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Abstract This article presents an axiomatic characterization of a new value for cooperative games with incomplete communication. The result is obtained by slight modifications of associated games proposed by Hamiache (1999, 2001). This new associated game can be expressed as a matrix formula. We generate a series of successive associated games and show that its limit is an inessential game. Three axioms (associated consistency, inessential game, continuity) characterize a unique sharing rule. Combinatorial arguments and matrix tools provide a procedure to compute the solution. The new sharing rule coincides with the Shapley value when the communication is complete.

Keywords cooperative games · graphs · associated consistency · Shapley Value.
JEL Classification: C71

1 Introduction

The present article contributes to the literature generalizing the Shapley value (Shapley, 1953) for situations of limited communication between players. We

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consider a type of communication that can be represented by graph, as initiated by Myerson (1977). Only pairwise meetings can occur, and some of them are not permitted.

The present article joins a family of works dealing with value on graphs: the Myerson value (1977), the position value (Borm *et al*, 1992), the average-tree value (Herings *et al*, 2008), the F-value (Hamiache, 1999) and the mean value (Hamiache, 2003). The extension of the Shapley value, for the games with communication structure discussed herein, is a brand new solution.

Here the associated consistency axiom is an direct extension of the associated game proposed by Hamiache (2001). Instead of considering the coalition $N \setminus S$, coalition S now considers only its immediate neighbors, namely $S^* \setminus S$. The general idea behind this kind of consistency is as follows. For a given game, agents may elaborate expectations of the game and may be willing to allow the computation of their payments to be based on these expectations. In game theory this approach, which involves auxiliary games, namely reduced games, was initially proposed by Davis and Maschler (1965), Sobolev (1975), and Peleg (1980). In the context of the Shapley value, Hart and Mas-Colell (1989) introduced a consistency axiom based on a modified reduced game.

From a technical point of view, the proposed associated game v^* can be represented by a matrix formula, $v^* = P_g\, M_c\, P_g\, v$, where P_g is a matrix translating the communication structure and matrix M_c is intimately linked to the Shapley value. The process of the associated game generates a sequence of games: the associated game of the game, the associated game of the associated game and so on. We show that this sequence converges to some inessential game. Using the inessential game axioms we characterize a unique sharing rule for cooperative games with communication structure which coincides with the Shapley value for complete communication structures. The proof partly develops matrix arguments and partly relies on combinatorial computations.

In the next chapter the general framework is presented. In chapter 3 the associated game and the axioms are detailed. The proof of the main theorem is assembled in Chapter 4. The most technical aspects of the proofs are relegated to the Appendix.

2 The General Framework

Let U be a non-empty and finite set of players. A coalition is a non-empty subset of U. A coalitional game with transferable utility (a TU game) is a pair (N, v) where N is a coalition and v is a function satisfying $v : 2^N \to \mathbb{R}$ and $v(\emptyset) = 0$. A game (N, v) is said to be inessential if for all pairs of disjoint coalitions $S \subseteq N$ and $T \subseteq N \setminus S$, $v(S \cup T) = v(S) + v(T)$. Note that a game (N, v) is inessential if and only if, for all coalitions $S \subseteq N$, $v(S) = \sum_{i \in S} v(\{i\})$. A unanimity game (N, u_R) is defined such that $u_R(S) = 1$ if $R \subseteq S$, and 0 otherwise.

In this article, structures of communication are represented by simple graphs. Given a coalition N, we denote by g_{N} the set of all unordered pairs of

N, $g_{_N} = \{\{i,j\} \mid i \neq j, i \in N, j \in N\}$. Note that we define only links between two distinct vertices, in other words, no loops are admitted. A graph is a pair $\langle N, g\rangle$ where $N \subseteq U$ is the set of vertices and $g \subseteq g_{_N}$ is the set of links. We denote by GR the set of all the graphs, $GR = \{\langle N, g\rangle \mid N \subseteq U$ and $g \subseteq g_{_N}\}$. Two vertices i and j such that $\{i,j\} \in g$ are said to be adjacent. Two players can communicate directly if and only if they are adjacent. The graph $\langle S, g(S)\rangle$ with $S \subseteq N$ is an induced graph of $\langle N, g\rangle$ when $g(S)$ is given by $g(S) = \{\{i,j\} \mid i \in S,\ j \in S,\ \{i,j\} \in g\}$. Given two vertices i and j in N, a path of graph $\langle N, g\rangle$ between vertices i and j is a series of vertices of N, $i = i_1, i_2, \ldots, i_k = j$ such that for all q, $1 \leq q \leq k-1$, $\{i_q, i_{q+1}\} \in g$. If a path of graph $\langle N, g\rangle$ exists between two players i and j of N, we say that they are connected by graph $\langle N, g\rangle$. The fact that two members of N, i and j, are connected by graph $\langle N, g\rangle$ will be symbolized by $i \rightarrow_{\langle N,g\rangle} j$. Moreover, we admit that for all players $i \in N$ we also have $i \rightarrow_{\langle N,g\rangle} i$. The binary relation $\rightarrow_{\langle N,g\rangle}$ is symmetric, transitive and reflexive. For a given graph $\langle N, g\rangle$, we denote by N/g the partition of set N defined by $N/g = \big\{\{i \in N \mid i \rightarrow_{\langle N,g\rangle} j\} \mid j \in N\big\}$. A member of N/g is called a component of graph $\langle N, g\rangle$. A component can also be defined as a maximal connected coalition. The induced graph $\langle S, g(S)\rangle$ is connected if any two players of S are connected by the graph $\langle S, g(S)\rangle$. In terms of components, the graph $\langle S, g(S)\rangle$ is connected if set S/g is a singleton ($\#(S/g) = 1$). We say that coalition S is connected if $\langle S, g(S)\rangle$ is a connected graph. We denote by $S^*_{\langle N,g\rangle}$ the closed neighborhood of coalition S. It is the set of all the vertices of the graph $\langle N, g\rangle$ which are adjacent to at least one of the vertices of the set S, $S^*_{\langle N,g\rangle} = \{i \in N \mid \exists\, j \in S$ such that $\{i,j\} \in g\} \cup S$. Where no confusion is possible, we omit subscript $\langle N, g\rangle$ and only write S^*.

A game with communication structure is a triplet (N, v, g), where N is a coalition, (N, v) is a game and $\langle N, g\rangle$ is a graph. Let us denote by G the set of all these games, and by G_N the set of games whose players' set is N. We define the game $(N, v/g, g)$ so that: $(v/g)(S) = \sum_{R \in S/g} v(R)$ for all $S \subseteq N$. In words, the value of a coalition in the new game is the sum of the values of the coalition's components. Therefore, the value $(v/g)(S)$ of coalition S reflects the fact that cooperation can only take place between players who can physically communicate.

A sharing rule, or a solution on G, is a function ϕ which associates with each game $(N, v, g) \in G$, a vector $\phi(N,\ v,\ g)$ of $\mathbb{R}^N$.

3 The Associated Game and Three Axioms

Given a game (N, v, g) and a positive real parameter τ, the associated game (N, v^*_τ, g) is defined for connected coalitions S as,

$$v^*_\tau(S) = v(S) + \tau \sum_{\substack{j \\ j \in S^* \setminus S}} [v(S \cup \{j\}) - v(S) - v(\{j\})], \tag{1}$$

and for non-connected coalitions S by $v^*_\tau(S) = \sum_{R \in S/g} v^*_\tau(R)$. The associated game $(N,\ v^*_\tau,\ g)$ is indeed a game, since $v^*_\tau(\emptyset) = 0$. It is also true that $v^*_\tau(N) =$

$v(N)$. Note that for complete graphs we have for all coalitions $S^* = N$, in that case Equation (1) coincides with the associated game[1] in Hamiache (2001).

For a given game, its associated game is assumed to translate formally how players understand their original situation. A given coalition may have designs on at least a part of the surplus generated by the cooperation with "satellite players" taken separately. In other words, coalition S may believe that the appropriation of at least a part of the surpluses $[v(S \cup \{j\}) - v(S) - v(\{j\})]$, generated by its cooperation with each of its immediate neighbors, $j \in S^* \setminus S$, is within reach. What we are actually doing here is applying a "divide and rule" approach to the set of players that are immediate neighbors of the coalition. This short-sighted associated game can be interpreted as coalition self-evaluation, and thus could give rise to new claims. The solution concept is said to be consistent if it gives the same payments to players in the original game and in the imaginary associated game.

