Value Function of Differential Games without Isaacs Conditions. An Approach with Non-Anticipative Mixed Strategies

Rainer Buckdahn[1,3]; Juan Li[2,3*]; Marc Quincampoix[1]

[1] Laboratoire de Mathématiques, CNRS-UMR 6205, Université de Bretagne Occidentale, 6, avenue Victor Le Gorgeu, CS 93837, 29238 Brest cedex 3, France.

[2] School of Mathematics and Statistics, Shandong University at Weihai, Weihai 264209, P.R. China;

[3] Institute for Advanced Study, Shandong University, Jinan 250100, P.R. China

E-mails: rainer.buckdahn@univ-brest.fr; marc.quincampoix@univ-brest.fr; juanli@sdu.edu.cn.

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Abstract In the present paper we investigate the problem of the existence of a value for differential games without Isaacs condition. For this we introduce a suitable concept of mixed strategies along a partition of the time interval, which are associated with classical nonanticipative strategies (with delay). Imposing on the underlying controls for both players a conditional independence property, we obtain the existence of the value in mixed strategies as the limit of the lower as well as of the upper value functions along a sequence of partitions which mesh tends to zero. Moreover, we characterize this value in mixed strategies as the unique viscosity solution of the corresponding Hamilton-Jacobi-Isaacs equation.

1 Introduction

In the present work we consider 2-person zero-sum differential games which dynamics is defined through the doubly controlled differential equation

$$\frac{d}{ds}X_s = f(s, X_s, u_s, v_s),\ s \in [t, T], \tag{1.1}$$

and which pay-off functional is described by

$$J := g(X_T). \tag{1.2}$$

The initial data (t, x) are in $[0, T] \times R^d$. Given two compact metric control state spaces U and V, the both players use control processes $u = (u_s)$ and $v = (v_s)$ with values in U and V, respectively. They control the state space process $X = (X_s)$ which takes its values in R^d; its dynamics is driven by a bounded, continuous function $f = (f(t, x, u, v)) : [0, T] \times R^d \times U \times V \to R^d$ which is Lipschitz in x, uniformly with respect to (u, v), and the terminal pay-off function $g : R^d \to R$ is supposed to be bounded and Lipschitz. Under these assumptions on f the above equation has a unique solution $X = (X_s)_{s\in[t,T]}$, denoted by $X^{t,xu,v}$ in order to indicate the dependence on the initial data (t, x) and the control processes $u = (u_s)$ and $v = (v_s)$ chosen by player 1 and 2, respectively; and for the associated pay-off functional we write $J(t, x; u, v)$. While the objective of the first player consists in maximizing the pay-off at terminal time T, the second player's objective is to minimize it.

One important issue in the theory of 2-person zero-sum differential games is the study of conditions under which the *value* of the game exists, i.e., under which the lower and the upper value functions of the game coincide. Indeed, with an appropriate concept of strategies, which will be introduced in Section 2, two

*Juan Li is the corresponding author.

value functions can be introduced, the lower and the upper one. For the case of a deterministic differential game with dynamics (1.1) and pay-off (1.2) the lower value function $V : [0,T] \times R^d$ and the upper one $U : [0,T] \times R^d$ are defined as follows:

$$V(t,x) = \sup_\alpha \inf_\beta J(t,x,\alpha,\beta), \quad U(t,x) = \inf_\beta \sup_\alpha J(t,x,\alpha,\beta), \quad (t,x) \in [0,T] \times R^d, \tag{1.3}$$

where α runs the set of admissible strategies for the first player, and β those for the second one. Given such a couple of admissible strategies (α, β), we define the associated pay-off functional $J(t,x,\alpha,\beta)$ through the unique couple of controls (u,v) such that $\alpha(v) = u$ and $\beta(u) = v : J(t,x,\alpha,\beta) := J(t,x,u,v)$.

In the literature, since the pioneering works of Isaacs, there have been many works showing the existence of the value of the game, this means the equality between the lower and the upper value functions, under the so-called Isaacs condition saying that, for all $(t,x,p) \in [0,T] \times R^d \times R^d$,

$$\sup_{u\in U} \inf_{v \in V} f(t,x,u,v)p = \inf_{v\in V} \sup_{u\in U} f(t,x,u,v)p. \tag{1.4}$$

Moreover, under this condition (1.4) the value function $V(= U)$ solves a partial differential equation, the so-called Hamilton-Jacobi-Isaacs equation. Such an existence result for the value was obtained in [13] in the context of nonanticipative Varaiya-Roxin-Elliot-Kalton strategies, see [12], [19] and [21], and also in [2], [7] and [18], but here for differential games with constraints. As concerns the context of positional strategies, we refer to [16] for similar results.

For 2-person zero-sum stochastic differential games the existence of a value was obtained in [14] and later revisited and generalized in [6]. We also refer the reader to [5] and the references therein for an overview and a more complete description of these approaches.

Our main goal in the present paper is to investigate the problem of the existence of a value without Isaacs condition. Having other approaches in the classical theory of differential games in mind, it is not surprising that we need a proper, suitable notion of mixed strategies. This proper notion of mixed strategies related with a suitable randomization allows to show that the lower and the upper value functions *defined in mixed strategies* coincide. Moreover, we prove that the value in mixed strategies $V = (V(t,x) = U(t,x))$ solves in viscosity sense the Hamilton-Jacobi-Isaacs equation

$$\begin{array}{rcll} \dfrac{\partial}{\partial t} V(t,x) + H(t, V(t,x), \nabla_x V(t,x)) & = & 0, & (t,x) \in [0,T] \times R^d, \\ V(T,x) & = & g(x), & x \in R^d, \end{array} \tag{1.5}$$

which Hamiltonian is given by

$$H(t,x,p) := \inf_{\nu \in \Delta V} \sup_{\mu \in \Delta V} \int_V \int_U f(t,x,u,v)\mu(du)\nu(dv)p, \quad (t,x,p) \in [0,T] \times R^d \times R^d. \tag{1.6}$$

Here ΔU and ΔV denote the set of probability measures on the set U and V (equipped with the Borel σ-field), respectively. It is worth pointing out that the supremum and the infimum in (1.6) commute due to the classical minmax theorem. This commutation between the supremum and the infimum in (1.6) constitutes also the key in the proof of the existence of the value in mixed strategies; it can be regarded as an automatically satisfied Isaacs condition concerning ΔU and ΔV interpreted as control state spaces. Having this in mind one could immediately define mixed strategies as nonanticipative strategies with delay for controls taking their values in ΔU and ΔV, respectively. This would lead to the same value of the game, given by (1.5).

But proceeding like that would mean to use relaxed controls. However, being interested in strong controls, i.e., controls taking their values in the given control state space U and V, respectively, we define controls and strategies, where the randomness–necessary for defining the concept of mixed strategies–appears in the choices of the players and not in the values of the controls. In this sense our work can be considered as an extension of the famous Kuhn Theorem for repeated games (cf [17] and also [1]) to the context of deterministic differential games.

To the best of our knowledge, the existence of the value for differential games without Isaacs condition was only investigated in the case of positional strategies in [16], but with different techniques. Moreover, the nonanticipative strategies used in [2, 7, 18] do not allow to write the game in a *normal form* (i.e., to play a strategy of one player against a strategy of the other one) and, consequently, they are not appropriate for the definition of mixed strategies. Here in our work we use the concept of nonanticipativity with delays (see [4, 5] and [8]) and we define a corresponding notion of mixed strategies.

Let us explain the organization of the paper and link it with some explanation concerning our approach: *Section 2* is devoted to some preliminaries. We introduce there, in particular, the underlying filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_j)_{j\geq 1}, P)$ which we use for the randomization of the controls and the strategies. Given an arbitrary partition Π of the interval $[0, T]$, we introduce the admissible controls for both players along this partition Π and the corresponding nonanticipative strategies with delay (for short NAD-strategies). The specificity of the choice of our admissible controls along the partition $\Pi = \{0 = t_0 < \cdots < t_n = T\}$ consists in the fact that, given the available information $\mathcal{F}_i$ at time t_i, the admissible control processes for player 1 restricted to the time interval $[t_i, t_{i+1})$ are independent of those for player 2. This conditional independence of the control processes on subintervals defined by the partition Π turns out to be the crucial element in our approach. We show that, along the partition Π, for every couple of NAD strategies α, β, there exists a unique couple of admissible controls u, v of player 1 and 2, respectively, such that $\alpha(v) = u$ and $\beta(u) = v$. This allows to give a sense to the pay-off functional $J(t, x; \alpha, \beta)$. Since the admissible controls are random, also the pay-off functionals are random, and so are, a priori, V^Π and U^Π, the lower and the upper value functions along the partition Π. In *Section 3* we show that V^Π and U^Π satisfy along the partition Π the dynamic programming principle. This principle allows to prove with the help of a backward iteration that V^Π and U^Π are deterministic. For this a key result is that V^Π and U^Π are invariant with respect to a certain class of bijective transformations $\tau : \Omega \to \Omega$ which law is equivalent to the underlying probability measure P, combined with a statement saying that any random variable with such an invariance property has to coincide P-almost surely with a constant. The proof extends an idea coming from [6], where it was developed for a Brownian framework. Furthermore, the fact that V^Π and U^Π are deterministic, allows to prove that

$$V^\Pi(t, x) = \sup_\alpha \inf_\beta E[J(t, x, \alpha, \beta)], \quad U^\Pi(t, x) = \inf_\beta \sup_\alpha E[J(t, x, \alpha, \beta)], \quad (t, x) \in [0, T] \times R^d, \tag{1.7}$$

where α runs the set of NAD-strategies along Π for the first player, and β those for the second player. This combined with standard estimates yields that V^Π and U^Π are jointly Lipschitz in (t, x), with a Lipschitz constant which does not depend on the partition Π. From there we deduce in *Section 4* that the lower and the upper value functions V^Π and U^Π converge uniformly on compacts to the unique solution of the Hamilton-Jacobi-Isaacs equation (1.5), as the maximal distance $|\Pi|$ between two neighbouring points of the partition Π tends to zero. Consequently, the limits of V^Π and U^Π, $V := \lim_{|\Pi|\to 0} V^\Pi$ and $U := \lim_{|\Pi|\to 0} U^\Pi$ exist and coincide: $V = U$ is the value in mixed strategies of the game.

2 Preliminaries

Let $\lambda_2(dx) = dx$ denote the two-dimensional Borel measure defined on the quadrate $[0, 1]^2 \subset R^2$ endowed with the Borel field $\mathcal{B}([0, 1]^2)$. Denoting by $\mathbb{N}$ the set of all positive integers we introduce our underlying probability space $(\Omega, \mathcal{F}, P)$ as product space

$$(\Omega, \mathcal{F}, P) := \left(([0, 1]^2)^{\mathbb{N}}, \mathcal{B}([0, 1]^2)^{\otimes\mathbb{N}}, \lambda_2^{\otimes\mathbb{N}}\right),$$

i.e., $\Omega = \{\omega = (\omega_j)_{j\geq 1} \,|\, \omega_j \in [0, 1]^2,\ j \geq 1\}$ is the space of all $[0, 1]^2$-valued sequences, endowed with the product Borel-field $\mathcal{F} = \mathcal{B}([0, 1]^2)^{\otimes\mathbb{N}}$ and the product probability measure $P = \lambda_2^{\otimes\mathbb{N}}$. Moreover, letting $\zeta_j = (\zeta_{j,1}, \zeta_{j,2}) : \Omega \longrightarrow [0, 1]^2$ denote the coordinate mapping on Ω :

$$\zeta_j(\omega) = (\zeta_{j,1}(\omega), \zeta_{j,2}(\omega)) = (\omega_{j,1}, \omega_{j,2}), \quad \omega = ((\omega_{j,1}, \omega_{j,2}))_{j\geq 1} \in \Omega,$$

we have that $\mathcal{F}$ is the smallest σ-field over Ω, with respect to which all coordinate mappings ζ_j, $j \geq 1$, are measurable. In what follows we will also need the σ-fields $\mathcal{G}_j := \zeta_{j,1}^{-1}(\mathcal{B}([0, 1])) = \{\{\zeta_{j,1} \in \Gamma\} \,|\, \Gamma \in \mathcal{B}([0, 1])\}$

and $\mathcal{H}_j := \zeta_{j,2}^{-1}(\mathcal{B}([0,1]))$ generated by $\zeta_{j,1}$ and $\zeta_{j,2}$, respectively, $j \geq 1$, as well as the σ-field

$$\mathcal{F}_j := \sigma\{\cup_{i\leq j}(\mathcal{G}_i \cup \mathcal{H}_i)\} = \sigma\{\zeta_i,\, 1 \leq i \leq j\},$$

generated by the coordinate mappings $\zeta_1, \dots, \zeta_j$, for $j \geq 1$. We remark that, for all $j \geq 1$, the σ-fields $\mathcal{G}_j, \mathcal{H}_j$ and $\mathcal{F}_{j-1}$ are independent. Moreover, $\mathbb{F} = (\mathcal{F}_j)_{j\geq 1}$ forms a time-discrete filtration, and $\mathcal{F} = \vee_{j\geq 1}\mathcal{F}_j$ $(:= \sigma\{\cup_{j\geq 1}\mathcal{F}_j\}\,)$. We also recall that a random time $\tau : \Omega \to \{0, 1, 2, \dots\}$ is an $\mathbb{F}$-stopping time, if $\{\tau = j\} \in \mathcal{F}_j$, $j \geq 0$.

Let U and V be compact metric spaces; by ΔU and ΔV we denote the space of probability measures on $(U, \mathcal{B}(U))$ and on $(V, \mathcal{B}(V))$, respectively. The fact that all probability measure $\mu \in \Delta U$ ($\nu \in \Delta V$, resp.) coincides with the law of a suitable U-valued random variable (V-valued random variable, resp.) defined over the space $([0,T], \mathcal{B}([0,T]))$ endowed with the one-dimensional Borel measure (it's an elementary consequence of Skorohod's Representation Theorem, refer to pp 70 in [3]), implies, in particular, that

$$\Delta U = \{P_\xi \,|\, \xi \in L^0(\Omega, \mathcal{G}_j, P; U)\}^1,\ \Delta V = \{P_\xi \,|\, \xi \in L^0(\Omega, \mathcal{H}_j, P; U)\},\ j \geq 1.$$

In order to introduce the dynamics of the controlled system we want to investigate, we shall begin with defining the admissible controls for the both players. We define them along a partition Π of the time interval $[0,T]$.

