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Abstract

Can we reconcile stability with non-manipulability in two-sided matching problems by selecting lotteries over matchings? We parameterize, through sets of utility functions, how ordinal preferences induce preferences over lotteries and develop corresponding notions of ex-ante stability and non-manipulability. For most sets, the properties are incompatible. However, for the set of utility functions with *increasing differences*, stability and non-manipulability characterize *Compromises and Rewards*. This novel rule is fundamentally different from the one that has attracted most attention in the literature, *Deferred Acceptance*. We then derive complementary negative results that show that increasing differences essentially is a necessary condition for the properties to be compatible.

1 Introduction

In designing a centralized procedure that elicits preferences to match agents, we are faced with an impossibility: if the rule used to match always selects a stable matching, then it is manipulable[1] (Roth, 1982; Alcalde and Barberà, 1994). We examine whether this can be overturned when we instead select a lottery over matchings and appropriately redefine the properties. The answer is: sometimes. We develop notions of ex-ante stability and non-manipulability and show that they often are incompatible. However, we also identify a condition under which *there is a non-manipulable rule that selects stable probabilistic matchings.*

Resorting to stochastic methods to solve matching problems is common in practice. Lotteries play a part in, for instance, the assignment of medical graduates to internships (Bronfman et al., 2015; Roth and Shorrer, 2015); the admission of students to courses and universities (Stasz and van Stolk, 2007); the introduction of new players in professional sports leagues (Taylor and Trogdon, 2007); the allocation of housing to undergraduates (Abdulkadiroğlu and Sönmez, 1999); the distribution of social housing to the population (see the New York City Department of Housing Preservation and Development); and in routing and connecting users to servers (Valiant, 1982).

In practice, a lottery is often used to gain equity: loosely speaking, it is a fair way to *ex-post* separate those *ex-ante* identical. Like the introduction of a time dimension or repetition, lotteries vastly extends the allocation space. These alterations to a model can overturn negative results: With a time dimension

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[1]A non-manipulable rule is sometimes referred to as "strategy-proof". In the preference revelation game associated with the rule, truth-telling is a weakly dominant strategy.

and sufficiently patient agents, we can overturn the non-existence of Condorcet winners (see Rubinstein, 1979; Bernheim and Slavov, 2009). Through repetition, an ex-ante efficient social choice function becomes implementable when copies of the same decision problem are linked together (Jackson and Sonnenschein, 2007). In Shapley and Scarf's (1974) object allocation problem, no deterministic rule satisfies either *equal treatment of equals*, *no-envy*, or *anonymity*; in contrast, there is a stochastic rule that satisfies all of these properties (Bogomolnaia and Moulin, 2001). These important results serve as inspiration for our analysis.

We start off where the literature on deterministic matching faced an obstacle. Namely, rules that always select stable matchings (Roth, 1982) or individually rational and efficient matchings (Alcalde and Barberà, 1994) are manipulable. A natural question is: what about rules that *randomize* between stable matchings? Proposition 1 shows that such "ex-post stable" rules fail a weak incentive requirement. We take this to imply that such rules fall outside the scope of this paper: indeed, the rule that we argue in favor of occasionally randomizes *only* between *unstable* matchings. This feature distinguishes our work sharply from the majority of papers on two-sided matching in which stable matchings are the key objects of study.

In the model, agents report ordinal preferences to a centralized clearing house. Conveniently, this is all the information that is used in defining the ex-post properties. In contrast, to define ex-ante properties, preferences over lotteries matter. We first use *stochastic dominance* to induce preferences over lotteries. We then note that each ordinal preference is consistent with infinitely many utility functions and, to each such function, we can associate an expected utility function over lotteries. These concept are connected: one lottery is not stochastically dominated by another whenever there is a utility function for which the expected utility of the former is at least that of the latter. We then parameterize how preferences over lotteries are induced by restricting the set of utility functions to consider to an arbitrary set $\mathcal{V}$. Specifically, one lottery is at least as good as another if its expected utility is no smaller for some utility function in $\mathcal{V}$. We then define a parameterized form of ex-ante stability, for which a larger $\mathcal{V}$ corresponds to a stronger "$\mathcal{V}$-stability". Generally, $\mathcal{V}$-stable probabilistic matchings need not exist unless the utility functions in $\mathcal{V}$ are "sufficiently similar". We then define "$\mathcal{V}$-non-manipulability" in the same spirit: reporting true preferences should be at least as good as telling a lie, no matter the utility function in $\mathcal{V}$ that is used to evaluate the resulting probabilistic matchings.

A set that turns out to be interesting is the set of utility functions with "increasing differences". A utility function u has *increasing differences* if, when j is preferred to k and k to m, the utility difference between j and k, $u(j)-u(k)$, exceeds the utility difference between k and m, $u(k)-u(m)$.[2] Loosely speaking, a utility function with increasing differences is bounded from below by a sufficiently steep exponential function. This type of utility function may represent that the agents differ severely in quality, like an Ivy League college versus a decent college versus an even worse college that still provides basic education.[3] Alternatively, we can focus on the preference over lotteries that these utility functions induce. In particular, a fair lottery between j and m is preferred to being matched to k with certainty. Most of our results have reference to the utility functions with increasing differences, $\mathcal{U}^{id}$.

Taking inspiration from David Gale's *Top Trading Cycles* mechanism (Shapley and Scarf, 1974) and the *Probabilistic Serial* mechanism (Bogomolnaia and Moulin, 2001), we introduce *Compromise and Rewards*, *CR*, a rule that selects probabilistic matchings.[4] Its essential feature is to make agent i's partner a coin flip between an agent who prefers i (the *compromise*) and an agent who i prefers (the *reward*). To achieve this, a graph is created in which each agent points to her preferred partner. This graph contains a cycle in which each agent is neighbor to an agent who prefers her and an agent who she prefers. In the lottery that *CR* selects, the agent is matched with probability one half to each of her neighbors. In this way, *CR* treats both "sides" equally, a feature that sharply distinguishes it from Gale and Shapley's (1962) *Deferred Acceptance* (*DA*). Moreover, as the agent has to "give up" her "reward" to affect her "compromise" (point to a less preferred agent), she does not benefit from misreporting her preferences.

[2]This is similar to a utility function that satisfies "uniformly relatively bounded indifference with respect to bound 1/2" in Mennle and Seuken (2017b,a).

[3]However, it does not apply if there is more than one Ivy League college as these would not "differ severly in quality".

[4]For the problem of allocating objects, probabilistic versions of *Top Trading Cycles* have been studied by Abdulkadiroğlu and Sönmez (1998), Kesten (2009), and Aziz (2015) among others.

Our most important result is Theorem 1. It shows that probabilistic rules can overturn the incompatibility of stability and non-manipulability in certain circumstances. Loosely speaking, stability first disqualifies rules that overlook too much of the preferences, like constant rules and dictatorships. This is reinforced by non-manipulability: the rule should benefit agents on both sides, as consistently favoring agents on one side may permit agents on the other side to manipulate. However, this then implies that agents may match not to someone they prefer but to someone who prefers them – a compromise. For the rule to nevertheless be stable, the utility functions have to weigh these compromises lightly compared to the corresponding "rewards". Along these lines, CR and $\mathcal{U}^{id}$ become natural candidates to examine. And indeed, Theorem 1 establishes that CR is $\mathcal{U}^{id}$*-stable* and $\mathcal{U}^{id}$*-non-manipulable.*

We then examine whether this result can be developed further – are there other rules that satisfy these properties, and what about other sets of utility functions? For the first question, Theorem 2 states that CR is the *only* rule to satisfy weaker versions of the properties in Theorem 1. Hence, *a rule is* $\mathcal{U}^{id}$*-stable and* $\mathcal{U}^{id}$*-non-manipulable if and only if it is* CR. For the second question, Theorem 3 shows that no rule is stable and non-manipulable for larger sets that contain $\mathcal{U}^{id}$. In addition, Theorem 4 establishes that positive results cannot be obtained for sets that "differ too much" from $\mathcal{U}^{id}$.

We offer two ways to interpret the results. The first is to say that, for an application in which the preferences naturally satisfy increasing differences, stability and non-manipulability are compatible. Indeed, for such an application, we present a novel rule with desirable properties. However, we do not have a real-world example in which restricting to increasing differences is compelling. The second is to turn it around: in an application that *lacks* a natural preference restriction, the properties are incompatible. But if we insist on *some form* of stability and non-manipulability, it is reasonable to go for the strongest forms compatible. In this way, CR and $\mathcal{U}^{id}$ provide a good solution: they impose minimal constraints on how agents' ordinal preferences induce preferences over stochastic outcomes (in the sense of Theorems 3 and 4), all the while guaranteeing the existence of an ex-ante stable and non-manipulable rule (Theorems 1 and 2).

We can put our results into context as follows. Gale and Shapley (1962) introduced the *two-sided matching problem* and showed that DA always selects a stable matching. The importance of stability was solidified through a series of papers that highlighted the benefit of selecting stable matchings when solving practical problems (see, for instance, Roth, 1984, 1991; Roth and Peranson, 1999; and the survey by Roth and Sotomayor, 1990). This encouraged researchers to get a better understanding of stable matchings; for our purposes, the work of Vande Vate (1989) and Rothblum (1992) is of particular interest. They developed a new way of describing stable matchings: namely, as the solutions to an integer program. This was followed up by Roth et al. (1993) who considered the program's relaxation, that is, the corresponding linear program. In short, the stable matchings are the extreme points of a polytope, appropriately termed "the stable matching polytope", and the linear program provides non-integer solutions that correspond to lotteries over stable matchings or "fractional stable matchings".[5] In a recent study, Manjunath (2013) shows that, if the agents compare lotteries using stochastic dominance, then lotteries over stable matchings are "weakly sd-stable". He also shows that, if the agents compare lotteries by their expected utility, then a lottery over unstable matchings may Pareto dominate a stable matching. Our work expands on this intuition.

An alternative interpretation of a lottery is time sharing: a coin flip between j and k can be interpreted as the agent spending half her time with j and half her time with k. Thus, our study is somewhat related to dynamic matching. This literature has been focused on developing notions of stability, like "strict self-sustaining stable plans" (Damiano and Lam, 2005), "dynamically-stable matchings" (Kadam and Kotowski, 2016), and "credible group stability" (Kurino, 2009). Preferences are generally modelled as dynamically changing and the results tend to rely on stable matchings and variations on DA. In further contrast to this paper, (full) non-manipulability has consistently been left out of the analysis. Indeed, this holds in general: presumably as a consequence of the discouraging results found by Roth (1982) and Alcalde and Barberà (1994), there have been few attempts to design rules that provide *all* agents with incentives to report preferences truthfully.[6] The success of DA in applications may also have contributed to the lack of

[5] See also Biró and Fleiner (2010), Chiappori et al. (2014), and Manjunath (2016).

[6] Some exceptions exist for restricted preference domains. Alcalde and Barberà (1994) show that, if preferences satisfy "top dominance", then there are stable and non-manipulable rules. Bogomolnaia and Moulin (2004) study dichotomous preferences.

attempts to reconcile stability and non-manipulability. In any case, our findings provide a novel contribution to the literature.

The paper is structured as follows. We present the model in Section 2. In Section 3, we examine ex-post properties. In Section 4, we introduce ex-ante properties. In Section 5, we present our main results and introduce *CR*. We discuss related questions in Section 6. Proofs are in the Appendix.

2 Model and definitions

There is a set of **agents** N partitioned into $M = \{m_1, \ldots, m_n\}$ and $W = \{w_1, \ldots, w_n\}$; we fix N, M, and W throughout. A **matching** $\mu\colon N \to N$ is a bijection that maps each agent to her partner such that, for $i \in N$, $\mu(i) = j \implies \mu(j) = i$. Agents may only match across sides, so $\mu(i) \in N^i$ where $N^i = M \cup \{i\}$ if $i \in W$ and $N^i = W \cup \{i\}$ if $i \in M$. If $\mu(i) = i$, then i is **single**. We also describe μ by its graph: $\mu = \{(i,j),(k,m),\ldots\}$ is equivalent to $\mu(i) = j$, $\mu(k) = m$, and so on. The **set of matchings** is $\mathcal{M}$. A **preference** for $i \in N$ is a binary relation R_i on N^i such that, for $\{j,k\} \subseteq N^i$, i finds j at least as desirable as k whenever $j \; R_i \; k$. Preferences are *complete*, *transitive*, and *antisymmetric*. The first- (top-) and second-ranked partners are $t(R_i)$ and $s(R_i)$: for each $j \in N^i \setminus \{t(R_i)\}$, $t(R_i) \; P_i \; s(R_i) \; R_i \; j$. The strict (*irreflexive*) relation is P_i. For $i \in N$, the **set of preferences** is $\mathcal{R}_i$. For $S \subseteq N$, $\mathcal{R}^S$ is the Cartesian product $\times_{i \in S} \mathcal{R}_i$. A profile of preferences, or simply a **profile**, is $R \equiv (R_i)_{i \in N} \in \mathcal{R}^N$.[7] A **two-sided matching problem** is completely described by a profile.