In the following we will deal with large square matrices of order $2^n - 1$. We will start by developing a matrix formula for the associated game v^*_τ. To ensure homogeneity in the notations of matrices and vectors, we have found it convenient to order the set of coalitions of N to label columns and rows of the square matrices of size $2^n - 1$. We define a lexicographic order for sets of same-size coalitions. Let us consider two coalitions of size ϵ, $K = \{\kappa_1, \kappa_2, \ldots, \kappa_\epsilon\}$ and $L = \{\ell_1, \ell_2, \ldots, \ell_\epsilon\}$ with $\kappa_1 < \kappa_2 < \cdots < \kappa_\epsilon$ and $\ell_1 < \ell_2 < \cdots < \ell_\epsilon$. The lexicographic order $\prec_{\ell ex}$ for the set of coalitions of size ϵ is defined as follows, $K \prec_{\ell ex} L$ if and only if $[\kappa_1 < \ell_1]$ or [there is a natural number γ, with $1 < \gamma \le \epsilon$, verifying $\kappa_\eta = \ell_\eta$ for all $1 \le \eta < \gamma$, and $\kappa_\gamma < \ell_\gamma$].

Let us consider two coalitions S and T. We will say that coalition S precedes coalition T, denoted $S \prec T$, if $[\#S < \#T]$ or $[\#S = \#T$ and $S \prec_{\ell ex} T]$. The order chosen induces an order for the values of coalitions of N, $\Big(v(\{1\}), \ldots, v(\{n\}), v(\{1,2\}), \ldots, v(\{n-1,n\}), v(\{1,2,3\}), \ldots, v(N)\Big)$. The first n coordinates of vector v are the values of singletons. The next $\binom{n}{2}$ coordinates of vector v are the values of doubletons and so on.

Let us consider M_c, the matrix form of the associated game in Hamiache (2001), as defined in Hamiache (2010), a square matrix of order $2^n - 1$, the lines and columns of which are labeled with coalitions respecting the above order. The generic element of matrix M_c for $\emptyset \neq S \subseteq N$ and $\emptyset \neq T \subseteq N$, is given by,

$$M_c[S,T] = \begin{cases} 1 - \#(N \setminus S)\,\tau & \text{if} \quad S = T, \\ \tau & \text{if} \quad \#S + 1 = \#T \quad \text{and} \quad S \subseteq T, \\ -\tau & \text{if} \quad \#T = 1 \quad \text{and} \quad T \nsubseteq S, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

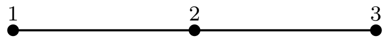
We know that matrix M_c is diagonalizable, and that 1 is an eigenvalue with algebraic multiplicity n. We also know that $1 - s\,\tau$ are eigenvalues of M_c with multiplicity $\binom{n}{s}$ for all s verifying $2 \le s \le n$.

[1] $v^*_\tau(S) = v(S) + \tau \sum_{j \in N \setminus S} [v(S \cup \{j\}) - v(S) - v(\{j\})]$.

Given a graph $\langle N, g\rangle$, we define P_g, a square matrix of order $2^n - 1$, so that for all non-empty coalitions S and $T \subseteq N$,

$$P_g[S,T] = \begin{cases} 1 \text{ if } & T \in S/g, \\ 0 \text{ if } & T \notin S/g. \end{cases} \tag{3}$$

Numerical Example 1: To illustrate the concepts presented in this article, we adopt the format "running numerical example". We shall focus on the simplest case of interest, namely a three player game $N = \{1,2,3\}$ when the communication system is described by the line graph $g = \{\{1,2\},\{2,3\}\}$.



Since $\#N = 3$, we have the matrix form of Eq. (2),

$$M_c = \begin{array}{cc} & \begin{array}{ccccccc} 1 & 2 & 3 & 12 & 13 & 23 & 123 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 12 \\ 13 \\ 23 \\ 123 \end{array} & \begin{bmatrix} 1-2\tau & -\tau & -\tau & \tau & \tau & 0 & 0 \\ -\tau & 1-2\tau & -\tau & \tau & 0 & \tau & 0 \\ -\tau & -\tau & 1-2\tau & 0 & \tau & \tau & 0 \\ 0 & 0 & -\tau & 1-\tau & 0 & 0 & \tau \\ 0 & -\tau & 0 & 0 & 1-\tau & 0 & \tau \\ -\tau & 0 & 0 & 0 & 0 & 1-\tau & \tau \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

And,

$$P_g = \begin{array}{cc} & \begin{array}{ccccccc} 1 & 2 & 3 & 12 & 13 & 23 & 123 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 12 \\ 13 \\ 23 \\ 123 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

Before going further, let us list a few properties of matrices P_g.

Property 1: $P_g\, v = (v/g)$.

Property 2: Matrix P_g is idempotent, $P_g\, P_g = P_g$.

Proof: For all games v we have $P_g\, P_g\, v = P_g\,(v/g) = (v/g) = P_g\, v$. □

Let us define matrix D_g as follows:

$$D_g[S,T] = \begin{cases} 1 \text{ if } S = T \text{ and } \#(S/g) = 1, \\ 0 \text{ otherwise.} \end{cases}$$

It is clear that $D_g\, D_g = D_g$. We also have $D_g\, P_g = D_g$ and $P_g\, D_g = P_g$.

Property 3: Matrix P_g is diagonalizable.

Proof: For matrix $R = P_g + I - D_g$ and its inverse $R^{-1} = -P_g + I + D_g$, direct computation shows that $P_g = (P_g + I - D_g)\, D_g\, (-P_g + I + D_g)$. □

Property 4: Let us consider the n vectors $x_{\{i\}}$ for $i \in N$, defined by,

$$x_{\{i\}}[T] = \begin{cases} 1 \text{ if } \ i \in T, \\ 0 \text{ otherwise.} \end{cases} \tag{4}$$

The pairs $<1,\, x_{\{i\}}>$ are eigenpairs of matrix P_g.
Proof: Let us compute $(P_g \cdot x_{\{i\}})[T]$ for all coalitions $T \subseteq N$,

$$(P_g \cdot x_{\{i\}})[T] = \sum_{\substack{K \\ \emptyset \neq K \subseteq N}} P_g[T,K] \cdot x_{\{i\}}[K] = \sum_{\substack{K \\ \emptyset \neq K \subseteq N \\ K \in T/g}} x_{\{i\}}[K] = \sum_{\substack{K \\ \emptyset \neq K \subseteq N \\ i \in K \in T/g}} 1.$$

If $i \in T$ there is only one coalition K verifying $i \in K \in T/g$ and $(P_g \cdot x_{\{i\}})[T] = 1 = x_{\{i\}}[T]$. If $i \notin T$ there is no coalition K verifying $i \in K \in T/g$ and $(P_g \cdot x_{\{i\}})[T] = 0$. We have thus proved that $P_g \cdot x_{\{i\}} = x_{\{i\}}$, which completes the proof of Property 4. □

Lemma 1: For all games (N, v, g) in G, a matrix form of the associated game (N, v^*_τ, g) is given by,

$$v^*_\tau = P_g\, M_c\, P_g\, v. \tag{5}$$

Proof:

$$\big(P_g\, M_c\, P_g\ v\big)[S] = \big(P_g\, M_c\, (v/g)\big)[S] = \sum_{\substack{T \\ T \subseteq N}} \big(P_g M_c\big)[S,T]\,(v/g)(T)$$

$$= \sum_{\substack{T \\ T \subseteq N}} \sum_{\substack{R \\ R \subseteq N}} \big(P_g\big)[S,R]\,\big(M_c\big)[R,T]\,(v/g)(T)$$

$$= \sum_{\substack{R \\ R \subseteq N}} \big(P_g\big)[S,R] \sum_{\substack{T \\ T \subseteq N}} \big(M_c\big)[R,T]\,(v/g)(T).$$

Using the fact that $(M_c\, w)[R] = w(R) + \tau \sum_{j \notin R}[w(R \cup \{j\}) - w(R) - w(\{j\}))$,

$$= \sum_{\substack{R \\ R \subseteq N}} \big(P_g\big)[S,R] \Big[(v/g)(R) + \tau \sum_{\substack{j \\ j \in N \setminus R}} \big[(v/g)(R \cup \{j\}) - (v/g)(R) - (v/g)(\{j\})\big]\Big]$$

$$= \sum_{\substack{R \\ R \in S/g}} \Big[(v/g)(R) + \tau \sum_{\substack{j \\ j \in N \setminus R}} [(v/g)(R \cup \{j\}) - (v/g)(R) - (v/g)(\{j\})]\Big].$$

For all connected coalitions R and all players $j \notin R^*$, we have $[(v/g)(R \cup \{j\}) - (v/g)(R) - (v/g)(\{j\})] = 0$,

$$\sum_{\substack{R \\ R \in S/g}} \big[v(R) + \tau \sum_{\substack{j \\ j \in R^* \setminus R}} v(R \cup \{j\}) - v(R) - v(\{j\})\big] = \sum_{\substack{R \\ R \in S/g}} v^*_\tau(R),$$

which completes the proof of Lemma 1. □

As a consequence of Lemma 1 the successive associated games can be expressed as $v^{**}_\tau = \big(P_g\, M_c\, P_g\big)\, v^*_\tau = \big(P_g\, M_c\, P_g\big)^2\, v,\ \ldots,\ v^{(k\,*)}_\tau = \big(P_g\, M_c\, P_g\big)^k\, v.$

We will show below that the series of powers of matrix $\left(P_g\, M_c\, P_g\right)^k$ is convergent as k tends to infinity.

