma 2 in the appendix). The implication of this result is that as long as a bidder fully reveals her type, with positive probability she will get hurt in the resale stage as resale proceeds with monopoly offer. Thus intuitively, anticipating this consequence, a bidder would not follow a strictly increasing bid function over any non-degenerate interval. The rest of the proof is to formalize this intuition. First, we derive a bidder's expected payoff as a function of her type and reported type (in the associated direct game, and given that the other bidder truthfully reports her type). We then demonstrate that truthful reporting over the interval characterized in Lemma 2 does not even guarantee a local maximum of the payoff: the payoff as a function of the reported type $\hat{x}$, though continuous, has a smaller left derivative than its right derivative. This implies that reporting truthfully is never optimal. Thus incentive compatibility never holds, which rules out the existence of the proposed equilibrium (that reveals someone's type with positive probability) in the original game. The discontinuity of the marginal benefit when varying the reported type $\hat{x}$ around one's true type is caused by the following feature of the monopoly resale mechanism: if a bidder underbids in a strictly increasing segment and loses, she will accept the offer in resale because the reseller underestimates her type; if she overbids in a strictly increasing segment and loses, she will reject the offer in resale because the reseller overestimates her type.

Our result is consistent with Krishna (2002, pp. 55-57) and Hafalir and Krishna (2008), who demonstrate that the auctioneer needs to be careful about the design of the auction. More specifically, in an asymmetric first-price auction with two bidders, they show that there does not exist any equilibrium with resale (monopoly mechanism) in which the outcome of the auction completely reveals the values (when the losing bid is announced).[4] The intuition is, to establish a strictly increasing equilibrium for the initial auction, the reseller should not be able to infer the buyer's exact value. Put differently, a player would be willing to fully reveal her type only when this information cannot be used to fully extract her surplus in a later stage. This insight is also consistent with Garratt and Tröger (2006), who consider a second-price sealed bid auction between a speculator and a regular bidder with resale. They show that the bidder will bid her true value only when that value is above some cutoff value such that the speculator does not find it profitable to overbid her (and hold a resale subsequently).

4 Resale with Monopsony Offer

We now consider the case in which the buyer makes the offer to the seller (monopsony offer). Again we consider second-price sealed bid auctions.

Suppose there exists a symmetric increasing equilibrium bidding strategy $\beta(x)$ for the first item. Then, the first-item loser, who makes a bid b, infers that the first-item winner's type (her first signal) follows the truncated distribution $F[\beta^{-1}(b), 1]$. We will first solve for the candidate $\beta(x)$ and then verify that no one has an incentive to deviate when the opponent follows this $\beta(x)$.

We assume that the regularity condition holds for $F(\cdot)$, that is, $(c - F(x))/f(x)$ is decreasing in x for any constant $c \in [0, F(x)]$.

In the equilibrium we are going to construct below, the only event in which resale occurs is when the first-item loser gets signal 1 in the second stage while her rival (the first-item winner) gets signal 0. If the first-item loser does not follow the equilibrium strategy in the first stage, resale may occur in other events. We will work backward.

[4] The losing bid does not need to be announced to establish the non-existence result in our case since we consider second-price auctions and the loser's value can be inferred from the payment.

4.1 The Resale

Suppose that in the resale stage, the first-item loser makes an offer r to buy the first item from the first-item winner. The optimal offer should solve:

$$\begin{aligned}&\max \frac{F(r)-F(\beta^{-1}(b))}{1-F(\beta^{-1}(b))}(x+\theta-r)+\frac{1-F(r)}{1-F(\beta^{-1}(b))}\\ &\text{s.t. } r\in[\beta^{-1}(b),1],\ x+\theta-r\geq 1\ .\end{aligned}$$

Let $\pi(x,\beta^{-1}(b),\theta)$ denote the value of the objective function when r is offered optimally. That is, $\pi(x,\beta^{-1}(b),\theta)$ is the expected payoff to the first-item loser, who possesses first signal x and bids b in the first stage, conditional on resale.

It is easily seen that when r is unconstrained, its optimal value should satisfy

$$x+\theta-1=r+\frac{F(r)-F(\beta^{-1}(b))}{f(r)}. \tag{1}$$

Given the regularity condition, the right hand side of the above equation is increasing in r. Hence the solution is unique. It is easily verified that as long as $r\geq\beta^{-1}(b)$, then $x+\theta-r\geq 1$. So the buyer can only be better off with resale.

Let $r(x,\beta^{-1}(b))$ be the solution to the unconstrained FOC. It can be verified that $r(x,\beta^{-1}(b))$ is increasing in both x and $\beta^{-1}(b)$. Let $r^*(x,\beta^{-1}(b))$ denote the solution with constraints. Then

$$r^*(x,\beta^{-1}(b))=\min\{1,\max\{r(x,\beta^{-1}(b)),\beta^{-1}(b)\}\}.$$

Given x, the LHS of equation (1) is increasing in θ. This implies that there exists a $\hat{\theta}$ such that when $\theta\geq\hat{\theta}$,

$$x+\theta-1>r+\frac{F(r)-F(\beta^{-1}(b))}{f(r)},$$

since r cannot exceed 1. Therefore $r^*(x,\beta^{-1}(b))=1$ for any x and $\beta^{-1}(b)$ when $\theta\geq\hat{\theta}$. By setting $r=1,x=0$,and $\beta^{-1}(b)=0$, it can be shown that

$$\hat{\theta}=2+\frac{1}{f(1)}. \tag{2}$$

When $\theta < \hat{\theta}$ and $\beta^{-1}(b) \geq g(x, \theta)$, where $g(x, \theta)$ solves $x+\theta-1 = 1+\frac{1-F(g(x,\theta))}{f(1)}$, $r^*(x, \beta^{-1}(b)) = 1$. When $\theta < \hat{\theta}$,and $\beta^{-1}(b) < g(x, \theta)$, $r^*(x, \beta^{-1}(b)) = \max\{r(x, \beta^{-1}(b)), \beta^{-1}(b)\}$.

To save notation, let $r(x) = r^*(x, x)$ denote the optimal reserve price when the first-item loser follows equilibrium $\beta(\cdot)$ in the first auction ($\beta^{-1}(b) = x$). When $\beta^{-1}(b) = x$, the condition $\beta^{-1}(b) \geq g(x, \theta)$ can be rewritten as $x \geq g(\theta)$, where $g(\theta)$ solves $g(\theta) + \theta - 1 = 1 + \frac{1-F(g(\theta))}{f(1)}$.

When we check deviations, we will consider off-equilibrium resales and assume that the potential seller comes to resale whenever the potential buyer holds the item. To consider the perfect Bayesian equilibrium (PBE), we also assume that the potential buyer's beliefs follow the Bayesian rule off the equilibrium path whenever it applies.

4.2 The Second Auction

We will consider the following strategies and beliefs in the second auction:

If the second signal is $y = 1$, then the first-item winner bids θ, and the first-item loser bids $\pi(x, \beta^{-1}(b), \theta)$.[5] If the second signal is $y = 0$, then both the first-item winner and loser bid 0.

The first-item loser's belief is that when the first-item winner loses the second item or wins it in a tie at price 0, the first-item winner's second signal is 0; when the first-item loser, with a second signal 1, loses against the first-item winner, she believes that the first-item winner's second signal is 1. The first-item winner's belief does not matter (since she does not make offers in the resale stage). The first-item loser holds a resale when her expected net gain from resale is positive and the potential seller accepts the offer whenever she is weakly better off.

Note that when $\beta^{-1}(b) = x$ or close to x, $\pi(x, \beta^{-1}(b), \theta) < \theta$. So when the first-item winner and loser both have second signal 1, the first-item winner will obtain the second item in equilibrium.

There seems to be an incentive for deviation where the first-item winner with $y = 0$ may overbid the first-item loser when the latter has $y = 1$ in order to resell the whole package back to the first-item loser. However, assuming that the first-item loser follows the equilibrium strategy, the first-item winner does not expect the loser to hold a resale in this case so there is no gain for the first-item winner to overbid.