A matching is individually rational if each agent finds her partner at least as desirable as being single. Thus, for $R \in \mathcal{R}^N$, $\mu \in \mathcal{M}$ is **individually rational** if, for each $i \in N$, $\mu(i) \; R_i \; i$. A matching is (Pareto) efficient if no other matching makes everyone at least as well off and someone better off. Thus, for $R \in \mathcal{R}^N$, $\mu \in \mathcal{M}$ is **efficient** if there is no $\mu' \in \mathcal{M}$ such that, for each $i \in N$, $\mu'(i) \; R_i \; \mu(i)$, and, for some $j \in N$, $\mu'(j) \; P_j \; \mu(j)$. The **set of efficient matchings at $\boldsymbol{R}$** is $\mathcal{E}(R) \subseteq \mathcal{M}$. A group of agents block a matching if they can pair up to make everyone in the group better off. Thus, for $R \in \mathcal{R}^N$, $S \subseteq N$ **blocks** $\mu \in \mathcal{M}$ if there is $\mu' \in \mathcal{M}$ such that, for each $i \in S$, $\mu'(i) \in S$ and $\mu'(i) \; P_i \; \mu(i)$. A matching that is not blocked is **stable**. The **set of stable matchings at $\boldsymbol{R}$** is $\mathcal{S}(R) \subseteq \mathcal{M}$.

A **probabilistic matching** $\pi \in \mathbb{R}^{N \times N}$ is such that, for $\{i,j\} \subseteq N$, $\pi_{ij} = \pi_{ji} \in [0,1]$ is the probability with which i and j are matched. If $j \notin N^i$, then $\pi_{ij} = 0$. Moreover, π is *doubly stochastic*: $\sum_{j \in N^i} \pi_{ij} = 1$. The **set of probabilistic matchings** is Π. For $\pi_i \in \mathbb{R}^N$, $\operatorname{supp}(\pi_i) \equiv \{j \in N^i \mid \pi_{ij} > 0\}$. We reserve $\mu^0 \in \mathcal{M}$ and $\pi^0 \in \Pi$ to be such that, for each $i \in N$, $\mu^0(i) = i$ and $\pi^0_{ii} = 1$. For $\mathcal{L} \subseteq \mathcal{M}$, $\Delta\mathcal{L} \subseteq \Pi$ is the set of probabilistic matchings that are induced by lotteries over matchings in $\mathcal{L}$. As an example, suppose that $\mu = \{(m_1,w_1),(m_2,w_2)\}$, $\mu' = \{(m_1,w_1),(m_2),(w_2)\}$, and $\mu'' = \{(m_1),(w_1),(m_2,w_2)\}$. Then $\Delta(\{\mu,\mu^0\}) = \Delta(\{\mu',\mu''\})$. Moreover, $\Pi = \Delta\mathcal{M}$ (compare Birkhoff, 1946; von Neumann, 1953). A **rule** $\varphi\colon \mathcal{R}^N \to \Pi$ maps to each profile a probabilistic matching. With $\pi \equiv \varphi(R)$, we define $\varphi_i(R) \equiv (\pi_{ij})_{j \in N}$ and $\varphi_{ij}(R) \equiv \pi_{ij}$. We reserve φ^0 to be such that, for each $R \in \mathcal{R}^N$, $\varphi^0(R) = \pi^0$.

3 Ex-post properties

A natural starting point is to examine ex-post properties of rules. These, as the properties studied in the deterministic setting, are based solely on the agents' ordinal preferences.

A rule is *ex-post individually rational* if the rule always selects a probabilistic matching that is induced by a lottery over individually rational matchings.

Definition 1 (Ex-post individual rationality)**.** For each $R \in \mathcal{R}^N$ and $\{i,j\} \subseteq N$,

$$\varphi_{ij}(R) > 0 \implies j \; R_i \; i.$$

Azevedo and Budish (2013) examine a large-market limit. In contrast, we study the full preference domain in the finite case.

[7] For $\{i,j\} \subseteq N$ and $S \subseteq N$, we denote $R_{-i} \equiv (R_k)_{k \in N \setminus \{i\}}$, $R_{-ij} \equiv (R_k)_{k \in N \setminus \{i,j\}}$, and $R_{-S} \equiv (R_k)_{k \in N \setminus S}$.

A rule is *ex-post efficient/stable* if the rule's selection always is induced by a lottery over the respective type of matchings. As different lotteries may induce the same probabilistic matching, we require only there to *exist* a lottery over the right type of matchings that induces the rule's selection. Hence, the selection of an *ex-post efficient* rule may also be induced by a lottery over inefficient matchings.

Definition 2 (Ex-post efficiency)**.** For each $R \in \mathcal{R}^N$,

$$\varphi(R) \in \Delta\mathcal{E}(R).$$

Definition 3 (Ex-post stability)**.** For each $R \in \mathcal{R}^N$,

$$\varphi(R) \in \Delta\mathcal{S}(R).$$

Respect mutual best is a substantial weakening of *ex-post stability*. Agents who top-rank each other should be matched with certainty.

Definition 4 (Respect mutual best)**.** For each $R \in \mathcal{R}^N$ and $\{i, j\} \subseteq N$,

$$j = t(R_i) \text{ and } i = t(R_j) \implies \varphi_{ij}(R) = 1.$$

Ex-post stability implies *ex-post individual rationality*, *ex-post efficiency*, and *respect mutual best*. The latter three properties are logically independent.

Next is a requirement on the agents' incentives to report true preferences. An agent ex-post manipulates a rule if she with certainty is at least as well off and with positive probability is better off telling a lie than telling the truth at some profile. Thus, $i \in N$ **ex-post manipulates** φ at $R \in \mathcal{R}^N$ through $R'_i \in \mathcal{R}_i$ if $\pi'_i \equiv \varphi_i(R'_i, R_{-i}) \neq \varphi_i(R) \equiv \pi_i$ and, for each $\{j, k\} \subseteq N^i$, $\pi'_{ij} > 0$ and $\pi_{ik} > 0$ imply $j \; R_i \; k$. A rule is *ex-post non-manipulable* if it never is susceptible to ex-post manipulation.[8,9]

We are now ready to present our first result. Proposition 1 shows that the aforementioned ex-post properties are incompatible.

Proposition 1. *Let φ be ex-post individually rational; if φ respects mutual best or if φ is ex-post efficient, then φ is ex-post manipulable.*

Other combinations of the properties can be satisfied. For instance, *DA*[10] is *ex-post stable*, so it *respects mutual best* and is *ex-post individually rational* and *ex-post efficient*. The rule φ^0 is *ex-post individually rational* and *ex-post non-manipulable*. *Compromises and Rewards* (Section 5) *respects mutual best* and is *ex-post efficient* and *ex-post non-manipulable*.[11]

4 Ex-ante properties

Motivated by Proposition 1, we turn to ex-ante properties. The ex-post properties are silent on how agents compare lotteries over matchings; now, in contrast, preferences over probabilistic matchings matter. Recall that agents report ordinal preferences. In the absence of richer information, it is common to use *stochastic dominance* to induce preferences over probabilistic matchings.[12]

[8] For completeness, we define also **standard non-manipulability** for rules $\phi\colon \mathcal{R}^N \to \mathcal{M}$ that select matchings. *Standard non-manipulability* requires that, for each $R \in \mathcal{R}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}_i$, $\phi_i(R) \; R_i \; \phi_i(R'_i, R_{-i})$.

[9] We can define **weak ex-post non-manipulability** by requiring a manipulation to leave the agent better off with certainty. Randomizing between the W- and the M-optimal stable matching is *ex-post stable* and *weakly ex-post non-manipulable*.

[10] Specifically, the rule that, for each profile, selects the probabilistic matching induced by the degenerate lottery that puts probability 1 on the W-optimal stable matching.

[11] In consequence, *Compromises and Rewards* is not *ex-post individually rational*. An interesting parallel is d'Aspremont and Gérard-Varet (1979): their "expected externality mechanism" is efficient, non-manipulable, and budget-balanced but not individually rational.

[12] See, for instance, Bogomolnaia and Moulin (2001); Manjunath (2013); Doğan and Yildiz (2015).

4.1 Stochastic dominance and expected utility

Agent i's **sd-preference** is the binary relation R_i^{sd} on Π induced by $R_i \in \mathcal{R}_i$ such that, for each $\{\pi_i, \pi_i'\} \subseteq \mathbb{R}^N$,[13]

$$\pi_i \; R_i^{\mathrm{sd}} \; \pi_i' \iff \forall k \in N^i, \; \sum_{j R_i k} \pi_{ij} \geq \sum_{j R_i k} \pi_{ij}'.$$

If $\pi_i \; R_i^{\mathrm{sd}} \; \pi_i'$ but not $\pi_i' \; R_i^{\mathrm{sd}} \; \pi_i$, then $\pi_i \; P_i^{\mathrm{sd}} \; \pi_i'$. Equivalently, $\pi_i \; P_i^{\mathrm{sd}} \; \pi_i'$ if $\pi_i \; R_i^{\mathrm{sd}} \; \pi_i'$ and $\pi_i \neq \pi_i'$. To better connect the sd-preference to the parameterized preference that we later define, it is useful to now define the "not stochastically dominated by"-relation:

$$\pi_i \; R_i^{\mathrm{nsd}} \; \pi_i' \iff \neg \left(\pi_i' \; P_i^{\mathrm{sd}} \; \pi_i \right).$$

A **utility function** is an injective function $u \colon N \to \mathbb{R}$. The **set of utility functions** is $\mathcal{U}$. Moreover, $u \in \mathcal{U}$ is **consistent** with $R_i \in \mathcal{R}_i$ if, for each $\{j, k\} \subseteq N^i$, $j \; R_i \; k \iff u(j) \geq u(k)$. The **set of utility functions consistent with $\boldsymbol{R_i}$** is $\mathcal{U}(R_i) \subseteq \mathcal{U}$. With some abuse of notation, a **profile of utility functions** is $u = (u_i)_{i \in N} \in \mathcal{U}^N$. For $R \in \mathcal{R}^N$, let $\mathcal{U}^N(R) = \times_{i \in N} \mathcal{U}(R_i) \subseteq \mathcal{U}^N$. Given $u \in \mathcal{U}$, the **expected utility** of $\pi_i \in \mathbb{R}^N$ is

$$E(u, \pi_i) = \sum_{j \in N^i} \pi_{ij} u(j).$$

There is a utility function consistent with R_i for which the expected utility of π_i is at least that of π_i' if and only if π_i is not stochastically dominated by π_i' (see Bogomolnaia and Moulin, 2001). That is, for each $R_i \in \mathcal{R}_i$ and $\{\pi_i, \pi_i'\} \subseteq \mathbb{R}^N$,

$$\pi_i \; R_i^{\mathrm{nsd}} \; \pi_i' \iff \exists u \in \mathcal{U}(R_i) : E(u, \pi_i) \geq E(u, \pi_i').$$

4.2 Parametrization

We create a parameterized preference by restricting the utility functions to be considered. A **set of utility functions** is $\mathcal{V} \subseteq \mathcal{U}$. Such sets are non-empty, closed under positive affine transformation,[14] and closed under permutation of the agents' names. Intuitively, if u is in the set, then so are all the utility functions that have the same image as u. For each $\{i, j\} \subseteq N$, $\{R_i, R_j'\} \subseteq \mathcal{R}_i \times \mathcal{R}_j$, and $\{u, u'\} \subseteq \mathcal{U}(R_i) \times \mathcal{U}(R_j')$ such that $\{u(k) \mid k \in N^i\} = \{u'(k) \mid k \in N^j\}$, $u \in \mathcal{V} \iff u' \in \mathcal{V}$.