The reader will observe that the previous matrix form of the associated game differs from the usual transformations of matrix M_c, where for some non-singular matrix Z we compute matrix $Z\, M_c\, Z^{-1}$. Here the transformation generates a matrix $M_g = P_g\, M_c\, P_g$ which is not similar to M_c. Moreover matrix P_g is generally singular, which makes the treatment of the convergence of the series of powers of M_g far more complicated.

Numerical example 2: For $\langle N, g\rangle = \langle\{1,2,3\}, \{\{1,2\},\{2,3\}\}\rangle$ we have,

$$M_g = \left(P_g\, M_c\, P_g\right) = \begin{bmatrix} 1-\tau & -\tau & 0 & \tau & 0 & 0 & 0 \\ -\tau & 1-2\,\tau & -\tau & \tau & 0 & \tau & 0 \\ 0 & -\tau & 1-\tau & 0 & 0 & \tau & 0 \\ 0 & 0 & -\tau & 1-\tau & 0 & 0 & \tau \\ 1-\tau & -2\,\tau & 1-\tau & \tau & 0 & \tau & 0 \\ -\tau & 0 & 0 & 0 & 0 & 1-\tau & \tau \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We are now ready to formulate our system of axioms:

Axiom 1: (Inessential Game) For all inessential games (N, v), the solution verifies $\phi_i(N, v, g) = v(\{i\})$ for all i in N.

Axiom 2: (Associated Consistency) For all games (N, v, g) in G, the associated game (N, v^*_τ, g) verifies $\phi(N, v, g) = \phi(N, v^*_\tau, g)$.

Axiom 3: (Continuity) For all convergent sequences $\{(N, v_k, g)\}_{k=1}^{\infty}$ the limit of which is game $(N, \tilde{v}, g)$ we have $\lim_{k\to\infty} \phi(N, v_k, g) = \phi(N, \tilde{v}, g)$. (The convergence of the games is point-wise).

This set of axioms is a direct adaptation of Hamiache's (2001) set of axioms to situations of incomplete communication. Naturally, the second axiom applies to the associated game defined by Equation (1).

Main Theorem:

There is one and only one solution ϕ verifying Axioms 1, 2, and 3, provided that τ is sufficiently small[2].

The remainder of this article is devoted to the proof of the main theorem.

[2] That threshold value depends on the characteristic values of matrix $(1/\tau)(P_g M_c P_g - P_g)$. Since those characteristic values are changing from graph to graph we do not have a sharp result for τ. In Hamiache (2001) we obtained $\tau < \frac{2}{n}$ for complete graphs.

4 The Proof

Lemma 2: Let us define matrix $A = \frac{1}{\tau}(P_g M_c P_g - P_g)$, the sequence $\{A^\theta\}_{\theta=1}^{\infty}$ of powers of a matrix A verifies for all integer m,

$$(P_g M_c P_g)^m = \sum_{\theta=1}^{m} \tau^\theta \binom{m}{\theta} A^\theta + P_g. \tag{6}$$

Proof:[3] From the definition of matrix A we have, $P_g M_c P_g = \tau\, A + P_g$. Since P_g is idempotent, $P_g P_g = P_g$, and $A P_g = P_g A = A$, using the binomial formula leads to the required result. □

Numerical example 3: For three player games when the communication system, $\langle N, g\rangle = \langle\{1,2,3\},\{\{1,2\},\{2,3\}\}\rangle$, matrix A is as follows,

$$A = \frac{1}{\tau}(P_g M_c P_g - P_g) = \begin{bmatrix} -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & -2 & -1 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 1 \\ -1 & -2 & -1 & 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The reader will note that A is the matrix of the coefficients of τ in matrix M_g. Below we present, on the basis of Eq. (6), a few examples of the polynomials which are elements of the successive powers of matrix M_g. These terms correspond to coalitions $S = \{1,2\}$ and $T = \{3\}$ (fourth line and third column).

$(M_g)[S,T] = -\mathbf{1}\,\tau,$

$(M_g)^2[S,T] = \mathbf{2}\,\tau^2 - \mathbf{1}\binom{2}{1}\,\tau,$

$(M_g)^3[S,T] = -\mathbf{4}\,\tau^3 + \mathbf{2}\binom{3}{2}\,\tau^2 - \mathbf{1}\binom{3}{1}\,\tau,$

$(M_g)^4[S,T] = \mathbf{8}\,\tau^4 - \mathbf{4}\binom{4}{3}\,\tau^3 + \mathbf{2}\binom{4}{2}\,\tau^2 - \mathbf{1}\binom{4}{1}\,\tau,$

$(M_g)^5[S,T] = -\mathbf{16}\,\tau^5 + \mathbf{8}\binom{5}{4}\,\tau^4 - \mathbf{4}\binom{5}{3}\,\tau^3 + \mathbf{2}\binom{5}{2}\,\tau^2 - \mathbf{1}\binom{5}{1}\,\tau,$

$(M_g)^6[S,T] = \mathbf{32}\,\tau^6 - \mathbf{16}\binom{6}{5}\,\tau^5 + \mathbf{8}\binom{6}{4}\,\tau^4 - \mathbf{4}\binom{6}{3}\,\tau^3 + \mathbf{2}\binom{6}{2}\,\tau^2 - \mathbf{1}\binom{6}{1}\,\tau,$

$(M_g)^7[S,T] = -\,\mathbf{64}\,\tau^7 + \mathbf{32}\binom{7}{6}\,\tau^6 - \mathbf{16}\binom{7}{5}\,\tau^5 + \mathbf{8}\binom{7}{4}\,\tau^4 - \mathbf{4}\binom{7}{3}\,\tau^3 + \mathbf{2}\binom{7}{2}\,\tau^2 - \mathbf{1}\binom{7}{1}\,\tau.$

Lemma 3: Given parameter $q = 2^n - 1$, there exists μ, $1 \le \mu \le q-1$ and a set of parameters $b_1, b_2, \ldots, b_{q-\mu}$ such that,

$$A^q = -b_{q-\mu}.A^\mu - b_{q-\mu-1}.A^{\mu+1} - \ldots - b_1.A^{q-1}. \tag{7}$$

[3] This proof has been proposed by a referee. It replaces advantageously a longer previous proof.

Note that μ is the algebraic multiplicity of the null eigenvalue of A.

Proof : Let us consider the characteristic polynomial of matrix A.

$$\text{charpoly}(A) = b_q + b_{q-1}.\, x + b_{q-2}.\, x^2 + ... + b_1.\, x^{q-1} + x^q. \tag{8}$$

The terms of the last line of matrix A being all zeroes, 0 is an eigenvalue of A and x is one of the factors of the characteristic polynomial. As a consequence, $b_q = 0$. Coefficient b_1 in Eq. (8) is the trace of matrix A. The terms of the trace of matrix A are as follows,

$$A[S, S] = \begin{cases} -(\#S^* - \#S) & \text{if } \#(S/g) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, the trace of A is strictly negative and $b_1 \neq 0$. So there must exist a parameter μ, which is in fact the algebraic multiplicity of the null eigenvalue of matrix A, such that the characteristic polynomial reduces to,

$$\text{charpoly}(A) = b_{q-\mu}.\, x^{\mu} + ... + b_1.\, x^{q-1} + x^q. \tag{9}$$

Since matrix A annihilates the characteristic polynomial (Cayley-Hamilton), isolating A^q, we obtain the required Equation (7), which completes the proof of Lemma 3. □

Numerical example 4: $charpoly(A, x) := x^7 + b_1\, x^6 + b_2\, x^5 + b_3\, x^4 = x^7 + 6\, x^6 + 12\, x^5 + 8\, x^4 = x^4\, (x + 2)^3$. In that case we have $b_1 = 6$, $b_2 = 12$, $b_3 = 8$.