[5] Note that the first-item winner's payment equals the first-item loser's bid in the first auction, and thus she can infer the loser's first signal in equilibrium.

4.3 The First Auction

We consider a bidder with a first signal x who makes a bid b for the first item. Suppose her rival has a first signal z and follows the equilibrium strategy. For illustration purpose we assume that $x \leq r(z)$, which implies that $r^{-1}(x) \leq z$. When $z \in [r^{-1}(x), \beta^{-1}(b)]$, the offer $r(z)$ will be accepted by the reseller (the bidder in question) when she loses the second item. Since $x \leq r(x)$ implies $r^{-1}(x) \leq x$, in equilibrium where $x = \beta^{-1}(b)$, we cannot have $\beta^{-1}(b) < r^{-1}(x)$. Also note that the first-item winner can infer the first-item loser's type; thus she knows whether or not she will accept the buyer's offer in resale (if any) immediately after the first auction.

Proposition 2 *The strategies and beliefs specified in the previous two subsections for the resale and the second auction, and the following symmetric bid function for the first auction, constitute a Perfect Bayesian Equilibrium:*

$$\beta(x) = p^2(x+\theta) + (1-p)^2 x + 2p(1-p)r(x) - p^2\pi(x,x,\theta), \tag{3}$$

where

$$r(x) = \begin{cases} 1, \text{ when } \theta \geq \hat{\theta}, \text{ or } \theta < \hat{\theta} \text{ and } x \geq g(\theta) \\ \text{the unique solution of } x+\theta-1 = r + \frac{F(r)-F(x)}{f(r)}, \text{ otherwise} \end{cases}, \text{ and}$$

$$\pi(x,x,\theta) = \begin{cases} x+\theta-1, \text{ when } \theta \geq \hat{\theta}, \text{ or } \theta < \hat{\theta} \text{ and } x \geq g(\theta) \\ \frac{F(r(x))-F(x)}{1-F(x)}(x+\theta-r(x)) + \frac{1-F(r(x))}{1-F(x)}, \text{ otherwise} \end{cases}.$$

Proof. See the appendix. ■

From the proof, one can see that $\beta(x)$ is the difference between the expected contingent payoff of winning the first item and that of losing the first item (which is the opportunity cost of winning). Therefore, in the equilibrium characterized in Proposition 2, a bidder bids her "effective value" of the first item in the first auction. This is analogous to a "bid-your-value" equilibrium in a standard second-price sealed bid auction, where the expected contingent payoff of winning is the value of the item while the expected contingent payoff of losing is zero.

The proof of Proposition 2 is tedious, so it is relegated to the appendix. Basically, given the strategies and beliefs in the resale and the second auction specified in the previous subsections, we first identify a candidate symmetric equilibrium bid function $\beta(\cdot)$ for the first item, and then proceed with the following steps. First, we show that if both bidders follow the proposed equilibrium bidding strategy in the first auction, no bidder has an incentive to deviate in the second auction and the resale stage. We then show that no bidder has an incentive to deviate from $\beta(\cdot)$ in the first auction and plays optimally, conditional on this deviation, in the subsequent stages (the second auction and the resale). The first step is straightforward to show, and our proof in the appendix focuses on the second step.

Note that there are multiple equilibria in this new setting with monopsony offers. For example, based on the PBE proposed in Proposition 2, one can obtain infinitely many other PBE's by perturbing the first-item winner's strategy, so that when her second signal is 0, she bids any positive amount less than $\pi(x_{-i}, x_{-i}, \theta)$ rather than 0 in the second auction, where x_{-i} is her competitor's first signal inferred from the first auction payment. It follows immediately that she does not have an incentive to deviate from this strategy, given that the other bidder follows the strategy specified in Proposition 2. Similarly, when the first-item loser's second signal is 0, she can bid any amount between 0 and θ as long as the first-item winner bids 0 for the second item when her second signal is 0. Note, however, this multiple equilibria problem is caused by the discreteness of the second auction signals.

In Proposition 2, we focus on the equilibrium in which both bidders bid 0 when their second signal is 0.[6] In the proof, when we consider the events off equilibrium path, we utilize the fact that as long as the first-item loser follows the proposed strategy and wins the second item when her second signal is 1, she believes that the first-item winner's second signal is 0 regardless of her payment for the second item.

4.4 Resale vs. Non-resale: A Comparison

First, although the first-item loser bids more aggressively when her second signal is 1, which results in a higher revenue for the seller in the second auction, the resale effect imposed on the bidding function for the first auction is ambiguous. The equilibrium bid functions in the first auction under the cases without resale (NR) and with resale (R) are as follows:

$$\begin{aligned}\beta^{NR}(x) &= x + p(\theta - 1), \\ \beta^{R}(x) &= p^2(x+\theta) + (1-p)^2 x + 2p(1-p)r(x) - p^2\pi(x, x, \theta).\end{aligned} \tag{4}$$

[6] Again, this corresponds to the "bid-your-value" strategy in the one-shot counterpart of the second-price sealed bid auction.

In the appendix (Lemma 3), we show that there is no general ranking between $\beta^{NR}(x)$ and $\beta^{R}(x)$. This suggests that the revenue comparison is also ambiguous, although the allocation efficiency is apparently higher when resale is allowed. To see this more specifically, let R^{NR} and R^{R} denote the original seller's expected revenues without and with resale, respectively. Then the auctioneer's expected revenue is the sum of the expected revenues from the two auctions.[7]

Let $X_{(2)}$ denote the second order statistic of the first signals. Its cumulative distribution function and density function are denoted by $F_{(2)}(x)$ and $f_{(2)}(x)$, respectively. It can be shown that

$$\begin{aligned} R^{NR} &= \int_0^1 [x + p(\theta - 1) + p^2] f_{(2)}(x) dx, \\ R^{R} &= \int_0^1 [p^2(x+\theta) + (1-p)^2 x + 2p(1-p) r(x)] f_{(2)}(x) dx, \end{aligned}$$

where $f_{(2)}(x) = 2f(x)[1 - F(x)]$. For comparison purposes, we define the difference in expected revenues between the two cases as follows:

$$DR = R^{NR} - R^{R}.$$

Despite the absence of a general ranking of expected revenue generated under the resale and the non-resale cases, we are able to identify some sufficient conditions in which one case dominates the other.

Proposition 3 *When $\theta \geq \hat{\theta}$, $R^{NR} \geq R^{R}$ if $EX_{(2)} \geq (3-\theta)/2$, and $R^{NR} < R^{R}$ if $EX_{(2)} < (3-\theta)/2$; When $\theta < \hat{\theta}$, $R^{NR} > R^{R}$ if $\theta \geq g^{-1}[F_{(2)}^{-1}(1/2)]$ or if $f(x)$ is a decreasing function.*[8]

Proof. See the appendix. ■

When θ is sufficiently large, $EX_{(2)} \geq (3-\theta)/2$ is also more likely to hold (it holds trivially when $\theta \geq 3$). Thus expected revenue is higher when resale is banned, as long as the synergy factor is sufficiently large. A rough intuition is as follows. Based on the comparison of bidding functions with and without resale, allowing resale lowers the first auction revenue for most cases, but raises the second auction revenue for all of the cases. When θ is sufficiently large and resale is not allowed, the effect from the first auction dominates the effect from the second auction, mainly because the second

[7]Since no reserve prices are imposed, the auctioneer earns positive revenue from the second auction if and only if both bidders have second signal 1.

[8]Note that $f'(x) < 0$ implies that $\frac{F(r)-F(x)}{f(r)}$ is increasing in r, which is our regularity condition.

auction revenue is positive only when both bidders obtain 1 as the second signal. Thus in this case expected revenue is higher when resale is banned. On the other hand, when $f'(x) < 0$, first-item values would be low in a statistical sense. The first-auction equilibrium bid functions (4) suggest that when x is small and θ is not large, bidders behave more aggressively with resale. This effect increases the first auction revenue with resale and makes the difference in the first-auction revenue with and without resale smaller. When x is small, however, the difference between the second-auction losers' bids would also be small, which suggests that the difference in the second-auction revenue with and without resale would also be small. The effect from the second auction dominates the effect from the first auction, giving rise to the result that $R^{NR} > R^{R}$ when θ is not sufficiently large and $f(\cdot)$ is decreasing.