The parameterized preference $R_i^{\mathcal{V}}$ is defined as follows. For each $R_i \in \mathcal{R}_i$, $\mathcal{V} \subseteq \mathcal{U}$, and $\{\pi_i, \pi_i'\} \subseteq \mathbb{R}^N$,

$$\pi_i \; R_i^{\mathcal{V}} \; \pi_i' \iff \exists u \in \mathcal{V}(R_i) : E(u, \pi_i) \geq E(u, \pi_i').$$

The strict relation $P_i^{\mathcal{V}}$ is such that

$$\pi_i \; P_i^{\mathcal{V}} \; \pi_i' \iff \exists u \in \mathcal{V}(R_i) : E(u, \pi_i) > E(u, \pi_i').$$

We are now ready to define our ex-ante properties. Informally, agents S "$\mathcal{V}$-block" π through π' if each $i \in S$ has a utility function in $\mathcal{V}$ for which she prefers π' to π. Thus, the smaller $\mathcal{V}$ the more difficult it is to block and the more "$\mathcal{V}$-stable" outcomes exist; "strong sd-stability" (see Manjunath, 2013) is "$\mathcal{U}$-stability". For $R \in \mathcal{R}^N$ and $\mathcal{V} \subseteq \mathcal{U}$, $S \subseteq N$ **$\boldsymbol{\mathcal{V}}$-blocks** $\pi \in \Pi$ if there is $\pi' \in \Pi$ such that, for each $i \in S$, $\mathrm{supp}(\pi_i') \subseteq S$ and $\pi_i' \; R_i^{\mathcal{V}} \; \pi_i$, and, for some $j \in N$, $\pi_j' \; P_j^{\mathcal{V}} \; \pi_j$. A probabilistic matching that is not $\mathcal{V}$-blocked is **$\boldsymbol{\mathcal{V}}$-stable**. The **set of $\boldsymbol{\mathcal{V}}$-stable probabilistic matchings at $\boldsymbol{R}$** is $\mathcal{S}^{\mathcal{V}}(R) \subseteq \Pi$.

[13] The notation $\sum_{j R_i k}$ is short for summing over the set $\{j \in N^i \mid j \; R_i \; k\}$.

[14] If $v \in \mathcal{U}$ is a positive affine transformation of $u \in \mathcal{V}$, then $v \in \mathcal{V}$.

Definition 5 ($\mathcal{V}$-stability)**.** For each $R \in \mathcal{R}^N$,

$$\varphi(R) \in \mathcal{S}^{\mathcal{V}}(R).$$

For $R \in \mathcal{R}^N$ and $\mathcal{V} \subseteq \mathcal{U}$, $\pi \in \Pi$ is **$\mathcal{V}$-efficient** if there is no $\pi' \in \Pi$ such that, for each $i \in N$, $\pi'_i \; R_i^{\mathcal{V}} \; \pi_i$, and, for some $j \in N$, $\pi'_j \; P_j^{\mathcal{V}} \; \pi_j$. The **set of $\mathcal{V}$-efficient probabilistic matchings at R** is $\mathcal{E}^{\mathcal{V}}(R) \subseteq \Pi$.

Definition 6 ($\mathcal{V}$-efficiency)**.** For each $R \in \mathcal{R}^N$,

$$\varphi(R) \in \mathcal{E}^{\mathcal{V}}(R).$$

Finally, reporting true preferences should be at least as good as telling a lie, no matter the utility function in $\mathcal{V}$ that is used to evaluate the resulting probabilistic matchings. For $\mathcal{V} \subset \mathcal{W} \subseteq \mathcal{U}$, *$\mathcal{W}$-non-manipulability* is stronger than *$\mathcal{V}$-non-manipulability* whereas *$\mathcal{U}$-non-manipulability* coincides with "strategy-proofness" in Bogomolnaia and Moulin (2001, page 309).

Definition 7 ($\mathcal{V}$-non-manipulability)**.** For each $R \in \mathcal{R}^N$, $i \in N$, and $R'_i \in \mathcal{R}_i$,

$$\neg \left(\varphi_i(R'_i, R_{-i}) \; P_i^{\mathcal{V}} \; \varphi_i(R) \right).$$

Definition 8 ($\mathcal{V}$-group non-manipulability)**.** For each $R \in \mathcal{R}^N$, $S \subseteq N$, and $R'_S \in R^S$, there is $i \in S$ such that

$$\neg \left(\varphi_i(R'_S, R_{-S}) \; P_i^{\mathcal{V}} \; \varphi_i(R) \right).$$

5 Results

We now present our main contributions. In Subsection 5.1, we define the set of utility functions with "increasing differences" $\mathcal{U}^{id}$. In Subsection 5.2, we introduce *Compromises and Rewards*, *CR*, shown in Theorem 1 to be *$\mathcal{U}^{id}$-stable* and *$\mathcal{U}^{id}$-group non-manipulable*. Thereafter, Theorem 2 shows that *CR* is the *only* rule that *respects mutual best* and is *$\mathcal{U}^{id}$-efficient* and *$\mathcal{U}^{id}$-non-manipulable*. Theorem 3 is a maximality result with regards to $\mathcal{U}^{id}$. Finally, Theorem 4 is a necessary condition on $\mathcal{V}$ for there to exist *$\mathcal{V}$-efficient* and *$\mathcal{V}$-non-manipulable* rules that *respect mutual best*.

5.1 Utility functions with increasing differences

A utility function has "increasing differences" if, for each triplet of agents, the utility difference between the most and the second most preferred exceeds the utility difference between the second most and the least preferred.[15] The **set of utility functions with increasing differences** is $\mathcal{U}^{id} \subseteq \mathcal{U}$.

Definition 9 ($u \in \mathcal{U}(R_i)$ has increasing differences)**.** For each $\{j, k, m\} \subseteq N^i$,

$$j \; P_i \; k \; P_i \; m \implies u(j) - u(k) > u(k) - u(m). \tag{id}$$

Inequality (id) applies to *every* triplet of agents and not merely consecutively ranked ones. As an example, say $u \in \mathcal{U}$ has increasing differences and takes on the values $\alpha_0 \leq \cdots \leq \alpha_n$, that is, $\{u(j) \mid j \in N^i\} = \{\alpha_0, \ldots, \alpha_n\}$. Increasing differences implies that the function is bounded from below by an exponential function; if, say, $\alpha_0 = 0$ and $\alpha_1 = 2$, then, for each $k \geq 1$, $\alpha_k \geq 2^k$.

[15] If we change the inequality to a weak inequality, then Theorem 1 no longer holds unless *$\mathcal{V}$-stability* is weakened.

R_{m_1}	R_{m_2}	R_{m_3}	R_{m_4}	R_{m_5}	R_{w_1}	R_{w_2}	R_{w_3}	R_{w_4}	R_{w_5}
w_1	w_2	w_3	w_4	w_5	m_2	m_3	m_1	m_4	m_4
									$\emptyset$

Table 1: Preferences for Example 1. Less preferred agents than those listed can be ranked in any way.

5.2 *Compromises and Rewards*

We now introduce a rule that satisfies several desirable properties for the set $\mathcal{U}^{id}$. A further appealing novelty is that the rule treats the two "sides" symmetrically.[16] In this way, it is fundamentally different from DA, which by design gives some agents preferential treatment.

Definition 10 (Compromises and Rewards, CR)**.** For each $R \in \mathcal{R}^N$, define $\pi \equiv CR(R)$ as follows.

1. Create a graph in which each agent points to her top-ranked partner in the graph.
2. Select a cycle in the graph and label it $C = 1, \ldots, m$.
3. If $m \leq 2$, set, for each $i \in C$, $\pi_{i,i+1} = 1 \quad (i+1 \mod m)$.
4. If $m > 2$, set, for each $i \in C$, $\pi_{i,i+1} = 1/2 \ (i+1 \mod m)$.
5. Remove the agents of C and reiterate the procedure until no agent remains.

A *cycle* is a list of agents $C = 1, \ldots, m$ such that 1 points to 2, 2 to 3, …, $(m-1)$ to m, and m to 1 (if $m = 1$, then agent 1 points to herself). As there is a finite number of agents, there always is a cycle. Indeed, there are as many cycles as there are components in the created graph. If there are multiple components, then the order in which we proceed is irrelevant: a cycle remains a cycle if another cycle is removed. We illustrate the rule in Example 1.

Example 1. Consider the profile R in Table 1. We reiterate the procedure four times to determine $CR(R)$; see Figure 1. There are two cycles in the first graph and the one not selected, m_4, w_4, remains a cycle in the second graph.

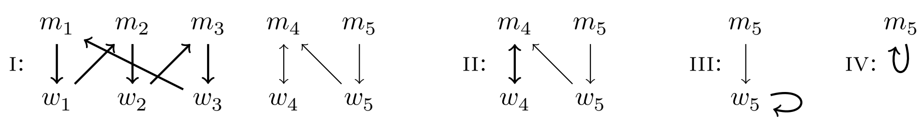

Figure 1: The four graphs encountered in determining $CR(R)$.

The non-zero entries of $\pi \equiv CR(R)$ are as follows:

$$\pi_{m_1 w_1} = \pi_{w_1 m_2} = \pi_{m_2 w_2} = \pi_{w_2 m_3} = \pi_{m_3 w_3} = \pi_{w_3 m_1} = 1/2$$
$$\pi_{m_4 w_4} = \pi_{w_5 w_5} = \pi_{m_5 m_5} = 1.$$

5.3 Main results

Theorem 1 is positive: it shows that probabilistic rules can overturn the incompatibility between stability and non-manipulability in the right circumstances. Thereafter, Theorem 2 pins down CR as the only rule to satisfy weaker versions of the properties for $\mathcal{U}^{id}$.

Theorem 1. *Compromises and Rewards is $\mathcal{U}^{id}$-stable and $\mathcal{U}^{id}$-group non-manipulable.*

[16] It is "gender fair" (Özkal-Sanver, 2004).

Theorem 2. *If a rule respects mutual best and is $\mathcal{U}^{id}$-efficient and $\mathcal{U}^{id}$-non-manipulable, then it is Compromises and Rewards.*

Next, we show that the properties are incompatible for larger sets that contain $\mathcal{U}^{id}$. To derive this maximality result, we parametrize inequality (id) by $\varepsilon \geq 0$ to define the set $\mathcal{U}_\varepsilon^{id} \supseteq \mathcal{U}^{id}$. For each $\varepsilon \geq 0$, $R_i \in \mathcal{R}_i$, and $u \in \mathcal{U}(R_i)$, $u \in \mathcal{U}_\varepsilon^{id}(R_i)$ if and only if, for each $\{j, k, m\} \subseteq N^i$,

$$j \; P_i \; k \; P_i \; m \implies (1+\varepsilon)(u(j) - u(k)) > u(k) - u(m).$$

The larger ε, the larger $\mathcal{U}_\varepsilon^{id}$; if $\varepsilon = 0$, then $\mathcal{U}_\varepsilon^{id} = \mathcal{U}^{id}$.

Theorem 3. *For each $\varepsilon > 0$, no rule respects mutual best and is $\mathcal{U}_\varepsilon^{id}$-efficient and $\mathcal{U}_\varepsilon^{id}$-non-manipulable.*

Finally, Theorem 4 shows that utility functions with increasing differences essentially are necessary for the properties to be compatible. Specifically, the utility functions have to be bounded from below by the exponential function with base $\beta \equiv (\sqrt{13} - 1)/2 \approx 1.3028$ (recall that the corresponding function for increasing differences has base 2).

Theorem 4. *Let $u \in \mathcal{V} \subseteq \mathcal{U}$ take on the values $\alpha_0 = 0 < \alpha_1 = \beta < \alpha_2 < \cdots < \alpha_n$. If there is a rule that respects mutual best and is $\mathcal{V}$-efficient and $\mathcal{V}$-non-manipulable, then, for each $k \geq 1$, $\alpha_k \geq \beta^k$.*

To see the implications of Theorem 4, we can apply it to two "standard" utility functions. Suppose first that $u \in \mathcal{U}$ takes on the values $0, 1, \ldots, n$. Then, if $n \geq 6$ and $u \in \mathcal{V}$, no rule *respects mutual best* and is *$\mathcal{V}$-efficient* and *$\mathcal{V}$-non-manipulable*. If u instead takes on the values $\sqrt{0}, \sqrt{1}, \ldots, \sqrt{n}$, then the properties are incompatible for $n \geq 3$.

6 Discussion

We conclude with a discussion on plausible changes to the model.

6.1 What if agents report cardinal preferences?

Instead of using utility functions to find the best "ordinal"[17] rule, agents may report utilities from $\mathcal{U}^{id}$ directly. The strategy spaces are extended immensely: this makes non-manipulability significantly stronger whereas stability gets significantly weaker. By Theorem 2, *CR* is the only *ordinal* rule that is *$\mathcal{U}^{id}$-stable* and *$\mathcal{U}^{id}$-non-manipulable*, but we can only conjecture that the properties characterize the rule. To support this conjecture, we refer to similar results for related models (like Ehlers et al., 2016): even though there is cardinal information available, this information cannot affect the selection of a non-manipulable rule.

