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OBITUARY

Bill Sandholm in Memoriam

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William H. Sandholm, 1970–2020

As were so many others, we were very saddened to hear that William "Bill" Sandholm passed away on July 6, 2020, at the all-too-young age of 49. Bill was an excellent and insightful researcher, an inspiring colleague, and a trusted friend.

Bill graduated from Dartmouth College in 1992, with a major in Economics, and received his Ph.D. from Northwestern's Kellogg Graduate School of Economics in 1998. He joined the Department of Economics at the University of Wisconsin that fall, and spent the next 22 years at Wisconsin, rising to become the Richard E. Stockwell Professor of Economics. In the process, he became a fixture in the economic theory and game theory communities. He served on the editorial boards of *Dynamic Games and Applications*, *Games and Economic Behavior*, the *International Journal of Game Theory*, the *Journal of Dynamics and Games*, the *Journal of Economic Behavior and Organization*, the *Journal of Economic Theory*, *Mathematics*

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of Operations Research, and *Theoretical Economics*. He served on several National Science Foundation panels. He was a member of the Council of the Game Theory Society. He was tireless as an unselfish meeting organizer and referee.

Bill had a wide range of interests—he was as interested in Dostoyevsky as in Dedekind, and as comfortable with Cartesian philosophy as with Cartesian coordinates—but his focus was on evolutionary game theory. Bill pursued his interest in evolutionary game theory with a wide range of coauthors, to whom we apologize for not mentioning in the text, trusting readers to refer to the following bibliography.

Three main themes appear in Bill's research: (1) the application of evolutionary game theory to implementation problems in economics, (2) the convergence of behavior in evolutionary game theory, and (3) the micro foundations of evolutionary dynamics. On the first theme Bill was a pioneer. In a series of papers ("Evolutionary Implementation and Congestion Pricing" (*Review of Economic Studies*, 2002), "Negative Externalities and Evolutionary Implementation" (*Review of Economic Studies*, 2005), and "Pigouvian Pricing and Stochastic Evolutionary Implementation" (*Journal of Economic Theory*, 2007), Bill provided a general framework demonstrating how a benevolent government can use simple pricing schemes to ensure efficient behavior in implementation problems with hidden information and hidden actions. The key to this analysis is the notion of evolutionary implementation, invented by Bill for this purpose. Evolutionary implementation is a strong form of the requirement that players who follow a reasonable and quite general myopic adjustment process eventually learn to behave as the planner desires. In "Evolutionary Implementation...," Bill solves a network planner's problem, showing that efficient behavior can be evolutionarily implemented using a simple, decentralized price scheme. (More precisely, he constructs a price scheme under which the set of efficient outcomes is globally stable under any reasonable evolutionary process.) In "Negative Externalities...", Bill extends these results to a general model of negative externalities, as occurs whenever (for example) my decision to use a highway imposes a cost on you via increased congestion. In "Pigouvian Pricing...", Bill obtains yet more general implementation results for a model in which externalities may be negative, positive, or some combination of the two. The pricing scheme he employs defines a game which may possess multiple equilibria, but he exploits his previous theoretical results to isolate a unique and efficient outcome.

The key to evolutionary implementation is the convergence of behavior. It is then no surprise that Bill was interested in the convergence properties of evolutionary dynamics, the second main theme in his research. In two papers, "Potential Games with Continuous Player Sets" (*Journal of Economic Theory*, 2001) and "Excess Payoff Dynamics and Other Well-Behaved Evolutionary Dynamics" (*Journal of Economic Theory*, 2005), Bill introduces general classes of population games which both have desirable convergence properties and which are relevant in economic applications. In the first paper, he shows that in potential games, for dynamics which satisfy a weak characteristic (payoff monotonicity) of coherent behavior, convergence occurs to equilibrium play. In "Excess Payoff Dynamics...", Bill introduces stable games, a more general class of games that includes many standard games as special cases. He shows that while payoff monotonicity of an evolutionary dynamic is insufficient to guarantee convergence to equilibrium in these games, supplementing this requirement with some mild technical conditions ensures that convergence occurs. A third paper, "On the Global Convergence of Stochastic Fictitious Play" (*Econometrica*, 2002) addresses a problem in fictitious play that first attracted the attention of economists and mathematicians in the 1960s, providing some of the strongest convergence results yet found. In "The Projection Dynamic and the Replicator Dynamic" (*Games and Economic Behavior* 2008), "The Projection Dynamic and the Geometry of Population Games" (*Games and Economic Behavior* 2008) and "Local Stability under Evolutionary Game Dynamics" (*Theoretical Economics*, 2010), Bill very nicely extends his results to new classes of dynamics. Together, these papers provide what are now the standard tools for studying convergence in evolutionary models.

At the same time, Bill was careful to identify the limits of his work. It is intuitively evident that some conditions are needed to ensure the convergence of an evolutionary system. However, it seems just as intuitive that evolutionary dynamics should quite generally purge strictly dominated strategies. Bill's 2011 *Theoretical Economics* paper shows that this need not be the case. The key to his example is a pure strategy that is strictly dominated by a mixture of three pure strategies. The population shares of these three strategies cycle perpetually over time, precluding convergence to the dominating mixed strategy, thereby allowing the strictly dominated pure strategy to survive.