Our result is different from Hafalir and Krishna (2009), who show that allowing resale increases the original seller's expected revenue *unambiguously* in a first-price auction with asymmetric bidders. This difference suggests that the effect of resale possibility on revenue may depend on a variety of factors, including the auction format, the symmetry of bidders, or the presence of synergy, etc.

To further understand the ambiguous nature of the general revenue ranking, we examine the generalized uniform distribution family $F(x) = x^{\alpha}, \alpha > 0$. When $\theta \geq \hat{\theta} = 2 + \frac{1}{\alpha}$, $DR = p(1-p)\frac{2\alpha^3-\alpha^2+2\alpha+1}{\alpha(\alpha+1)(2\alpha+1)} > 0$. Therefore, for any generalized uniform distribution, the seller's expected revenue is higher without resale when $\theta \geq 2 + \frac{1}{\alpha}$. When $\theta < \hat{\theta}$, $f'(\cdot) \leq 0$ if and only if $\alpha \leq 1$; when $\alpha > 1$, $\theta \geq g^{-1}[F_{(2)}^{-1}(\frac{1}{2})]$ if and only if $\alpha(2-\theta-g(\theta)) \leq -\frac{\sqrt{2}}{2}$. It can be shown that $g(\theta)$ is decreasing in α. Therefore, $R^{NR} > R^{R}$ unless $EX_{(2)} < \frac{3-\theta}{2}$ for $\theta \geq \hat{\theta}$.

We next turn to the comparison of probabilities of the exposure problem. When resale is not allowed, the exposure problem arises when the first-item winner's second signal is 0, while her payment for the first item is higher than her stand-alone value for the first item. When resale is allowed, the exposure problem arises when the first-item winner ends up overpaying for the first item when both bidders' second signals are 0, or when her expected payoff from resale is less than her payment for the first item.

We define $P_{\exp}^{NR}(x)$ and $P_{\exp}^{R}(x)$ to be the probabilities of exposure for the cases without and with resale, respectively, given the bidder's first signal x. Let z be the other bidder's first signal. These two probabilities can be expressed as

$$\begin{aligned} P^{NR}_{\exp}(x) &= (1-p)\cdot P(z<x, \beta^{NR}(z)>x), \\ P^{R}_{\exp}(x) &= (1-p)^2\cdot P(z<x, \beta^{R}(z)>x) + p(1-p)\cdot P(z<x, \beta^{R}(z)>\max\{x, r(z)\}). \end{aligned}$$

While it is complicated to make a general comparison, we can relate the comparison of exposure probabilities to the bidding functions in the first stage:

Proposition 4 *If $\beta^{NR}(z) \geq \beta^{R}(z)$ for $z \in [0, x)$, $P^{NR}_{\exp}(x) > P^{R}_{\exp}(x)$; In particular, when the synergy effect is sufficiently large (θ sufficiently large), the probability of exposure is higher when resale is banned.*

Proof. See the appendix. ■

The intuition is straightforward. The exposure problem occurs when the first-item winner's overall payoff is negative. If a bidder's value is such that whenever she wins the first item her payment is higher without resale, then whenever her overall payoff is negative with resale, it will also be negative without resale because the first-item winner has the option of reselling the item when resale is allowed. When the synergy effect is sufficiently strong, $\beta^{NR}(\cdot) > \beta^{R}(\cdot)$. Hence the probability of exposure is higher when resale is not allowed.

It turns out that the condition $\theta + p \geq 3$ is "tight" in the sense that when $\theta + p < 3$, we can identify conditions under which $P^{NR}_{\exp}(x) \leq P^{R}_{\exp}(x)$. To see this, we consider $\theta > \hat{\theta} = 2 + \frac{1}{f(1)}$ and $\theta + p < 3$. In this case we have

$$\begin{aligned} P^{NR}_{\exp}(2p-p^2) &= (1-p)[F(2p-p^2) - F(2p-p^2-p(\theta-1))], \\ P^{R}_{\exp}(x) &= (1-p)^2 F(2p-p^2). \end{aligned}$$

Clearly, when p is very close to 0, there exist θ and f such that $\theta > \hat{\theta} = 2 + \frac{1}{f(1)}$ and $\theta + p < 3$. Meanwhile, it is trivial to show that if $p = 0$, there is no exposure problem because there is no chance of getting second signal 1, which means $P^{NR}_{\exp}(0) = 0 = P^{R}_{\exp}(0)$. We can also demonstrate that

$$\lim_{p\to 0^+} P^{NR}_{\exp}(p) = (\theta-1)f(0) < 2f(0) = \lim_{p\to 0^+} P^{R}_{\exp}(p),$$

since $\theta + p < 3$ implies that $\theta < 3$. Thus $P^{NR}_{\exp}(p) < P^{R}_{\exp}(p)$ when p is sufficiently close to 0, and that $\theta > \hat{\theta}$ and $\theta + p < 3$.

To help understand our comparison results, we next focus on the special case in which the first-item value is distributed uniformly over $[0,1]$.

Example: The Uniform Distribution Case

When the first signal follows $U[0,1]$, it can be verified that $\hat{\theta}=3$, $g(\theta)=\frac{3-\theta}{2}$, and $r(x)$ and $\pi(x,x,\theta)$ are given below.

$$
\begin{aligned}
r(x) &= \begin{cases} 1, \text{when } \theta \geq 3 \text{ or } \theta < 3 \text{ \& } x \geq \frac{3-\theta}{2} \\ x+\frac{\theta-1}{2}, \text{ otherwise} \end{cases} \\
\pi(x,x,\theta) &= \begin{cases} x+\theta-1, \text{when } \theta \geq 3 \text{ or } \theta < 3 \text{ \& } x \geq \frac{3-\theta}{2} \\ 1+\frac{(\theta-1)^2}{4(1-x)}, \text{ otherwise} \end{cases}
\end{aligned}
$$

Therefore, the equilibrium bid function for the first item is:

$$
\beta^R(x) = \begin{cases} (1-p)^2x - p^2 + 2p, \text{ when } \theta \geq 3 \text{ or } \theta < 3 \text{ \& } x \geq \frac{3-\theta}{2} \\ x+p(\theta-1)-p^2\frac{(\theta-1)^2}{4(1-x)}, \text{ otherwise} \end{cases}
$$

Since $\beta^{NR}(x)=x+p(\theta-1)$, it can be easily verified that $\beta^R(x)<\beta^{NR}(x)$ for any $p\in(0,1)$ and $\theta>1$, which means that with the opportunity of resale, bidders bid less aggressively for the first item. From Proposition 4, it follows that the exposure probability is lower when resale is allowed. Note that with the uniform distribution, we do not require θ to be sufficiently large in order for the probability of exposure to be lower with resale.

Define $\pi^{NR}(x)$ and $\pi^R(x)$ to be the expected payoff for a bidder with the first signal x under the no-resale and resale scenarios. With some derivations, it can be shown that:

$$
\begin{aligned}
\pi^{NR}(x) &= \int_0^x (x-z)dz + p(1-p) \\
\pi^R(x) &= \begin{cases} (1-2p+2p^2)\int_0^x (x-z)dz + p(1-p)(x+\theta-1), \ \theta \geq \hat{\theta} \\ (1-p+p^2)\frac{x^2}{2} + p(1-p)(3-\theta)\frac{x}{2} + p(1-p)\frac{\theta^2+1}{4}, \ \theta < \hat{\theta} \text{ and } x \geq g(\theta) \\ \frac{x^2}{2} + p(1-p)(\frac{3}{8}\theta^2 - \frac{3}{4}\theta + \frac{11}{8}), \text{ otherwise} \end{cases}
\end{aligned}
$$

It can be verified that $\pi^{NR}(x)<\pi^R(x)$ holds for any $x\in[0,1]$.