Remark: In the above calculations the characteristic polynomial could be replaced by the minimal polynomial. Indeed, this step would significantly reduce the number of parameters b.

Let us consider a real function $f(x)$. The Taylor series of f (or more precisely the Maclaurin series) is given by,

$$f(x) = f(0) + \frac{f'(0)}{1!}\, x + \frac{f''(0)}{2!}\, x^2 + \frac{f'''(0)}{3!}\, x^3 + \dots \tag{10}$$

Given a sequence of real numbers, can we find any function such that the successive coefficients of the Maclaurin series coincide with the successive elements of the sequence in question? For example, in the case of the infinite constant sequence $\{1, 1, 1, \dots\}$ there is indeed such a function $f(x) = (1 - x)^{-1} = 1 + x + x^2 + x^3 + \dots$. We will say that $(1 - x)^{-1}$ is the generating function of the sequence $\{1, 1, 1, \dots\}$. An obvious advantage of this concept is an economy of means in describing infinite sequences. The following proposition gives the general form of generating functions for the sequences $\{A^{\theta}[\cdot, \cdot]\}_{\theta=1}^{\infty}$.

Lemma 4: For all coalitions S, $T \subseteq N$, there is a set of parameters a_1, a_2, $\dots$ $a_{q-\mu}$ such that the generating function F of $\{A^{\theta}[S, T]\}_{\theta=1}^{\infty}$ is:

$$F(x) = \sum_{i=1}^{\infty} A^i[S, T]\, x^i = \frac{R(x)}{Q(x)} = \frac{a_1\, x + a_2\, x^2 + a_3\, x^3 + ... + a_{q-\mu}\, x^{q-\mu}}{1 + b_1\, x + b_2\, x^2 + b_3\, x^3 + ... + b_{q-\mu}\, x^{q-\mu}}, \tag{11}$$

where parameters b_1, b_2, $\dots$ $b_{q-\mu}$ are those found in Lemma 3.
Proof[4]: To define the coefficients a, let us consider the following system of equations,

$$A^1[S,T]\,x^1 = a_1\,x^1$$
$$A^2[S,T]\,x^2 + b_1\,A^1[S,T]\,x^2 = a_2\,x^2$$
$$\vdots$$
$$A^{q-\mu}[S,T]\,x^{q-\mu} + b_1\,A^{q-\mu-1}[S,T]\,x^{q-\mu}\cdots + b_{q-\mu-1}\,A^1[S,T]\,x^{q-\mu} = a_{q-\mu}\,x^{q-\mu}$$
$$A^{q-\mu+1}[S,T]\,x^{q-\mu+1} + b_1\,A^{q-\mu}[S,T]x^{q-\mu+1} + \cdots + b_{q-\mu}A^1[S,T]x^{q-\mu+1} = 0$$
$$A^{q-\mu+2}[S,T]\,x^{q-\mu+2} + b_1\,A^{q-\mu+1}[S,T]x^{q-\mu+2} + \cdots + b_{q-\mu}A^2[S,T]x^{q-\mu+2} = 0$$
$$\vdots$$

Summing up the above expressions we get,

$$F(x) + b_1\,x\,F(x) + \cdots + b_{q-\mu}\,x^{q-\mu}F(x) = a_1\,x^1 + \dots a_{q-\mu}\,x^{q-\mu},$$

which concludes the proof of Lemma 4. $\square$

Numerical Example 5: Direct computations give the following values: $(A)[S,T] = -1$, $(A)^2[S,T] = 2$, $(A)^3[S,T] = -4$, $(A)^4[S,T] = 8$, $(A)^5[S,T] = -16$, $(A)^6[S,T] = 32$, $(A)^7[S,T] = -64$. We know that in our particular case $b_1 = 6$ and $b_2 = 12$. Below we compute the relevant coefficients a.

$a_1 = (A)^1[S,T] = -1$,
$a_2 = (A)^2[S,T] + b_1\,(A)^3[S,T] = 2 + 6 \times (-1) = -4$,
$a_3 = (A)^3[S,T] + b_1\,(A)^2[S,T] + b_2\,(A)^1[S,T] = (-4) + 6 \times (2) + 12 \times (-1) = -4$.

As a result we have,

$$F(x) = \frac{R(x)}{Q(x)} = \frac{-4\,x^3 - 4\,x^2 - x}{8\,x^3 + 12\,x^2 + 6\,x + 1}.$$

Reader will note that we have $F(x) = -x/(1 + 2\,x)$, but we are not interested at this stage by the simplified form.

Lemma 5: If $<\lambda,\, v>$ is an eigenpair of matrix $M_g = P_g\,M_c\,P_g$ with $\lambda \neq 0$, then $<1,\, v>$ is an eigenpair of matrix P_g.
Proof: Suppose that $P_g\,M_c\,P_g\,v = \lambda\,v$. Multiplying both sides of the last equation by P_g, and using the fact that P_g is idempotent, we get, $P_g\,M_c\,P_g\,v = \lambda\,P_g\,v = \lambda\,v$. As a result, when $\lambda \neq 0$, $<1,\, v>$ is an eigenpair of matrix P_g. $\square$
Lemma 6: For all $i \in N$, $<1,\, x_{\{i\}}>$ where $x_{\{i\}}$ is defined by Eq. (4) are eigenpairs of matrix $M_g = P_g\,M_c\,P_g$.

[4] We thank an anonymous referee for this extremely concise proof.

Proof: It is well known (Hamiache, 2010) that 1 is an eigenvalue of matrix M_c with multiplicity n, and that the n vectors $x_{\{i\}}$ for $i \in N$ are the corresponding eigenvectors.

$$P_g\, M_c\, P_g\, x_{\{i\}} = P_g\, M_c\, x_{\{i\}} = P_g\, x_{\{i\}} = x_{\{i\}}.$$

The first and the last equalities follow from Property 4. The second equality is true since $< 1,\, x_{\{i\}} >$ is an eigenpair of matrix M_c. □

Lemma 7: The eigenvalues of matrix $P_g\, M_c\, P_g$ are of the form $\lambda = 1 - \sigma\tau$ where σ is a real or complex parameter.

Proof: Let us decompose matrix M_c as $M_c = Id - \tau\, \widetilde{M_c}$. Let us consider an eigenpair $< \lambda,\, w >$ of matrix $P_g\, M_c\, P_g$.

$$P_g\, M_c\, P_g\, w = P_g\, \left(Id - \tau\, \widetilde{M_c} \right)\, P_g\, w = P_g\, w - \tau\, P_g\, \widetilde{M_c}\, P_g\, w = \lambda\, w.$$

Since we know that $< 1,\, w >$ is an eigenpair of matrix P_g we have,

$$P_g\, w - \tau\, P_g\, \widetilde{M_c}\, P_g\, w = w - \tau\, P_g\, \widetilde{M_c}\, P_g\, w = \lambda\, w,$$

re-ordering the previous expression leads to,

$$P_g\, \widetilde{M_c}\, P_g\, w = -\frac{\lambda - 1}{\tau}\, w = \sigma\, w,$$

which completes the proof of Lemma 7. □

Lemma 8: The spectral radius of $P_g\, M_c\, P_g$ is equal to 1.

Proof: In appendix. □

Lemma 9: The moduli of complex eigenvalues of matrix $P_g\, M_c\, P_g$ are strictly smaller than 1.

Proof: In appendix. □

The fact that $\lambda = 1 - \sigma\, \tau$ combined with Lemma 8 and Lemma 9, means that 1 is the sole eigenvalue of M_g on the unit circle.

Lemma 10: The sequence $(P_g M_c P_g)^m[S, T]$ is convergent and

$$\lim_{m \to \infty} (P_g M_c P_g)^m[S, T] = \frac{a_{q-\mu}}{b_{q-\mu}} + P_g[S, T], \tag{12}$$

with $a_{q-\mu}$ and $b_{q-\mu}$ being the coefficients of terms x^{q-1} in $R(x)$ and $Q(x)$, respectively as defined by Equation (11) (see Lemma 4).