Definition 2.1. *(admissible control) A process $u \in L^0_{\mathcal{F}}(0,T;U)^2$ is said to be an admissible control for Player 1 along a partition $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ of the interval $[0,T]$, if, for any j $(1 \leq j \leq n)$, its restriction $u_{|[t_{j-1},t_j)}$ to the interval $[t_{j-1}, t_j)$ is of the form $u_{|[t_{j-1},t_j)} = \sum_{k\geq 1} I_{\Gamma_{j,k}} u^{j,k}$, where $(\Gamma_{j,k})_{k\geq 1} \subset \mathcal{F}_{j-1}$ is a partition of Ω and $(u^{j,k})_{k\geq 1} \subset L^0_{\mathcal{G}_j}(t_{j-1}, t_j; U)$. If this is the case, we write $u \in \mathcal{U}^{\Pi}_{0,T}$.*

Similarly, we say that $v \in L^0_{\mathcal{F}}(0,T;V)$ is an admissible control along the partition Π for Player 2, if, for any j $(1 \leq j \leq n)$, its restriction $v_{|[t_{j-1},t_j)}$ to the interval $[t_{j-1}, t_j)$ is of the form $v_{|[t_{j-1},t_j)} = \sum_{k\geq 1} I_{\Gamma_{j,k}} v^{j,k}$, where $(\Gamma_{j,k})_{k\geq 1} \subset \mathcal{F}_{j-1}$ is a partition of Ω and $(v^{j,k})_{k\geq 1} \subset L^0_{\mathcal{H}_j}(t_{j-1}, t_j; V)$. If this is the case, we write $v \in \mathcal{V}^{\Pi}_{0,T}$.

Finally, for $0 \leq t \leq t_l \in \Pi$, we put

$$\mathcal{U}^{\Pi}_{t,t_l} := \{(u_s)_{s\in[t,t_l]} | u \in \mathcal{U}^{\Pi}_{0,T}\} \text{ and } \mathcal{V}^{\Pi}_{t,t_l} := \{(v_s)_{s\in[t,t_l]} | v \in \mathcal{V}^{\Pi}_{0,T}\}.$$

Let us describe now the dynamics of our differential game along a partition Π of the interval $[0,T]$. For this we consider a bounded continuous function $f = (f(t,x,u,v)) : [0,T] \times R^d \times U \times V \longrightarrow R^d$ which is supposed to be Lipschitz in x, uniformly with respect to (t,u,v). Given initial data $(t,x) \in [0,T] \times R^d$ and two controls $u \in \mathcal{U}^{\Pi}_{t,T}$ and $v \in \mathcal{V}^{\Pi}_{t,T}$, we define the continuous process $X^{t,x,u,v} = (X^{t,x,u,v}_s)_{s\in[t,T]}$ as the unique solution of the following pathwise differential equation:

$$X^{t,x,u,v}_s = x + \int_t^s f(r, X^{t,x,u,v}_r, u_r, v_r)\mathrm{d}r, \quad s \in [t,T], \quad (u,v) \in \mathcal{U}^{\Pi}_{t,T} \times \mathcal{V}^{\Pi}_{t,T}. \tag{2.1}$$

We remark that standard estimates show

Lemma 2.1. *For a suitable real constant C independent of the partition Π we have, for all $(u,v) \in \mathcal{U}^{\Pi}_{t,T} \times \mathcal{V}^{\Pi}_{t,T}$, for all $(t,x), (t',x') \in [0,T] \times R^d$ and all $s \in [t \vee t', T]$,*

$$\begin{array}{l} \text{(i) } |X^{t,x,u,v}_s - x| \leq CT, \\ \text{(ii) } |X^{t,x,u,v}_s - X^{t',x',u,v}_s| \leq C(|t-t'| + |x-x'|). \end{array} \tag{2.2}$$

Let $g : R^d \to R$ be a bounded Lipschitz function. For a game over the time interval $[t,T]$ along the partition $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$, with $0 \leq i \leq n-1$ such that $t_i \leq t < t_{i+1}$, we consider the payoff functional $E[g(X^{t,x,u,v}_T)|\mathcal{F}_i]$ which Player 1 tries to maximize through the control $u \in \mathcal{U}^{\Pi}_{t,T}$ and Player 2 tries to minimize through his choice of $v \in \mathcal{V}^{\Pi}_{t,T}$. However, in order to guarantee the existence of a value of the game, we consider a game in which both players use non-anticipative strategies with delay (NAD-strategies).

[1] As usual, $L^0(\Omega, \mathcal{G}_j, P; U)\}$ denotes the space of all U-valued random variables defined on $(\Omega, \mathcal{G}_j, P)$.

[2] $L^0_{\mathcal{F}}(0,T;U)$ denotes the space of all measurable U-valued processus $u = (u_t)_{t\in[0,T]}$ such that u_t is $\mathcal{F}$-measurable, for all $t \in [0,T]$.

Definition 2.2. *(NAD-strategies) Let $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ be a partition of the interval $[0,T]$ and $0 \le t \le t_l \in \Pi$. We say that $\beta : \mathcal{U}^{\Pi}_{t,t_l} \longrightarrow \mathcal{V}^{\Pi}_{t,T}$ is an NAD-strategy for Player 2 over the time interval $[t, t_l]$ along the partition Π, if for all $\mathbb{F}$-stopping time $\tau : \Omega \to \{0, 1, \ldots, n-1\}$ and all controls $u, u' \in \mathcal{U}^{\Pi}_{t,t_l}$ with $u = u'$, $dsdP$-a.s. on $[[t, t_\tau]]$[3], it holds $\beta(u) = \beta(u')$, $dsdP$-a.s. on $[[t, t_{\tau+1}]]$. The set of all NAD-strategy for Player 2 over $[t, t_l]$ along Π is denoted by $\mathcal{B}^{\Pi}_{t,t_l}$.*

In an obvious symmetric way we also introduce for Player 1 the set of all NAD-strategies over the interval $[t, t_l]$ along Π, and we denote it by $\mathcal{A}^{\Pi}_{t,t_l}$.

The following result is crucial; it permits to associate couples of NAD-strategies with couples of admissible controls.

Lemma 2.2. *For all couple of NAD strategies $(\alpha, \beta) \in \mathcal{A}^{\Pi}_{t,t_l} \times \mathcal{B}^{\Pi}_{t,t_l}$, there exists unique couple of admissible controls $(u, v) \in \mathcal{U}^{\Pi}_{t,t_l} \times \mathcal{V}^{\Pi}_{t,t_l}$ such that $\alpha(v) = u$, $\beta(u) = v$, $dsdP$-a.s. on $[t, t_l] \times \Omega$.*

Although such a result is well-known for deterministic and stochastic differential games (see, for instance, [4] and [5]), we want to sketch here the proof for the convenience of the reader, because the context we study differs a bit from that of [4] and [5].

Proof. Let $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ be a partition of the interval $[0,T]$, $0 \le t_i \le t < t_{i+1} \le t_l \in \Pi$, and $(\alpha, \beta) \in \mathcal{A}^{\Pi}_{t,t_l} \times \mathcal{B}^{\Pi}_{t,t_l}$. Then, due to the definition of NAD strategies, $\alpha(v), \beta(u)$ restricted to the interval $[t, t_{i+1}]$ depend only on the restrictions of the controls $v \in \mathcal{V}^{\Pi}_{t,t_l}$ and $u \in \mathcal{U}^{\Pi}_{t,t_l}$ to the interval $[t, t_i]$. But since this interval is empty or at most a singleton (and, hence, of Lebesgue measure zero), $\alpha(v), \beta(u)$ restricted to the interval $[t, t_{i+1}]$ don't depend on v and u. Consequently, given arbitrary $u^0 \in \mathcal{U}^{\Pi}_{t,t_l}, v^0 \in \mathcal{V}^{\Pi}_{t,t_l}$, we put $u^1 := \alpha(v^0), v^1 := \beta(u^0)$, and we have

$$\alpha(v^1) = u^1, \quad \beta(u^1) = v^1, \text{ on } [t, t_{i+1}].$$

Supposing that we have constructed $(u^{j-1}, v^{j-1}) \in \mathcal{U}^{\Pi}_{t,t_l} \times \mathcal{V}^{\Pi}_{t,t_l}$ such that $\alpha(v^{j-1}) = u^{j-1}$ and $\beta(u^{j-1}) = v^{j-1}$, $dsdP$-a.s. on $[t, t_{i+j-1}]$, we put $u^j := \beta(v^{j-1})$, $v^j := \alpha(u^{j-1})$. Then, obviously, $(u^j, v^j) \in \mathcal{U}^{\Pi}_{t,t_l} \times \mathcal{V}^{\Pi}_{t,t_l}$, $(u^j, v^j) = (u^{j-1}, v^{j-1})$, $dsdP$-a.s. on $[t, t_{i+j-1}]$, and because of the NAD property of the strategies α, β we have $u^j = \beta(v^j)$, $v^j = \alpha(u^j)$, $dsdP$-a.s. on $[t, t_{i+j}]$. By iterating the argument up to $j = l - i$ we obtain the assertion of the lemma. ∎

Remark 2.1. *Given a couple of NAD strategies $(\alpha, \beta) \in \mathcal{A}^{\Pi}_{t,t_l} \times \mathcal{B}^{\Pi}_{t,t_l}$ the above Lemma 2.2 allows to define the dynamics $X^{t,x,\alpha,\beta} = (X^{t,x,\alpha,\beta}_s)_{s \in [t,t_l]}$ along the partition Π over the interval $[t, t_l]$ ($t_l \in \Pi$) through that of the couple of admissible controls $(u, v) \in \mathcal{U}^{\Pi}_{t,t_l} \times \mathcal{V}^{\Pi}_{t,t_l}$ associated with by the relation $\alpha(v) = u$, $\beta(u) = v$, $dsdP$-a.s. on $[t, t_l] \times \Omega$.*

After the above preliminary discussion we can now introduce the value functions of the game along a partition $\Pi = \{0 = t_0 < \cdots < t_n = T\}$ of the interval $[0,T]$. For the initial data $(t, x) \in [0,T] \times R^d$ we define the lower value function V and the upper value function U along a partition $\Pi = \{0 = t_0 < \cdots < t_n = T\}$ as follows:

$$\begin{array}{rcl} V^{\Pi}(t,x) & := & \mathrm{esssup}_{\alpha \in \mathcal{A}^{\Pi}_{t,t_l}} \mathrm{essinf}_{\beta \in \mathcal{B}^{\Pi}_{t,t_l}} E[g(X^{t,x,\alpha,\beta}_T) | \mathcal{F}_i], \\ U^{\Pi}(t,x) & := & \mathrm{essinf}_{\beta \in \mathcal{B}^{\Pi}_{t,t_l}} \mathrm{esssup}_{\alpha \in \mathcal{A}^{\Pi}_{t,t_l}} E[g(X^{t,x,\alpha,\beta}_T) | \mathcal{F}_i], \\ & & \text{for } \ t_i \le t < t_{i+1} < T \ \ (0 \le i \le n-1). \end{array} \tag{2.3}$$

We emphasize that, since the lower and the upper value functions are defined as a combination of essential supremum and essential infimum over an indexed family of uniformly bounded, $\mathcal{F}_i$-measurable random variables, also they themselves are a priori bounded, $\mathcal{F}_i$-measurable random variables (Recall the definition of the essential supremum and infimum, e.g., in Dunford and Schwartz [11], Dellacherie [10] or in the appendix of Karatzas and Shreve [15], where a detailed discussion is made.). However, in the next section we will show that the lower and the upper value functions are deterministic (The interested reader

[3]The stochastic interval $[[t, t_\tau]]$ is defined as $\{(s, \omega) \in [t,T] \times \Omega \,|\, t \le s \le t_{\tau(\omega)}\}$.

is also referred to [6], where a comparable approach, but in a completely different framework is done for stochastic differential games with Isaacs condition.)

We also remark that we have the following statement as an immediate consequence of Lemma 2.1 and the fact the the function g is bounded and Lipschitz:

Lemma 2.3. *Under our standard assumptions on the coefficients f and g we have that there is some constant L such that, for all $(t,x),(t',x')\in[0,T]\times R^d$ and for all partition Π,*

$$\begin{array}{l}\text{(i)}|V^{\Pi}(t,x)|\le L,\\ \text{(ii)}|V^{\Pi}(t,x)-V^{\Pi}(t,x')|\le L|x-x'|,\ \ P\text{-}a.s.\end{array}\tag{2.4}$$

3 Lower and upper value functions along a partition

The objective of this section is to study the properties of the above introduced lower and upper value functions along a partition $\Pi=\{0=t_0<t_1<\cdots<t_n=T\}$ of the interval $[0,T]$. More precisely, we first establish a dynamic programming principle (DPP) which on its part will allow to prove that the both value functions are deterministic.

Theorem 3.1. *(Dynamic Programming Principle) Let $\Pi=\{0=t_0<\cdots<t_n=T\}$ be an arbitrary partition of the interval $[0,T]$ and $(t,x)\in[0,T]\times R^d$. Then, for i,l such that $t_i\le t<t_{i+1}\le t_l$,*

$$\begin{array}{rcl}V^{\Pi}(t,x)&=&\mathrm{esssup}_{\alpha\in\mathcal{A}^{\Pi}_{t,t_l}}\mathrm{essinf}_{\beta\in\mathcal{B}^{\Pi}_{t,t_l}}E[V^{\Pi}(t_l,X^{t,x,\alpha,\beta}_{t_l})\mid\mathcal{F}_i],\\ U^{\Pi}(t,x)&=&\mathrm{essinf}_{\beta\in\mathcal{B}^{\Pi}_{t,t_l}}\mathrm{esssup}_{\alpha\in\mathcal{A}^{\Pi}_{t,t_l}}E[U^{\Pi}(t_l,X^{t,x,\alpha,\beta}_{t_l})\mid\mathcal{F}_i],\ \ P\text{-}a.s.\end{array}\tag{3.1}$$

For the proof which will be split in two lemmas, we will restrict to the lower value function along a partition; the proof for the upper value function along a partition uses a symmetric argument. Keeping the notations introduced in the above theorem we put

$$\widetilde{V}^{\Pi}_l(t,x)=\mathrm{esssup}_{\alpha\in\mathcal{A}^{\Pi}_{t,t_l}}\mathrm{essinf}_{\beta\in\mathcal{B}^{\Pi}_{t,t_l}}E[V^{\Pi}(t_l,X^{t,x,\alpha,\beta}_{t_l})\mid\mathcal{F}_i].\tag{3.2}$$

We remark that $\widetilde{V}^{\Pi}_l(t,x)$ is an $\mathcal{F}_i$-measurable random variable.