For the uniform distribution, bidders always bid less aggressively for the first item. However, bidders bid more aggressively for the second item. With some derivations, we can show that

$$R^{NR} = p\theta + p^2 - p + \frac{1}{3},$$
$$R^R = \begin{cases} p^2\theta + \frac{-4p^2+4p+1}{3}, \text{when } \theta \geq 3 \\ \frac{-p(1-p)\theta^3}{12} + \frac{p(1-p)\theta^2}{4} + \frac{(p^2+3p)\theta}{4} + \frac{11p^2-11p+4}{12}, \text{ when } \theta < 3 \end{cases}.$$

Thus, we have

$$R^{NR} - R^R = \begin{cases} p(1-p)(\theta - \frac{5}{3}), \text{ if } \theta \geq 3 \\ \frac{p(1-p)}{12}(\theta-1)^3, \text{ if } 1 < \theta < 3 \end{cases}$$

So $R^{NR} - R^R > 0$, which implies that in the uniform distribution case, the effect of underbidding for the first item dominates the effect of overbidding for the second item, and the revenue comparison is unambiguous.

5 Discussion

Our model is stylized, and our results hinge on several important assumptions. This section is thus devoted to a discussion on the restrictions of our modeling and the robustness of our results.

First, in our model, the synergy enters the value of a package as a multiplier to the value of the second item, while the previous literature in general assumes that synergies enter the value of a package as a term to be added to the stand-alone value of a package. However, we do not think that our major results would be altered if we adopt the "summation" version instead: Inefficiency would still persist when resale was banned, and in the resale stage, the seller or buyer would still make an optimal offer, taking into account the synergy.

Second, in our model, we assume that the value for the second item is a binary variable drawn from a Bernoulli distribution. This is for ease of tractability. If we assume that the second signal, like the first signal, is also a continuous variable distributed according to the distribution function $F(\cdot)$, our analysis would become too complicated. First, the events of resale would be too tedious to identify, and second we will have to work with the convolution of two continuous distributions, which is simply intractable. With the Bernoulli distribution, however, the need for resale (in equilibrium) has an intuitive interpretation: resale occurs if the winner of the first item has no value for the second item, and the loser of the first auction has a positive value for the second item. Moreover, since the equilibrium probability of resale is $p(1-p)$, to some extent, different probabilities of resale can be obtained by varying p. Thus, though restrictive, our special value structure captures the essence of the resale opportunity in auctions with synergy.

Finally, our analysis focuses on two bidders only. If we extend our analysis to the general case with n bidders, the analysis for the monopoly case turns out to be quite involved. With $Y = 0$, a bidder will bid 0 in equilibrium regardless of winning the first auction or not. With $Y = 1$, the winner of the first item will bid θ as in the two-bidder case, but for the losers of the first item, they would not bid according to a strictly increasing bid function in equilibrium. The reason is that the winner of the first item may want to resell the whole package if the allocation of the first item is inefficient. So pooling or partial pooling would occur in the second auction for bidders who lost the first auction with $Y = 1$. Thus after two auctions, it might be the case that neither of the two items is owned by a bidder who values the package the most. So it is possible that the second item owner would try to buy the first item, then resell the whole package to a third party, a complication that is absent with $n = 2$. Given this complication, it is very difficult to analyze the cases in which resale occurs with $n > 3$.[9] This complication, however, is absent in the symmetric increasing equilibrium in which all bidders follow a (common) strictly increasing bid function in the first auction. It can be established that as long as there is positive probability in which full extraction in resale may occur, the non-existence result continues to hold.

For the monopsony case with $n > 3$, the analysis becomes intractable as the second-item winner's expected payoff from resale depends on her payment for the second item. The intuition is that when $n > 3$ and the second-item winner makes a positive payment for the second item, it does not mean that she has the highest first signal among all the first-auction losers since some loser with a higher first signal might have obtained second signal 0. Therefore, the payment is positively correlated with the first-item winner's first signal, making the analysis intractable. This is different from the case $n = 3$, where a positive payment always indicates that the second-item winner has the highest first signal between both losers. Basically we can show that our equilibrium analysis can be extended

[9] To the best of our knowledge, the similar non-existence result in auctions with monopoly resale has only been established in the context of two bidders (Krishna, 2002; Hafalir and Krishna, 2008; and Garratt and Tröger, 2006).

to the case $n = 3$ when the synergy factor, θ, is sufficiently large. When θ is not sufficiently large, however, some additional conditions are needed to support the same equilibrium structure as constructed in the two-bidder case.

To summarize, extending our analysis with two bidders to the general case with n bidders presents some technical challenges. While specific results may be subject to change, we believe that the general features of the equilibrium, should it exist, remain the same.

6 Conclusion

This paper offers the first analysis of auctions with synergies by explicitly considering the post-auction resale opportunity, which is arguably more natural if we believe that secondary markets cannot always be banned.

Our analysis suggests that when the initial auctions take the form of second-price sealed bids, there is a potential problem of information revelation, even if bids are not announced. In particular, a monopoly mechanism in resale can destroy any equilibrium in which a bidder has to fully reveal her type over some range with positive probability. Since typically an auctioneer cannot control the selling mechanism in the resale stage, our results have some immediate policy implication for how secondary markets should be regulated.

When resale proceeds with a monopsony offer, we are able to identify symmetric increasing equilibrium strategies in auctions for both items. While allowing resale always improves efficiency, the effect of resale on expected revenue and the probability of exposure are both ambiguous. We identify sufficient conditions under which allowing resale leads to higher expected revenue or lower probability of exposure.

Our model is one of the simplest that captures essentials of auctions with synergy and resale. Despite our modeling restrictions, we believe that the main insights of our results are robust, as discussed in the previous section. Nevertheless, working out a more general framework to study the interactions between auctions with synergy and resale should be a direction for future research.

Appendix

Proof of Proposition 1: To show this non-existence result, we will need to rule out any equilibrium in which at least one bidder follows a first-auction bid function that is strictly increasing over a non-degenerate interval (so that when she loses with a value in this interval, she fully reveals her type). Formally, we start with the assumption that there exist a pair of equilibrium bidding strategies in the first auction, which are represented by $\beta_i(\cdot) : [0,1] \to \mathbf{R}_+$, $i = 1, 2$. Suppose $\beta_i(x_i)$ is a piecewise function over disjoint intervals $[x_{i,n-1}, x_{i,n})$, $n = 1, 2, ..., N-1$, and $[x_{i,N-1}, 1]$, where $x_{i,0} = 0$; and that $\beta_i(x_i)$ is weakly increasing over $[0,1]$.[10] Without loss of generality, we assume that $\beta_1(\cdot)$ is strictly increasing over some non-degenerate interval $(\underline{v}_1, \bar{v}_1)$ and $\beta_1(\bar{v}_1) \leq \beta_2(1)$. Our non-existence proof will be based on two lemmas.

Our first lemma establishes that in such an equilibrium, there exists an interval $(\underline{v}_2, \bar{v}_2)$ over which $\beta_2(\cdot)$ is strictly increasing and $\beta_1(\underline{v}_1, \bar{v}_1) \cap \beta_2(\underline{v}_2, \bar{v}_2)$ is a non-degenerate interval, where $\beta_i(\underline{v}_i, \bar{v}_i) = (\beta_i(\underline{v}_i), \beta_i(\bar{v}_i))$; in other words, there exists some non-degenerate overlap bid interval that can be mapped back onto some subsets over which each bidder's bidding function is strictly increasing.