6.2 What if agents may be indifferent?

An interesting generalization is to allow agents to be indifferent between partners, that is, preferences need not be antisymmetric. Each agent partitions her potential partners into "tiers" of agents between whom she is indifferent. So, at the top tier may be the most successful students and companies; at the second top tier are the ones slightly worse; and so on. It is plausible that a college student would take a gamble for an internship at a world-leading company at the risk of ending somewhere else over a certain but only decent internship. Unfortunately, the impossibility cannot be overturned.[18,19]

[17] A rule is *ordinal* if, for each pair of utility profiles consistent with the same preference profile, the rule makes the same selection.

[18] Non-manipulability is here strengthened in two ways: (i) there are more profiles at which to manipulate, and (ii) there are more lies to manipulate through. Our proof uses only (i). That is, an agent manipulates at a profile where his preference includes an indifference by telling a lie that includes no indifferences.

[19] *Respect mutual best* is redefined as follows. Suppose j is the only one out of i's top-ranked partners who top-ranks i. Moreover, i is j's only top-ranked partner. Then i and j should be matched with certainty.

R_{m_1}	R_{m_2}	R_{m_3}	R_{w_1}	R_{w_2}	R_{w_3}
w_1	w_1	w_3	m_3	m_1	m_1
w_2	w_2	w_2	m_2	m_2	m_2
w_3	w_3	w_1	m_1	m_3	m_3

Table 2: Preferences for Subsection 6.4.

Proposition 2. *When agents may be indifferent, if a rule respects mutual best and is $\mathcal{U}^{id}$-efficient, then it is $\mathcal{U}^{id}$-manipulable.*

6.3 What if there is no real outside option?

The combination of individual rationality and non-manipulability is generally very strong, as, for instance, shown in Proposition 1. However, we may also consider a setting in which individual rationality is vacuous. More precisely, suppose that each agent prefers a partner to being single, regardless of the partner. This yields a new domain of profiles $\mathcal{R}_0^N \subset \mathcal{R}^N$ such that, for each $m \in M$, $w \in W$, $R_m \in (\mathcal{R}_0)_m$, and $R_w \in (\mathcal{R}_0)_w$, $w \; P_m \; m$ and $m \; P_w \; w$. Roth (1982) focused on this domain and showed that rules that select stable matchings are *standard manipulable*. As a side note, we strengthen his result:

Proposition 3. *If $\phi \colon \mathcal{R}_0^N \to \mathcal{M}$ respects mutual best, then ϕ is standard manipulable.*

Of course, our focus is on rules that select probabilistic matchings. As a corollary to Theorem 1, *CR* is $\mathcal{U}^{id}$*-stable* and $\mathcal{U}^{id}$*-non-manipulable* in this restricted setting. However, it is no longer the only rule to satisfy these properties.[20]

6.4 How is *CR* related to the optimal stable matchings?

It is natural to think of *CR* as randomizing between two matchings: one that consistently favors the women and one that favors the men. From the perspective of the men, the former matching is the "compromise", the latter the "reward". Denote these matchings μ_c and μ_r, respectively. The profile R in Table 2 shows that there is no systematic relation between μ_c and μ_r and the woman- and man-optimal stable matchings, μ_w and μ_m. At R, m_1 prefers $\mu_w(m_1) = w_2$ to $\mu_c(m_1) = w_3$ and $\mu_r(m_1) = w_1$ to $\mu_m(m_1) = w_2$. In contrast, m_2 prefers $\mu_c(m_2) = w_2$ to $\mu_w(m_2) = w_3$ and $\mu_m(m_2) = w_1$ to $\mu_r(m_2) = w_2$.

6.5 What about matching in general?

The general matching problem, or the "roommate problem", is studied by Tan (1991), Abeledo and Rothblum (1994, 1996), Teo and Sethuraman (1998), Chung (2000), and Gudmundsson (2014b), among others. The agents are no longer partitioned into two sides but are instead part of a single group. Preliminary results indicate that *CR* can be adapted to be non-manipulable and select probabilistic matchings that cannot be blocked by groups of fewer than six agents (Gudmundsson, 2014a). Whether the rule is fully stable is an open question.

References

Abdulkadiroğlu, A. and Sönmez, T. (1998). Random Serial Dictatorship and the Core from Random Endowments in House Allocation Problems. *Econometrica*, 66:689–701.

[20] Let $\varepsilon > 0$. Let $\pi \equiv \varphi(R) = CR(R)$ unless $R \in \mathcal{R}^N$ is such that, when defining $CR(R)$, there is a cycle that contains all agents. If so, for each $i \in N \cap W$, set $\pi_{i,i+1} = 1/2 + \varepsilon$ and $\pi_{i,i-1} = 1/2 - \varepsilon$. For sufficiently small ε, φ is $\mathcal{U}^{id}$*-stable* and $\mathcal{U}^{id}$*-group non-manipulable.*

Abdulkadiroğlu, A. and Sönmez, T. (1999). House Allocation with Existing Tenants. *Journal of Economic Theory*, 88:233–260.

Abeledo, H. G. and Rothblum, U. G. (1994). Stable Matchings and Linear Inequalities. *Discrete Applied Mathematics*, 54(1):1–27.

Abeledo, H. G. and Rothblum, U. G. (1996). Stable Matchings and Linear Programming. *Linear Algebra and its Applications*, 245:321–333.

Alcalde, J. and Barberà, S. (1994). Top Dominance and the Possibility of Strategy-proof Stable Solutions to Matching Problems. *Economic Theory*, 4(3):417–435.

Azevedo, E. M. and Budish, E. (2013). Strategy-proofness in the Large. *Research Paper 13-35, Chicago Booth*.

Aziz, H. (2015). Generalizing Top Trading Cycles for Housing Markets with Fractional Endowments. *mimeo*.

Bernheim, B. D. and Slavov, S. N. (2009). A Solution Concept for Majority Rule in Dynamic Settings. *Review of Economic Studies*, 76(1):33–62.

Birkhoff, G. (1946). Three Observations on Linear Algebra. *Revi. Univ. Nac. Tucuman, ser A*, 5:147–151.

Biró, P. and Fleiner, T. (2010). Fractional Solutions for NTU-games. *In Proceedings of COMSOC 2010: 3rd International Workshop on Computational Social Choice*, pages 283–294.

Bogomolnaia, A. and Moulin, H. (2001). A New Solution to the Random Assignment Problem. *Journal of Economic Theory*, 100(2):295–328.

Bogomolnaia, A. and Moulin, H. (2004). Random Matching Under Dichotomous Preferences. *Econometrica*, 72(1):257–279.

Bronfman, S., Hassidim, A., Afek, A., Romm, A., Sherberk, R., Hassidim, A., and Massler, A. (2015). Assigning Israeli medical graduates to internships. *Israel Journal of Health Policy Research*, 4:6.

Chiappori, P.-A., Galichon, A., and Salanié, B. (2014). The Roommate Problem - Is More Stable Than You Think. CESifo Working Paper Series 4676, CESifo Group Munich.

Chung, K.-S. (2000). On the Existence of Stable Roommate Matchings. *Games and Economic Behavior*, 33:9–15.

Damiano, E. and Lam, R. (2005). Stability in Dynamic Matching Markets. *Games and Economic Behavior*, 52:34–53.

d'Aspremont, C. and Gérard-Varet, L.-A. (1979). Incentives and Incomplete Information. *Journal of Public Economics*, 11:25–45.

Doğan, B. and Yildiz, K. (2015). Efficiency and Stability of Probabilistic Assignments in Marriage Problems. *Games and Economic Behavior*, 95:47–58.

Ehlers, L., Majumdar, D., Mishra, D., and Sen, A. (2016). Continuity and Incentive Compatibility in Cardinal Voting Mechanisms. *Working Paper*.

Gale, D. and Shapley, L. S. (1962). College Admissions and the Stability of Marriage. *American Mathematical Monthly*, 69:9–15.

Gudmundsson, J. (2014a). Sequences in Pairing Problems: A New Approach to Reconcile Stability with Strategy-Proofness for Elementary Matching Problems. Working Papers 2014:40, Lund University, Department of Economics.

Gudmundsson, J. (2014b). When Do Stable Roommate Matchings Exist? A Review. *Review of Economic Design*, 18(2):151–161.

Jackson, M. O. and Sonnenschein, H. F. (2007). Overcoming Incentive Constraints by Linking Decisions. *Econometrica*, 75(1):241–257.

Kadam, S. V. and Kotowski, M. H. (2016). Time Horizons, Lattice Structures, and Welfare in Multi-Period Matching Markets. *mimeo.*

Kesten, O. (2009). Why do popular mechanisms lack efficiency in random environments? *Journal of Economic Theory*, 144(5):2209–2226.

Kurino, M. (2009). Credibility, Efficiency, and Stability: A Theory of Dynamic Matching Markets. *mimeo.*

Manjunath, V. (2013). Stability and the Core of Probabilistic Marriage Problems. *Working Paper.*

Manjunath, V. (2016). Markets for Fractional Partnerships. *Forthcoming in Games and Economic Behavior.*

Mennle, T. and Seuken, S. (2017a). Hybrid Mechanisms: Trading Off Strategyproofness and Efficiency of Random Assignment Mechanisms. *Working paper.*

Mennle, T. and Seuken, S. (2017b). Partial Strategyproofness: Relaxing Strategyproofness for the Random Assignment Problem. *Working paper.*

Özkal-Sanver, I. (2004). A Note on Gender Fairness in Matching Problems. *Mathematical Social Sciences*, 47(2):211–217.

Roth, A. E. (1982). The Economics of Matching: Stability and Incentives. *Mathematics of Operations Research*, 7(4):617–628.

Roth, A. E. (1984). The Evolution of the Labor Market for Medical Interns and Residents: A Case Study in Game Theory. *Journal of Political Economy*, 92(6):991–1016.

Roth, A. E. (1991). A Natural Experiment in the Organization of Entry-Level Labor Markets: Regional Markets for New Physicians and Surgeons in the United Kingdom. *American Economic Review*, 81(3):415–440.

Roth, A. E. and Peranson, E. (1999). The Redesign of the Matching Market for American Physicians: Some Engineering Aspects of Economic Design. *American Economic Review*, 89(4):748–780.

Roth, A. E., Rothblum, U. G., and Vande Vate, J. H. (1993). Stable Matchings, Optimal Assignments, and Linear Programming. *Mathematics of Operations Research*, 18:803–828.

Roth, A. E. and Shorrer, R. I. (2015). The Redesign of the Medical Intern Assignment Mechanism in Israel. *Israel Journal of Health Policy Research*, 4:11.

Roth, A. E. and Sotomayor, M. (1990). *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis.* Econometric Society Monographs, Cambridge University Press, New York.

Rothblum, U. G. (1992). Characterization of Stable Matchings as Extreme Points of a Polytope. *Mathematical Programming*, 54:57–67.

Rubinstein, A. (1979). A Note about the "Nowhere Denseness" of Societies Having an Equilibrium under Majority Rule. *Econometrica*, 47(2):511–514.

Shapley, L. S. and Scarf, H. E. (1974). On cores and indivisibility. *Journal of Mathematical Economics*, 1:23–27.

R_{m_1}	R_{m_2}	R_{w_1}	R_{w_2}	R'_{m_1}	R'_{m_2}	R'_{w_1}	R'_{w_2}	R''_{m_1}	R''_{m_2}	R''_{w_1}	R''_{w_2}
w_1	w_2	m_2	m_1	w_1	w_2	m_2	m_1	w_2	w_1	m_1	m_2
w_2	w_1	m_1	m_2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Table 3: Preferences for the proof of Proposition 1.

Stasz, C. and van Stolk, C. (2007). The Use of Lottery Systems in School Admissions. Working Papers 460, RAND Europe.

Tan, J. J. M. (1991). A Necessary and Sufficient Condition for the Existence of a Complete Stable Matching. *Journal of Algorithms*, 12:154–178.

Taylor, B. A. and Trogdon, J. G. (2007). Losing to Win: Tournament Incentives in the National Basketball Association. *Journal of Labor Economics*, 20(1):23–41.

Teo, C.-P. and Sethuraman, J. (1998). The Geometry of Fractional Stable Matchings and its Application. *Mathematics of Operations Research*, 23:874–891.

Valiant, L. G. (1982). A Scheme for Fast Parallel Communication. *SIAM Journal on Computing*, 11(2):350–361.

Vande Vate, J. H. (1989). Linear Programming Brings Marital Bliss. *Operations Research Letters*, 8:147–153.

von Neumann, J. (1953). *A Certain Zero-Sum Two-Person Game Equivalent to the Optimal Assignment Problem*, volume 2. Contributions to the theory of games.

A Proofs

In the proofs, we abbreviate *ex-post* by *xp*, $\mathcal{U}^{id}$ by *id*, *standard* by *st*, *individual rationality* by *IR*, *respect mutual best* by *RMB*, *efficiency* by *EFF*, *non-manipulability* by *NM*, and *stability* by *STAB*.