Lemma 3.1. *Under our standard assumptions we have $\widetilde{V}^{\Pi}_l(t,x)\le V^{\Pi}(t,x)$, P-a.s.*

Proof. Step 1. Let us fix arbitrarily $\varepsilon>0$. Then, there exists $\alpha^{\varepsilon}_1\in\mathcal{A}^{\Pi}_{t,t_l}$ such that

$$\widetilde{V}^{\Pi}_l(t,x)\le\mathrm{essinf}_{\beta_1\in\mathcal{B}^{\Pi}_{t,t_l}}E[V^{\Pi}(t_l,X^{t,x,\alpha^{\varepsilon}_1,\beta_1}_{t_l})\mid\mathcal{F}_i]+\varepsilon,\ \text{P-a.s.}\tag{3.3}$$

Indeed, setting $I_1(\alpha)=\mathrm{essinf}_{\beta_1\in\mathcal{B}^{\Pi}_{t,t_l}}E[V^{\Pi}(t_l,X^{t,x,\alpha,\beta_1}_{t_l})\mid\mathcal{F}_i]$, we know from the properties of the essential supremum over a family of random variables that there is a countable sequence $(\alpha_k)_{k\ge1}\subset\mathcal{A}^{\Pi}_{t,t_l}$ such that

$$\widetilde{V}^{\Pi}_l(t,x)=\mathrm{esssup}_{\alpha_1\in\mathcal{A}^{\Pi}_{t,t_l}}I_1(\alpha_1)=\sup_{k\ge1}I_1(\alpha_k),\ P\text{-a.s.}\tag{3.4}$$

Then, obviously, $\triangle_k:=\{\widetilde{V}^{\Pi}_l(t,x)\le I_1(\alpha_k)+\varepsilon\}\in\mathcal{F}_i,\ \ k\ge1$, and putting $\Gamma_k:=\Delta_k\setminus(\bigcup_{i<k}\Delta_i),\ k\ge1$, we define an $(\Omega,\mathcal{F}_i)$-partition, i.e., a partition of Ω, composed of elements of the σ-field $\mathcal{F}_i$. Let us now introduce the mapping $\alpha^{\varepsilon}_1:=\Sigma_{k\ge1}I_{\Gamma_k}\alpha_k(\cdot):\mathcal{V}^{\Pi}_{t,t_l}\to\mathcal{U}^{\Pi}_{t,t_l}$. It can be easily checked that such defined mapping belongs to $\mathcal{A}^{\Pi}_{t,t_l}$, and standard arguments (see, e.g., [6]) allow to show that

$$E[V^{\Pi}(t_l,X^{t,x,\alpha^{\varepsilon}_1,\beta_1}_{t_l})\mid\mathcal{F}_i]=\sum_{j\ge1}I_{\Gamma_j}E[V^{\Pi}(t_l,X^{t,x,\alpha_j,\beta_1}_{t_l})\mid\mathcal{F}_i],\ \text{for all }\beta_1\in\mathcal{B}^{\Pi}_{t,t_l}.$$

Therefore, again for all $\beta_1\in\mathcal{B}^{\Pi}_{t,t_l}$,

$$\begin{array}{l}\widetilde{V}^{\Pi}_l(t,x)\le\sum_{k\ge1}I_{\Gamma_k}I_1(\alpha_k)+\varepsilon\\ \le\displaystyle\sum_{k\ge1}I_{\Gamma_k}E[V^{\Pi}(t_l,X^{t,x,\alpha_k,\beta_1}_{t_l})\mid\mathcal{F}_i]+\varepsilon=E[V^{\Pi}(t_l,X^{t,x,\alpha^{\varepsilon}_1,\beta_1}_{t_l})\mid\mathcal{F}_i]+\varepsilon.\end{array}$$

Given now an arbitrary $\beta \in \mathcal{B}^{\Pi}_{t,T}$ and any $u_2 \in \mathcal{U}^{\Pi}_{t_l,T}$ we make the following particular choice of β_1:

$$\beta_1(u_1)(s) := \beta(u)(s),\ s \in [t, t_l],\ u_1 \in \mathcal{U}^{\Pi}_{t,t_l},$$

where

$$u(s) := \begin{cases} u_1(s), & s \in [t, t_l] \\ u_2(s), & s \in (t_l, T]. \end{cases}$$

Abbreviating, in what follows we will write for such a process composed over different intervals:

$$u = u_1 \oplus u_2,\ \beta_1(u_1) = \beta(u_1 \oplus u_2)_{|[t,t_l]}.$$

We observe that $\beta_1 \in \mathcal{B}^{\Pi}_{t,t_l}$, and as consequence of its nonanticipativity property, it is independent of the particular choice of u_2. Consequently,

$$\widetilde{V}^{\Pi}_l(t,x) \le \varepsilon + E[V^{\Pi}(t_l, X^{t,x,\alpha_1^{\varepsilon},\beta_1}_{t_l}) \mid \mathcal{F}_i],\ P\text{-a.s.}, \tag{3.5}$$

for our particular choice of β_1, since we have seen that this relation holds true for all $\beta_1 \in \mathcal{B}^{\Pi}_{t,t_l}$.

Step 2. Let us now continue by discussing the expression $V^{\Pi}(t_l, X^{t,x,\alpha_1^{\varepsilon},\beta_1}_{t_l})$ inside the above conditional expectation in (3.5). For this end we consider a partition $(O_j)_{j\ge 1}$ of R^d, composed of nonempty Borel sets, such that, for all $j \ge 1$, the maximal distance between two elements of $\mathcal{O}_j$ is less than or equal to ε. Let us fix in all $\mathcal{O}_j$ an arbitrary element y_j.

In analogy to Step 1 we see also here that, for every $j \ge 1$, there exists $\alpha_2^{\varepsilon,j} \in \mathcal{A}^{\Pi}_{t_l,T}$ such that

$$\begin{aligned} V^{\Pi}(t_l, y_j) &= \mathrm{esssup}_{\alpha_2 \in \mathcal{A}^{\Pi}_{t_l,T}} \mathrm{essinf}_{\beta_2 \in \mathcal{B}^{\Pi}_{t_l,T}} E[g(X^{t_l,y_j,\alpha_2,\beta_2}_T) \mid \mathcal{F}_l] \\ &\le \varepsilon + \mathrm{essinf}_{\beta_2 \in \mathcal{B}^{\Pi}_{t_l,T}} E[g(X^{t_l,y_j,\alpha_2^{\varepsilon,j},\beta_2}_T) \mid \mathcal{F}_l],\ P\text{-a.s.} \end{aligned}$$

In dependence of our $\beta \in \mathcal{B}^{\Pi}_{t,T}$ already chosen in the preceding Step 1 we want to make now a particular choice of $\beta_2 \in \mathcal{B}^{\Pi}_{t_l,T}$. For this end we notice that, since $(\alpha_1^{\varepsilon}, \beta_1) \in \mathcal{A}^{\Pi}_{t,t_l} \times \mathcal{B}^{\Pi}_{t,t_l}$, due to Lemma 2.2 there exists a unique couple $(u_1^{\varepsilon}, v_1^{\varepsilon}) \in \mathcal{U}^{\Pi}_{t,t_l} \times \mathcal{V}^{\Pi}_{t,t_l}$ such that $\alpha_1^{\varepsilon}(v_1^{\varepsilon}) = u_1^{\varepsilon}$, and $\beta_1(u_1^{\varepsilon}) = v_1^{\varepsilon}$. With the notations introduced in Step 1 we define now

$$\beta_2(u_2) := \beta(u_1^{\varepsilon} \oplus u_2)_{|[t_l,T]},\ u_2 \in \mathcal{U}^{\Pi}_{t_l,T}.$$

It is straight-forward to check that $\beta_1 \in \mathcal{B}^{\Pi}_{t_l,T}$, and, consequently,

$$V^{\Pi}(t_l, y_j) \le \varepsilon + E[g(X^{t_l,y_j,\alpha_2^{\varepsilon,j},\beta_2}_T) \mid \mathcal{F}_l],\ P\text{-a.s.} \tag{3.6}$$

Thus, from the Lipschitz continuity of $V^{\Pi}(t_l, .)$ (see Lemma 2.3) we obtain

$$\begin{aligned} V^{\Pi}(t_l, X^{t,x,\alpha_1^{\varepsilon},\beta_1}_{t_l}) &\le C\varepsilon + \sum_{j\ge 1} V^{\Pi}(t_l, y_j) I_{\{X^{t,x,\alpha_1^{\varepsilon},\beta_1}_{t_l} \in O_j\}} \\ &\le (C+1)\varepsilon + \sum_{j \ge 1} I_{\{X^{t,x,\alpha_1^{\varepsilon},\beta_1}_{t_l} \in O_j\}} E[g(X^{t_l,y_j,\alpha_2^{\varepsilon,j},\beta_2}_T) \mid \mathcal{F}_l]. \end{aligned} \tag{3.7}$$

Let us introduce now $\alpha_2^{\varepsilon} := \sum_{j\ge 1} I_{\{X^{t,x,\alpha_1^{\varepsilon},\beta_1}_{t_l} \in O_j\}} \alpha_2^{\varepsilon,j}$. It is easy to verify that α_2^{ε} belongs to $\mathcal{A}^{\Pi}_{t_l,T}$. On the other hand, for every $(\alpha_2^{\varepsilon,j}, \beta_2) \in \mathcal{A}^{\Pi}_{t_l,T} \times \mathcal{B}^{\Pi}_{t_l,T}$, there exists a unique couple $(u_2^{\varepsilon,j}, v_2^{\varepsilon,j}) \in \mathcal{U}^{\Pi}_{t_l,T} \times \mathcal{V}^{\Pi}_{t_l,T}$, such that

$$\alpha_2^{\varepsilon,j}(v_2^{\varepsilon,j}) = u_2^{\varepsilon,j},\quad \beta_2(u_2^{\varepsilon,j}) = v_2^{\varepsilon,j},$$

and with its help we define

$$(u_2^{\varepsilon}, v_2^{\varepsilon}) := \sum_{j\ge 1} I_{\{X^{t,x,\alpha_1^{\varepsilon},\beta_1}_{t_l} \in O_j\}} (u_2^{\varepsilon,j}, v_2^{\varepsilon,j}) \in \mathcal{U}^{\Pi}_{t_l,T} \times \mathcal{V}^{\Pi}_{t_l,T}.$$

Then, according to the definition of α_2^{ε} and the nonanticipativity of the elements of $\mathcal{A}^{\Pi}_{t_l,T}$ (see Definition 2.2 for nonanticipative strategies), since $v_2^{\varepsilon} = v_2^{\varepsilon,j}$ on $\{X^{t,x,\alpha_1^{\varepsilon},\beta_1}_{t_l} \in O_j\} \times [t_l, T]$, we also have

$$\alpha_2^{\varepsilon}(v_2^{\varepsilon}) = \alpha_2^{\varepsilon,j}(v_2^{\varepsilon}) = \alpha_2^{\varepsilon,j}(v_2^{\varepsilon,j}) = u_2^{\varepsilon,j} = u_2^{\varepsilon} \text{ on } \{X^{t,x,\alpha_1^{\varepsilon},\beta_1}_{t_l} \in O_j\} \times [t_l, T],\ j \ge 1.$$

Consequently, since $(\mathcal{O}_j)_{j\geq 1}$ forms a partition of R^d, it holds $\alpha_2^\varepsilon(v_2^\varepsilon) = u_2^\varepsilon$. Analogously, we obtain $\beta_2(u_2^\varepsilon) = v_2^\varepsilon$. Moreover, recalling that $(u_1^\varepsilon, v_1^\varepsilon) \in \mathcal{A}_{t,t_l}^\Pi \times \mathcal{B}_{t,t_l}^\Pi$ has been introduced such that $\alpha_1^\varepsilon(v_1^\varepsilon) = u_1^\varepsilon,\ \beta_1(u_1^\varepsilon) = v_1^\varepsilon$, we define a couple of controls $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}_{t,T}^\Pi \times \mathcal{V}_{t,T}^\Pi$ by putting $u^\varepsilon := u_1^\varepsilon \oplus u_2^\varepsilon$ and $v^\varepsilon := v_1^\varepsilon \oplus v_2^\varepsilon$. Furthermore, we introduce

$$\alpha^\varepsilon(v) := \alpha_1^\varepsilon(v_1) \oplus \alpha_2^\varepsilon(v_2), \text{ for } v_1 := v_{|[t,t_l]},\ v_2 := v_{|[t_l,T]},\ v \in \mathcal{V}_{t,T}^\Pi.$$

Then $\alpha^\varepsilon \in \mathcal{A}_{t,T}^\Pi$, and $\alpha^\varepsilon(v^\varepsilon) = \alpha_1^\varepsilon(v_1^\varepsilon) \oplus \alpha_2^\varepsilon(v_2^\varepsilon) = u_1^\varepsilon \oplus u_2^\varepsilon = u^\varepsilon$, and, on the other hand, recalling the definition of β_1 and β_2, we have

$$\beta(u^\varepsilon) = \beta(u_1^\varepsilon \oplus u_2^\varepsilon) \mid_{[t,t_l)} \oplus \beta(u_1^\varepsilon \oplus u_2^\varepsilon) \mid_{[t_l,T]} = \beta_1(u_1^\varepsilon) \oplus \beta_2(u_2^\varepsilon) = v_1^\varepsilon \oplus v_2^\varepsilon = v^\varepsilon.$$