Lemma 1 *There exists $(\underline{b}_1, \bar{b}_1) \subset \beta_1(\underline{v}_1, \bar{v}_1)$, such that $\beta_2^{-1}(\underline{b}_1, \bar{b}_1)$ is a non-degenerate interval and $\beta_2(\cdot)$ is strictly increasing over this interval.*

Proof. Suppose not. Then, either bidder 2 never bids over the interval $\beta_1(\underline{v}_1, \bar{v}_1)$, or all types of bidder 2 who bid over this interval are pooling or partial pooling (i.e., this part of the bidding function is a step-wise function). In the first case, bidder 1 will be at least weakly better off by pooling at $\beta_1(\underline{v}_1)$ because it does not switch her from winning to losing or vice versa, but she benefits from hiding her first signal. In the second case, by a similar argument, bidder 1 with a type in $(\underline{v}_1, \bar{v}_1)$ has an incentive to pool, a contradiction. ■

Based on lemma 1, let $(\underline{x}_1, \bar{x}_1)$ be a sub-interval of $(\underline{v}_1, \bar{v}_1)$ such that $\beta_2(\cdot)$ is strictly increasing over $\beta_2^{-1}(\beta_1(\underline{x}_1), \beta_1(\bar{x}_1))$. Redefine $\beta_2^{-1}(\beta_1(\underline{x}_1), \beta_1(\bar{x}_1)) = (\underline{x}_2, \bar{x}_2)$. We will next prove that there exists a bidder k, $k = 1, 2$, such that when she follows $\beta_k(\cdot)$ and loses with a first signal lying within some subinterval of $(\underline{x}_k, \bar{x}_k)$, she expects the resale to be held with positive probability.

[10] Our analysis should apply to other possible cases, i.e., one bidder follows a weakly increasing bidding function while the other bidder follows a weakly decreasing bidding function.

LEMMA 2 *For bidder k, $k = 1, 2$, there exists an interval $(x_{k*}, x^{k*}) \subset (\underline{x}_k, \bar{x}_k)$ such that when she follows $\beta_k(\cdot)$ and loses with $x_k \in (x_{k*}, x^{k*})$, she expects a resale held by the other bidder with positive probability.*

Proof. We consider the two intervals $(\underline{x}_1, \bar{x}_1)$ and $(\underline{x}_2, \bar{x}_2)$. We examine three possibilities in order:

1. $\bar{x}_1 \leq \underline{x}_2$. In this case, if bidder 2 with the first signal in $\beta_2^{-1}(\beta_1(\underline{x}_1), \beta_1(\bar{x}_1))$ loses, she knows there is a positive probability that bidder 1's first signal is below hers; thus she expects that there will be a resale with positive probability. Therefore, $k = 2$ and $(x_{k*}, x^{k*}) = (\underline{x}_2, \bar{x}_2)$.

2. $\underline{x}_1 \geq \bar{x}_2$. In this case, $k = 1$ and $(x_{k*}, x^{k*}) = (\underline{x}_1, \bar{x}_1)$.

3. Otherwise, $(\underline{x}_1, \bar{x}_1) \cap (\underline{x}_2, \bar{x}_2)$ is a non-degenerate interval. Without loss of generality, assume that $\bar{x}_2 \leq \bar{x}_1$.

   (a) If $\underline{x}_1 = \underline{x}_2, \bar{x}_1 = \bar{x}_2$ and there does not exist an $\check{x} \in (\underline{x}_1, \bar{x}_1)$ such that $\beta_1(\check{x}) \neq \beta_2(\check{x})$, we know these two bidding functions coincide on this interval. In this case, when bidder 1 loses, there is a positive probability for bidder 2 to hold a resale in the event that bidder 1 obtains second signal 1 while bidder 2 obtains second signal 0. Therefore, $k = 1$ and $(x_{k*}, x^{k*}) = (\underline{x}_1, \bar{x}_1)$.

   (b) When there exists an $\check{x} \in (\underline{x}_1, \bar{x}_1)$ such that $\beta_1(\check{x}) \neq \beta_2(\check{x})$, without loss of generality, we assume that $\beta_1(\check{x}) < \beta_2(\check{x})$. By continuity, there exists a sufficiently small $\xi > 0$, such that $\beta_1(x) < \beta_2(x)$ for $x \in (\check{x} - \xi, \check{x} + \xi)$. When bidder 1 with a first signal in $(\check{x} - \xi, \check{x} + \xi)$ loses, there is a positive probability that she loses to bidder 2 with a first signal lower than hers; thus she expects a resale with positive probability. Therefore, $k = 1$ and $(x_{k*}, x^{k*}) = (\check{x} - \xi, \check{x} + \xi)$.

   (c) In all the other cases, at least one of the equality does not hold. Without loss of generality, we assume that $\bar{x}_2 < \bar{x}_1$. Clearly, when bidder 1 with a first signal in $(\bar{x}_2, \bar{x}_1)$ loses, she expects the resale to occur with positive probability. Therefore, $k = 1$ and $(x_{k*}, x^{k*}) = (\bar{x}_2, \bar{x}_1)$.

So in all the cases, the conclusion of the lemma is true. ■

We are now ready to show the non-existence result. Without loss of generality we will consider the following strategies and beliefs for the second auction and the resale stages: the first-item winner bids θ when her second signal is 1 and bids δ, a sufficiently small positive amount, when her second signal is 0; the first-item loser bids 1 when her second signal is 1 and 0 otherwise. The first-item winner's beliefs about the first-item loser's second signal are that it is 0 if she wins the second item at zero price, and 1 otherwise. In the resale, the first-item winner's offer is a vector of the item(s) offered for resale and the corresponding optimal price(s).[11] She does not offer the second item for resale when she believes that the first-item loser's second signal is 0. The first-item loser's beliefs about the first-item winner's signals do not matter and she accepts any offer in resale as long as she is weakly better off. It can be verified that the above pair of the strategy profile and belief system constitutes a perfect Bayesian equilibrium (PBE) for the second auction and resale stages. Moreover, in such a PBE, the first-item loser plays a pure and separating strategy so that after the second auction, the first-item winner can infer the first-item loser's second-item signal.[12]

Without loss of generality, we let $k = 1$ in Lemma 2 and consider the case in which, conditional on losing the first auction, bidder 1 expects resale to occur with positive probability only when her second signal is 1 and bidder 2's signal is 0. We follow this case for ease of illustration; the logic of our analysis below should go through for all the other cases discussed in the proof of Lemma 2.

Suppose bidder 1, with the first signal x, plays as if her type is $\hat{x}$ in the first auction, but plays optimally, conditional on x and $\hat{x}$, in the following two stages (i.e., the second auction and the resale). For resale, this means that if bidder 1 underbids in a strictly increasing segment and loses, she will accept the offer in resale (if any) because the reseller underestimates her type; if she overbids in a strictly increasing segment and loses, she will reject the offer in resale because the

[11] Because of the asymmetry in the first auction bid functions, it is possible that even if both bidders' second signals are the same, there is a potential gain from resale. Taking this into account, the first-item winner calculates the optimal offer vector given her beliefs about the first-item loser's two signals.

[12] There are other equilibria in which the first-item winner can infer the first-item loser's second signal after the second auction. Using backward induction, one can verify that the contingent payoffs in resale for the first-item winner and loser when their second signals are 1's are at least θ and 1, respectively. Moreover, we do not require the first-item winner to infer the loser's second signal in cases where conditional on winning the first auction and her beliefs about the loser's first signal, the winner does not find it profitable to hold a resale for all possible subsequent events (in these cases, the first-item winner can bid 0 when her second signal is 0). Working with those equilibria will not alter our analysis.

reseller overestimates her type. Clearly, for $\beta_1(\cdot)$ to be an equilibrium bidding strategy, we require that $\hat{x} = x$ for $x \in (x_{1*}, x^{1*})$, where the existence of an interval (x_{1*}, x^{1*}) is established in Lemma 2. Below we only consider local deviations so that $\hat{x}$ is sufficiently close to x. By continuity, Lemma 2 implies that with a sufficiently small deviation, conditional on losing the first auction, bidder 1 still expects bidder 2 to hold a resale with positive probability.

