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Irresolute mechanism design: a new path to possibility

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Abstract
Often preferences in a group of agents are such that any sensible goal must admit a tie between all alternatives. The standard formulation in mechanism design demands that in this case all alternatives must be equilibrium outcomes of the decision making mechanism. However, as far as the idea of an equilibrium is to predict the outcome, we could equally well require that there are no equilibria at all. Although this may seem innocent, it allows the mechanism designer to implement goals that are impossible to enforce with any other implementation concept, like mixed Nash implementation, subgame perfect implementation, or Nash implementation using undominated strategies.

Keywords Condorcet rule · Collective decision making · Implementation · Impossibility results · Nash equilibrium · Social choice theory

JEL Classification C72 · D71

1 Introduction

The fact that social choice theory was born in the aftermath of Arrow's impossibility theorem (Arrow 1963) was an omen of things to come: Results in this field have had a negative connotation ever since, either saying that no goal of society can satisfy certain desiderata (Arrow 1963; Fishburn 1973; Sen 1970), or that there would be no reliable way to collect the information that is needed anyway (Barberá 1997;

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Ching and Zhou 2002; Duggan and Schwarz 2000; Gibbard 1997, 1973; Gärdenfors 1976; Kelly 1977; Sato 2008; Satterthwaite 1975).[1] The second problem is more fundamental in the sense that, at the end of the day, society has to make a decision.

To formulate the second problem exactly, and to also define concepts that are need later on, let $N = \{1, \dots, n\}$ be the set of agents, A the set of alternatives, $\Theta = \Theta_1 \times \cdots \times \Theta_n$ the set of states, and $\succeq_i^\theta$ the preference relation of agent i at state θ. Furthermore, suppose that the state space is unrestricted,[2] and the goal of the mechanism designer, or the choice rule (CR), can be represented as a mapping $f : \Theta \to A$ that associates an acceptable alternative $f(\theta) \in A$ to each state $\theta \in \Theta$. Then, either the process of collecting private information is not reliable in the sense that some individual i has an incentive to misrepresent his or her information at some state $\theta = (\theta_1, \theta_2, \dots, \theta_n)$, that is

$$f(\theta_i', \theta_{-i}) \succ_i^\theta f(\theta_i, \theta_{-i}) \text{ for some } \theta_i' \in \Theta_i,$$

or choice rule f has other undesirable features − it is dictatorial (selects the best alternative of the same agent at all states) or it has only 2 alternatives in the range i.e. $| f(\Theta) |= 2$.[3] This is the famous Gibbard-Satterthwaite-theorem (Gibbard 1973; Satterthwaite 1975), and a choice rule that is not prone to this type of misrepresentation is called *strategy-proof*.

Two possible ways to escape this impossibility suggest themselves immediately. The outcome could be random, or it could be an entire set of alternatives. Unfortunately, both generalizations arrive at a similar conclusion as the GS-theorem. In the first case dictatorship is just replaced with random dictatorship (Gibbard 1997),[4] and in the second case, a similar conclusion holds for all sensible ways to generalize the concept of misrepresentation so that it applies for correspondences (Barberá et al. 2001; Ching and Zhou 2002; Duggan and Schwarz 2000).

After the birth of mechanism design in the late 1960s and early 1970s, pioneered by Leonid Hurwicz, Stanley Reiter, Eric Maskin, and Roger Myerson, new possibilities began to emerge.[5] Unfortunately, in the case of unrestricted state space, this approach has not lead that far. Now we know that if the Nash equilibrium correspondence of a decision mechanism is a function, then it is either dictatorial, constant, or selects between two alternative only, and in all other cases, it tends to be too large in the sense of selecting too many alternatives at each state. On the other hand, while some refinements of Nash equilibrium admittedly give more permissive results, like virtual implementation (Abreu and Sen 1990), subgame perfect implementation (Abreu and Sen 1990; Moore and Repullo 1998; Vartiainen 2007), and

[1] To name put a few from a large and still expanding literature.