This shows that $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}_{t,T}^\Pi \times \mathcal{V}_{t,T}^\Pi$ is the unique couple of controls which is associated with $(\alpha^\varepsilon, \beta) \in \mathcal{A}_{t,T}^\Pi \times \mathcal{B}_{t,T}^\Pi$. Hence,

$$X_T^{t_l, X_{t_l}^{t,x,\alpha_1^\varepsilon,\beta_1}, u_2^\varepsilon, v_2^\varepsilon} = X_T^{t_l, X_{t_l}^{t,x,u_1^\varepsilon,v_1^\varepsilon}, u_2^\varepsilon, v_2^\varepsilon} = X_T^{t,x,u^\varepsilon,v^\varepsilon} = X_T^{t,x,\alpha^\varepsilon,\beta}, \tag{3.8}$$

and, taking into account in addition the Lipschitz property of g , we get

$$\begin{aligned}
&\textstyle\sum_{j\geq 1} I_{\{X_{t_l}^{t,x,\alpha_1^\varepsilon,\beta_1} \in O_j\}} g(X_T^{t_l,y_j,\alpha_2^{\varepsilon,j},\beta_2}) \\
&= \textstyle\sum_{j\geq 1} I_{\{X_{t_l}^{t,x,\alpha_1^\varepsilon,\beta_1} \in O_j\}} g(X_T^{t_l,y_j,u_2^{\varepsilon,j},v_2^{\varepsilon,j}}) \\
&= \textstyle\sum_{j\geq 1} I_{\{X_{t_l}^{t,x,\alpha_1^\varepsilon,\beta_1} \in O_j\}} g(X_T^{t_l,y_j,u_2^\varepsilon,v_2^\varepsilon}) \\
&\leq g(X_T^{t_l, X_{t_l}^{t,x,\alpha_1^\varepsilon,\beta_1}, u_2^\varepsilon, v_2^\varepsilon}) + C\varepsilon = g(X_T^{t,x,\alpha^\varepsilon,\beta}) + C\varepsilon.
\end{aligned} \tag{3.9}$$

Consequently, from (3.7) and (3.9),

$$V^\Pi(t_l, X_{t_l}^{t,x,\alpha_1^\varepsilon,\beta_1}) \leq C\varepsilon + E[g(X_T^{t,x,\alpha^\varepsilon,\beta}) \mid \mathcal{F}_l],\ P\text{-a.s.} \tag{3.10}$$

Furthermore, from (3.5),

$$\widetilde{V}_l^\Pi(t,x) \leq \varepsilon + E[V^\Pi(t_l, X_{t_l}^{t,x,\alpha_1^\varepsilon,\beta_1}) \mid \mathcal{F}_i] \leq C\varepsilon + E[g(X_T^{t,x,\alpha^\varepsilon,\beta}) \mid \mathcal{F}_i],\ P\text{-a.s.} \tag{3.11}$$

This relation holds true for our arbitrarily chosen and, hence, for all $\beta \in \mathcal{B}_{t,T}^\Pi$. It follows that

$$\begin{aligned}
\widetilde{V}_l^\Pi(t,x) &\leq C\varepsilon + \operatorname{essinf}_{\beta\in\mathcal{B}_{t,t_l}^\Pi} E[g(X_T^{t,x,\alpha^\varepsilon,\beta}) \mid \mathcal{F}_i] \\
&\leq C\varepsilon + \operatorname{esssup}_{\alpha\in\mathcal{A}_{t,t_l}^\Pi} \operatorname{essinf}_{\beta\in\mathcal{B}_{t,t_l}^\Pi} E[g(X_T^{t,x,\alpha,\beta}) \mid \mathcal{F}_i] \\
&= C\varepsilon + V^\Pi(t,x), \quad P\text{-a.s.},
\end{aligned} \tag{3.12}$$

and the statement follows by letting ε tend to zero. ∎

Let us now come the converse statement to Lemma 3.1.

Lemma 3.2. *Under our standard assumptions we have* $\widetilde{V}_l^\Pi(t,x) \geq V^\Pi(t,x)$, *$P$-a.s.*

Proof. Because of the symmetry of some arguments to those in the proof of Lemma 3.1, this proof here will be kept shorter.

Let us fix any $\alpha \in \mathcal{A}_{t,T}^\Pi$ and, for some arbitrarily chosen $v_2 \in \mathcal{V}_{t_l,T}$, we put $\alpha_1(v_1) := \alpha(v_1 \oplus v_2) \mid_{|[t,t_l)}$, $v_1 \in \mathcal{V}_{t,t_l}$. Obviously, such defined mapping α_1 belongs to $\mathcal{A}_{t,t_l}^\Pi$ and, as a consequence of its nonanticipativity, it doesn't depend on the choice of v_2. Thus, from the definition of $\widetilde{V}_l^\Pi(t,x)$ it follows that

$$\widetilde{V}_l^\Pi(t,x) \geq \operatorname{essinf}_{\beta_1\in\mathcal{B}_{t,t_l}^\Pi} E[V^\Pi(t_l, X_{t_l}^{t,x,\alpha_1,\beta_1}) \mid \mathcal{F}_i],\ P\text{-a.s.}, \tag{3.13}$$

and, similarly to (3.5), we can show that, for any given $\varepsilon > 0$, there exists some $\beta_1^\varepsilon \in \mathcal{B}^\Pi_{t,t_l}$ such that

$$\widetilde{V}_l^\Pi(t,x) \geq E[V^\Pi(t_l, X_{t_l}^{t,x,\alpha_1,\beta_1^\varepsilon})|\mathcal{F}_i] - \varepsilon,\ P\text{-a.s.} \tag{3.14}$$

In analogy to the proof of Lemma 3.1 we discuss now the expression $V^\Pi(t_l, X_{t_l}^{t,x,\alpha_1,\beta_1^\varepsilon})$ inside the above conditional expectation in (3.14). For this we let $(u_1^\varepsilon, v_1^\varepsilon) \in \mathcal{U}^\Pi_{t,t_l} \times \mathcal{V}^\Pi_{t,t_l}$ denote the unique couple of admissible controls associated with $(\alpha_1, \beta_1^\varepsilon) \in \mathcal{A}^\Pi_{t,t_l} \times \mathcal{B}^\Pi_{t,t_l}$ by Lemma 2.2, i.e., such that $\alpha_1(v_1^\varepsilon) = u_1^\varepsilon$, $\beta_1^\varepsilon(u_1^\varepsilon) = v_1^\varepsilon$, and we introduce the NAD-strategy $\alpha_2^\varepsilon \in \mathcal{A}^\Pi_{t_l,T}$ by putting $\alpha_2^\varepsilon(v_2) := \alpha(v_1^\varepsilon \oplus v_2)\ |_{[t_l,T]}$, $v_2 \in \mathcal{V}^\Pi_{t_l,T}$. In order to construct an appropriate NAD-strategy $\beta_2^\varepsilon \in \mathcal{B}^\Pi_{t_l,T}$, we use the Borel partition $(\mathcal{O}_j)_{j\geq 1}$ and the sequence $y_j \in \mathcal{O}_j$, $j \geq 1$, introduced in the second step of the proof of Lemma 3.1. Choosing $\beta_2^{\varepsilon,j} \in \mathcal{B}^\Pi_{t_l,T}$ such that

$$\begin{array}{rcl} V^\Pi(t_l, y_j) & \geq & \operatorname{essinf}_{\beta_2 \in \mathcal{B}^\Pi_{t_l,T}} E[g(X_T^{t_l,y_j,\alpha_2^\varepsilon,\beta_2})|\mathcal{F}_l] \\ & \geq & E[g(X_T^{t_l,y_j,\alpha_2^\varepsilon,\beta_2^{\varepsilon,j}})|\mathcal{F}_l] - \varepsilon,\ P\text{-a.s.},\ j \geq 1, \end{array} \tag{3.15}$$

we define $\beta_2^\varepsilon \in \mathcal{B}^\Pi_{t_l,T}$ and $\beta^\varepsilon \in \mathcal{B}^\Pi_{t,T}$ by putting

$$\begin{array}{l} \beta_2^\varepsilon(u_2) := \sum_{j\geq 1} I_{\{X_{t_l}^{t,x,\alpha_1,\beta_1^\varepsilon} \in \mathcal{O}_j\}} \beta_2^{\varepsilon,j}(u_2),\ u_2 \in \mathcal{U}^\Pi_{t_l,T}, \\ \beta^\varepsilon(u) := \beta_1^\varepsilon(u_1) \oplus \beta_2^\varepsilon(u_2),\ \ u_1 := u_{|[t,t_l)},\ u_2 := u_{|[t_l,T]},\ u \in \mathcal{U}^\Pi_{t,T}. \end{array} \tag{3.16}$$

Consequently, taking into account the Lipschitz property of $V^\Pi(t_l,.)$ and using (3.15), we have similarly to Step 2 of the proof of the preceding Lemma 3.1

$$\begin{array}{rcl} \widetilde{V}_l^\Pi(t,x) & \geq & E[V^\Pi(t_l, X_{t_l}^{t,x,\alpha_1,\beta_1^\varepsilon})|\mathcal{F}_i] - \varepsilon \\ & \geq & \sum_{j\geq 1} E[I_{\{X_{t_l}^{t,x,\alpha_1,\beta_1^\varepsilon} \in \mathcal{O}_j\}} V^\Pi(t_l, y_j)|\mathcal{F}_i] - C\varepsilon \\ & \geq & \sum_{j\geq 1} E[I_{\{X_{t_l}^{t,x,\alpha_1,\beta_1^\varepsilon} \in \mathcal{O}_j\}} g(X_T^{t_l,y_j,\alpha_2^\varepsilon,\beta_2^{\varepsilon,j}})|\mathcal{F}_i] - C\varepsilon \\ & = & \sum_{j\geq 1} E[I_{\{X_{t_l}^{t,x,\alpha_1,\beta_1^\varepsilon} \in \mathcal{O}_j\}} g(X_T^{t_l,y_j,\alpha_2^\varepsilon,\beta_2^\varepsilon})|\mathcal{F}_i] - C\varepsilon \\ & \geq & E[g(X_T^{t_l,X_{t_l}^{t,x,\alpha_1,\beta_1^\varepsilon},\alpha_2^\varepsilon,\beta_2^\varepsilon})|\mathcal{F}_i] - C\varepsilon,\ P\text{-a.s.} \end{array} \tag{3.17}$$

Let $(u_2^\varepsilon, v_2^\varepsilon) \in \mathcal{U}^\Pi_{t_l,T} \times \mathcal{V}^\Pi_{t_l,T}$ be the unique couple of controls associated with $(\alpha_2^\varepsilon, \beta_2^\varepsilon) \in \mathcal{A}^\Pi_{t_l,T} \times \mathcal{B}^\Pi_{t_l,T}$ by Lemma 2.2. Then, it is straight-forward to show that the couple $(u^\varepsilon, v^\varepsilon) = (u_1^\varepsilon \oplus u_2^\varepsilon, v_1^\varepsilon \oplus v_2^\varepsilon) \in \mathcal{U}^\Pi_{t,T} \times \mathcal{V}^\Pi_{t,T}$ verifies $\alpha(v^\varepsilon) = u^\varepsilon$, $\beta^(u^\varepsilon) = v^\varepsilon$. Consequently,

$$\begin{array}{rcl} \widetilde{V}_l^\Pi(t,x) & \geq & E[g(X_T^{t_l,X_{t_l}^{t,x,\alpha_1,\beta_1^\varepsilon},\alpha_2^\varepsilon,\beta_2^\varepsilon})|\mathcal{F}_i] - C\varepsilon \\ & \geq & E[g(X_T^{t_l,X_{t_l}^{t,x,u_1^\varepsilon,v_1^\varepsilon},u_2^\varepsilon,v_2^\varepsilon})|\mathcal{F}_i] - C\varepsilon \\ & = & E[g(X_T^{t,x,u^\varepsilon,v^\varepsilon})|\mathcal{F}_i] - C\varepsilon \\ & = & E[g(X_T^{t,x,\alpha,\beta^\varepsilon})|\mathcal{F}_i] - C\varepsilon \\ & \geq & \operatorname{essinf}_{\beta \in \mathcal{B}^\Pi_{t,T}} E[g(X_T^{t,x,\alpha,\beta})|\mathcal{F}_i] - C\varepsilon,\ \ P\text{-a.s.} \end{array} \tag{3.18}$$

Taking into account the arbitrariness of $\alpha \in \mathcal{A}^\Pi_{t,T}$ and of $\varepsilon > 0$, we conclude

$$\widetilde{V}_l^\Pi(t,x) \geq V^\Pi(t,x),\ P\text{-a.s.}, \tag{3.19}$$

and the proof is complete. ∎

Obviously, the proof of the DPP for V^Π is an immediate consequence of the both preceding lemmas, and the proof for U^Π is analogous.

After having established the DPP for the lower and the upper value functions along a partition V^Π and U^Π, we can now show that these a priori random fields are deterministic. More precisely, we have the following

Theorem 3.2. *For all partition Π of the interval $[0,T]$, the lower value function along a partition V^Π as well as the upper one U^Π is deterministic, i.e., for all $(t,x)\in[0,T]\times R^d$,*

$$V^\Pi(t,x)=E\left[V^\Pi(t,x)\right] \text{ and } U^\Pi(t,x)=E\left[U^\Pi(t,x)\right],\ P\text{-a.s.}$$

Remark 3.1. *The above theorem allows to identify the lower and the upper value functions along a partition with their deterministic versions: $V^\Pi(t,x):=E\left[V^\Pi(t,x)\right]$ and $U^\Pi(t,x):=E\left[U^\Pi(t,x)\right]$, $(t,x)\in[0,T]\times R^d$.*

In view of the symmetry of the arguments we will restrict the proof to the case of the lower value function along a partition V^Π. We consider a partition of the interval $[0,T]$ of the form $\Pi=\{0=t_0<\dots<t_{n-1}<t_n=T\}$ and prove by backward iteration that the lower value function along a partition V^Π is deterministic. For this we note that, for the first step of the backward iteration, we have the following

Lemma 3.3. *For the above introduced partition Π and with the above notations we have that*

$$V^\Pi(t_{n-1},x)=esssup_{\alpha\epsilon\mathcal{A}^\Pi_{t_{n-1},t_n}}essinf_{\beta\epsilon\mathcal{B}^\Pi_{t_{n-1},t_n}}E[g(X^{t_{n-1},x,\alpha,\beta}_{t_n})|\mathcal{F}_{n-1}]$$

is deterministic, i.e., $V^\Pi(t_{n-1},x)=E\left[V^\Pi(t_{n-1},x)\right]$, P-a.s., for all $x\in R^d$.