Proof: In appendix. □

The Main Theorem stipulates that parameter τ should be small enough. From the proof of Lemma 10 we learn that τ should verify that $\mid 1 + \frac{\tau}{z_i} \mid < 1$ for all the roots z_i of polynomial $Q(x)$, where $1 \leq i \leq q - \mu$. Let us consider a complete communication structure. The eigenvalues of the corresponding matrix A that can be expressed as $(M_c - Id)/\tau$ are $0, -2, -3, \ldots, -n$. It is easy to see that the roots of $Q(x)$ are $z_i = -\frac{1}{2}, -\frac{1}{3}, \ldots, -\frac{1}{n}$. The constraint $\mid 1 + \frac{\tau}{z_i} \mid < 1$ leads to $0 < \tau < \frac{2}{n}$, as required in Hamiache (2001).

Lemma 11: For all games $(N,\, v,\, g) \in G$, the limit game $(N,\, \tilde{v},\, g)$ defined by

$$\tilde{v} = \lim_{k\to\infty} (P_g\, M_c\, P_g)^k\, v, \tag{13}$$

is an inessential game.
Proof: In appendix. □

Numerical Example 6: Since $Q(x) = 8\,x^3 + 12\,x^2 + 8\,x = (1 + 2\,x)^3$, polynomial $Q(x)$ has three identical roots $z_i = -\frac{1}{2}$. To ensure convergence we must have $\left|1 + \frac{\tau}{z_i}\right| = |1 - 2\,\tau| < 1$ which is true if and only if $0 < \tau < 1$.

$$\lim_{m\to\infty} (P_g M_c P_g)^m = (M_g)^\infty = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 & -\frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} & 0 & -\frac{1}{4} & 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note that $\frac{a_3}{b_3} = \frac{-4}{8} = -\frac{1}{2}$ as shown at line 4 column 3.

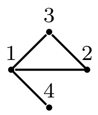
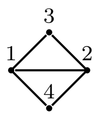
We have now all the elements permitting to conclude the main proof. By the associated consistency, the continuity and the inessential game axioms, we have for sufficiently small parameters τ,

$$\phi_i(N,\, v,\, g) = \lim_{m\to\infty} \phi_i(N,\, v_\tau^{(m*)},\, g) = \phi_i(N,\, \tilde{v},\, g) = \tilde{v}(\{i\}).$$

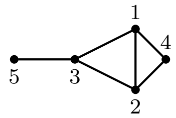
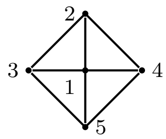
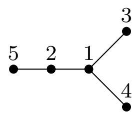
It is also true that for the unanimity game u_N we have, $\phi_i(N,\, u_N,\, g) = (M_g)^\infty[\{i\},\, N]$. The proof of the main theorem is thus complete. □

5 Conclusion

As a conclusion, we provide some comparisons of the New Value and the Mean Value[5] (Hamiache, 2003) for unanimity games u_N on a selection of graphs.

	New Value	Mean Value
	$(\frac{4}{9}, \frac{7}{36}, \frac{7}{36}, \frac{1}{6})$	$(\frac{4}{9}, \frac{7}{36}, \frac{7}{36}, \frac{1}{6})$
	$(\frac{5}{18}, \frac{5}{18}, \frac{2}{9}, \frac{2}{9})$	$(\frac{7}{24}, \frac{7}{24}, \frac{5}{24}, \frac{5}{24})$

[5] The mean value of the unanimity game (N, u_N, g) can be computed with the following formula, $MV(N, u_N, g) = \frac{1}{k(N)} \sum_{i\in N:\#(N\setminus\{i\})=1} MV(N \setminus \{i\}, u_{N\setminus\{i\}}, g(N \setminus \{i\}))$, where $k(N) = \#\{i \in N \mid N \setminus \{i\} \text{ connected}\}$.

	New Value	Mean Value
	$\left(\frac{7}{36}, \frac{7}{36}, \frac{10}{27}, \frac{7}{54}, \frac{1}{9}\right)$	$\left(\frac{31}{144}, \frac{31}{144}, \frac{101}{288}, \frac{11}{96}, \frac{5}{48}\right)$
	$\left(\frac{3}{14}, \frac{11}{56}, \frac{11}{56}, \frac{11}{56}, \frac{11}{56}\right)$	$\left(\frac{7}{30}, \frac{23}{120}, \frac{23}{120}, \frac{23}{120}, \frac{23}{120}\right)$
	$\left(\frac{5}{12}, \frac{11}{36}, \frac{7}{72}, \frac{7}{72}, \frac{1}{12}\right)$	$\left(\frac{5}{12}, \frac{11}{36}, \frac{7}{72}, \frac{7}{72}, \frac{1}{12}\right)$

The reader will note the propinquity between these two solutions. At least in the exhibited cases the results do not contradict the basic intuition that better connected players are better rewarded. This is not the case for all the values in the literature. For example, the Myerson value (Myerson, 1977), in the case of unanimity games over connected coalition N, gives $\frac{1}{n}$ to each one of the player independently of the graph. The average tree value (ATV) (Herings *et al*, 2008) coincides with the Myerson value for unanimity games over connected trees. In our numerical example it would then give $\frac{1}{3}$ to each player. The position value (Borm *et al*, 1992) rewards particularly players on links that contribute more to the connectedness of the grand coalition and may lead to counter-intuitive results.

The new value proposed in this paper has several advantages over existing values. The knowledge of the values of sub-games is not needed. As already pointed out, it is not the case for the mean value and the F-value (Hamiache, 1999).

This new value offers payoffs for the unanimity games that are rational numbers, so we have exact values without any approximation. This is usually not the case for values requiring the computation of characteristic vectors such as Hamiache (1999).

The associated game presented in this paper is simpler than the associated game that leads to the Mean Value (Hamiache, 2003). Indeed, in the case of the Mean Value when a coalition “attacks” a player, this player can call on other coalitions for protection. This point adds a significant amount of complexity to the construction of the associated game.

Finally, the axiomatic characterization developed in this article is based on the same axioms as those found in Hamiache (2001) to characterize the Shapley value.

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Appendix

Proof of Lemma 8: First of all it is easy to see that $< 0, y_T >$ are eigenpairs of matrix M_g for non-connected coalitions T, when $y_T[S] = 1$ if $S = T$ and $y_T[S] = 0$ if $S \neq T$. In the following, we will assume that $\lambda \neq 0$.

We already know that n vectors $x_{\{i\}}$ for $i \in N$, as defined by Eq. (4), are independent eigenvectors of matrix M_c related to eigenvalue 1 and that they are also eigenvectors of M_g. We will denote the other eigenpairs of matrix M_c by $< \lambda_S, x_S >$ for $S \subseteq N$ and $\#S \geq 2$. Let $< \lambda, w >$ be an eigenpair of matrix $M_g = P_g\, M_c\, P_g$. Vector x can be expressed as a linear combination of eigenvectors of matrix M_c, $w = \sum_{\emptyset \neq S \subseteq N} c_S\, x_S$. We will prove below that the eigenvalues of matrix M_g have a norm smaller than or equal to one.

$$P_g\, M_c\, P_g \Big[\sum_{\substack{i \\ i \in N}} c_{\{i\}}\, x_{\{i\}} + \sum_{\substack{S \\ S \subseteq N \\ \#S \geq 2}} c_S\, x_S \Big] = P_g\, M_c \Big[\sum_{\substack{i \\ i \in N}} c_{\{i\}}\, x_{\{i\}} + \sum_{\substack{S \\ S \subseteq N \\ \#S \geq 2}} c_S\, x_S \Big]$$

$$= P_g \Big[\sum_{\substack{i \\ i \in N}} c_{\{i\}}\, x_{\{i\}} + \sum_{\substack{S \\ \#S \geq 2}} \lambda_S\, c_S\, x_S \Big] = \sum_{\substack{i \\ i \in N}} c_{\{i\}}\, x_{\{i\}} + P_g \sum_{\substack{S \\ \#S \geq 2}} \lambda_S\, c_S\, x_S$$

$$= \lambda \Big[\sum_{\substack{i \\ i \in N}} c_{\{i\}}\, x_{\{i\}} + \sum_{\substack{S \\ S \subseteq N \\ \#S \geq 2}} c_S\, x_S \Big].$$