A.1 Proof of Proposition 1

Let $M = \{m_1, m_2\}$, $W = \{w_1, w_2\}$, and refer to preferences in Table 3. Let φ be *xp-IR*, $\pi \equiv \varphi(R)$, $\pi' \equiv \varphi(R'_{m_1}, R_{-m_1})$, and $\pi'' \equiv \varphi(R'_{w_1}, R_{-w_1})$.

Part 1: Let φ *RMB*. To obtain a contradiction, suppose φ is *xp-NM*. By *RMB*, $\varphi_{w_2 m_2}(R'_{m_1}, R''_{w_2}, R_{-m_1 w_2}) = 1$. By *xp-IR*, $\pi'_{m_1 w_2} = 0$. By *xp-NM*, $\pi'_{w_2 m_2} = 1$, else w_2 manipulates at (R'_{m_1}, R_{-m_1}) through R''_{w_2}. So $\pi'_{w_1 m_2} = 0$. By *RMB*, $\varphi_{w_1 m_1}(R'_{m_1}, R''_{w_1}, R_{-m_1 w_1}) = 1$. By *xp-NM*, $\pi'_{w_1 m_1} = 1$, else w_1 manipulates at (R'_{m_1}, R_{-m_1}) through R''_{w_1}. By *xp-NM*, $\pi_{m_1 w_1} = 1$, else m_1 manipulates at R through R'_{m_1}.

By *RMB*, $\varphi_{m_1 w_2}(R'_{w_1}, R''_{m_1}, R_{-m_1 w_1}) = 1$. By *xp-IR*, $\pi''_{w_1 m_1} = 0$. By *xp-NM*, $\pi''_{m_1 w_2} = 1$, else m_1 manipulates at (R'_{w_1}, R_{-w_1}) through R''_{m_1}. So $\pi'_{m_2 w_2} = 0$. By *RMB*, $\varphi_{m_2 w_1}(R'_{w_1}, R''_{m_2}, R_{-m_2 w_1}) = 1$. By *xp-NM*, $\pi''_{m_2 w_1} = 1$, else m_2 manipulates at (R'_{w_1}, R_{-w_1}) through R''_{m_2}. By *xp-NM*, $\pi_{w_1 m_2} = 1$, else w_1 manipulates at R through R'_{w_1}. Therefore $\pi_{w_1 m_1} = \pi_{w_1 m_2} = 1$, which is a contradiction.

Part 2: Let φ be *xp-EFF*. To obtain a contradiction, suppose φ is *xp-NM*. By *xp-IR* and *xp-EFF*, $\varphi_{m_2 w_2}(R'_{m_1}, R'_{m_2}, R_{-m_1 m_2}) = 1$. By *xp-NM*, $\pi'_{m_2 w_2} = 1$, else m_2 manipulates at (R'_{m_1}, R_{-m_1}) through R'_{m_2}. By *xp-EFF*, $\pi'_{m_1 w_1} = 1$. By *xp-NM*, $\pi_{m_1 w_1} = 1$, else m_1 manipulates at R through R'_{m_1}.

By *xp-IR* and *xp-EFF*, $\varphi_{w_2 m_1}(R'_{w_1}, R'_{w_2}, R_{-w_1 w_2}) = 1$. By *xp-NM*, $\pi''_{w_2 m_1} = 1$, else w_2 manipulates at (R'_{w_1}, R_{-w_1}) through R'_{w_2}. By *xp-EFF*, $\pi''_{w_1 m_2} = 1$. By *xp-NM*, $\pi_{w_1 m_2} = 1$, else w_1 manipulates at R through R''_{w_1}. Therefore $\pi_{w_1 m_1} = \pi_{w_1 m_2} = 1$, which is a contradiction. □

A.2 Proof of Theorem 1

We proceed in two separate parts that have similar setups. Let $R \in \mathcal{R}^N$, $S \subseteq N$, and $\pi \equiv CR(R)$. In Part 1, suppose there is $\pi' \in \Pi$ such that π is blocked by S through π'. In Part 2, suppose there is $R'_S \in \mathcal{R}^S$ such that S manipulates at R through R'_S; let $\pi' \equiv \varphi(R'_S, R_{-S})$. Furthermore, assume S is minimal in this regard: that is, no $T \subset S$ blocks or manipulates. Let $C^0, C^1, \dots$ denote the cycles encountered when determining $CR(R)$. Moreover, let t be such that, for each $x < t$, $S \cap C^x = \emptyset$. Let $C^t = 1, \dots, m$. (Our proof is for $m > 2$; the case $m \leq 2$ can be handled along the same lines but is much simpler.) To obtain a contradiction, suppose $S \cap C^t \neq \emptyset$.

Part 1: *id-STAB.*

Let $i \in S \cap C^t$. As i points to $i+1$ in the cycle C^t, for each $j \in N \setminus (C^0 \cup C^1 \cup \dots \cup C^{t-1}) \supseteq S$, $(i+1)\ R_i\ j$.

Suppose first $(i+1) \notin S$. Then, for each $j \in S$, $(i+1)\ P_i\ j$. Let j denote i's most preferred agent in S: for each $k \in S$, $j\ R_i\ k$. If $(i-1)\ R_i\ j$, then i is clearly worse off at π' than at π, a contradiction. Otherwise, if $j\ P_i\ (i-1)$, then, for each $u \in \mathcal{U}^{id}(R_i)$,

$$E(u, \pi_i) - E(u, \pi'_i) \geq \frac{u(i+1) + u(i-1)}{2} - u(j) = \frac{(u(i+1) - u(j)) - (u(j) - u(i-1))}{2} > 0,$$

so i is worse off at π' than at π. This is a contradiction. Hence, $(i+1) \in S$. By repeating the argument for each $i \in C^t$, $C^t \subseteq S$.

Suppose $i \in C^t$ is such that no $j \in C^t$ has a lower probability of being matched with her neighbor of C^t at π': for each $j \in C^t$, $\pi'_{i,i+1} \leq \pi'_{j,j+1}$. This includes $j = i-1$, so $\pi'_{i,i+1} \leq \pi'_{i-1,i}$. As $\pi'_{i,i+1} + \pi'_{i,i-1} \leq 1$, $\pi'_{i,i+1} \leq 1/2$.

To obtain a contradiction, suppose $\pi_{i,i+1} < 1/2$. Let j denote i's most preferred agent in S excluding $i+1$: for each $k \in S \setminus \{i+1\}$, $j\ R_i\ k$. As $C^t \subseteq S$, $(i-1) \in S$, so $j\ R_i\ (i-1)$. For each $u \in \mathcal{U}^{id}(R_i)$ such that $u(j) = 0$,[21]

$$u(i+1) + u(i-1) = \big((u(i+1) - u(j)) - (u(j) - u(i-1))\big) > 0.$$

As first $u(i-1) \leq 0$ and $0 \leq \pi'_{i,i+1} \leq \pi'_{i,i-1}$ by the choice of i, and second $\pi'_{i,i+1} < 1/2$ by assumption and $u(i+1) + u(i-1) > 0$, i is not better off at π' than at π:

$$\begin{aligned} E(u, \pi'_i) &\leq \pi'_{i,i+1} u(i+1) + \pi'_{i,i-1} u(i-1) + (1 - \pi'_{i,i+1} - \pi'_{i,i-1}) \cdot u(j) \\ &\leq \pi'_{i,i+1} u(i+1) + \pi'_{i,i+1} u(i-1) + (1 - \pi'_{i,i+1} - \pi'_{i,i-1}) \cdot 0 \\ &= \pi'_{i,i+1}\big(u(i+1) + u(i-1)\big) \\ &< (1/2) \cdot \big(u(i+1) + u(i-1)\big) = E(u, \pi_i). \end{aligned}$$

This is a contradiction, so $\pi'_{i,i+1} = 1/2$. But then, for each $i \in C^t$, $\pi'_{i,i+1} = 1/2$ and $\pi'_i = \pi_i$. Hence, all agents in C^t are matched under π' as under π and no one is better off. Then $S \setminus C^t \subset S$ can block, a contradiction to S being minimal.

Part 2: *id-group NM.* Note first that the misreport has no effect on the cycles $C^0, C^1, \dots, C^{t-1}$: they occur when defining $CR(R'_S, R_{-S})$ as well. Let $N^t \equiv N \setminus (C^0 \cup C^1 \cup \dots \cup C^{t-1})$. For each $i \in S$ and $j \notin N^t$, $\pi'_{ij} = 0$. By a similar argument as in Part 1, we can show that, for each $i \in S \cap C^t$, $\pi'_{i,i+1} \geq 1/2$, else there is a utility function in $\mathcal{U}^{id}(R_i)$ for which i is worse off at π' than at π. So if i is selected as part of the cycle C' when defining $CR(R'_S, R_{-S})$, then i is a neighbor of $(i+1)$ in C'.

Suppose $(i+1) \notin S$. For each $j \in N^t$, $(i+2)\ R_{i+1}\ j$. Hence, $(i+1)$ points to $(i+2)$ when C' is selected, so $C' = \dots, i, (i+1), (i+2), \dots$. Otherwise, if $(i+1) \in S$, then $\pi'_{i+1,i+2} \geq 1/2$, else there is a utility function in $\mathcal{U}^{id}(R_{i+1})$ for which $(i+1)$ is worse off at π' than at π. Therefore, $C' = \dots, i, (i+1), (i+2), \dots$ or $C' = \dots, (i+2), (i+1), i, \dots$. Repeating the argument, we find that either $C' = i, (i+1), \dots, (i-1) = C^t$ or $C' = i(i-1) \dots (i+1)$ (that is, C^t in reverse). But then, for each $i \in C^t$, $\pi'_i = \pi_i$, so no agent in C^t is better off at π' than at π. Again, then $S \setminus C^t \subset S$ can manipulate, a contradiction to S being minimal. $\square$

[21] As $\mathcal{U}^{id}$ is closed under affine transformations, such a u exists.

Lemma 1. *Let φ be id-NM. Consider $R \in \mathcal{R}^N$, $\pi \equiv \varphi(R)$, and $\{i, j\} \subseteq N$ such that $\pi_{ij} > 0$ and, for each $k \in N^i$ such that $k \; P_i \; j$, $\pi_{ik} = 0$. Let $R'_i \in \mathcal{R}_i$ be such that $j = t(R'_i)$ and let $\pi' \equiv \varphi(R'_i, R_{-i})$. Then $\pi'_{ij} = \pi_{ij}$.*

Proof. To obtain a contradiction, suppose first $\pi'_{ij} > \pi_{ij} > 0$. Let $S = \{k \in N^i \mid j \; P_i \; k\}$ and $u \in \mathcal{U}^{id}(R_i)$ be such that $\max_{k \in S} u(k) = 1$, $\min_{k \in S} u(k) = 0$, and

$$u(j) = \frac{1 - \pi_{ij} + \pi'_{ij}}{\pi'_{ij} - \pi_{ij}} = 1 + \frac{1}{\pi'_{ij} - \pi_{ij}} > 2.$$

Then i manipulates at R through R'_i, contradicting *id-NM*:

$$\begin{aligned} E(u, \pi'_i) - E(u, \pi_i) &\geq \Big(\pi'_{ij} u(j) + (1 - \pi'_{ij}) \cdot 0\Big) - \Big(\pi_{ij} u(j) + (1 - \pi_{ij}) \cdot 1\Big) \\ &= (\pi'_{ij} - \pi_{ij}) u(j) - (1 - \pi_{ij}) \\ &= (1 - \pi_{ij} + \pi'_{ij}) - (1 - \pi_{ij}) = \pi'_{ij} > 0. \end{aligned}$$

Analogously, if $\pi'_{ij} < \pi_{ij}$, then use $u'(j) \equiv 2 - u(j)$ to show that $E(u', \pi_i) > E(u', \pi'_i)$. So i manipulates at (R'_i, R_{-i}) through R_i, contradicting *id-NM*. □

Lemma 2. *Let φ RMB and be id-NM. Consider $R \in \mathcal{R}^N$, $\pi \equiv \varphi(R)$, and $\{i, j\} \subseteq N$ such that $j = t(R_i)$ and $i = s(R_i)$. Then $\pi_{ij} \geq 1 - \pi_{ij} - \pi_{ii} = \sum_{k \in N \setminus \{i,j\}} \pi_{ik}$.*

Proof. Let $R'_i \in \mathcal{R}_i$ be such that $t(R'_i) = i$. By *RMB*, $\pi' \equiv \varphi(R'_i, R_{-i})$ is such that $\pi'_{ii} = 1$.