Given that bidder 2 follows the proposed equilibrium, bidder 1 solves the following problem:

$$\max_{\hat{x} \in (x_{1*}, x^{1*})} \int_0^{\beta_2^{-1}(\beta_1(\hat{x}))} [w(x,y) - \beta_2(y)] f(y) dy + \int_{\beta_2^{-1}(\beta_1(\hat{x}))}^1 l(x, y, \hat{x}) f(y) dy,$$

where $w(x,y)$ is bidder 1's contingent payoff if she obtains the first item given her first signal x and her rival's first signal y, and $l(x, y, \hat{x})$ is bidder 1's contingent payoff if she loses the first item in the first auction given that she mimics type $\hat{x}$.

By backward induction, $l(x, y, \hat{x}) = p(1-p)[(x - \hat{x}) I_{\{x > \hat{x}\}} I_{\{y \leq \hat{x} + \theta - 1\}} + 1 - \delta]$, where $y \leq \hat{x} + \theta - 1$ represents the event where bidder 2 finds it profitable to hold a resale when her second signal is 0, bidder 1's second signal is 1, and $\hat{x} + \theta - 1$ is the optimal offer given her beliefs.

Now we consider the case in which bidder 1 mimics type $\hat{x}$, which is sufficiently close to x. When this deviation is downward (i.e., $x > \hat{x}$), we have

$$\int_{\beta_2^{-1}(\beta_1(\hat{x}))}^1 l(x, y, \hat{x}) f(y) dy = \int_{\beta_2^{-1}(\beta_1(\hat{x}))}^{\min\{\hat{x}+\theta-1,1\}} p(1-p)[x - \hat{x} + 1 - \delta] f(y) dy + \int_{\min\{\hat{x}+\theta-1,1\}}^1 p(1-p) f(y)(1-\delta) dy.$$

If this deviation is upward (i.e., $x < \hat{x}$), we have

$$\int_{\beta_2^{-1}(\beta_1(\hat{x}))}^1 l(x, y, \hat{x}) f(y) dy = \int_{\beta_2^{-1}(\beta_1(\hat{x}))}^1 p(1-p) f(y)(1-\delta) dy.$$

Define FOC_{down} (FOC_{up}) to be the derivative w.r.t $\hat{x}$ in the downward (upward) deviation case. Letting $\hat{x} \uparrow x$ in the downward case and $\hat{x} \downarrow x$ in the upward case, we can verify that

$$\lim_{\hat{x} \uparrow x} FOC_{down} = \lim_{\hat{x} \downarrow x} FOC_{up} - p(1-p) \int_{\beta_2^{-1}(\beta_1(x))}^{\min\{\hat{x}+\theta-1,1\}} f(y) dy.$$

Thus $FOC_{down}(\hat{x} \uparrow x) < FOC_{up}(\hat{x} \downarrow x)$ for $p \in (0,1)$.

It can be easily verified that the objective function is continuous at $\hat{x} = x$. For the left derivative to be smaller than the right derivative of a continuous function, it is either $FOC_{down} < FOC_{up} \leq 0$, $0 \leq FOC_{down} < FOC_{up}$, or $FOC_{down} < 0 < FOC_{up}$. For all these cases, it can be easily verified that $\hat{x} = x$ cannot even be a local maximum. This suggests that the incentive compatibility condition

$\hat{x} = x$ fails. Hence, the proposed equilibrium in which at least one bidder reveals her type with positive probability does not exist. ■

Proof of Proposition 2: Suppose that bidding for the second item follows the proposed equilibrium strategies. We will first compute the candidate equilibrium bid function for the first item ($\beta(\cdot)$) and then verify that it is indeed the equilibrium bid function for the first item.

Case 1: $\theta \geq \hat{\theta}$, or $\theta < \hat{\theta}$ and $\beta^{-1}(b) = x \geq g(\theta)$, which implies that the offer made by this bidder in resale should be 1.

When she wins and $\theta \geq \hat{\theta}$, her rival will make an offer $r(z) = 1$ in resale if resale occurs, where z is the rival's first auction signal;

When she wins and $\theta < \hat{\theta}$, we have

$$\begin{aligned}
&\int_0^{\beta^{-1}(b)} E(\text{payoff}|\text{winning the first item}) f(z) dz \\
= &\int_0^{g(\theta)} E(\text{payoff}|\text{winning the first item}) f(z) dz + \int_{g(\theta)}^{\beta^{-1}(b)} E(\text{payoff}|\text{winning the first item}) f(z) dz.
\end{aligned}$$

However, to derive FOCs we only need to consider the event $z \geq g(\theta)$, since the term $\int_0^{g(\theta)} E(\text{payoff}|\text{winning}$ the first item$) f(z) dz$ does not contain b. When $z \geq g(\theta)$, she accepts the offer since $r(z) = 1$.

The expected gross payoff upon winning is given by

$$p^2[x + \theta - (z + \theta - 1)] + (1 - p)p(x + \theta) + p(1 - p) + (1 - p)^2 x,$$

and the expected payoff upon losing is given by

$$p(1 - p)(x + \theta - 1).$$

It can be easily verified that the implication of the FOCs in this case is that a bidder bids an amount equal to her maximal willingness to pay, which is the difference between the expected payoffs conditional on winning and not winning the item when a tie occurs.

Straightforward calculations lead to

$$\beta(x) = (1 - p)^2 x - p^2 + 2p. \tag{5}$$

Clearly, in this case, $\beta'(x) > 0$.

Case 2: $\theta < \hat{\theta}$ and $\beta^{-1}(b) = x < g(\theta)$.

In this case, the first-item loser will make an offer less than 1 in resale. Again, when a bidder with $x < g(\theta)$ wins, the only event that is relevant for the FOCs is when her rival's signal z satisfies $r(z) \geq x$.

The gross contingent payoff upon winning for the relevant event is given by

$$p^2[x + \theta - \pi(z, z, \theta)] + (1-p)p(x+\theta) + p(1-p)r(z) + (1-p)^2 x.$$

The expected payoff upon losing is given by

$$p(1-p)\pi(x, \beta^{-1}(b), \theta).$$

Taking the derivative of the expected payoff w.r.t b and evaluating this derivative at $\beta^{-1}(b) = x$, we have

$$\beta(x) = p^2(x+\theta) + (1-p)^2 x + 2p(1-p)r(x) - p^2\pi(x, x, \theta), \tag{6}$$

and hence

$$\begin{aligned} \beta'(x) &= p - p(1-p) + (1-p)^2 + 2p(1-p)r'(x) \\ &\quad - p^2 \frac{F(r(x)) - F(x)}{1 - F(x)} \left[-\frac{f(x)}{f(r(x))} \frac{1 - F(r(x))}{1 - F(x)} + 1 \right]. \end{aligned}$$

Note that $\frac{1-F(v)}{f(v)}$ decreases in v and $x \leq r(x)$ implies that

$$\frac{f(x)}{f(r(x))} \frac{1 - F(r(x))}{1 - F(x)} \in [0, 1],$$

which further implies

$$\frac{F(r(x)) - F(x)}{1 - F(x)} \left[-\frac{f(x)}{f(r(x))} \frac{1 - F(r(x))}{1 - F(x)} + 1 \right] \in [0, 1].$$

Therefore,

$$\begin{aligned} \beta'(x) &\geq p - p(1-p) + (1-p)^2 + 2p(1-p)r'(x) - p^2 \\ &= (1-p)^2 + 2p(1-p)r'(x) \\ &> 0. \end{aligned}$$

We can thus summarize (5) and (6) into (3), which is the candidate equilibrium bid function in the first auction.

Next, we prove that no bidder can deviate from the proposed equilibrium profitably. We will proceed in two steps. The first step is to show that if both bidders follow the proposed bidding strategy in the first auction, no bidder has an incentive to deviate in the second auction or in the resale stage. The second step is to show that no bidder has an incentive to deviate in the first auction and plays optimally subsequently. The first step is straightforward to verify, hence we will focus on the second step below.

Denote $V(x, \hat{x})$ to be the expected payoff for a bidder of type (first signal) x who mimics type $\hat{x}$ in the first auction and plays optimally in the second auction and the resale stage.