Proof. A crucial role will be played by the following auxiliary statement:
Let $\tau:\Omega\to\Omega,\ \ \omega\to\tau(\omega)=(\tau(\omega)_k)_{k\geq1}$, be an arbitrary measurable bijection which law $P\circ[\tau]^{-1}$ is equivalent to the underlying probability measure P, such that $\tau'(\omega):=(\tau(\omega)_1,\dots,\tau(\omega)_{n-1})$, $\omega\in\Omega$, is $\mathcal{F}_{n-1}-\mathcal{B}(([0,1]^2)^{n-1})$-measurable, and $\tau(\omega)_k=\omega_k,\ k\geq n,\omega\in\Omega$. Then

$$V^\Pi(t_{n-1},x)\circ\tau=V^\Pi(t_{n-1},x),\ P\text{-a.s.}$$

Let us prove this assertion. For this we notice first that, using the equivalence between $P\circ[\tau]^{-1}$ and P as well as the bijectivity of τ, we can change the order between $\text{esssup}_{\alpha\in\mathcal{A}^\Pi_{t_{n-1},t_n}}\text{essinf}_{\beta\in\mathcal{B}^\Pi_{t_{n-1},t_n}}$ and the transformation τ (The reader interested in details is referred to the corresponding proof in [6].), i.e., we have

$$V^\Pi(t_{n-1},x)\circ\tau=\text{esssup}_{\alpha\in\mathcal{A}^\Pi_{t_{n-1},t_n}}\text{essinf}_{\beta\in\mathcal{B}^\Pi_{t_{n-1},t_n}}\left(E[g(X^{t_{n-1},x,\alpha,\beta}_{t_n})|\mathcal{F}_{n-1}]\circ\tau\right),\quad P\text{-a.s.}$$

Let us study now the expression $E[g(X^{t_{n-1},x,\alpha,\beta}_{t_n})|\mathcal{F}_{n-1}](\tau)$, occurring in the above formula. For this we recall first that, due to the definition, for any couple of admissible control processes $(u,v)\in\mathcal{U}^\Pi_{t_{n-1},t_n}\times\mathcal{V}^\Pi_{t_{n-1},t_n}$, there exists an $(\Omega,\mathcal{F}_{n-1})$-partition $(\Gamma_j)_{j\geq1}$ and an associated sequence of couples of control processes $(u_j,v_j)\in L^0_{\mathcal{G}_n}(t_{n-1},t_n;U)\times L^0_{\mathcal{H}_n}(t_{n-1},t_n;V),\ \ j\geq1$, such that $(u,v)=\sum_{j\geq1}I_{\Gamma_j}(u^j,v^j)$. Since $\Gamma_j\in\mathcal{F}_{n-1}$, we can find a Borel function f_j with $f_j(\zeta_1,\dots,\zeta_{n-1})=I_{\Gamma_j},\ j\geq1$. Then the relation

$$I_{\tau^{-1}(\Gamma_j)}(\omega)=f_j(\tau'(\omega)),\ \omega\in\Omega,$$

proves that $\tau^{-1}(\Gamma_j)\in\mathcal{F}_{n-1},\ j\geq1$. Hence, taking into account that the mapping $\tau:\Omega\to\Omega$ is bijective and $\tau(\omega)_n=\omega_n,\ \omega\in\Omega$, we see that also $(\tau^{-1}(\Gamma_j))_{j\geq1}$ forms an $(\Omega,\mathcal{F}_{n-1})$-partition, and

$$(u(\tau),v(\tau))=\sum_{j\geq1}I_{\Gamma_j}(\tau)\cdot(u^j,v^j)=\sum_{j\geq1}I_{\tau^{-1}(\Gamma_j)}\cdot(u^j,v^j)\in\mathcal{U}^\Pi_{t_{n-1},t_n}\times\mathcal{V}^\Pi_{t_{n-1},t_n}. \tag{3.20}$$

(Recall that u^j is $\mathcal{G}_n$-measurable and, hence, a measurable function of $\zeta_{n,1}$, while v^j is is $\mathcal{H}_n$-measurable and, thus, a measurable function of $\zeta_{n,2}$.)
On the other hand, a straight-forward application of the transformation $\tau:\Omega\to\Omega$ to equation (2.1) yields

$$X^{t_1,x,u,v}_{t_2}(\tau)=X^{t_1,x,u(\tau),v(\tau)}_{t_2}.$$

Indeed, the only random processes in the equation (2.1) of the dynamics are the control processes u and v.

Let us now consider an arbitrary couple of nonanticipative strategies $(\alpha, \beta) \in \mathcal{A}^{\Pi}_{t_{n-1},t_n} \times \mathcal{B}^{\Pi}_{t_{n-1},t_n}$ with which we associate the mappings $\alpha_\tau : \mathcal{V}^{\Pi}_{t_{n-1},t_n} \to \mathcal{U}^{\Pi}_{t_{n-1},t_n}$ and $\beta_\tau : \mathcal{U}^{\Pi}_{t_{n-1},t_n} \to \mathcal{V}^{\Pi}_{t_{n-1},t_n}$ defined as follows:

$$\alpha_\tau(v) = \alpha(v(\tau^{-1}))(\tau),\ \beta_\tau(u) = \beta(u(\tau^{-1}))(\tau),$$

for $u \in \mathcal{U}^{\Pi}_{t_{n-1},t_n}, v \in \mathcal{V}^{\Pi}_{t_{n-1},t_n}$. It can be easily checked that such defined mappings are themselves again nonanticipative strategies: $\alpha_\tau \in \mathcal{A}^{\Pi}_{t_{n-1},t_n}$, $\beta \in \mathcal{B}^{\Pi}_{t_{n-1},t_n}$. Moreover, from the bijectivity of τ it can be easily deduced that

$$\{\alpha_\tau | \alpha \in \mathcal{A}^{\Pi}_{t_{n-1},t_n}\} = \mathcal{A}^{\Pi}_{t_{n-1},t_n},\ \{\beta_\tau | \beta \in \mathcal{B}^{\Pi}_{t_{n-1},t_n}\} = \mathcal{B}^{\Pi}_{t_{n-1},t_n}.$$

Given an arbitrary couple of nonanticipative strategies $(\alpha, \beta) \in \mathcal{A}^{\Pi}_{t_{n-1},t_n} \times \mathcal{B}^{\Pi}_{t_{n-1},t_n}$ we consider the couple of admissible controls $(u, v) \in \mathcal{U}^{\Pi}_{t_{n-1},t_n} \times \mathcal{V}^{\Pi}_{t_{n-1},t_n}$, associated with by the relations $\alpha(v) = u,\ \beta(u) = v$. Since τ' is $\mathcal{F}_{n-1}$-measurable and $\tau(\omega)_n = \omega_n,\ \omega \in \Omega$, we obtain

$$\begin{aligned}&E[g(X^{t_{n-1},x,\alpha,\beta}_{t_n})|\mathcal{F}_{n-1}] \circ \tau = E[g(X^{t_{n-1},x,u,v}_{t_n})|\mathcal{F}_{n-1}] \circ \tau = E[g(X^{t_{n-1},x,u,v}_{t_n} \circ \tau)|\mathcal{F}_{n-1}]\\ &= E[g(X^{t_{n-1},x,u(\tau),v(\tau)}_{t_n})|\mathcal{F}_{n-1}].\end{aligned}$$

On the other hand, we observe that, due to the definition of the strategies α_τ and β_τ we have

$$u = \alpha(v) = \alpha(v(\tau) \circ \tau^{-1}),\ \text{i.e.,}\ u(\tau) = \alpha(v(\tau) \circ \tau^{-1})(\tau) = \alpha_\tau(v(\tau)),$$

and the symmetric argument yields $v(\tau) = \beta_\tau(u(\tau))$. Consequently, the unique couple of admissible controls associated with $(\alpha_\tau, \beta_\tau)$ is $(u(\tau), v(\tau))$, and we can conclude that

$$E[g(X^{t_{n-1},x,\alpha,\beta}_{t_n})|\mathcal{F}_{n-1}](\tau) = E[g(X^{t_{n-1},x,\alpha_\tau,\beta_\tau}_{t_n})|\mathcal{F}_{n-1}].$$

Using this together with the fact that

$$\{\alpha_\tau | \alpha \in \mathcal{A}^{\Pi}_{t_{n-1},t_n}\} = \mathcal{A}^{\Pi}_{t_{n-1},t_n},\ \{\beta_\tau | \beta \in \mathcal{B}^{\Pi}_{t_{n-1},t_n}\} = \mathcal{B}^{\Pi}_{t_{n-1},t_n},$$

we obtain

$$\begin{aligned}&V^{\Pi}(t_{n-1}, x) \circ \tau = \text{esssup}_{\alpha \in \mathcal{A}^{\Pi}_{t_{n-1},t_n}} \text{essinf}_{\beta \in \mathcal{B}^{\Pi}_{t_{n-1},t_n}} \Big(E[g(X^{t_{n-1},x,\alpha,\beta}_{t_n})|\mathcal{F}_{n-1}] \circ \tau\Big)\\ &= \text{esssup}_{\alpha \in \mathcal{A}^{\Pi}_{t_{n-1},t_n}} \text{essinf}_{\beta \in \mathcal{B}^{\Pi}_{t_{n-1},t_n}} E[g(X^{t_{n-1},x,\alpha_\tau,\beta_\tau}_{t_n})|\mathcal{F}_{n-1}]\\ &= \text{esssup}_{\alpha \in \mathcal{A}^{\Pi}_{t_{n-1},t_n}} \text{essinf}_{\beta \in \mathcal{B}^{\Pi}_{t_{n-1},t_n}} E[g(X^{t_{n-1},x,\alpha,\beta}_{t_n})|\mathcal{F}_{n-1}]\\ &= V^{\Pi}(t_{n-1}, x).\end{aligned}$$

Hence, $V^\pi(t_1, x) \circ \tau = V^\pi(t_1, x)$, P-a.s., and the proof of Lemma 3.3 will be completed by the following result. ∎

Lemma 3.4. *Let $\xi \in L^1(\Omega, \mathcal{F}_{n-1}, P)$ be a random variable which is invariant with respect to all measurable bijection $\tau : \Omega \to \Omega$ which law $P \circ [\tau]^{-1}$ is equivalent to the underlying probability measure P, such that $\tau'(\omega) := (\tau(\omega)_1, \dots, \tau(\omega)_{n-1})$, $\omega \in \Omega$, is $\mathcal{F}_{n-1} - \mathcal{B}(([0, 1]^2)^{n-1})$-measurable, and $\tau(\omega)_k = \omega_k$, $k \geq n, \omega \in \Omega$. Then ξ is almost surely constant, i.e., $\xi = E[\xi]$.*

Proof. We begin with noting that it suffices to prove this lemma under the additional assumption that ξ is nonnegative. Otherwise, we can always decompose ξ as a difference of its positive and its negative part, and observing that both parts are invariant with respect to τ on their turn we can make the proof for them separately.

Given $1 \leq i \leq n-1$, $j = 1, 2$, let us denote by $\theta_{i,j}$ the vector of all coordinate mappings $(\zeta_{1,1}, \zeta_{1,2}, \zeta_{2,1}, \zeta_{2,2}, \dots)$ but without the component $\zeta_{i,j}$. Then, putting $\zeta(\omega) := \omega$, $\omega \in \Omega$, we can identify $\zeta \equiv (\theta_{i,j}, \zeta_{i,j})$, and with this identification we can write $\xi(\omega) = \xi(\theta_{i,j}(\omega), \zeta_{i,j}(\omega))$, $\omega \in \Omega$.

Recalling that $\xi \geq 0$, let us now introduce the following $\mathcal{F}_{n-1}$-measurable mapping $\varphi : \Omega \to [0,1]$:

$$\varphi(\omega) = \varphi(\theta_{i,j}(\omega), \zeta_{i,j}(\omega)) = \frac{\int_0^{\zeta_{i,j}(\omega)} (\xi(\theta_{i,j}(\omega), s) + 1)\, ds}{\int_0^1 \xi(\theta_{i,j}(\omega), s) ds + 1},\ \omega \in \Omega.$$

Obviously, $\varphi(\theta_{i,j}(\omega), .) : [0,1] \to [0,1]$ is a continuous, strictly increasing bijection which derivative is

$$\frac{\partial}{\partial s}\varphi(\theta_{i,j}(\omega), s) = \frac{\xi(\theta_{i,j}(\omega), s) + 1}{\int_0^1 \xi(\theta_{i,j}(\omega), r) dr + 1},\ s \in [0,1],\ \omega \in \Omega.$$

We now put

$$\tau(\omega) := (\theta_{i,j}(\omega), \varphi(\theta_{i,j}(\omega), \zeta_{i,j}(\omega))),\ \omega \in \Omega.$$

Such defined mapping $\tau : \Omega \to \Omega$ satisfies the assumptions of the lemma. Indeed, due to the definition τ is a bijection, $\tau(\omega)_k = \omega_k,\ k \geq n,\ \omega \in \Omega$, and τ' is $\mathcal{F}_{n-1}$-measurable. Moreover, the law $P \circ [\tau]^{-1}$ is equivalent to the underlying probability measure P. Indeed, for any nonnegative random variable η over $(\Omega, \mathcal{F}, P)$ we have

$$\begin{aligned}
&E\left[\eta(\tau)\frac{\partial}{\partial s}\varphi(\theta_{i,j}, \zeta_{i,j})\right] = E\left[\int_0^1 \eta(\theta_{i,j}, \varphi(\theta_{i,j}, s))\frac{\partial}{\partial s}\varphi(\theta_{i,j}, s) ds\right] \\
&= E\left[\int_0^1 \eta(\theta_{i,j}, s) ds\right] = E[\eta],
\end{aligned}$$

where $\frac{\partial}{\partial s}\varphi(\theta_{i,j}, s) > 0$, for all $s \in [0,1]$. Consequently, we know from our assumption that the random variable ξ is invariant under the transformation τ, and, thus, observing that $\int_0^1 \xi(\theta_{i,j}(\omega), s) ds$ does not depend on $\zeta_{i,j}(\omega)$, we have

$$\begin{aligned}
&E[\xi^2] + E[\xi] = E[\xi(\xi + 1)] \\
&= E[\xi(\tau)(\xi + 1)] = E\left[\xi(\tau)\frac{\partial}{\partial s}\varphi(\theta_{i,j}, \zeta_{i,j})\left(\int_0^1 \xi(\theta_{i,j}, s) ds + 1\right)\right] \\
&= E\left[\xi\left(\int_0^1 \xi(\theta_{i,j}, s) ds + 1\right)\right] = E\left[\xi \int_0^1 \xi(\theta_{i,j}, s) ds\right] + E[\xi] \\
&= E\left[\left(\int_0^1 \xi(\theta_{i,j}, s) ds\right)^2\right] + E[\xi].
\end{aligned}$$