The initial work in evolutionary game theory focused on models of random matching, in which members of a population are repeatedly matching into pairs to play an underlying game. This model grew naturally out of interest in using evolutionary arguments to evaluate equilibria in games, and has the advantage of endowing payoffs with a linear structure. Bill was among the first to extend this reasoning to more general population games, in which individual payoffs can be quite general functions of the population strategy profile, as, for example, in networks. Perhaps more important than the increased generality *per se*, this approach allowed Bill to simplify and identify the essence of the central arguments. This approach culminated in his 2010 MIT Press book, *Population Games and Evolutionary Dynamics*. A notable aspect of the reaction to this book, now a standard reference in this area, is that it is routinely characterized simply as "beautiful."

While evolutionary population models commonly take the form of differential equations characterizing the aggregate behavior of a population, this aggregate behavior presumably reflects an underlying vision of the individual. As the third main theme in Bill's research, he investigated the behavioral foundations of evolutionary dynamics, examining Markov processes based on specifications of individual behavior. The formulation of such models is relatively straightforward, but there is an art to extracting useful results. In "Evolution and Equilibrium under Inexact Information" (*Games and Economic Behavior*, 2003), Bill uses approximation results from probability theory to show that under broad assumptions, aggregate behavior in such models follows an almost deterministic trajectory (given by the solution to an ordinary differential equation derived from the expected motion of the process) for a very long time. This result provides a useful description of disequilibrium adjustments in individual behavior while retaining much of the convenience of the population-level approach. However, in some circumstances it reveals little about the sorts of behaviors that emerge in the long run, in what is referred to as the steady state equilibrium of the model. Bill therefore identifies another stochastic process that nicely approximates the equilibrium behavior. This approximation leads him to introduce a new notion of evolutionary stability, local probabilistic stability (LPS), which requires that a population which begins play in equilibrium settle into a fixed stochastic pattern around the equilibrium. Bill shows that LPS is an appealing way to understand long run behavior in population models, in particular that it provides a criterion for understanding what sorts of long run behavioral patterns may be expected to emerge. In a series of subsequent papers, including papers in *Games and Economic Behavior* in 2007 ("Simple Formulas for Stationary Distributions and Stochastically Stable States") and two 2009 *Journal of Economic Theory* papers ("Stable Games and their Dynamics" and "Large-Population Potential Games"), Bill brings these techniques to fruition by providing complete analyses of important classes of problems. The concept of LPS has become a standard tool in evolutionary game theory.

Much of Bill's more recent work was concerned with exploring further the connections between individual behavior and population dynamics. Throughout this work, Bill exhibited his talent for constructive abstraction, isolating the essence of an idea by stripping away confounding detail. A number of Bill's early papers exploited the fact that if a population is large, then aggregate behavior will be almost deterministic, even if individual behavior is stochastic. In "Stochastic Approximations with Constant Step Size and Differential Inclusions" (*SIAM Journal on Control and Optimization*, 2013), a technically impressive paper, he extends existing approximation results, making precise the sense in which deterministic dynamics can be useful in characterizing the random behavior of a population of agents. "Sampling Best Response Dynamics and Deterministic Equilibrium Selection" (*Theoretical Economics*, 2015), building on a 2001 *IJGT* paper, takes these ideas in a more applied direction, showing that if members of a population choose strategies by first observing the behavior of a few others in the population, and then playing a best response to this sample, then the resulting population behavior can be described by a deterministic system that selects so-called *p*-dominant equilibria. Models that link stochastic individual strategy choice to the aggregate selection of risk-dominant equilibria are familiar, but are often viewed as generating problematically long waiting times. Bill's paper is striking for the fact that it does not rely on ever-longer waiting times.

The behavior of evolutionary processes built up from stochastic individual behavior often hinges on "large deviations", or rare but critical bursts of behavior. Results from large deviation theory again appear in Bill's work, stretching from his 2010 paper in *Theoretical Economics* to his 2018 paper in *Mathematics of Operations Research*. In "Large Deviations and Stochastic Stability in the Small Noise Double Limit" (*Theoretical Economics*, 2016), Bill identifies conditions under which large deviations can be analyzed within optimal control theory. His forthcoming *Mathematics of Operations Research* paper, unfortunately to appear posthumously, extends this idea, again bringing impressive technique to bear.

There are yet more of Bill's papers that warrant attention. Evident throughout his work was a sense of curiosity. Bill was a leader in extending the purview of evolutionary game theory from random matching games in fixed populations to the study of quite general population games. While many others were content to consider the replicator dynamic, Bill developed and examined a host of alternatives, with "Riemannian" dynamics (*Journal of Economic Theory*, 2018) being one of his recent contributions. While many others were content to focus on best response dynamics, Bill again considered alternatives, with "best experienced payoff dynamics" (*Journal of Economic Theory*, 2020) being the most recent.

Beyond his research, Bill will be remembered for his collegiality, generosity, and humility. He was kind and generous, just as eager to talk about, or work on, others' research as his own. He was always more concerned with finding the truth than with worrying about getting credit for having found it. He was devoted to his students, with whom he was invariably gentle and encouraging, while insisting on exacting standards. He wrote innovative software for evolutionary dynamics, and then not only made it freely available but patiently answered questions from those seeking help with it. He put enormous energy into his teaching, drawing rave reviews for his undergraduate statistics course, otherwise usually viewed as the epitome of drudgery, and that teaching culminated in his recent *Vital Statistics* textbook. Bill made any group or project he was a part of work better. We miss him a lot.

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