The first equality uses the fact that the $< 1, w >$ is an eigenpair of matrix P_g. The second equality is true since $< \lambda_S, x_S >$ is an eigenpair of matrix M_c. Let us consider the norm of the two last terms,

$$\| P_g \Big[\sum_{\substack{i \\ i\in N}} c_{\{i\}}\, x_{\{i\}} + \sum_{\substack{S \\ S\subseteq N \\ \#S\geq 2}} \lambda_S\, c_S\, x_S \Big] \| = \mid \lambda \mid \cdot \| \Big[\sum_{\substack{i \\ i\in N}} c_{\{i\}}\, x_{\{i\}} + \sum_{\substack{S \\ S\subseteq N \\ \#S\geq 2}} c_S\, x_S \Big] \|.$$

Matrix P_g is diagonalizable (Property 3), its eigenvalues are thus semi-simple. It is well known that in that case there exists a matrix norm $\| \cdot \|$ verifying $\| P_g \| = \rho(P_g)$ where $\rho(P_g)$ is the spectral radius which is equal to one. Moreover, there exists a vector norm $\| \cdot \|$ compatible with the considered matrix norm (theorem 5.7.13 p. 324 Horn and Johnson), which means that

$$\begin{aligned} \| P_g \Big[\sum_{\substack{i \\ i\in N}} c_{\{i\}}\, x_{\{i\}} + \sum_{\substack{S \\ S\subseteq N \\ \#S\geq 2}} \lambda_S\, c_S\, x_S \Big] \| &\leq \| P_g \| \cdot \| \Big[\sum_{\substack{i \\ i\in N}} c_{\{i\}}\, x_{\{i\}} + \sum_{\substack{S \\ S\subseteq N \\ \#S\geq 2}} \lambda_S\, c_S\, x_S \Big] \| \\ &= \| \Big[\sum_{\substack{i \\ i\in N}} c_{\{i\}}\, x_{\{i\}} + \sum_{\substack{S \\ S\subseteq N \\ \#S\geq 2}} \lambda_S\, c_S\, x_S \Big] \|. \end{aligned}$$

We thus obtain,

$$\mid \lambda \mid \cdot \| \Big[\sum_{\substack{i \\ i\in N}} c_{\{i\}}\, x_{\{i\}} + \sum_{\substack{S \\ S\subseteq N \\ \#S\geq 2}} c_S\, x_S \Big] \| \leq \| \Big[\sum_{\substack{i \\ i\in N}} c_{\{i\}}\, x_{\{i\}} + \sum_{\substack{S \\ S\subseteq N \\ \#S\geq 2}} \lambda_S\, c_S\, x_S \Big] \|,$$

for $\#S = s \geq 2$ the eigenvalues of M_c are $\lambda_S = 1 - s\tau$,

$$\begin{aligned} \mid \lambda \mid \cdot \| \Big[\sum_{\substack{i \\ i\in N}} c_{\{i\}}\, x_{\{i\}} + \sum_{\substack{S \\ S\subseteq N \\ \#S\geq 2}} c_S\, x_S \Big] \| &\leq \| \Big[\sum_{\substack{i \\ i\in N}} c_{\{i\}}\, x_{\{i\}} + \sum_{\substack{S \\ S\subseteq N \\ \#S\geq 2}} (1 - s\tau)\, c_S\, x_S \Big] \| \\ &\leq \| \Big[\sum_{\substack{i \\ i\in N}} c_{\{i\}}\, x_{\{i\}} + \sum_{\substack{S \\ S\subseteq N \\ \#S\geq 2}} c_S\, x_S \Big] \| + \tau \| \sum_{\substack{S \\ S\subseteq N \\ \#S\geq 2}} s\, c_S\, x_S \| \\ (\mid \lambda \mid -1) \cdot \| \Big[\sum_{\substack{i \\ i\in N}} c_{\{i\}}\, x_{\{i\}} + \sum_{\substack{S \\ S\subseteq N \\ \#S\geq 2}} c_S\, x_S \Big] \| &\leq \tau \cdot \| \sum_{\substack{S \\ \#S\geq 2}} s\, c_S\, x_S \|. \end{aligned}$$

Parameter τ is positive and arbitrarily small. Thus if $(\mid \lambda \mid -1)$ is positive, the term of the left hand side of the inequality is positive and we can choose parameter τ to be sufficiently small to contradict the inequality. So $(\mid \lambda \mid -1)$ cannot be strictly positive, which leads to $\mid \lambda \mid \leq 1$. □

Proof of Lemma 9: Let $< \lambda,\, w >$ be an eigenpair of matrix M_g when $w = \sum_{\emptyset\neq S\subseteq N} c_S\, x_S$,

$$P_g M_c P_g \sum_{\substack{S \\ \emptyset\neq S\subseteq N}} c_S\, x_S = \lambda \sum_{\substack{S \\ \emptyset\neq S\subseteq N}} c_S\, x_S.$$

We know that $< 1,\, w >$ is an eigenpair of matrix P_g,

$$P_g M_c \sum_{\substack{S \\ \emptyset\neq S\subseteq N}} c_S\, x_S = \lambda\, P_g \sum_{\substack{S \\ \emptyset\neq S\subseteq N}} c_S\, x_S.$$

Separating the singletons,

$$P_g M_c \sum_{\substack{S \\ \emptyset\neq S\subseteq N}} c_S\, x_S = P_g M_c \sum_{\substack{i \\ i\in N}} c_{\{i\}}\, x_{\{i\}} + P_g M_c \sum_{\substack{S \\ \#S\geq 2}} c_S\, x_S = P_g \lambda \sum_{\substack{S \\ \emptyset\neq S\subseteq N}} c_S\, x_S.$$

Since $< 1, x_{\{i\}} >$ for all $i \in N$ and $< 1 - s\,\tau, x_S >$ for all coalitions S verifying $\#S = s \geq 2$ are eigenpairs of matrix M_c,

$$P_g \sum_{\substack{i\\ i\in N}} c_{\{i\}} x_{\{i\}} + P_g \sum_{\substack{S\\ S\subseteq N\\ \#S\geq 2}} (1 - s\,\tau) c_S x_S = P_g \sum_{\substack{S\\ \emptyset\neq S\subseteq N}} c_S x_S - P_g \tau \sum_{\substack{S\\ S\subseteq N\\ \#S\geq 2}} s c_S x_S = P_g \lambda \sum_{\substack{S\\ \emptyset\neq S\subseteq N}} c_S x_S.$$

For all connected coalitions T we have, after assembling a few terms,

$$\tau \sum_{\substack{S\\ S\subseteq N\\ \#S\geq 2}} s\, c_S\, x_S[T] = (1-\lambda) \sum_{\substack{S\\ \emptyset\neq S\subseteq N}} c_S\, x_S[T].$$

Since $w = \sum_{\emptyset\neq S\subseteq N} c_S\, x_S$ is an eigenvector, there exists at least a connected coalition T such that $w[T] = \sum_{\emptyset\neq S\subseteq N} c_S\, x_S[T] \neq 0$. Isolating λ,

$$\lambda = 1 - \tau \frac{\displaystyle\sum_{\substack{S\\ \#S\geq 2}} s\, c_S\, x_S[T]}{\displaystyle\sum_{\substack{S\\ \emptyset\neq S\subseteq N}} c_S\, x_S[T]}.$$

Since matrix $P_g M_c P_g$ is real, $\overline{\lambda}$, the complex conjugate of λ, is also one of its eigenvalues, and the next equality is true,

$$\overline{\lambda} = 1 - \tau \frac{\displaystyle\sum_{\substack{S\\ \#S\geq 2}} s\, \overline{c_S}\, x_S[T]}{\displaystyle\sum_{\substack{S\\ \emptyset\neq S\subseteq N}} \overline{c_S}\, x_S[T]}.$$

$$\lambda\,\overline{\lambda} = \mid \lambda \mid^2 = \frac{1}{\mid w[T] \mid^2} \Big[w[T] - \tau \sum_{\substack{S\\ \#S\geq 2}} s\, c_S\, x_S[T] \Big] \Big[\overline{w}[T] - \tau \sum_{\substack{S\\ \#S\geq 2}} s\, \overline{c_S}\, x_S[T] \Big].$$

Expanding the two last terms of the previous expression,

$$\mid \lambda \mid^2 = \frac{1}{\mid w[T] \mid^2} \Big[\mid w[T] \mid^2 - 2\,\tau Re\Big(w[T] \sum_{\substack{S\\ \#S\geq 2}} s\, \overline{c_S}\, x_S[T] \Big) + \tau^2 \mid \sum_{\substack{S\\ \#S\geq 2}} s\, c_S\, x_S[T] \mid^2 \Big].$$