[2] Unrestricted state space means that for any preference profile $\succeq = (\succeq_1 \succeq_2, \dots, \succeq_n)$, there exists a state $\theta \in \Theta$, such that $\succeq_i^\theta = \succeq_i$ for all $i \in N$.

[3] As usual, θ_{-i} is the profile $(\theta_1, \dots, \theta_{i-1}, \theta_{i+1}, \dots, \theta_n)$ that specifies the preference relation of each agent except i, and $f(\Theta)$ is the set $\{f(\theta) \mid \theta \in \Theta\}$.

[4] Although, to be exact, also a few other not so good choice rules emerge in this case.

[5] For a review of the main contributions see (Baliga and Sjöström 2007; Chorchoón 1996, 1996; Jackson 2001; Moore and Repullo 1990; Maskin 1999; Palfrey and Srivastava 1991; Serrano 2004).

implementation using undominated strategies (Palfrey and Srivastava 1991), they all have well-known problems, and moreover, Nash equilibrium is certainly the most natural solution concept since it demands the least amount of cognitive power from agents.

In retrospect it seems that about the only way to realize a choice rule with good properties is to identify logical connections in the set of preferences − Black's singlepeaked domain with the median voter rule being a case in point (Black 1948; Blin and Satterthwaite 1946; Moulin 1980).[6] This is often unsatisfactory, and utterly so as a general solution, since there is nothing to guarantee that such a logic will suggest itself or even be there. Although the common explanation that preferences of voters are single-peaked over the left-right -axis is intuitively compelling, is there any strong reason why voters would conceptualize things like this, or is it rather so that in most voting situations a natural assumption is that the state space is unrestricted. Of course, if we admit this, then we have to admit that social choice theory is really facing an impossibility that seems inescapable.

In this paper we argue that this would be a hasty conclusion. There is a little shortcoming in the original definition of Nash implementation that has substantial implications for the impossibility result; when all alternatives of A are acceptable, rather than demanding all alternatives to be equilibrium outcomes, as the standard formulation does, we could equally well require that none of them are. If the mechanism designer does not care what the outcome is, then he does not need to predict it. The reason why this innocent change expands the set of implementable choice rules is simple. Each equilibrium in a decision making mechanism under one preference profile implies constraints on what can be selected at other preference profiles through monotonicity which is a necessary condition for implementation (Maskin (1999)). When a choice rule regards all alternatives to be equally good, a large bundle of constraints is generated, some of which are necessarily strong. For if all alternatives are equally good for the collective, then those alternatives that are valued highly by some of the agents, must usually be valued little by others. However, selecting a low ranked alternative at some state generates a lot of constraints for other states through monotonicity. In this paper we reformulate the definition of implementation in these lines and show that the impossibility result does not hold anymore.

The rest of this paper is organized as follows. In Sect. 2 we propose a modification of the standard mechanism design problem that is more in line with the interpretation of equilibrium as a prediction. Although we do not want to rename old concept, but since this serves us well, we call the standard formulation *resolute mechanism design* and the new formulation *irresolute mechanism design*. Then, in Sect. 3, we derive some general results. In particular, we show that a condition called *irresolute monotonicity* is necessary for implementation, and when combined with a condition called *strict no-veto power*, a strengthening of the standard no-veto power condition, it becomes sufficient. Section 4 shows that our modification expands the set of Nash

[6] Another one is a quasi-linear environment with the VCG -mechanism. See (Aswal etal. 2003) for general conditions that guarantee GS -theorem can be avoided.

implementable choice rules even in the domain of all strict preferences; a specific Condorcet extension is now implementable in the case of 3 agents and 3 alternatives, something that was not possible in the standard sense. In Sect. 5 we show that in some specific cases irresolute mechanism design is able to go beyond the most powerful implementation concepts like mixed strategy Nash implementation, subgame perfect implementation, and even Nash implementation using undominated strategies. Section 6 concludes with a short discussion.