Define $\Delta = 1 - 2\pi_{ij} - \pi_{ii}$. To obtain a contradiction, suppose $\Delta > 0$. Note that this excludes $\pi_{ii} = 1$. Define

$$\varepsilon = \frac{\Delta}{2(2\pi_{ij} + \Delta)} = \frac{1 - 2\pi_{ij} - \pi_{ii}}{2(1 - \pi_{ii})} = \frac{1}{2} - \frac{\pi_{ij}}{1 - \pi_{ii}}.$$

As $\Delta > 0$ and $\pi_{ij} \geq 0$, $\varepsilon > 0$; as $\pi_{ii} < 1$, $\varepsilon \leq 1/2$. Let $u \in \mathcal{U}^{id}(R_i)$ be such that $u(j) = 2 + \varepsilon$, $u(i) = 1$, and, for each $k \in N^i$ such that $i \; P_i \; k$, $u(k) < \varepsilon$. Then, using $u(k) < \varepsilon$ in the first step and $\Delta > 0$ in the last,

$$\begin{aligned} E(u, \pi_i) &< \pi_{ij} u(j) + \pi_{ii} u(i) + (1 - \pi_{ij} - \pi_{ii}) \cdot \varepsilon \\ &= \pi_{ij}(2 + \varepsilon) + \pi_{ii} \cdot 1 + (1 - 2\pi_{ij} - \pi_{ii}) \cdot \varepsilon + \pi_{ij}\varepsilon \\ &= 2\pi_{ij}(1 + \varepsilon) + \pi_{ii} + \Delta\varepsilon \\ &= (2\pi_{ij} + \Delta) \cdot \varepsilon - (1 - 2\pi_{ij} - \pi_{ii}) + 1 \\ &= \Delta/2 - \Delta + 1 = 1 - \Delta/2 < 1 = E(u, \pi'_i). \end{aligned}$$

Then i manipulates at R through R'_i, a contradiction. □

A.3 Proof of Theorem 2

Let φ *RMB* and be *id-EFF* and *id-NM*. To obtain a contradiction, suppose there is $R \in \mathcal{R}^N$ such that $\pi \equiv CR(R) \neq \varphi(R) \equiv \pi'$. Let $C^0, C^1, \ldots$ denote the cycles encountered when determining $CR(R)$. Moreover, let t be such that, for each $x < t$ and each $i \in C^x$, $\pi'_i = \pi_i$. Let $C^t = 1, \ldots, m$. Let $N^t \equiv N \setminus (C^0 \cup C^1 \cup \cdots \cup C^{t-1})$ be the agents that remain when C^t is chosen. To obtain a contradiction, suppose there is $i \in C^t$ such that $\pi'_i \neq \pi_i$.

Part 1: If $m = 1$, then, for each $j \in \text{supp}(\pi'_1) \setminus \{1\} \neq \emptyset$, $1 \; P_1 \; j$. Then agent 1 manipulates by top-ranking being single.

Part 2: If $m = 2$, then $\pi_{12} = 1 > \pi'_{12}$. Let $R'_1 \in \mathcal{R}_1$ be such that $2 = t(R'_1)$. By Lemma 1, $\varphi_{12}(R'_1, R_{-1}) = \pi'_{12}$. Let $R'_2 \in \mathcal{R}_2$ be such that $1 = t(R'_2)$. By Lemma 1, $\varphi_{12}(R'_1, R'_2, R_{-1,2}) = \pi'_{12} < 1$. This contradicts *RMB*.

Part 3: Suppose $m > 2$. Let $\pi^1 \equiv \pi'$, and, without loss of generality, $i \in C^t$ be such that $\pi^1_{i,i+1} < \pi_{i,i+1} = 1/2$. Consider $R'_i \in \mathcal{R}_i$ such that $t(R'_i) = i+1$ and $s(R'_i) = i$. Let $\pi^2 \equiv \varphi(R'_i, R_{-i})$. By Lemma 1, $\pi^2_{i,i+1} = \pi^1_{i,i+1} < 1/2$. By Lemma 2, $\pi^2_{i,i-1} \leq \pi^2_{i,i+1}$. Consider $R'_{i-1} \in \mathcal{R}_{i-1}$ such that $t(R'_{i-1}) = i$ and $s(R'_{i-1}) = i-1$. Let $\pi^3 \equiv \varphi(R'_i, R'_{i-1}, R_{-i,(i-1)})$. By Lemma 1, $\pi^3_{i-1,i} = \pi^2_{i-1,i}$. By Lemma 2, first applied from the perspective of $(i-1)$ and then from i, $\pi^3_{i-1,i-2} \leq \pi^3_{i-1,i} \leq \pi^3_{i,i+1}$. Repeat the argument along the cycle and let $R^* \equiv (R'_{C^t}, R_{-C^t})$ and $\pi^* \equiv \varphi(R^*)$. Then

$$\pi^*_{i,i+1} \leq \pi^*_{i+1,i+2} \leq \cdots \leq \pi^*_{i-1,i} \leq \pi^*_{i,i+1}.$$

Hence, for each $i \in C^t$, $\pi^*_{i,i-1} = \pi^*_{i,i+1} < 1/2$. As a consequence of Lemma 2, $\text{supp}(\pi^*_i) = \{(i-1), i, (i+1)\} \subseteq C^t$.

For each $i \in C^t$,

$$\begin{aligned} E(u, \pi^*_i) &= \pi^*_{i,i+1} u(i+1) + \pi^*_{i,i+1} u(i-1) + (1 - 2\pi^*_{i,i+1}) u(i) \\ &= \pi^*_{i,i+1} \big((u(i+1) - u(i)) - (u(i) - u(i-1)) \big) + u(i). \end{aligned}$$

As $u \in \mathcal{U}^{id}$, this is increasing in $\pi^*_{i,i+1}$. Hence, *id-EFF* implies $\pi^*_{i,i+1} = 1/2$, a contradiction. □

A.4 Independence of properties

Removing any one of the properties of Theorem 2 opens up the scope for other rules:

Without *RMB*: Let $\omega\colon \{1, \ldots, n\} \to N$ be an ordering of the agents and $\omega^{-1}(i)$ denote agent i's position in ω. Let $N_1 = N$. At step $t \in \mathbb{N}$, let $\{i, j\} \subseteq N_t$ be such that, for each $k \in N_t$, $\omega^{-1}(i) \leq \omega^{-1}(k)$ and $j \; R_i \; k$. Set $\varphi^\omega_{ij}(R) = 1$ and $N_{t+1} = N_t \setminus \{i, j\}$. The rule φ^ω is *id-EFF* and *id-NM*.

Without *id-EFF*: Let $\varphi^x(R) = CR(R)$ unless $R \in \mathcal{R}^N$ is such that, when defining $CR(R)$, there is a cycle that contains all agents. For such R and $\pi \equiv \varphi^x(R)$, set, for each $i \in C$, $\pi_{i,i+1} = \pi_{ii} = \pi_{i,i-1} = 1/3$. The rule φ^x *RMB* and is *id-NM* (Proposition 4).

Without *id-NM*: Let $\varphi(R) = WDA(R)$ unless $R \in \mathcal{R}^N$ is such that $WDA(R)$ is inefficient. For such R, set $\varphi(R) = CR(R)$. The rule differs from CR and WDA: for R and R' in the proof of Proposition 1, $\varphi(R) = WDA(R) \neq CR(R)$ and $\varphi(R') = CR(R') \neq WDA(R')$. Moreover, φ *RMB* and is *id-EFF*.

Proposition 4. *The rule φ^x defined in Subsection A.4 is id-NM.*

Proof. By Theorem 1, CR is *id-NM*. For an agent to manipulate φ^x, she needs to do so at profiles where the selection of CR differs from that of φ^x.

To obtain a contradiction, suppose $i \in N$ can manipulate φ^x. Let $R \in \mathcal{R}^N$ and $R'_i \in \mathcal{R}_i$ be as follows. When determining $CR(R)$, there is one cycle that contains all agents, so $\varphi^x(R) \neq CR(R)$. When determining $CR(R'_i, R_{-i})$, there is no such cycle, so $\varphi^x(R'_i, R_{-i}) = CR(R'_i, R_{-i})$. In Part 1, i's manipulation is through R'_i at R; in Part 2, through R_i at (R'_i, R_{-i}). Let $\pi \equiv \varphi^x(R)$ and $\pi' \equiv \varphi^x(R'_i, R_{-i})$. As in Figure 2, let $\{j, k, m\} \subseteq N^i$ be such that $t(R_i) = j$, $t(R'_i) = m$, and $t(R_k) = i$. Hence, $\pi_{ij} = \pi_{ik} = \pi_{ii} = 1/3$ and $\pi'_{ik} = \pi'_{im} = 1/2$. We treat $m \in \{i, k\}$ separately, that is, when $\pi'_{im} = 1$.

$$\begin{tikzcd}
k \arrow[r] & i \arrow[r] \arrow[d, dashed] & j \arrow[d, bend left] \\
\cdots \arrow[u, bend left] & m \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Figure 2: At R, there is a cycle $i, j, \ldots, m, \ldots, k$ that contains all agents (solid). The preference R'_i is such that m is i's most preferred agent (dashed).

Part 1: To obtain a contradiction, suppose i manipulates φ at R through R'_i. Note that $j \; P_i \; i$ and $j \; P_i \; k$. Consider $u \in \mathcal{U}^{id}(R_i)$ such that $u(i) = 0$ and $u(j) = 2$. If $i \; P_i \; k$, then $u(k) < u(i) = 0$. If $k \; P_i \; i$, as

R_{m_1}	R_{m_2}	R_{w_1}	R_{w_2}
w_1	w_2	m_2	m_1
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Table 4: Preferences for the proof of Theorem 3.

$u \in \mathcal{U}^{id}(R_i)$,

$$j \; P_i \; k \; P_i \; i \implies u(j) - u(k) > u(k) - u(i) \iff 2 - u(k) > u(k) - 0 \iff u(k) < 1.$$

Similarly, $u(m) < 1$. Therefore,

$$\begin{aligned} E(u, \pi_i') - E(u, \pi_i) &= \frac{1}{2}\big(u(k) + u(m)\big) - \frac{1}{3}\big(u(j) + u(k) + u(i)\big) \\ &= \frac{1}{6}\big(u(k) + 3u(m) - 2u(j) - 2u(i)\big) \\ &< \frac{1}{6}\big(1 + 3 \cdot 1 - 2 \cdot 2 - 2 \cdot 0\big) = 0. \end{aligned}$$

This contradicts that i manipulates φ^x at R through R_i'.

We obtain the same contradiction for $m \in \{i, k\}$. If $m = i$, then $\pi_{ii}' = 1$ and $E(u, \pi_i') = u(i) = 0 \leq E(u, \pi_i)$. This is clear if $k \; P_i \; i$, whereas if $i \; P_i \; k$, as $u \in \mathcal{U}^{id}(R_i)$,

$$u(j) - u(i) > u(i) - u(k) \iff E(u, \pi) = \frac{1}{3}\big(u(j) + u(k) + u(i)\big) > u(i) = 0.$$

If $m = k$, then $\pi_{ik}' = 1$, so

$$E(u, \pi_i') - E(u, \pi_i) = \frac{1}{6}\big(4u(k) - 2u(j) - 2u(i)\big) < \frac{1}{6}\big(4 \cdot 1 - 2 \cdot 2 - 2 \cdot 0\big) = 0.$$

Part 2: To obtain a contradiction, suppose i manipulates φ at (R_i', R_{-i}) through R_i. Note that $m \; P_i' \; i$ and $m \; P_i' \; j$. Consider $u \in \mathcal{U}^{id}(R_i')$ such that $u(i) = 0$ and $u(m) = 2$. As $m \; P_i' \; j$, $u(j) < u(m) = 2$. Therefore,

$$\begin{aligned} E(u, \pi_i) - E(u, \pi_i') &= \frac{1}{3}\big(u(j) + u(k) + u(i)\big) - \frac{1}{2}\big(u(k) + u(m)\big) \\ &= \frac{1}{6}\big(2u(j) - u(k) + 2u(i) - 3u(m)\big) \\ &< \frac{1}{6}\big(2 \cdot 2 - 0 + 2 \cdot 0 - 3 \cdot 2\big) < 0. \end{aligned}$$

This contradicts that i manipulates φ^x at (R_i', R_{-i}) through R_i. We obtain the same contradiction for $m \in \{i, k\}$ as i then is matched with her most preferred partner at π'. $\square$

A.5 Proof of Theorem 3

Let $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$, and refer to preferences in Table 4. It suffices to show the incompatibility for small $\varepsilon > 0$ as this implies that the properties are incompatible for all larger ε. Fix $0 < \varepsilon < 1$ and let $u \in \mathcal{U}_\varepsilon^{id} \setminus \mathcal{U}^{id}$ and $v \in \mathcal{U}^{id}$ take on the values $0, 1, 2 - \varepsilon/2$ and $0, 1, 4$, respectively. Let φ *RMB* and be $\mathcal{U}_\varepsilon^{id}$-*EFF* and $\mathcal{U}_\varepsilon^{id}$-*NM*. Finally, let $\pi \equiv \varphi(R)$.