We can choose an eigenvector w such that $\mid w[T] \mid = 1$,

$$\mid \lambda \mid^2 = 1 - 2\,\tau Re\Big(w[T] \sum_{\substack{S\\ \#S\geq 2}} s\, \overline{c_S}\, x_S[T] \Big) + \tau^2 \mid \sum_{\substack{S\\ \#S\geq 2}} s\, c_S\, x_S[T] \mid^2 .$$

Since $\mid \lambda \mid^2 \leq 1$ we have,

$$2\,\tau Re\Big(w[T] \sum_{\substack{S\\ \#S\geq 2}} s\, \overline{c_S}\, x_S[T] \Big) - \tau^2 \mid \sum_{\substack{S\\ \#S\geq 2}} s\, c_S\, x_S[T] \mid^2 \geq 0.$$

$$0 < \tau \leq \frac{2\, Re\Big(w[T] \sum_{S:\#S\geq 2} s\, \overline{c_S}\, x_S[T] \Big)}{\mid \sum_{S:\#S\geq 2} s\, c_S\, x_S[T] \mid^2}.$$

Choosing parameter τ sufficiently small will ensure that $\mid \lambda \mid^2 < 1$. $\square$

Auxiliary result:

$$\sum_{\theta=1}^{m}\binom{j+\theta-1}{\theta}\binom{m}{\theta}\left(\frac{\tau}{z_i}\right)^{\theta}=-1$$
$$+\sum_{\theta=0}^{j-1}\binom{j-1}{j-1-\theta}\binom{m}{j-1-\theta}\left(\frac{\tau}{z_i}\right)^{j-1-\theta}\left(1+\frac{\tau}{z_i}\right)^{m-j+1+\theta}.$$

Proof:

$$\sum_{\theta=0}^{m}\binom{\theta+j-1}{\theta}\binom{m}{\theta}\left(\frac{\tau}{z_i}\right)^{\theta}=\sum_{\theta=0}^{m}\frac{(\theta+j-1)\dots(\theta+1)}{(j-1)!}\binom{m}{\theta}\left(\frac{\tau}{z_i}\right)^{\theta}$$
$$=\frac{1}{(j-1)!}\sum_{\theta=0}^{m}\frac{d^{j-1}}{d\left(\frac{\tau}{z_i}\right)^{j-1}}\left[\binom{m}{\theta}\left(\frac{\tau}{z_i}\right)^{\theta+j-1}\right]$$
$$=\frac{1}{(j-1)!}\frac{d^{j-1}}{d\left(\frac{\tau}{z_i}\right)^{j-1}}\left[\left(\frac{\tau}{z_i}\right)^{j-1}\sum_{\theta=0}^{m}\binom{m}{\theta}\left(\frac{\tau}{z_i}\right)^{\theta}\right]$$
$$=\frac{1}{(j-1)!}\frac{d^{j-1}}{d\left(\frac{\tau}{z_i}\right)^{j-1}}\left[\left(\frac{\tau}{z_i}\right)^{j-1}\left(1+\frac{\tau}{z_i}\right)^{m}\right]. \tag{14}$$

Applying Leibnitz's Theorem for differentiation of a product,

$$\frac{d^t}{dx^t}(u\cdot v)=\sum_{\theta=0}^{t}\binom{t}{\theta}\frac{d^\theta}{dx^\theta}(u)\frac{d^{t-\theta}}{dx^{t-\theta}}(v),$$

to Eq. (14), with $u=(\tau/z_i)^{j-1}$, $v=(1+\tau/z_i)^m$ and $t=j-1$ we obtain,

$$=\frac{1}{(j-1)!}\sum_{\theta=0}^{j-1}\binom{j-1}{\theta}\frac{d^\theta}{d\left(\frac{\tau}{z_i}\right)^{\theta}}\left(\frac{\tau}{z_i}\right)^{j-1}\frac{d^{j-1-\theta}}{d\left(\frac{\tau}{z_i}\right)^{j-1-\theta}}\left(1+\frac{\tau}{z_i}\right)^{m}$$

$$=\frac{1}{(j-1)!}\sum_{\theta=0}^{j-1}\binom{j-1}{\theta}\frac{(j-1)!}{(j-\theta-1)!}\left(\frac{\tau}{z_i}\right)^{(j-1-\theta)}\frac{(m)!}{(m-j+\theta+1)!}\left(1+\frac{\tau}{z_i}\right)^{(m-j+1+\theta)}$$
$$=\sum_{\theta=0}^{j-1}\binom{j-1}{j-\theta-1}\binom{m}{j-\theta-1}\left(\frac{\tau}{z_i}\right)^{(j-1-\theta)}\left(1+\frac{\tau}{z_i}\right)^{(m-j+1+\theta)},$$

which proves the Auxiliary result. □

Proof of Lemma 10: If T is non-connected, all the terms of the series are equal to 0 and Lemma 10 is true. So let us consider instead that T is connected.

$$(P_gM_cP_g)^m[S,T]=\sum_{\theta=1}^{m}A^\theta[S,T]\binom{m}{\theta}\tau^\theta+P_g[S,T],$$

where the parameters $A^\theta[S,T]$ are the successive coefficients of the powers of x in the Maclaurin development of the generating function $F(x)$. Those coefficients are the value of the relevant derivatives of $F(x)$ at point $x=0$.

$$A^\theta[S,T]=\frac{1}{\theta!}\frac{d^\theta F(0)}{d\,x^\theta}. \tag{15}$$

Performing the euclidean division of Eq. (11), we can rewrite the generating function of Lemma 4 as,

$$F(x) = \frac{R(x)}{Q(x)} = \frac{\alpha_0 + \alpha_1\, x + ... + \alpha_{q-\mu-1}\, x^{q-\mu-1}}{1 + b_1\, x + b_2\, x^2 + ... + b_{q-\mu+1}\, x^{q-\mu+1}} + \frac{a_{q-\mu}}{b_{q-\mu}},$$

where $\alpha_i = a_i - \frac{a_{q-\mu}}{b_{q-\mu}} b_i$ for $1 \leq i \leq q - \mu - 1$ and $\alpha_0 = -\frac{a_{q-\mu}}{b_{q-\mu}}$.

From the partial fraction decomposition theorem, we can write the rational function, the first term of the right hand side of the previous equation, as a finite linear combination of terms of the form, $E(x) = (x - z)^{-w}$, where z is a root of the denominator and w is an integer at most equal to the algebraic multiplicity of z. Note that z could be a complex number.

$$F(x) = \sum_{i=1}^{p} \sum_{j=1}^{w_i} \beta_{i,j}\, \frac{1}{(x - z_i)^j} + \frac{a_{q-\mu}}{b_{q-\mu}},$$

where $z_1, \ldots, z_p$ are the roots of $Q(x)$, $w_1, \ldots, w_p$ are their respective algebraic multiplicities and $\beta_{i,j}$ are the coefficients of the linear combination. The derivative of order θ is given by,

$$\frac{d^\theta\, F(x)}{d\, x^\theta} = \sum_{i=1}^{p} \sum_{j=1}^{w_i} \beta_{i,j}\, \frac{(-j)(-j-1)\ldots(-j-\theta+1)}{(x - z_i)^{j+\theta}},$$

and the coefficients $A^\theta[S, T]$ are given by,

$$A^\theta[S, T] = \frac{1}{\theta!}\frac{d^\theta\, F(0)}{d\, x^\theta} = \sum_{i=1}^{p} \sum_{j=1}^{w_i} \beta_{i,j}\, \frac{(-1)^\theta}{(-z_i)^{j+\theta}} \binom{j+\theta-1}{\theta}.$$

$$(P_g M_c P_g)^m[S, T] = \sum_{\theta=1}^{m} \sum_{i=1}^{p} \sum_{j=1}^{w_i} \beta_{i,j}\, \frac{(-1)^\theta}{(-z_i)^{j+\theta}} \binom{j+\theta-1}{\theta} \binom{m}{\theta} \tau^\theta + P_g[S, T].$$

Inverting the order of summations,

$$= \sum_{i=1}^{p} \sum_{j=1}^{w_i} \beta_{i,j}\, \frac{1}{(-z_i)^j} \sum_{\theta=1}^{m} \binom{j+\theta-1}{\theta} \binom{m}{\theta} \left(\frac{\tau}{z_i}\right)^\theta + P_g[S, T]. \tag{16}$$