2 The devil is in the details: resolute vs. irresolute mechanism design

Hurwicz (1960, 1972) was the first to give an explicit formulation of the idea that the goal of a society can be separated from the mechanism that is used to realize it.[7] Given n message spaces $M_1, \dots, M_n$, one for each agent, a mechanism g is a mapping

$$g : M_1 \times \cdots \times M_n \to A.$$

Let us denote $M = M_1 \times \cdots \times M_n$ and write this mechanism as $G = (M, g)$. In contrast to strategy-proofness, where the only concern is whether individuals have an incentive to lie or not, we need to be more exact on what kind of behavior is expected. A natural assumption is that a Nash equilibrium will be played.

Naturally, whether a given message is a Nash equilibrium or not, will depend on the true state. Once a state $\theta \in \Theta$ has been given, and preferences are therefore fixed, mechanism G becomes a game $\Gamma(\theta) = (G, \theta)$. A message profile $m^* = (m_1^*, \dots, m_n^*)$ is a pure strategy Nash equilibrium of this game if, and only if, $g(m^*) \succeq_i^\theta g(m_i, m_{-i}^*)$ holds for all $i \in N$ and all $m_i \in M_i$.[8] The set of all pure strategy Nash equilibrium profiles of $\Gamma(\theta)$ is denoted by $NE(G, \theta)$. Now we can formulate what Hurwicz meant;

Definition 1 (*Resolute Mechanism Design*) Choice rule $f : \Theta \to A$ is Nash implementable by a resolute mechanism if there exists a mechanism $G = (M, g)$ such that $g(NE(G, \theta)) = f(\theta)$ holds for all $\theta \in \Theta$.

In words, exactly those alternatives that choice rule f regards as acceptable are Nash equilibrium outcomes of G at all admissible states. The path-breaking result of Maskin (1999) says that if a choice rule is Nash implementable, then it is (Maskin) *monotonic*, and if it is monotonic and satisfies also *no-veto power* (NVP), then it is Nash implementable.[9] Let $L_i(x, \theta) \equiv \{y \in A \mid x \succeq_i^\theta y\}$ be the *lower contour set* of x for agent i at state θ. Choice rule f is monotonic, if for all $\theta, \psi \in \Theta$, and all $x \in f(\theta)$, if $L_i(x, \theta) \subseteq L_i(x, \psi)$ holds for all $i \in N$, then $x \in f(\psi)$. It satisfies no-veto power, if

[7] Historical details are given in Jackson (2001); Maskin (1999), and Moore (1996).

[8] Here $m_{-i}^* = (m_1^*, \dots, m_{i-1}^*, m_{i+1}^*, \dots, m_n^*)$ and $(m_i, m_{-i}^*) = (m_1^*, \dots, m_{i-1}^*, m_i, m_{i+1}^*, \dots, m_n^*)$ as usual.

[9] A full characterization (a necessary and sufficient condition) was later given by Moore and Repullo (1990) and Sjöström (1991).

for all $\theta \in \Theta$, and all $x \in A$, if x is the best alternative of at least $n - 1$ agents at state θ, then $x \in f(\theta)$.

Although this approach helps, it does not get us far in the case of unrestricted domain, which, as we have argued, is often the most natural assumption. According to Maskin (1999) a choice rule that satisfies Definition 1 must be monotonic, and if it is single-valued as well, then the result of Muller and Satterthwaite (1977) says that it must be strategy- proof. Therefore, by the GS -theorem, the choice rule must be either dictatorial, constant, or select between two alternatives only. However, although correspondences do not help with strategy-proofness, they do now. Maskin (1999) shows that the Pareto correspondence, which selects all Pareto optimal alternatives at each state, and also the individually rational correspondence, which for a fixed alternative, selects all those alternatives that are considered at least as good by all, are both Nash implementable by a resolute mechanism. Unfortunately, we are still left with two well-known problems: (1) The set of alternatives that these correspondences regard as acceptable are too large (even a dictatorial rule is Pareto optimal) and (2) once the mechanism has multiple equilibria at each state this will almost certainly lead to a coordination failure (what is the equilibrium that one anticipates others to play).