Consequently,

$$E\left[\int_0^1 \xi(\theta_{i,j}, s)^2 ds\right] = E[\xi^2] = E\left[\left(\int_0^1 \xi(\theta_{i,j}, s) ds\right)^2\right],$$

from where we see that

$$E\left[\int_0^1 \left(\xi(\theta_{i,j}, s) - \int_0^1 \xi(\theta_{i,j}, s) ds\right)^2 ds\right] = 0.$$

It follows that

$$\xi = \int_0^1 \xi(\theta_{i,j}, s) ds,\ \text{P-a.s.}, 1 \leq i \leq n-1,\ j = 1, 2.$$

Therefore, taking into account that ξ is $\mathcal{F}_{n-1}$-measurable and iterating the above result, we get

$$\begin{aligned}
&\xi = \int_0^1 \xi(\theta_{1,1}, s_{1,1}) ds_{1,1} = \int_0^1 \left(\int_0^1 \xi(\theta_{1,2}, s_{1,2}) ds_{1,2}\right)(\theta_{1,1}, s_{1,1}) ds_{1,1} \\
&= \int_{[0,1]^2} \xi(s_1, (\zeta_{2,1}, \zeta_{2,2}, \ldots, \zeta_{n-1,1}, \zeta_{n-1,2}) ds_1 = \cdots = \int_{[0,1]^{2(n-1)}} \xi(s) ds,\ \text{P-a.s.}
\end{aligned}$$

The proof of the lemma is complete now. ∎

By iterating the argument developed in the both preceding lemmas, we can prove now Theorem 3.2.

Proof. From the both preceding lemmas we see that together with $V^\Pi(t_n,.) := g(.)$ also the function $V^\Pi(t_{n-1},.)$ is deterministic. On the other hand, from the DPP satisfied by V^Π we obtain

$$V^\Pi(t_{n-2},x) = \text{esssup}_{\alpha\epsilon\mathcal{A}^\Pi_{t_{n-2},t_{n-1}}} \text{essinf}_{\beta\epsilon\mathcal{B}^\Pi_{t_{n-2},t_{n-1}}} E[V^\Pi(t_{n-1},X^{t_{n-2},x,\alpha,\beta}_{t_{n-1}})|\mathcal{F}_{n-2}],\ \text{P-a.s.},$$

for all $x \in R^d$. Hence, applying the argument of the both preceding lemmas again, but now for the deterministic function $V^\Pi(t_{n-1},.)$ instead of g (recall that due to Lemma 2.1 also the function $V^\Pi(t_{n-1},.)$ is bounded and Lipschitz), we conclude that also the function $V^\Pi(t_{n-2},.)$ is deterministic. Iterating this argument, we see that all $V^\Pi(t_l,.)\,(0 \le l \le n)$ are deterministic. This implies that V^Π is a deterministic function. Indeed, let $t_i \le t < t_{i+1}$. For the conclusion that the non-randomness of $V^\Pi(t_{i+1},.)$ involves that of $V^\Pi(t,.)$, it suffices to replace the driving coefficient $f(s,x,u,v)$ of the controlled dynamics by $f(s,x,u,v)I_{[t,T]}(s)$. This substitution doesn't change the values of $V^\Pi(t_l,x)$, $(i+1 \le l \le n)$ and $V^\Pi(t,x)$, $x \in R^d$, but now $V^\Pi(t,x)$ coincides with the deterministic function $V^\Pi(t_i,.)$ associated with the driver $f(s,x,u,v)I_{[t,T]}(s)$. The proof of Theorem 3.2 is complete. ∎

The both preceding major results, the DPP as well as the statement of non-randomness yield the following important characterization of the lower and the upper value functions along a partiton.

Theorem 3.3. *For all partition* Π *of the time interval* $[0,T]$, *and all* $(t,x) \in [0,T] \times R^d$, *we have*

$$\begin{array}{rcl} V^\Pi(t,x) &=& \sup_{\alpha\in\mathcal{A}^\Pi_{t,T}}\inf_{\beta\in\mathcal{B}^\Pi_{t,T}} E[g(X^{t,x,\alpha,\beta}_T)], \\ U^\Pi(t,x) &=& \inf_{\beta\in\mathcal{B}^\Pi_{t,T}}\sup_{\alpha\in\mathcal{A}^\Pi_{t,T}} E[g(X^{t,x,\alpha,\beta}_T)]. \end{array} \tag{3.21}$$

Proof. Let $\Pi = \{0 = t_0 < \cdots < t_n = T\}$, $t_i \le t < t_{i+1}\,(0 \le i \le n-1)$ and $x \in R^d$. Moreover, fix an arbitrary $\varepsilon > 0$. Then, due to (3.11) from the the proof of the DPP we know that there exists $\alpha^\varepsilon \in \mathcal{A}^\Pi_{t,T}$ such that, for all $\beta \in \mathcal{B}^\Pi_{t,T}$,

$$V^\Pi(t,x)(= \widetilde{V}^\Pi_l(t,x)) \le E[g(X^{t,x,\alpha^\varepsilon,\beta}_T) \mid \mathcal{F}_i] + \varepsilon,\ P\text{-a.s.}, \tag{3.22}$$

and from (3.18) we get for all $\alpha \in \mathcal{A}^\Pi_{t,T}$ the existence of $\beta^{\alpha,\varepsilon} \in \mathcal{B}^\Pi_{t,T}$ such that

$$V^\Pi(t,x)(= \widetilde{V}^\Pi_l(t,x)) \ge E[g(X^{t,x,\alpha,\beta^{\alpha,\varepsilon}}_T)|\mathcal{F}_i] - \varepsilon,\ P\text{-a.s.} \tag{3.23}$$

Consequently, considering that the function V^Π is deterministic and taking the expectation on both sides of (3.22) and (3.23), we get

$$E[g(X^{t,x,\alpha,\beta^{\alpha,\varepsilon}}_T)] - \varepsilon \le V^\Pi(t,x) \le E[g(X^{t,x,\alpha^\varepsilon,\beta}_T) \mid \mathcal{F}_i] + \varepsilon,$$

for all $(\alpha,\beta) \in \mathcal{A}^\Pi_{t,T} \times \mathcal{B}^\Pi_{t,T}$. Thus, taking into account the arbitrariness of $\varepsilon > 0$, the statement for V^Π follows directly, and that for U^Π can be verified analogously. The proof is complete. ∎

We observe that the latter Theorem3.3 combined with (2.2) provides directly the following statement:

Lemma 3.5. *There is some real constant* L, *only depending on the bound of* f *and the Lipschitz constants of* $f(s,.,u,v)$ *and of* g, *such that, for all partition* Π *of the interval* $[0,T]$ *and* $(t,x),(t',x') \in [0,T] \times R^d$,

$$|V^\Pi(t,x) - V^\Pi(t',x')| + |U^\Pi(t,x) - U^\Pi(t',x')| \le L(|t-t'| + |x-x'|). \tag{3.24}$$

4 Value in mixed strategies and associated Hamilton-Jacobi-Isaacs equation

The objective of this section is to show that the lower and the upper value functions along a partition V^{Π}, U^{Π} converge, as the maximal distance $|\Pi_n|$ between two neighbouring points of Π_n tends to zero as $n \to +\infty$, and that their common limit function V is the viscosity solution of the Hamilton-Jacobi-Isaacs equation

$$\left\{ \begin{array}{rcl} \frac{\partial}{\partial t}W(t,x) + \sup_{\mu\in\Delta U}\inf_{\nu\in\Delta V}\left(\widetilde{f}(t,x,\mu,\nu)\nabla W(t,x)\right) & = & 0; \\ W(T,x) & = & g(x), \end{array} \right. \tag{4.1}$$

where

$$\widetilde{f}(x,\mu,\nu) := \int_U \int_V f(x,u,v)\mu(du)\nu(dv),\ \ \mu \in \Delta U,\ \ \nu \in \Delta V.$$

More precisely, our main result of this section is the following

Theorem 4.1. *Under our standard assumptions on the coefficients f and g, the above Hamilton-Jacobi-Isaacs equation (4.1) possesses in the class of bounded continuous functions a unique viscosity solution V. Moreover, for any sequence of partitions Π_n, $n \geq 1$, of the interval $[0,T]$ with $|\Pi_n| \to 0$ as $n \to +\infty$, both the sequence of the lower value functions along a partition V^{Π_n} as well as that of the upper value functions along a partition U^{Π_n}, $n \geq 1$, converge uniformly on compacts to the function V.*

The definition of a continuous viscosity solution is by now standard, and the reader interested can find many literatures, e.g., refer to [9].

Definition 4.1. *A real-valued continuous function $W \in C([0,T]\times\mathbb{R}^d)$ is called*
(i) *a viscosity subsolution of equation (4.1) if $W(T,x) \leq \Phi(x)$, for all $x \in \mathbb{R}^d$, and if for all functions $\varphi \in C^1([0,T]\times\mathbb{R}^d)$ and $(t,x) \in [0,T)\times\mathbb{R}^d$ such that $W-\varphi$ attains its local maximum at (t,x),*

$$\frac{\partial\varphi}{\partial t}(t,x) + \sup_{\mu\in\Delta U}\inf_{\nu\in\Delta V}\left(\widetilde{f}(t,x,\mu,\nu)\nabla\varphi(t,x)\right) \geq 0;$$

(ii) *a viscosity supersolution of equation (4.1) if $W(T,x) \geq \Phi(x)$, for all $x \in \mathbb{R}^d$, and if for all functions $\varphi \in C^1([0,T]\times\mathbb{R}^d)$ and $(t,x) \in [0,T)\times\mathbb{R}^d$ such that $W-\varphi$ attains its local minimum at (t,x),*

$$\frac{\partial\varphi}{\partial t}(t,x) + \sup_{\mu\in\Delta U}\inf_{\nu\in\Delta V}\left(\widetilde{f}(t,x,\mu,\nu)\nabla\varphi(t,x)\right) \leq 0;$$

(iii) *a viscosity solution of equation (4.1) if it is both a viscosity sub- and a supersolution of equation (4.1).*

The whole section is devoted to the proof of the above theorem. The proof will be split in a sequel of auxiliary statements. Let us begin with observing that the equi-Lipschitz continuity of the families of lower and upper value functions along a partition, indexed with the help of the partitions Π of the interval $[0,T]$, stated in Lemma 3.5, is crucial for the application of the Arzelà-Ascoli Theorem. Let us arbitrarily fix a sequence of partitions $(\Pi_n)_{n\geq1}$ of the interval $[0,T]$, such that for the mesh of the partition Π_n it holds: $|\Pi_n| \to 0$ as $n \to +\infty$. Then we have

Lemma 4.1. *There exists a subsequence of partitions, again denoted by $(\Pi_n)_{n\geq1}$, and there are bounded Lipschitz functions $V,U : [0,T]\times R^d \to R$ such that $(V^{\Pi_n}, U^{\Pi_n}) \to (V,U)$, uniformly on compacts in $[0,T]\times R^d$.*

Later we will see that the function (V,U) defined by this Lemma 4.1 coincide and are independent of the choice of the sequence of partitions.

Proof. Indeed, from the Arzelà-Ascoli Theorem we know that, for any compact subset K of $[0,T]\times R^d$ and for any subsequence of partitions of $[0,T]$, there exist a subsequence (Π'_n) and functions $U', V' : K\to R$ such that $(V^{\Pi'_n}, U^{\Pi'_n})\to (V', U')$ uniformly on K, as $n\to+\infty$. By combining this result with a standard diagonalisation argument we can easily prove the stated assertion. ∎

Let us fix the subsequence $(\Pi_n)_{n\geq 1}$ related with U, V by Lemma 4.1. From Lemma 3.5 we have

Corollary 4.1. *For the real constant L introduced in Lemma 3.5 we have, for all $(t,x),(t',x')\in[0,T]\times R^d$,*

$$|V(t,x)-V(t',x')|+|U(t,x)-U(t',x')|\leq L(|t-t'|+|x-x'|). \tag{4.2}$$

By taking into account the uniform boundedness of the functions V^Π, U^Π, parameterized by Π-partition of the interval $[0,T]$ (Indeed, they are bounded by the bound of g.), this shows, in particular, that $V,\ U\in C_b([0,T]\times R^d)$ are bounded continuous functions. We are able to prove that V and U are viscosity solutions of equation (4.1). For this let us begin with

Proposition 4.1. *The function V is a viscosity solution of the Hamilton-Jacobi-Isaacs equation (4.1).*

In order to prove this statement, we show in a first step that

Lemma 4.2. *The function V is a viscosity subsolution of the Hamilton-Jacobi-Isaacs equation (4.1).*

Proof. Since we know that, by definition (2.3) of V^Π, $V^\Pi(T,x)=g(x)$, $x\in R^d$, for all partition Π, we also have $V(T,x)=g(x)$, $x\in R^d$. Let $(t,x)\in[0,T)\times R^d$ and $\varphi\in C^1([0,T]\times\mathbb{R}^d)$ be an arbitrary test function such that $\varphi(t,x)-V(t,x)=0\leq\varphi(s,y)-V(s,y)$, $(s,y)\in[0,T]\times R^d$. Since $V\in C_b([0,T];R^d)$ is bounded, we can assume without loss of generality that $\varphi\in C_b^1([0,T]\times\mathbb{R}^d)$, i.e., that φ itself as well as its first order derivatives are bounded. Recall that verifying that V is a viscosity subsolution is equivalent with showing that

$$\frac{\partial}{\partial t}\varphi(t,x)+\sup_{\mu\in\Delta U}\inf_{\nu\in\Delta V}\widetilde{f}(t,x,\mu,\nu)\nabla\varphi(t,x)\geq 0. \tag{4.3}$$

For proving the above relation we note that for any $\rho>0$ and $M>0$ we can find a positive integer $n_{\rho,M}$ such that, for all $n\geq n_{\rho,M}$,

$$|\varphi(t,x)-V^{\Pi_n}(t,x)|\leq\rho,\ \text{ and } V^{\Pi_n}(s,y)\leq\varphi(s,y)+\rho,\ \text{ for all } s\in[0,T],\ |y|\leq M.$$

Indeed, recall that $V^{\Pi_n}\to V$ converges uniformly on compacts, $V(t,x)=\varphi(t,x)$ and $V\leq\varphi$ on $[0,T]\times R^d$.