First, we show that, when each agent's preference is represented by u, then φ has to select π^0 at R. By *RMB*, if an agent i top-ranks being single, then she is guaranteed utility $u(i) = 1$. By $\mathcal{U}_\varepsilon^{id}$-*NM*, for this manipulation not to be beneficial for m_1:

$$\begin{aligned} E(u,\pi_{m_1}) \geq 1 &\iff \pi_{m_1w_1}(2-\varepsilon/2) + \pi_{m_1m_1} \geq 1 \\ &\iff \pi_{m_1w_1}(2-\varepsilon/2) + (1-\pi_{m_1w_1} - \pi_{m_1w_2}) \geq 1 \\ &\iff \pi_{m_1w_1}(1-\varepsilon/2) \geq \pi_{m_1w_2}. \end{aligned}$$

As $1-\varepsilon/2 < 1$, if $\pi_{m_1w_2} > 0$, then $\pi_{m_1w_1} > \pi_{m_1w_2}$. Repeating the argument for each agent, we derive a contradiction:

$$\pi_{m_1w_1} > \pi_{w_2m_1} > \pi_{m_2w_2} > \pi_{w_1m_2} > \pi_{m_1w_1}.$$

Hence, $\pi_{m_1w_2} = 0$. Apply the argument for w_2 to find $\pi_{w_2m_2} = 0$, and continue for each agent to reach $\pi = \pi^0$.

Second, we show that, when each agent's preference is represented by v, then φ cannot select π^0 at R. Let $\pi' \equiv CR(R)$. Then, for each $i \in N$, $2 = E(v,\pi'_i) > E(v,\pi^0_i) = 1$. By $\mathcal{U}^{id}_\varepsilon$-*EFF*, $\pi \neq \pi^0$. This is a contradiction. □

Lemma 3. *Let $\mathcal{V} \subseteq \mathcal{U}$ and φ RMB and be $\mathcal{V}$-NM. Let $p_1 \equiv \{i_1,j_1\}, p_2 \equiv \{i_2,j_2\}, \ldots, p_T \equiv \{i_T,j_T\}$ be such that, for each $t \in \{1,\ldots,T\}$,*

$$\forall k \in N \setminus \bigcup_{x=1}^{t-1} p_x,\ j_t\ R_{i_t}\ k \text{ and } i_t\ R_{j_t}\ k.$$

Then, for each $t \in \{1,\ldots,T\}$, $\varphi_{i_tj_t}(R) = 1$.

Proof. By *RMB*, $\varphi_{i_1j_1}(R) = 1$, so the statement is true for $T = 1$.

Assume the statement to be true for $T = 1,\ldots,t-1$. We only need to verify the statement for the final pair of the sequence: for instance, the second to last pair is itself the final pair in a shorter sequence, for which, by assumption, the statement is true. Let $R'_{i_t} \in \mathcal{R}_{i_t}$ be such that $t(R'_{i_t}) = j_t$. Similarly, let $R'_{j_t} \in \mathcal{R}_{j_t}$ be such that $t(R'_{j_t}) = i_t$. By *RMB*, $\varphi_{i_tj_t}(R'_{i_t}, R'_{j_t}, R_{-i_tj_t}) = 1$. By assumption, for each $x < t$, $\varphi_{i_xj_x}(R'_{i_t}, R_{-i_t}) = 1$, so $\varphi_{j_ti_x}(R'_{i_t}, R_{-i_t}) = \varphi_{j_tj_x}(R'_{i_t}, R_{-i_t}) = 0$. By $\mathcal{V}$-*NM*, $\varphi_{i_tj_t}(R'_{i_t}, R_{-i_t}) = 1$, else j_t manipulates at (R'_{i_t}, R_{-i_t}) through R'_{j_t}. By assumption, for each $x < t$, $\varphi_{i_xj_x}(R) = 1$, so $\varphi_{j_ti_x}(R) = \varphi_{j_tj_x}(R) = 0$. By $\mathcal{V}$-*NM*, $\varphi_{i_tj_t}(R) = 1$, else i_t manipulates at R through R'_{i_t}. Hence, the statement is true for $T = t$. By induction, it holds for each T. □

A.6 Proof of Theorem 4

Let φ *RMB* and be $\mathcal{V}$-*EFF* and $\mathcal{V}$-*NM*. Fix the number of agents n and the values $\alpha_0 = 0 < \alpha_1 < \cdots < \alpha_n$. Select an arbitrary $k \geq 2$. We show that $\alpha_k \geq \beta\alpha_{k-1}$. Preferences are in Table 5. Let $\pi \equiv \varphi(R)$, $\pi' \equiv \varphi(R'_{m_1}, R_{-m_1})$, $\pi'' \equiv \varphi(R''_{m_1}, R_{-m_1})$, and $\pi''' \equiv \varphi(R'_{m_1}, R''_{w_2}, R_{-m_1w_2})$. Moreover, let $u \in \mathcal{V}(R_{m_1})$, $u' \in \mathcal{V}(R'_{m_1})$, and $u'' \in \mathcal{V}(R''_{w_2})$ all take on the values $\alpha_0, \alpha_1, \ldots, \alpha_n$. Hence, u, u', and u'' are equal in terms of their range, but not in terms of which preference they represent. For instance, $u(w_{k+1}) = u''(m_2) = \alpha_n$.

Part 1: Incentive constraints at (R'_{m_1}, R_{-m_1}) We first examine the effect that $\mathcal{V}$-*NM* has on π'. In particular, m_1 should not benefit from reporting R''_{m_1} nor should w_2 from reporting R''_{w_2}.

By *RMB*, $\pi''_{m_1m_1} = 1$. By Lemma 3, $\pi'_{m_3w_3} = \cdots = \pi'_{m_nw_n} = 1$, so $\pi'_{m_1w_1} + \pi'_{m_1w_2} + \pi'_{m_1m_1} = 1$. By $\mathcal{V}$-*NM*, $E(u',\pi'_{m_1}) \geq E(u',\pi''_{m_1})$, else m_1 manipulates at (R'_{m_1}, R_{-m_1}) through R''_{m_1}:

$$\begin{aligned} E(u',\pi'_{m_1}) \geq E(u',\pi''_{m_1}) &\iff \alpha_k\pi'_{m_1w_1} + \alpha_{k-1}\pi'_{m_1m_1} \geq \alpha_{k-1} \\ &\iff \pi'_{m_1w_2} \leq \frac{\alpha_k - \alpha_{k-1}}{\alpha_{k-1}} \cdot \pi'_{m_1w_1}. \end{aligned}$$

u	R_{m_1}	R_{m_2}	R_{m_3}	$\dots$	R_{m_n}	R_{w_1}	R_{w_2}	R_{w_3}	$\dots$	R_{w_n}	R'_{m_1}	R''_{m_1}	R''_{w_2}
α_n	w_{k+1}	w_{k+1}	w_3	$\dots$	w_n	m_{k+1}	m_{k+1}	m_3	$\dots$	m_n	w_{k+1}	$\emptyset$	m_2
$\vdots$	$\vdots$	$\vdots$				$\vdots$	$\vdots$				$\vdots$		
	w_n	w_n				m_n	m_n				w_n		
α_k	w_1	w_2				m_2	m_1				w_1		
α_{k-1}	w_2	w_1				m_1	m_2				$\emptyset$		
α_{k-2}	$\emptyset$						m_3				w_3		
$\vdots$							$\vdots$				$\vdots$		
							m_k				w_k		
α_0							$\emptyset$				w_2		

Table 5: Preferences for the proof of Theorem 4. For $k = 2$, the lower contour set of R_{w_2} at m_2 is $\{w_2\}$. For $k = 3$, it is $\{m_3, w_2\}$. The preference R'_{m_1} is defined analogously.

By Lemma 3, $\pi'''_{m_3w_3} = \cdots = \pi'''_{m_nw_n} = 1$ and $\pi'''_{w_2m_2} = 1$. By Lemma 3, $\pi'_{m_3w_3} = \cdots = \pi'_{m_nw_n} = 1$. So $\pi'_{w_2m_1} + \pi'_{w_2m_2} + \pi'_{w_2w_2} = 1 = \pi'_{m_1w_1} + \pi'_{m_1w_2} + \pi'_{m_1m_1}$. By $\mathcal{V}$-*EFF*, $\pi'_{m_2m_2} = \pi'_{w_1w_1} = 0$, so $\pi'_{m_1m_1} = \pi'_{w_2w_2}$ and hence $\pi'_{w_2m_2} = \pi'_{m_1w_1}$. By $\mathcal{V}$-*NM*, $E(u'', \pi'_{w_2}) \geq E(u'', \pi'''_{w_2})$, else w_2 manipulates at (R'_{m_1}, R_{-m_1}) through R''_{w_2}:

$$\begin{aligned} E(u'', \pi'_{w_2}) \geq E(u'', \pi'''_{w_2}) &\iff \alpha_k \pi'_{w_2m_1} + \alpha_{k-1}\pi'_{w_2m_2} \geq \alpha_{k-1} \\ &\iff \pi'_{m_1w_2} \geq \frac{\alpha_{k-1}}{\alpha_k} \cdot (1 - \pi'_{m_1w_1}). \end{aligned}$$

Moreover, $\pi'_{m_1w_1} \geq 0$, $\pi'_{m_1w_2} \geq 0$, and $\pi'_{m_1w_1} + \pi'_{m_1w_2} \leq 1$. Therefore, π' is in the triangle ABC in Figure 3.

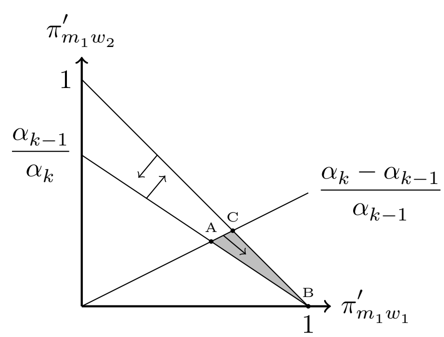

Figure 3: The constraints imposed on π' to ensure that m_1 and w_2 cannot manipulate at (R'_{m_1}, R_{-m_1}) illustrated graphically. Arrows indicate directions of constraints.

Part 2: Incentive constraint at R By Lemma 3, $\pi_{m_3w_3} = \cdots = \pi_{m_nw_n} = 1$. Then, by $\mathcal{V}$-*EFF*, $\pi_{m_1w_1} + \pi_{m_1w_2} = 1$.

We impose $\pi_{m_1w_1} = 1/2$ and argue that this is without loss. In brief, we show that α_k needs to exceed α_{k-1} by a certain factor. If not, then agent m_1 can manipulate the rule. The larger we make $\pi_{m_1w_1}$, the smaller this factor becomes. That is, the more difficult it is for m_1 to manipulate. However, symmetrically, we can show that α_k has to exceed α_{k-1} by a certain factor for w_2 not to manipulate. The larger we make $\pi_{m_1w_1}$, the smaller we make $\pi_{w_2m_1} = 1 - \pi_{m_1w_1}$, and the larger we make the factor. We want to find the smallest factor such that neither m_1 nor w_2 can manipulate. This requires $\pi_{m_1w_1} = \pi_{m_1w_2} = 1/2$.

Hence, $E(u, \pi_{m_1}) = (\alpha_k + \alpha_{k-1})/2$. By $\mathcal{V}$-*NM*, $E(u, \pi_{m_1}) \geq E(u, \pi'_{m_1})$, else m_1 manipulates at R through R'_{m_1}. We relax the right hand side by a lower bound $b \leq E(u, \pi'_{m_1})$ defined as the minimum of $E(u, \pi'_{m_1})$ over π' in ABC. If $E(u, \pi_{m-1}) < b$, then m_1 is able to manipulate at R no matter π'; if $E(u, \pi_{m-1}) \geq b$, then $\mathcal{V}$-*NM* can still be satisfied for some choice of π' in ABC.