Using the Auxiliary result we get,

$$\begin{aligned}(P_g M_c P_g)^m[S, T] = &- \sum_{i=1}^{p} \sum_{j=1}^{w_i} \beta_{i,j}\, \frac{1}{(-z_i)^j} + P_g[S, T] \\ &+ \sum_{i=1}^{p} \sum_{j=1}^{w_i} \beta_{i,j}\, \frac{1}{(-z_i)^j} \sum_{\theta=0}^{j-1} \binom{j-1}{j-1-\theta} \binom{m}{j-1-\theta} \left(\frac{\tau}{z_i}\right)^{j-1-\theta} \left(1 + \frac{\tau}{z_i}\right)^{m-j+1+\theta}.\end{aligned} \tag{17}$$

Since,

$$F(0) = \frac{R(0)}{Q(0)} = \sum_{i=1}^{p} \sum_{j=1}^{w_i} \beta_{i,j} \left(\frac{1}{-z_i}\right)^j + \frac{a_{q-\mu}}{b_{q-\mu}} = 0,$$

we have,

$$\begin{aligned}(P_g M_c P_g)^m[S, T] = &\frac{a_{q-\mu}}{b_{q-\mu}} + P_g[S, T] \\ &+ \sum_{i=1}^{p} \sum_{j=1}^{w_i} \beta_{i,j}\, \frac{1}{(-z_i)^j} \left[\sum_{\theta=0}^{j-1} \binom{j-1}{j-1-\theta} \binom{m}{j-1-\theta} \left(\frac{\tau}{z_i}\right)^{j-1-\theta} \left(1 + \frac{\tau}{z_i}\right)^{m-j+1+\theta}\right].\end{aligned}$$

Let us consider the characteristic polynomial in Eq. (8) and $Q(x)$ as defined by Eq. (11). If z_i is a root of $Q(x)$, it is true that $\frac{1}{z_i}$ is a root of Eq. (8) and thus an eigenvalue of matrix A (note that $z_i \neq 0$). We are now ready to show that $1 + \frac{\tau}{z_i}$ is an eigenvalue of matrix M_g.

Since the columns of matrix $(P_g M_c P_g)$ corresponding to non-connected coalitions are zero, the non-zero eigenvalues are preserved when we delete from matrix $(P_g M_c P_g)$ the lines and columns corresponding to non-connected coalitions. Let us denote by M_g^s that "simplified" matrix. The corresponding sub-matrix of A is thus equal to $(A^s) = (M_g^s - Id)/\tau$, which leads to $Id + \tau\, A^s = M_g^s$. As a consequence, the eigenvalues of M_g^s are given by $1 + \frac{\tau}{z_i}$, which are also non-zero eigenvalues of matrix M_g. Since the spectral radius of M_g is one, and since $z_i \neq 0$, the moduli of $1 + \frac{\tau}{z_i}$ are strictly smaller than 1.

Let us focus now on the terms $\binom{m}{j-1-\theta}(1 + \frac{\tau}{z_i})^m$ as m tends to infinity. If $\theta = j-1$, the corresponding term reduces to $(1 + \frac{\tau}{z_i})^m$ and thus converges to 0. Let us assume now that $\theta = 0, 1, \ldots, j-2$.

$$\binom{m}{j-1-\theta} \left|\left(1 + \frac{\tau}{z_i}\right)\right|^m = \frac{m(m-1)(m-2)...(m-j+2+\theta)}{(j-1-\theta)!} \left|\left(1 + \frac{\tau}{z_i}\right)\right|^m$$

$$\leq \frac{m^{j-1-\theta}}{(j-1-\theta)!} \left|\left(1 + \frac{\tau}{z_i}\right)\right|^m. \tag{18}$$

The logarithm of the last expression converges if and only if $(j-1-\theta)\ \log(m) + m\ \log\left|\left(1 + \frac{\tau}{z_i}\right)\right|$ converges as $m \to \infty$. Using the fact that $(\log m)/m \to 0$ as $m \to \infty$, the term in Expression (18) converges to 0 as m tends to infinity, which completes the proof of Lemma 10. □

Proof of Lemma 11:

Let us write $(P_g\, M_c\, P_g)^n = (P_g\, M_c\, P_g)\,(P_g\, M_c\, P_g)^{n-1}$, and let W be the limit of the series $\{(P_g\, M_c\, P_g)^k\}_{k=1}^{\infty}$. It is thus true that $(P_g\, M_c\, P_g)\, W = W$. In words, the columns of matrix W are eigenvectors of matrix $(P_g\, M_c\, P_g)$ related to eigenvalue 1. We shall show that these eigenvectors are "inessential" vectors. Let $w = (w_S)_{\emptyset \neq S \subseteq N}$ be an eigenvector associated to $\lambda = 1$. We will solve the following system of linear equations, $(P_g\, M_c\, P_g)\, w = w$. Since we have, for non-connected coalitions S, $w_S = \sum_{K \in S/g} w_K$, we will concentrate only on connected coalitions. Considering Eq. (1), we obtain after few cancellations,

$$\sum_{\substack{j \\ j \in S^* \setminus S}} \left(w_{S \cup \{j\}} - w_S - w_{\{j\}}\right) = 0. \tag{19}$$

So we learn that for all connected coalitions S, the related coefficient w_S is defined uniquely as a linear combination of $w_{N \cup \{j\}}$ and $w_{\{j\}}$ for $j \in S^* \setminus S$. Noting that for $S = N \setminus \{i\}$, we have $w_{N \setminus \{i\}} = w_N - w_{\{i\}}$, we can conclude that w_S is a linear combination of w_N and a selection of $w_{\{j\}}$. We show below that for all connected coalitions S such that $1 \leq \#S \leq n-1$, we have,

$$w_S = w_N - \sum_{\substack{i \\ i \notin S}} w_{\{i\}}. \tag{20}$$

Equation (20) is of course true for $S = N$ and for all the connected coalitions with $n-1$ elements. Let us assume that Eq. (20) is true for all connected coalitions of size s and above. We will show that Eq. (20) is also true for connected coalitions T verifying $\#T = s-1$.

$$-\sum_{\substack{j \\ j \in T^* \setminus T}} w_{\{j\}} - (\#T^* - \#T)\, w_T + \sum_{\substack{j \\ j \in T^* \setminus T}} w_{T \cup \{j\}} = 0.$$

Using the induction hypothesis,

$$-\sum_{\substack{j \\ j \in T^* \setminus T}} w_{\{j\}} - (\#T^* - \#T)\, w_T + \sum_{\substack{j \\ j \in T^* \setminus T}} \Big[w_N - \sum_{\substack{m \\ m \in N \setminus (T \cup \{j\})}} w_{\{m\}}\Big] = 0.$$

Taking into account that $N \setminus (T \cup \{j\}) = (N \setminus T) \setminus \{j\}$,

$$- \sum_{\substack{j \\ j \in T^* \setminus T}} w_{\{j\}} - (\#T^* - \#T)\, w_T + (\#T^* - \#T)\, w_N + \sum_{\substack{m \\ m \in T^* \setminus T}} w_{\{m\}} - (\#T^* - \#T) \sum_{\substack{m \\ m \in N \setminus (T)}} w_{\{m\}} = 0.$$

After relevant cancellations,

$$w_T = w_N - \sum_{\substack{m \\ m \in N \setminus T}} w_{\{m\}},$$

which proves that Equation (20) is true for all non-empty connected coalitions. As a consequence,

$$w_{\{j\}} = w_N - \sum_{\substack{i \\ i \in N \setminus \{j\}}} w_{\{i\}}, \tag{21}$$

$$w_N = \sum_{\substack{i \\ i \in N}} w_{\{i\}}. \tag{22}$$

Combining Eqs. (20) and (22),

$$w_S = w_N - \sum_{\substack{i \\ i \in N \setminus S}} w_{\{i\}} = \sum_{\substack{i \\ i \in N}} w_{\{i\}} - \sum_{\substack{i \\ i \in N \setminus S}} w_{\{i\}} = \sum_{\substack{i \\ i \in S}} w_{\{i\}}. \tag{23}$$

The eigenvectors of matrix $P_g\, M_c\, P_g$ related to eigenvalue 1 are "inessential" vectors. We have thus proved that $(P_g\, M_c\, P_g)^{\infty}[S,\, T] = \sum_{i \in S} (P_g\, M_c\, P_g)^{\infty}[\{i\},\, T]$. Which ensures that the limit game $\tilde{v}$ is inessential. $\square$