Let $n\geq n_{\rho,M}$, $\Pi_n=\{0=t_0^n<\cdots<t_N^n=T\}$, and let $i=i_n$ be such that $t_i^n\leq t<t_{i+1}^n\leq t_l^n$. Then, from the DPP (Theorem 3.1) with respect to the partition Π_n and since V^{Π_n} is bounded by some constant C, uniformly with respect to $n\geq 1$, we have

$$\begin{array}{rcl}\varphi(t,x)-\rho & \leq & V^{\Pi_n}(t,x)\\ & = & \operatorname{esssup}_{\alpha\in\mathcal{A}^{\Pi_n}_{t,t_l^n}}\operatorname{essinf}_{\beta\in\mathcal{B}^{\Pi_n}_{t,t_l^n}}E[V^{\Pi_n}(t_l^n,X^{t,x,\alpha,\beta}_{t_l^n})|\mathcal{F}_i]\\ & \leq & \operatorname{esssup}_{\alpha\in\mathcal{A}^{\Pi_n}_{t,t_l^n}}\operatorname{essinf}_{\beta\in\mathcal{B}^{\Pi_n}_{t,t_l^n}}E[\varphi(t_l^n,X^{t,x,\alpha,\beta}_{t_l^n})|\mathcal{F}_i]\\ & & \qquad +CP\{\,|X^{t,x,\alpha,\beta}_{t_l^n}|>M|\mathcal{F}_i\}+\rho,\ P\text{-a.s.}\end{array} \tag{4.4}$$

However,

$$P\{\,|X^{t,x,\alpha,\beta}_{t_l^n}|>M|\mathcal{F}_i\}\leq\frac{1}{M}E[|X^{t,x,\alpha,\beta}_{t_l^n}|\,|\mathcal{F}_i]\leq\frac{1}{M}(|x|+TC_f),$$

where we have used that $|X^{t,x,\alpha,\beta}_{t_l^n}|\leq|x|+TC_f$, with C_f denoting the bound of f. Thus, by choosing $M=M_\rho$ large enough, such that $\frac{C}{M}(|x|+TC_f)\leq\rho$, and recalling the equation for the dynamics of $X^{t,x,\alpha,\beta}_\cdot$, we have for $n\geq n_\rho\ (:=n_{\rho,M_\rho})$

$$\begin{array}{rl} & -3\rho\leq\operatorname{esssup}_{\alpha\in\mathcal{A}^{\Pi_n}_{t,t_l^n}}\operatorname{essinf}_{\beta\in\mathcal{B}^{\Pi_n}_{t,t_l^n}}E\Big[\varphi(t_l^n,X^{t,x,\alpha,\beta}_{t_l^n})-\varphi(t,x)|\mathcal{F}_i\Big]\\ = & \operatorname{esssup}_{\alpha\in\mathcal{A}^{\Pi_n}_{t,t_l^n}}\operatorname{essinf}_{\beta\in\mathcal{B}^{\Pi_n}_{t,t_l^n}}E\Bigg[\displaystyle\int_t^{t_l^n}\Big(\frac{\partial}{\partial r}\varphi(r,X_r^{t,x,\alpha,\beta})+f(r,X_r^{t,x,\alpha,\beta},(\alpha,\beta)_r)\nabla\varphi(r,X_r^{t,x,\alpha,\beta})\Big)dr|\mathcal{F}_i\Bigg].\end{array} \tag{4.5}$$

Here we have denoted by $(\alpha,\beta)_r$ the unique couple of control processes $(u,v)\in\mathcal{U}^{\Pi_n}_{t,t^n_l}\times\mathcal{V}^{\Pi_n}_{t,t^n_l}$ at time r, associated with $(\alpha,\beta)\in\mathcal{A}^{\Pi_n}_{t,t^n_l}\times\mathcal{B}^{\Pi_n}_{t,t^n_l}$ by Lemma 2.2. Let us introduce the continuity modulus

$$m(\delta):=\sup_{|r-t|+|y-x|\le\delta,u\in U,v\in V}\left|\left((\frac{\partial}{\partial r}\varphi)(r,y)+f(r,y,u,v)\nabla\varphi(r,y)\right)-\left((\frac{\partial}{\partial r}\varphi(t,x)+f(t,x,u,v)\nabla\varphi(t,x)\right)\right|,$$

$\delta>0$. Recalling that the function $f(.,.,u,v)$ is bounded and uniformly continuous, uniformly with respect to $(u,v)\in U\times V$, and that the first order derivatives of φ are bounded continuous functions, we see that $m:R_+\to R_+$ is an increasing function with $m(\delta)\to 0$, as $\delta\downarrow 0$. Thus, taking into account that $|X^{t,x,\alpha,\beta}_r-x|\le C_f|r-t|\le C_f|t^n_l-t|,\ r\in[t,t^n_l]$, we obtain

$$\begin{aligned}&\left|(\frac{\partial}{\partial r}\varphi)(r,X^{t,x,u,v}_r)+f(r,X^{t,x,u,v}_r,u_r,v_r)\nabla\varphi(r,X^{t,x,u,v}_r))-(\frac{\partial}{\partial t}\varphi(t,x)+f(t,x,u_r,v_r)\nabla\varphi(t,x))\right|\\&\le m(C|t^n_l-t|),\quad r\in[t,t^n_l].\end{aligned}\tag{4.6}$$

(The constant C depends on x, fixed in this proof.) Consequently, thanks to (4.5),

$$\begin{aligned}&-3\rho-(t^n_l-t)\left(\frac{\partial}{\partial t}\varphi(t,x)+m(C|t^n_l-t|)\right)\\&\le \operatorname{esssup}_{\alpha\in\mathcal{A}^{\Pi_n}_{t,t^n_l}}\operatorname{essinf}_{\beta\in\mathcal{B}^{\Pi_n}_{t,t^n_l}}E\left[\int_t^{t^n_l}f(t,x,(\alpha,\beta)_r)\nabla\varphi(t,x)dr|\mathcal{F}_i\right],\ P\text{-a.s.}\end{aligned}\tag{4.7}$$

Similarly to the argument of (3.3) in Step 1 of the proof of Lemma 3.1 we can show there is an NAD strategy $\alpha^\rho\in\mathcal{A}^{\Pi_n}_{t,t^n_l}$ such that

$$\begin{aligned}&-4\rho-(t^n_l-t)\left(\frac{\partial}{\partial t}\varphi(t,x)+m(C|t^n_l-t|)\right)\\&\le \operatorname{essinf}_{\beta\in\mathcal{B}^{\Pi_n}_{t,t^n_l}}E\left[\int_t^{t^n_l}f(t,x,(\alpha^\rho,\beta)_r)\nabla\varphi(t,x)dr|\mathcal{F}_i\right].\end{aligned}\tag{4.8}$$

Thus, since $\mathcal{V}^{\Pi_n}_{t,t^n_l}\subset\mathcal{B}^{\Pi_n}_{t,t^n_l}$ (Indeed, the controls $v\in\mathcal{V}^{\Pi_n}_{t,t^n_l}$ are identified with $\beta^v\in\mathcal{B}^{\Pi_n}_{t,t^n_l}$, where $\beta^v(u):=v,\ u\in\mathcal{U}^{\Pi_n}_{t,t^n_l}$.), we obtain from (4.8) that, for all $v\in\mathcal{V}^{\Pi_n}_{t,t^n_l}$,

$$-4\rho-(t^n_l-t)\left(\frac{\partial}{\partial t}\varphi(t,x)+m(C|t^n_l-t|)\right)\le E\left[\int_t^{t^n_l}f(t,x,(\alpha^\rho,v)_r)\nabla\varphi(t,x)dr|\mathcal{F}_i\right].\tag{4.9}$$

Let $v\in\mathcal{V}^{\Pi_n}_{t,t^n_l}$ be now of the special form $v:=\sum_{j=i+1}^l\xi_jI_{[t\vee t^n_{j-1},t\vee t^n_j)},\ \ \xi_j\in L^0(\Omega,\mathcal{H}_j,P;V)$. Then,

$$E\left[\int_t^{t^n_l}f(t,x,(\alpha^\rho,v)_r)\nabla\varphi(t,x)\mathrm{d}r|\mathcal{F}_i\right]=\sum_{j=i+1}^{l}E\left[\int_{t\vee t^n_{j-1}}^{t\vee t^n_j}f(t,x,\alpha^\rho(v)_r,\xi_j)\nabla\varphi(t,x)\mathrm{d}r|\mathcal{F}_i\right].\tag{4.10}$$

Let us put $u^\rho:=\alpha^\rho(v)\in\mathcal{U}^{\Pi_n}_{t,t^n_l}$, and let $i+1\le j\le l$. Then, due to the definition of the controls from $\mathcal{U}^{\Pi_n}_{t,t^n_l}$, there exist an partition $(\Gamma_k)_{k\ge1}\subset\mathcal{F}_{j-1}$ of Ω and a sequence $(u^k)_{k\ge1}\subset L^0_{\mathcal{G}_j}(t\vee t^n_{j-1},t\vee t^n_j;U)$ such that, for the restriction of u^ρ to $[t\vee t^n_{j-1},t\vee t^n_j)$,

$$u^\rho_{|[t\vee t^n_{j-1},t\vee t^n_j)}=\sum_{k\ge1}I_{\Gamma_k}u^k.$$

Consequently, recalling that $\xi_j\in L^0(\Omega,\mathcal{H}_j,P;V)$ and that the three σ-fields $\mathcal{G}_j,\mathcal{H}_j$ and $\mathcal{F}_{j-1}$ are mutually independent, we have

$$
\begin{aligned}
& E\left[\int_{t\vee t_{j-1}^n}^{t\vee t_j^n} f(t,x,\alpha^\rho(v)_r,\xi_j)\nabla\varphi(t,x)\mathrm{d}r|\mathcal{F}_i\right] \\
= \;& E\left[\sum_{k\geq 1} I_{\Gamma_k}\int_{t\vee t_{j-1}^n}^{t\vee t_j^n} E\left[f(t,x,u_r^k,\xi_j)\nabla\varphi(t,x)|\mathcal{F}_{j-1}\right]\mathrm{d}r|\mathcal{F}_i\right] \\
= \;& E\left[\sum_{k\geq 1} I_{\Gamma_k}\int_{t\vee t_{j-1}^n}^{t\vee t_j^n}\left(\int_{U\times V} f(t,x,u,v)\nabla\varphi(t,x)P_{u_r^k}(\mathrm{d}u)\otimes P_{\xi_j}(\mathrm{d}v)\right)\mathrm{d}r|\mathcal{F}_i\right] \\
\leq \;& (t\vee t_j^n - t\vee t_{j-1}^n)\cdot \sup_{\mu\in\triangle U}\left(\int_{U\times V} f(t,x,u,v)\nabla\varphi(t,x)\mu(\mathrm{d}u)\otimes P_{\xi_j}(\mathrm{d}v)\right).
\end{aligned}
\tag{4.11}
$$

Recall that $\widetilde{f}(t,x,\mu,\nu) := \int_{U\times V} f(t,x,u,v)\mu(\mathrm{d}u)\nu(\mathrm{d}v)$. Hence, from (4.9), (4.10) and (4.11),

$$
\begin{aligned}
& -4\rho - (t_l^n - t)\left(\frac{\partial}{\partial t}\varphi(t,x) + m(C|t_l^n - t|)\right) \\
\leq \;& E\left[\int_t^{t_l^n} f(t,x,(\alpha^\rho,v)_r)\nabla\varphi(t,x)dr|\mathcal{F}_i\right] \\
\leq \;& \sum_{j=i+1}^{l}(t\vee t_j^n - t\vee t_{j-1}^n)\cdot \sup_{\mu\in\Delta U}\widetilde{f}(t,x,\mu,P_{\xi_j})\nabla\varphi(t,x),
\end{aligned}
\tag{4.12}
$$

and from the arbitrariness of the random variables $\xi_j \in L^0(\Omega,\mathcal{H}_j,P;V)$, $i+1\leq j\leq l$ and the fact that $\Delta V = \{P_\xi \,|\, \xi\in L^0(\Omega,\mathcal{H}_j,P;V)$, we conclude

$$
\begin{aligned}
& -4\rho - (t_l^n - t)\left(\frac{\partial}{\partial t}\varphi(t,x) + m(C|t_l^n - t|)\right) \\
\leq \;& \sum_{j=i+1}^{l}(t\vee t_j^n - t\vee t_{j-1}^n)\cdot \inf_{\nu\in\Delta V}\sup_{\mu\in\Delta U}\widetilde{f}(t,x,\mu,\nu)\nabla\varphi(t,x) \\
= \;& (t_l^n - t)\cdot \inf_{\nu\in\Delta V}\sup_{\mu\in\Delta U}\widetilde{f}(t,x,\mu,\nu)\nabla\varphi(t,x).
\end{aligned}
\tag{4.13}
$$

We choose now $\varepsilon > 0$ arbitrarily small and we put $\rho = \varepsilon^2$. For $n \geq n_\rho$ large enough we can find some $l\,(i+1\leq l\leq n)$, such that $\varepsilon \leq t_l^{(n)} - t \leq 2\varepsilon$. Indeed, recall that the mesh of Π_n converges to zero, as $n\to+\infty$. Then it follows from (4.13) that

$$
\begin{aligned}
& -4(t_l^n - t)^2 - (t_l^n - t)\left(\frac{\partial}{\partial t}\varphi(t,x) + m(C|t_l^n - t|)\right) \\
\leq \;& (t_l^n - t)\cdot \inf_{\nu\in\Delta V}\sup_{\mu\in\Delta U}\widetilde{f}(t,x,\mu,\nu)\nabla\varphi(t,x).
\end{aligned}
\tag{4.14}
$$