We first consider off-equilibrium events. Again, we assume that the potential seller shows up in the resale and accepts the offer when she is weakly better off by doing so. The potential buyer makes an offer in resale when her expected net payoff is positive. Off-equilibrium resales might involve selling both items.

If the bidder overbids and wins while $x < z$, where z is her rival's first signal (she knows z conditional on winning), she may want to pretend that her second signal is 0 when it is actually 1 in order to resell the first item to her rival when her rival also has a second signal of 1. She can do this by bidding ε, which is close to 0, when her second signal is 1; thus she wins against a rival with signal 0, but loses against a rival with signal 1 (her rival still believes her second signal is 0 even if the rival pays ε for the second item). Note that when both bidders have a signal 1, if she bids θ and wins, her rival does not hold a resale and her payoff is $x + \theta - \pi(z, z, \theta)$; if she loses the second item, she accepts $r(z)$ in resale because $z > x > r^{-1}(x)$. Therefore, when $x + \theta - \pi(z, z, \theta) < r(z)$, she will bid ε when her second signal is 1. Also note that $x + \theta - \pi(z, z, \theta) > r(z)$ when x is just a little below z, since $z + \theta - \pi(z, z, \theta) > r(z)$. When both bidders get second signal 0, although the overbidding bidder wants to sell the first item to her rival, the rival does not hold a resale.

Therefore, when $x < \hat{x}$, the gross winning payoff is given by

$$p^2 \max\{x + \theta - \pi(z, z, \theta), r(z)\} + p(1-p)(x + \theta) + p(1-p)\max\{r(z), x\} + (1-p)^2 x.$$

The expression for the losing payoff remains unchanged.

On the other hand, if the bidder underbids and loses, she holds a resale even when they both have the same second signal, because it might be the case that $x > z$, and hence her expected net payoff from resale is positive. Note that when they both have second signal 1, the bidder who underbids in the first auction would prefer to buy the whole package in the resale rather than win the second item. In the case with underbidding, the expression for the gross winning payoff is the same as before, which is given by

$$p^2[x+\theta-\pi(z,z,\theta)]+p(1-p)(x+\theta)+p(1-p)\max\{r(z),x\}+(1-p)^2x.$$

However, the losing payoff is now

$$p^2l^1(x,\hat{x},\theta)+p(1-p)\pi(x,\hat{x},\theta)+(1-p)^2l^0(x,\hat{x},\theta),$$

where $l^1(x,\hat{x},\theta)$ and $l^0(x,\hat{x},\theta)$ denote the off-equilibrium payoffs when she holds a resale, knowing that both bidders have the same second signal 1 and 0, respectively. It is easy to verify that in these two off-equilibrium events, if $r^0(x,\hat{x})$ $(r^1(x,\hat{x}))$ denotes the optimal resale offer when they both have a 0(1) second signal, $r^1(x,\hat{x})=r^0(x,\hat{x})+\theta$. Also, $r^0(x,\hat{x})\in[\hat{x},x]$.

We now show that neither overbidding nor underbidding is profitable.

Overbidding case: $x<\hat{x}$.

When $x+\theta-\pi(z,z,\theta)<r(z)$ (overbids by a relatively large amount),

$$\begin{aligned}\frac{\partial V(x,\hat{x})}{\partial\hat{x}} &= f(\hat{x})[p^2r(\hat{x})+(1-p)p(x+\theta)+p(1-p)r(\hat{x})\\ &\quad +(1-p)^2x-\beta(\hat{x})-p(1-p)(x+\theta-r(x,\hat{x})]\\ &= f(\hat{x})\{(1-p)^2(x-\hat{x})+p(1-p)[r(x,\hat{x})-r(\hat{x})]+p^2[r(\hat{x})+\pi(\hat{x},\hat{x},\theta)-\hat{x}-\theta]\}\\ &< 0.\end{aligned}$$

The inequality results from the fact that $x<\hat{x}$, $r(x,\hat{x})\leq r(\hat{x},\hat{x})=r(\hat{x})$ and $\pi(\hat{x},\hat{x},\theta)\leq\hat{x}+\theta-r(\hat{x})$.

When $x+\theta-\pi(z,z,\theta)\geq r(z)$ (overbids by a relatively small amount),

$$\begin{aligned}\frac{\partial V(x,\hat{x})}{\partial\hat{x}} &= f(\hat{x})\{(1-p)^2(x-\hat{x})+p(1-p)[r(x,\hat{x})-r(\hat{x})]+p^2(x-\hat{x})\}\\ &< 0\end{aligned}$$

Therefore, overbidding is not a profitable deviation.

Underbidding case: $x > \hat{x}$.

$$\begin{aligned}
\frac{\partial V(x,\hat{x})}{\partial \hat{x}} &= f(\hat{x})\{p^2[x+\theta-\pi(\hat{x},\hat{x},\theta)]+(1-p)p(x+\theta) \\
&\quad +p(1-p)\max\{r(\hat{x}),x\}+(1-p)^2x-\beta(\hat{x})-p^2[x+\theta-r^1(x,\hat{x})] \\
&\quad -p(1-p)[x+\theta-r(x,\hat{x})]-(1-p)^2[x-r^0(x,\hat{x})]\} \\
&= f(\hat{x})\{p(1-p)r(x,\hat{x})-2p(1-p)r(\hat{x})+[p+(1-p)^2][r^0(x,\hat{x})-\hat{x}] \\
&\quad +p(1-p)\max\{r(\hat{x}),x\}\}
\end{aligned}$$

When $r(\hat{x}) \geq x$,

$$\frac{\partial V(x,\hat{x})}{\partial \hat{x}} = f(\hat{x})\{p(1-p)[r(x,\hat{x})-r(\hat{x})]+[p+(1-p)^2][r^0(x,\hat{x})-\hat{x}]\} > 0$$

The inequality results from the fact that $r(x,\hat{x}) \geq r(\hat{x},\hat{x}) = r(\hat{x})$ and $r^0(x,\hat{x}) > \hat{x}$.

When $r(\hat{x}) < x$,

$$\frac{\partial V(x,\hat{x})}{\partial \hat{x}} = f(\hat{x})\{p(1-p)[r(x,\hat{x})-2r(\hat{x})+x]+[p+(1-p)^2][r^0(x,\hat{x})-\hat{x}]\} > 0 \ ,$$

because $r(x,\hat{x})-2r(\hat{x})+x = r(x,\hat{x})-r(\hat{x})+x-r(\hat{x}) > 0$ and $r^0(x,\hat{x}) > \hat{x}$.

Therefore, underbidding is not a profitable deviation either. ■

Lemma 3 *The ranking between the equilibrium bid functions for the first auction with and without resale is ambiguous.*

Proof. The equilibrium bid functions in the first auction under the cases without resale (NR) and with resale (R) are as follows:

$$\begin{aligned}
\beta^{NR}(x) &= x+p(\theta-1), \\
\beta^{R}(x) &= p^2(x+\theta)+(1-p)^2x+2p(1-p)r(x)-p^2\pi(x,x,\theta).
\end{aligned}$$

Denote $\beta_1^R(x)$ for the case $\theta < \hat{\theta}$ and $x < g(\theta)$, and $\beta_2^R(x)$ for all the other cases. We will compare $\beta^{NR}(x)$ and $\beta^R(x)$ in order.

Case 1: $\theta \geq \hat{\theta}$.

Since $\beta^{NR}(x)$ and $\beta_2^R(x)$ are linear, if they have one intersection, it must be unique; thus it suffices to check the low and high ends of these functions. At the high ends,

$$\beta^{NR}(1) = 1 + p(\theta - 1) > 1 = \beta_2^R(1).$$

When $p \leq 1 - \frac{1}{f(1)}$, $3 - p \geq \hat{\theta}$. If $\theta \in (3-p, +\infty)$, then $\beta^{NR}(0) > \beta_2^R(0)$; thus $\beta^{NR}(x) > \beta_2^R(x)$ for $x \in [0, 1]$.