Part 3: Minimization We minimize $E(u, \pi'_{m_1}) = \alpha_k \pi'_{m_1 w_1} + \alpha_{k-1}\pi'_{m_1 w_2} + \alpha_{k-2}\pi'_{m_1 m_1}$ over π' in ABC. As $\pi'_{m_1 w_1} + \pi'_{m_1 w_2} + \pi'_{m_1 m_1} = 1$, $E(u, \pi'_{m_1}) = (\alpha_k - \alpha_{k-2})\pi'_{m_1 w_1} + (\alpha_{k-1} - \alpha_{k-2})\pi'_{m_1 w_2} + \alpha_{k-2}$. As $\alpha_k > \alpha_{k-2}$ and $\alpha_{k-1} > \alpha_{k-2}$, this is increasing in $\pi'_{m_1 w_1}$ and $\pi'_{m_1 w_2}$. So the minimizer is on AB, where

$$\pi'_{m_1 w_2} = \frac{\alpha_{k-1}}{\alpha_k} \cdot (1 - \pi'_{m_1 w_1}).$$

Therefore, the lower bound b satisfies the following inequality:

$$b \leq \frac{\alpha_k(\alpha_k - \alpha_{k-2}) - \alpha_{k-1}(\alpha_{k-1} - \alpha_{k-2})}{\alpha_k}\pi'_{m_1 w_1} + \frac{\alpha_{k-1}(\alpha_{k-1} - \alpha_{k-2})}{\alpha_k} + \alpha_{k-2}.$$

As $\alpha_k > \alpha_{k-1} > \alpha_{k-2} \geq 0$, this is increasing in $\pi'_{m_1 w_1}$. So the minimizer is at the point A, where

$$\begin{aligned} &\frac{\alpha_k - \alpha_{k-1}}{\alpha_{k-1}} \cdot \pi'_{m_1 w_1} = \frac{\alpha_{k-1}}{\alpha_k} \cdot (1 - \pi'_{m_1 w_1}) \\ \iff &\pi'_{m_1 w_1} = \frac{\alpha_{k-1}^2}{\alpha_k(\alpha_k - \alpha_{k-1}) + \alpha_{k-1}^2}. \end{aligned}$$

After some simplification, we find

$$b = \frac{\alpha_{k-1}(2\alpha_k\alpha_{k-1} - \alpha_k\alpha_{k-2} - \alpha_{k-1}^2)}{\alpha_k^2 - \alpha_k\alpha_{k-1} + \alpha_{k-1}^2} + \alpha_{k-2}.$$

In the next step, we minimize α_k subject to $E(u, \pi_{m_1}) = (\alpha_k + \alpha_{k-1})/2 \geq b$. Hence, we look for the smallest value of α_k such that m_1 with certainty cannot manipulate the rule at R through R'_{m_1}. We rewrite the constraint to arrive at

$$\alpha_k^2 + \alpha_k\alpha_{k-1} - 2\alpha_k\alpha_{k-2} - 3\alpha_{k-1}^2 + 2\alpha_{k-1}\alpha_{k-2} \geq 0.$$

As $\alpha_k > \alpha_{k-1} > \alpha_{k-2}$, the left hand side is increasing in α_k. Hence, the constraint binds at the minimum. Therefore, as the solution of a second order equation,

$$\begin{aligned} \alpha_k &= \frac{1}{2}\left(-\alpha_{k-1} + 2\alpha_{k-2} + \sqrt{13\alpha_{k-1}^2 - 12\alpha_{k-1}\alpha_{k-2} + 4\alpha_{k-2}^2}\right) \\ &\geq \frac{1}{2}\left(-\alpha_{k-1} + 2\alpha_{k-2} + \sqrt{13\alpha_{k-1}^2 - 4\sqrt{13}\alpha_{k-1}\alpha_{k-2} + 4\alpha_{k-2}^2}\right). \end{aligned}$$

Here, $4\sqrt{13} > 4\sqrt{9} = 12$ and $\alpha_{k-1} > \alpha_{k-2} \geq 0$, so $-4\sqrt{13}\alpha_{k-1}\alpha_{k-2} \leq -12\alpha_{k-1}\alpha_{k-2}$. In the next step, $\sqrt{13} > \sqrt{4} = 2$ and $\alpha_{k-1} > \alpha_{k-2}$ imply $\sqrt{13}\alpha_{k-1} > 2\alpha_{k-2}$, so

$$\begin{aligned} \alpha_k &\geq \frac{1}{2}\left(-\alpha_{k-1} + 2\alpha_{k-2} + \sqrt{(\sqrt{13}\alpha_{k-1} - 2\alpha_{k-2})^2}\right) \\ &= \frac{1}{2}\left(-\alpha_{k-1} + 2\alpha_{k-2} + \sqrt{13}\alpha_{k-1} - 2\alpha_{k-2}\right) \\ &= \frac{1}{2}\left(-\alpha_{k-1} + \sqrt{13}\alpha_{k-1}\right) = \beta\alpha_{k-1}. \end{aligned}$$

With $\alpha_0 = 0$ and $\alpha_1 = \beta$, $\alpha_2 \geq \beta\alpha_1 = \beta^2$, $\alpha_3 \geq \beta\alpha_2 = \beta^3$, and so on. This completes the proof.

R_{m_1}	R_{m_2}	R_{m_3}	R_{m_4}	R_{w_1}	R_{w_2}	R_{w_3}	R_{w_4}	R'_{m_1}	R'_{w_1}	R'_{m_2}	R'_{w_3}
$w_1 w_2$	w_3	w_4	$\emptyset$	m_2	m_3	m_1	m_1	w_1	m_1	$\emptyset$	$\emptyset$
$w_3 w_4$	$\emptyset$	$\emptyset$		m_1	m_1	$\emptyset$	$\emptyset$	w_2	$\emptyset$		
				$\emptyset$	$\emptyset$			w_3			
								w_4			

Table 6: Preferences for the proof of Proposition 2; the preference R_{m_1} contains indifference between agents w_1 and w_2 and agents w_3 and w_4.

Additional notes In the proof, we provide only a lower bound that relates α_k to α_{k-1}. This bound actually only applies to α_2: as k gets large, the relation between α_k and α_{k-1} approaches another value. Starting from

$$\alpha_k = \frac{1}{2}\left(-\alpha_{k-1} + 2\alpha_{k-2} + \sqrt{13\alpha_{k-1}^2 - 12\alpha_{k-1}\alpha_{k-2} + 4\alpha_{k-2}^2}\right),$$

we make the intelligent guess that this recursive relation is solved by an exponential function, so $\alpha_k = a^k$ for some $a > 1$. This guess is inconsistent with $\alpha_0 = 0$, but works well for large k. Then

$$a^k = \frac{1}{2}\left(-a^{k-1} + 2a^{k-2} + \sqrt{13a^{2k-2} - 12a^{2k-3} + 4a^{2k-4}}\right).$$

(i) Factoring out a^{k-2}, (ii) isolating the radical, and (iii) squaring both sides:

$$\begin{aligned} a^2 &= \frac{1}{2}\left(-a + 2 + \sqrt{13a^2 - 12a + 4}\right) \\ 2a^2 + a - 2 &= \sqrt{13a^2 - 12a + 4} \\ 4a^4 + 4a^3 - 7a^2 - 4a + 4 &= 13a^2 - 12a + 4. \end{aligned}$$

After simplification, we arrive at $a^3 + a^2 - 5a + 2 = 0$. The polynomial has three real roots, but only one that exceeds 1, namely

$$a = \frac{8}{3}\cos\left(\frac{1}{3}\arccos\left(-\frac{101}{128}\right)\right) - \frac{1}{3} \approx 1.4728.$$

This shows that the utility function eventually has to grow considerably faster than as stated in Theorem 4.

A.7 Proof of Proposition 2

Let $M = \{m_1, m_2, m_3, m_4\}$, $W = \{w_1, w_2, w_3, w_4\}$, and refer to preferences in Table 6. Let φ *RMB* and be *id-EFF* and $\pi \equiv \varphi(R)$. To obtain a contradiction, suppose φ is *id-NM*.

By *RMB*, $\varphi_{w_3w_3}(R'_{w_3}, R_{-w_3}) = \varphi_{m_2m_2}(R'_{m_2}, R_{-m_2}) = \varphi_{w_1m_1}(R'_{w_1}, R_{-w_1}) = 1$. By *id-NM*, $\pi_{w_3m_1} \geq \pi_{w_3m_2}$, $\pi_{m_2w_3} \geq \pi_{m_2w_1}$, and $\pi_{w_1m_2} \geq \pi_{w_1w_1}$, else there is $u \in \mathcal{U}^{id}$ such that w_3, m_2, or w_1 manipulates at R. In conclusion,

$$\pi_{m_1w_3} \geq \cdots \geq \pi_{w_1m_2} \geq \pi_{w_1w_1}.$$

By *id-EFF*, $\pi_{w_1m_3} = 0$. By *RMB*, $\pi_{m_4m_4} = 1$ so $\pi_{w_1m_4} = 0$. Therefore,

$$\pi_{w_1m_1} = 1 - \pi_{w_1w_1} - \pi_{w_1m_2} \geq 1 - \pi_{m_1w_3} - \pi_{m_1w_3} = 1 - 2\pi_{m_1w_3}.$$

R_{m_1}	R_{m_2}	R_{m_3}	R_{w_1}	R_{w_2}	R_{w_3}	R'_{m_1}	R'_{m_2}	R'_{w_1}	R''_{w_1}
w_1	w_2	w_1	m_2	m_1	m_1	w_2	w_1	m_2	m_3
w_2	w_1	w_2	m_1	m_2	m_2	w_1	w_2	m_3	m_2
w_3	w_3	w_3	m_3	m_3	m_3	w_3	w_3	m_1	m_1

Table 7: Preferences for the proof of Proposition 3.

As $\pi_{m_1w_1} \geq 0$,

$$\pi_{m_1w_1} + \pi_{m_1w_3} \geq 1 - 2\pi_{m_1w_3} + \pi_{m_1w_3} = 1 - \pi_{m_1w_3} \geq \max\{\pi_{m_1w_3}, 1 - \pi_{m_1w_3}\} \geq 1/2,$$

where $\pi_{m_1w_1} + \pi_{m_1w_3} = 1/2 \iff \pi_{m_1w_3} = 1/2$.

By a symmetric line of arguments,

$$\pi_{m_1w_2} + \pi_{m_1w_4} \geq \max\{\pi_{m_1w_4}, 1 - \pi_{m_1w_4}\} \geq 1/2,$$

where $\pi_{m_1w_2} + \pi_{m_1w_4} = 1/2 \iff \pi_{m_1w_4} = 1/2$.

As $(\pi_{m_1w_1} + \pi_{m_1w_3}) + (\pi_{m_1w_2} + \pi_{m_1w_4}) \leq 1$, we have $\pi_{m_1w_3} = \pi_{m_1w_4} = 1/2$. As a consequence of Theorem 2, for each $\hat{R} \in \mathcal{R}^N$ such that $\hat{R}$ has no indifferences, $\varphi(\hat{R}) = CR(R)$. In particular, $\varphi(R'_{m_1}, R_{-m_1}) = CR(R'_{m_1}, R_{-m_1})$. Then m_1 manipulates at R through R'_{m_1}. This contradicts *id-NM*. □

A.8 Proof of Proposition 3

Let $M = \{m_1, m_2, m_3\}$, $W = \{w_1, w_2, w_3\}$, and refer to preferences in Table 7. Let $\phi\colon \mathcal{R}_0^N \to \mathcal{M}$ *RMB*, $\mu \equiv \phi(R)$, and $\mu' \equiv \phi(R'_{w_1}, R_{-w_1})$. To obtain a contradiction, suppose ϕ is *st-NM*.

Let $(i,j) \in \{(1,2),(2,1)\}$. By *RMB*, $\phi_{m_i}(R'_{m_i}, R_{-m_i}) = w_j$. By *st-NM*, $\mu(m_i) \neq w_3$, else m_i manipulates at R through R'_{m_i}. Therefore, $\mu(m_3) = w_3$ and $\mu \in \mathcal{S}(R)$. We may assume $\mu = \{(m_1,w_1),(m_2,w_2),(m_3,w_3)\}$ as the case $\mu = \{(m_1,w_2),(m_2,w_1),(m_3,w_3)\}$ can be treated symmetrically.

By *st-NM*, $\mu'(w_1) \neq m_2$, else w_1 manipulates at R through R'_{w_1}. Let $(i,j) \in \{(1,2),(2,1)\}$. By *RMB*, $\phi_{m_i}(R'_{m_i}, R'_{w_1}, R_{-m_iw_1}) = w_j$. By *st-NM*, $\mu'(m_i) \neq w_3$, else m_i manipulates at (R'_{w_1}, R_{-w_1}) through R'_{m_i}. By *RMB*, $\phi_{w_1}(R''_{w_1}, R_{-w_1}) = m_3$. By *st-NM*, $\mu'(w_1) = m_3$, else w_1 manipulates at (R'_{w_1}, R_{-w_1}) through R''_{w_1}. Therefore $\mu'(m_3) \neq w_3$, so w_3 is without partner. This is a contradiction. □