Consequently, dividing both sides of this latter relation by $t_l^n - t$ and taking the limit as $\varepsilon\to 0$, we obtain

$$
\frac{\partial}{\partial t}\varphi(t,x) + \inf_{\nu\in\Delta V}\sup_{\mu\in\Delta U}\widetilde{f}(t,x,\mu,\nu)\nabla\varphi(t,x) \geq 0. \tag{4.15}
$$

In order to conclude, we remark that, for all $(t,x,p)\in[0,T]\times R^d\times R^d$, the function $H(t,x,\mu,\nu,p) := \widetilde{f}(t,x,\mu,\nu)p$ is bilinear in $(\mu,\nu)\in\Delta U\times\Delta V$. The spaces ΔU and ΔV are compact and convex. Consequently,

$$
\inf_{\nu\in\Delta V}\sup_{\mu\in\Delta U}\widetilde{f}(t,x,\mu,\nu)p = \sup_{\mu\in\Delta U}\inf_{\nu\in\Delta V}\widetilde{f}(t,x,\mu,\nu)p,\;\; (t,x,p)\in[0,T]\times R^d\times R^d, \tag{4.16}
$$

and relation (4.3) follows from (4.15). The proof is complete. ∎

In order to complete the proof of Proposition 4.1 we also have to prove the following

Lemma 4.3. *The function V is the viscosity supersolution of the Hamilton-Jacobi-Isaacs equation (4.1).*

Proof. In the proof of Lemma 4.2 we have already noticed that $V(T,x)=g(x)$, $x\in R^d$. Let us fix again $(t,x)\in[0,T)\times R^d$ and consider a test function $\varphi\in C_b^1([0,T]\times\mathbb{R}^d)$ which is bounded together with its first order derivatives, such that $V(t,x)-\varphi(t,x)=0\le V-\varphi$ on $[0,T]\times R^d$. In order to prove the statement we have to show that

$$\frac{\partial}{\partial t}\varphi(t,x)+\sup_{\mu\in\Delta U}\inf_{\nu\in\Delta V}\widetilde{f}(t,x,\mu,\nu)\nabla\varphi(t,x)\le 0. \tag{4.17}$$

Let us suppose that this latter relation doesn't hold true. Then, there exist $\delta>0$, and $\mu^*\in\Delta U$ such that

$$\begin{aligned} 0<\delta &< \frac{\partial}{\partial t}\varphi(t,x)+\sup_{\mu\in\Delta U}\inf_{\nu\in\Delta V}\widetilde{f}(t,x,\mu,\nu)\nabla\varphi(t,x)\\ &= \frac{\partial}{\partial t}\varphi(t,x)+\inf_{\nu\in\Delta V}\widetilde{f}(t,x,\mu^*,\nu)\nabla\varphi(t,x)\\ &\le \frac{\partial}{\partial t}\varphi(t,x)+\widetilde{f}(t,x,\mu^*,\nu)\nabla\varphi(t,x), \end{aligned} \tag{4.18}$$

for all $\nu\in\Delta V$. On the other hand, given an arbitrarily small $\rho>0$ and $M\ge C\rho^{-1}(|x|+C_fT)$, there exists $n_\rho\ge 1$, such that for all $n\ge n_\rho$,

$$|\varphi(t,x)-V^{\Pi_n}(t,x)|\le\rho,\quad V^{\Pi_n}(s,y)\ge\varphi(s,y)-\rho,\ s\in[0,T],\ |y|\le M.$$

Let $n\ge n_\rho$. Adapting the argument of the proof of the preceding Lemma 4.2 and using the notations introduced there, we first deduce from the DPP (Theorem 3.1) with respect to the partition Π_n that

$$\begin{aligned} \varphi(t,x)+\rho &\ge V^{\Pi_n}(t,x)\\ &= \mathrm{esssup}_{\alpha\in\mathcal{A}^{\Pi_n}_{t,t_l^n}}\,\mathrm{essinf}_{\beta\in\mathcal{B}^{\Pi_n}_{t,t_l^n}}\,E[V^{\Pi_n}(t_l^n,X^{t,x,\alpha,\beta}_{t_l^n})|\mathcal{F}_i]\\ &\ge \mathrm{esssup}_{\alpha\in\mathcal{A}^{\Pi_n}_{t,t_l^n}}\,\mathrm{essinf}_{\beta\in\mathcal{B}^{\Pi_n}_{t,t_l^n}}\,E[\varphi(t_l^n,X^{t,x,\alpha,\beta}_{t_l^n})|\mathcal{F}_i]\\ &\qquad -CP\{\,|X^{t,x,\alpha,\beta}_{t_l^n}|>M|\mathcal{F}_i\}-\rho\\ &\ge \mathrm{esssup}_{\alpha\in\mathcal{A}^{\Pi_n}_{t,t_l^n}}\,\mathrm{essinf}_{\beta\in\mathcal{B}^{\Pi_n}_{t,t_l^n}}\,E[\varphi(t_l^n,X^{t,x,\alpha,\beta}_{t_l^n})|\mathcal{F}_i]-2\rho,\ P\text{-a.s.} \end{aligned} \tag{4.19}$$

Consequently,

$$3\rho\ge\mathrm{esssup}_{\alpha\in\mathcal{A}^{\Pi_n}_{t,t_l^n}}\,\mathrm{essinf}_{\beta\in\mathcal{B}^{\Pi_n}_{t,t_l^n}}\,E\left[\int_t^{t_l^n}\left(\frac{\partial}{\partial r}\varphi(r,X_r^{t,x,\alpha,\beta})+f(r,X_r^{t,x,\alpha,\beta},(\alpha,\beta)_r)\nabla\varphi(r,X_r^{t,x,\alpha,\beta})\right)dr|\mathcal{F}_i\right], \tag{4.20}$$

and using the continuity modulus $m(.)$ introduced in the proof of Lemma 4.2 we obtain

$$\begin{aligned} &3\rho-(t_l^n-t)\left(\frac{\partial}{\partial t}\varphi(t,x)-m(C|t_l^n-t|)\right)\\ &\ge\mathrm{esssup}_{\alpha\in\mathcal{A}^{\Pi_n}_{t,t_l^n}}\,\mathrm{essinf}_{\beta\in\mathcal{B}^{\Pi_n}_{t,t_l^n}}\,E\left[\int_t^{t_l^n}f(t,x,(\alpha,\beta)_r)\nabla\varphi(t,x)dr|\mathcal{F}_i\right],\ P\text{-a.s.} \end{aligned} \tag{4.21}$$

In the next step, observing that we can identify $\mathcal{U}^{\Pi_n}_{t,t_l^n}$ as a subset of $\mathcal{A}^{\Pi_n}_{t,t_l^n}$, and choosing $u\in\mathcal{U}^{\Pi_n}_{t,t_l^n}$ of the form $u=\sum_{j=i+1}^l\xi_jI_{[t\vee t_{j-1}^n,t\vee t_j^n)}$, with $\xi_j\in L^0(\Omega,\mathcal{G}_j,P;U)$ such that $P_{\xi_j}=\mu^*$ $(i+1\le j\le l)$, we get

$$\begin{aligned} &3\rho-(t_l^n-t)\left(\frac{\partial}{\partial t}\varphi(t,x)-m(C|t_l^n-t|)\right)\\ \ge\ &\mathrm{essinf}_{\beta\in\mathcal{B}^{\Pi_n}_{t,t_l^n}}\,E\left[\int_t^{t_l^n}f(t,x,(u,\beta(u)_r))\nabla\varphi(t,x)dr|\mathcal{F}_i\right],\ P\text{-a.s.} \end{aligned} \tag{4.22}$$

In analogy to the argument of (3.3) in Step 1 of the proof of Lemma 3.1 we now can construct some $\beta^\rho\in\mathcal{B}^{\Pi}_{t,t_l^n}$ (depending on the control process u) such that

$$
\begin{aligned}
& 4\rho-(t_l^n-t)\left(\frac{\partial}{\partial t}\varphi(t,x)-m(C|t_l^n-t|)\right)\\
\geq\ & E\left[\int_t^{t_l^n} f(t,x,(u,\beta^\rho(u)_r))\nabla\varphi(t,x)dr|\mathcal{F}_i\right]\\
\geq\ & E\left[\sum_{j=i+1}^{l}\int_{t\vee t_{i-1}^n}^{t\vee t_j^n} f(t,x,(\xi_j,\beta^\rho(u)_r))\nabla\varphi(t,x)dr|\mathcal{F}_i\right].
\end{aligned}
\tag{4.23}
$$

We put now $v:=\beta^\rho(u)$, and we observe that, for any $i+1\le j\le n$, the restriction of v to the interval $[t\vee t_{j-1}^n,t_j^n)$ belongs to $\mathcal{V}_{t\vee t_{j-1}^n,t_j^n}^{\Pi_n}$. Consequently, by definition, $v|_{[t\vee t_{j-1}^n,t\vee t_j^n)}$ is of the form $v|_{[t\vee t_{j-1}^n,t\vee t_j^n)}=\sum\limits_{k\geq 1} I_{\Gamma_k}v^k$, where $(\Gamma_k)_{k\geq 1}\subset\mathcal{F}_{j-1}$ is a partition of Ω and $v_k\in L^0_{\mathcal{H}_j}(t_{j-1}^n,t_j^n;V)$, $k\geq 1$ (For simplicity of notations we have suppressed in this representation the dependence on j). Thus, the independence of the three σ-fields $\mathcal{H}_j$, $\mathcal{G}_j$ and $\mathcal{F}_{j-1}$yields

$$
\begin{aligned}
& E\left[\int_{t\vee t_{j-1}^n}^{t\vee t_j^n} f(t,x,\xi_j,\beta^\rho(u)_r)\nabla\varphi(t,x)\mathrm{d}r|\mathcal{F}_i^n\right]\\
=\ & E\left[\sum_{k\geq 1} I_{\Gamma_k}\int_{t\vee t_{j-1}^n}^{t\vee t_j^n} E\left[f(t,x,\xi_j,v_r^k)\nabla\varphi(t,x)|\mathcal{F}_{j-1}\right]\mathrm{d}r|\mathcal{F}_i\right]\\
=\ & E\left[\sum_{k\geq 1} I_{\Gamma_k}\int_{t\vee t_{j-1}^n}^{t\vee t_j^n} \widetilde{f}(t,x,\mu^*,P_{v_r^k})\nabla\varphi(t,x)\mathrm{d}r|\mathcal{F}_i\right]\\
\geq\ & (t\vee t_j^n-t\vee t_{j-1}^n)\cdot E\left[\sum_{k\geq 1} I_{\Gamma_k}\inf_{\nu\in\Delta V}(\widetilde{f}(t,x,\mu^*,\nu)\nabla\varphi(t,x))|\mathcal{F}_i\right]\\
=\ & (t\vee t_j^n-t\vee t_{j-1}^n)\cdot\inf_{\nu\in\Delta V}\widetilde{f}(x,\mu^*,\nu)\nabla\varphi(t,x)).
\end{aligned}
\tag{4.24}
$$

Therefore, summing up (4.24) with respect to j and substituting the result in (4.23) we obtain

$$
\begin{aligned}
& 4\rho+(t_l^n-t)m(C|t_l^n-t|)\\
\geq\ & (t_l^n-t)\cdot\left(\frac{\partial}{\partial t}\varphi(t,x)+\inf_{\nu\in\Delta V}\widetilde{f}(x,\mu^*,\nu)\nabla\varphi(t,x)\right)\\
\geq\ & \delta(t_l^n-t).
\end{aligned}
\tag{4.25}
$$

Let now $\varepsilon>0$, $\rho=\varepsilon^2$ and $|\Pi_n|>0$ be small enough, such that t_l^n can be chosen such that $\frac{\varepsilon}{2}\le t_l^n-t\le\varepsilon$. Then, from (4.25) we have

$$
4\varepsilon^2+\varepsilon m(C\varepsilon)\geq\frac{\varepsilon}{2}\delta. \tag{4.26}
$$

Thus, first dividing this latter relation by ε and after letting $\varepsilon\to 0$, we get $\delta\le 0$, which contradicts $\delta>0$ in (4.18). Therefore, our hypothesis is wrong and we have (4.17). The proof is complete. ∎

In analogy to Proposition 4.1 we can prove the following

Proposition 4.2. *Also the function $U\in C_b([0,T]\times R^d)$ is a viscosity solution of the Hamilton-Jacobi-Isaacs equation (4.1).*

Finally, we are able to prove Theorem 4.1.

Proof. Due to relation (4.16) we know that the bounded continuous functions V and U are viscosity solutions of the same Hamilton-Jacobi-Isaacs equation. On the other hand, since the Hamiltonian of this equation

$$
H(t,x,p)=\inf_{\nu\in\Delta V}\sup_{\mu\in\Delta U}(\widetilde{f}(t,x,\mu,\nu)p),\ (t,x,p)\in[0,T]\times R^d\times R^d,
$$

is bounded and continuous, Lipschitz in z, uniformly with respect to $(t,x)\in[0,T]\in R^d$, and

$$|H(t,x,p)-H(t,x',p)|\le C|p||x-x'|,\quad x,x'\in R^d,(t,p)\in[0,T]\times R^d,$$

it is by now well-known, that the viscosity solution of the Hamilton-Jacobi-Isaacs equation (4.1) is unique in the class of continuous functions with at most polynomial growth. Consequently, $V=U$. On the other hand, recall that we have got V and U as limit over a converging subsequence of the sequence V^{Π_n} and U^{Π_n}, respectively, where $(\Pi_n)_{n\ge1}$ is an arbitrarily chosen sequence of partitions of $[0,T]$ such that $|\Pi_n|\to 0$ $(n\to+\infty)$. Therefore, since the limit of the converging subsequence doesn't depend on the choice of the sequence, it follows that V^Π and U^Π converge along all sequence of partitions Π with $|\Pi|\to 0$, and the limit is $V=U$. The proof of Theorem 4.1 is complete. ∎

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