If $\theta \in [\hat{\theta}, 3-p]$, then $\beta^{NR}(0) \leq \beta_2^R(0)$; thus there exists a unique $x^* \in [0, 1)$ such that $\beta^{NR}(x^*) = \beta_2^R(x^*)$.

When $p > 1 - \frac{1}{f(1)}$, $3 - p < \hat{\theta}$. For $\theta \in [\hat{\theta}, +\infty)$, $\beta^{NR}(x) > \beta_2^R(x)$ for $x \in [0, 1]$.

Case 2: $\theta < \hat{\theta}$.

In this case,

$$\beta^R(x) = \begin{cases} \beta_2^R(x), & x \geq g(\theta) \\ \beta_1^R(x), & x < g(\theta) \end{cases}$$

Note that when $x < g(\theta)$, $r(x) < 1$ and $\pi(x, x, \theta) > x + \theta - 1$. The first inequality follows immediately and the second inequality results from the fact that $\pi(x, x, \theta)$ is the contingent resale payoff for the first-item loser when she offers $r(x)$ in resale, and $x+\theta-1$ is the contingent resale payoff when she offers 1 instead. That she prefers to offer $r(x)$ indicates that $\pi(x, x, \theta) > x + \theta - 1$.

Therefore, $\beta_1^R(x) < \beta_2^R(x)$ for $x < g(\theta)$. For the first order comparison, we have shown earlier that $\beta_1'^R(x) > \beta_2'^R(x)$. However, it is generally impossible to compare $\beta_1'^R(x)$ with $\beta'^{NR}(x) = 1$ based only on explicit restrictions of parameters and the distribution $F(\cdot)$.

When $p \leq 1 - \frac{1}{f(1)}, 3 - p \geq \hat{\theta}$. We have $\beta^{NR}(0) \leq \beta_2^R(0)$ and there exists a unique $x^* \in [0, 1)$ such that $\beta^{NR}(x^*) = \beta_2^R(x^*)$.

If $x^* \geq g(\theta)$ and $\beta^{NR}(0) \in [\beta_1^R(0), \beta_2^R(0)]$, there exists at least one $\tilde{x} < g(\theta)$ such that $\beta^{NR}(\tilde{x}) = \beta_1^R(\tilde{x})$. Or equivalently, there exist at least two intersections between $\beta^{NR}$and β^R.

If $x^* \geq g(\theta)$ and $\beta^{NR}(0) \in (0, \beta_1^R(0))$, there exists at least one intersection, which is x^*.

If $x^* < g(\theta)$ and $\beta^{NR}(0) \in [\beta_1^R(0), \beta_2^R(0)]$, there is ambiguity.

If $x^* < g(\theta)$ and $\beta^{NR}(0) \in (0, \beta_1^R(0))$, there exists at least one $\tilde{x} < g(\theta)$ such that $\beta^{NR}(\tilde{x}) = \beta_1^R(\tilde{x})$.

When $p > 1 - \frac{1}{f(1)}, 3 - p < \hat{\theta}$.

If $\theta \in [3-p, \hat{\theta})$,then $\beta^{NR}(0) \geq \beta_2^R(0)$, $\beta^{NR}(x) > \beta^R(x)$ for $x \in [0,1]$.

If $\theta \in (1, 3-p)$,then $\beta^{NR}(0) < \beta_2^R(0)$ and the cases are similar to those under $p \leq 1 - \frac{1}{f(1)}$.

It is thus clear that there is no general ranking between the bidding functions for the first auction with and without resale. ■

Proof of Proposition 3: $DR = R^{NR} - R^R$ is a piecewise function:

$$DR = \begin{cases} p(1-p)\int_0^1 (\theta + 2x - 3) f_{(2)}(x)dx, \text{ when } \theta \geq \hat{\theta} \\ p(1-p)\left[\int_0^1 (\theta + 2x - 3) f_{(2)}(x)dx + \int_0^{g(\theta)} (2 - 2r(x)) f_{(2)}(x)dx\right], \text{when } \theta < \hat{\theta} \end{cases}$$

Clearly, when $\theta \geq \hat{\theta}$, $DR = p(1-p)\left[(\theta - 3) + 2EX_{(2)}\right]$; thus $R^{NR} \geq R^R$ if $EX_{(2)} \geq (3-\theta)/2$, and $R^{NR} < R^R$ if $EX_{(2)} < (3-\theta)/2$.

When $\theta < \hat{\theta}$, we consider $\frac{\partial DR}{\partial \theta}$. Note that when $\theta = 1$, there is no synergy so there is no resale on equilibrium path, and the bidding functions for both auctions should be the same as in the no resale benchmark. Therefore, when $\theta = 1$, $DR = 0$. If we can identify conditions under which $\frac{\partial DR}{\partial \theta}$ is positive, then $DR > 0$ for all $\theta < \hat{\theta}$.

When $x = g(\theta)$, since $r(x) = 1$, we have

$$\frac{\partial DR}{\partial \theta} = p(1-p)\left[1 - 2\int_0^{g(\theta)} f_{(2)}(x)\frac{\partial r(x)}{\partial \theta}dx\right],$$

where $\frac{\partial r(x)}{\partial \theta} = \frac{1}{2 - \frac{f'(r(x))[F(r(x))-F(x)]}{f^2(r(x))}}$. The denominator can also be expressed as $1 + 1 - \frac{f'(r(x))[F(r(x))-F(x)]}{f^2(r(x))}$, where $1 - \frac{f'(r(x))[F(r(x))-F(x)]}{f^2(r(x))}$ is the derivative of $\frac{F(r)-F(x)}{f(r)}$ w.r.t r evaluated at $r(x)$. By the regularity assumption, this term should be positive; thus $\frac{\partial r(x)}{\partial \theta} \in (0,1)$.

If $F_{(2)}[g(\theta)] \leq \frac{1}{2}$, or equivalently if $\theta \geq g^{-1}[F_{(2)}^{-1}(\frac{1}{2})]$ (since $g(\theta)$ decreases in θ), $\int_0^{g(\theta)} f_{(2)}(x)\frac{\partial r(x)}{\partial \theta}dx < \int_0^{g(\theta)} f_{(2)}(x)dx = F_{(2)}[g(\theta)] \leq \frac{1}{2}$; thus $\frac{\partial DR}{\partial \theta} > 0$.

If $f(x)$ is a decreasing function, the term $\frac{f'(r(x))[F(r(x))-F(x)]}{f^2(r(x))} \leq 0$; thus $\frac{\partial r(x)}{\partial \theta} \leq \frac{1}{2}$ and $\int_0^{g(\theta)} f_{(2)}(x)\frac{\partial r(x)}{\partial \theta}dx \leq \frac{1}{2}\int_0^{g(\theta)} f_{(2)}(x)dx < \frac{1}{2}\int_0^1 f_{(2)}(x)dx = \frac{1}{2}$. In this case, we also have $\frac{\partial DR}{\partial \theta} > 0$. ■

Proof of Proposition 4: From the definition of $P^R_{\exp}(x)$, we know that since $P(z < x, \beta^R(z) > \max\{x, r(z)\}) < P(z < x, \beta^R(z) > x)$, we have $P^R_{\exp}(x) < (1-p) \cdot P(z < x, \beta^R(z) > x)$. If $\beta^{NR}(z) \geq \beta^R(z)$ for $z \in [0, x)$, $P(z < x, \beta^R(z) > x) < P(z < x, \beta^{NR}(z) > x)$, hence $P^R_{\exp}(x) < P^{NR}_{\exp}(x)$. As shown in the proof of Lemma 3, a sufficient condition for $\beta^{NR}(\cdot) > \beta^R(\cdot)$ is $\theta + p \geq 3$. Thus, when θ is sufficiently large, $P^{NR}_{\exp}(x) > P^R_{\exp}(x)$ for all $x \in [0, 1]$, which also implies that ex ante, the probability of exposure is higher when resale is not allowed. ■

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