We propose a novel approach to overcome some of these difficulties. To introduce the idea, let $N = \{1, 2, 3\}$, $A = \{x, y, z\}$, and suppose that preferences at state θ are given below:

Agent 1: $x \succ_1^\theta y \succ_1^\theta z$

Agent 2: $z \succ_2^\theta x \succ_2^\theta y$

Agent 3: $y \succ_3^\theta z \succ_3^\theta x$

These preferences exhibit what is known as a *majority- or Condorcet cycle* (Fishburn 1977; Young 1988) – but this is not the point. The point is that while it is natural to insist that choice rule f must selects all alternatives at state θ, that is $f(\theta) = A$, it is not equally natural to insist that $g(NE(G, \theta)) = A$ as in Definition 1. We could just as well allow $NE(G, \theta) = \emptyset$ or equivalently $g(NE(G, \theta)) = \emptyset$.[10] After all, if the mechanism designer does not care what is the final outcome, what difference does it make if the mechanism does not have an equilibrium? It is not like agent refuse to participate simply because an equilibrium does not exist. This is even more evident if the mechanism treats all alternatives equally.

OBSERVATION: As far as the idea of an equilibrium is to predict the outcome of a mechanism, there is no need for a decision making mechanism to have an equilibrium when all alternatives are considered equally good. ⋄

[10] This means that infinite message spaces must be allowed, otherwise there would exist at least one mixed strategy equilibrium (Nash 1950, 1951). For all practical purposes, however, this may only require that individuals see the message space as potentially infinite, which could be generated by a waiting time for example (see Artemov (2015)).

Whether this is the purview of game theory community in general, and opinions to the contrary have certainly been presented, it is clear that without this interpretation the enterprise of mechanism design would be pretty much void.[11] Despite of what the commonly accepted view is, or whether there even exist one, this observation does suggest that a certain amount of slack is possible in Definition 1.

For a given CR $f : \Theta \to A$, let us define the set of *resolute states* $\Theta^{f,R} \subseteq \Theta$ as

$$\Theta^{f,R} \equiv \{\theta \in \Theta \mid f(\theta) \neq A\}.$$

In words, $\Theta^{f,R}$ is the set of those states where all alternatives are not considered equally good. Accordingly, let $\Theta^{f,I} \equiv \Theta \setminus \Theta^{f,R}$ be the set of *irresolute states*.

Definition 2 (*Irresolute Mechanism Design*). Choice rule $f : \Theta \to A$ is Nash implementable by an irresolute mechanism if there exists a subset of states $\Theta^{f,R} \subseteq \Theta^D \subseteq \Theta$, and a mechanism $G = (M, g)$, such that (1) $f : \Theta^D \to A$ is implementable in the standard sense (Definition 1), and (2) $NE(G, \theta) = \emptyset$ for all $\theta \in \Theta \backslash \Theta^D \subseteq \Theta^I$.

In words, if all possible outcomes in the range of the mechanism are not desirable, then we have to make sure that we restrict Nash equilibrium outcomes to those alternatives that we like, and if the range of the mechanism is equal to the set of desirable outcomes, then it is possible, but not necessary, to allow that there are no Nash equilibria at all. This definition is not directly related to what is known as partial implementation. A CR $f : \Theta \to A$ is called *partially implementable* if there exists an implementable CR $g : \Theta \to A$ such that $g(\theta) \subseteq f(\theta)$ holds for all $\theta \in \Theta$. In other words, f is partially implementable if there is a mechanism where at least one socially optimal alternative is a Nash equilibrium outcome, while it is not allowed that there are no equlibria at all.[12] Upon reflection we may argue that it does not matter whether the mechanism has other solution besides Nash equilibrium in case (2) of the above definition. If it does we still accomplish implementation in the partial sense.

Definition 2 goes directly against an old tradition in social choice theory that consider consistency as an important property of a mechanism (Abdou and Keiding 1991; Dufwenberg and Stegman 2002; Dutta 1984; Dutta and Pattanaik 1978; Peleg 1984, 1978). Consistency means that at least one equilibrium must exists at all states. On the other hand, since we violate this property in the weakest possible way, we should rather worry whether it make any difference at all. Two things indicate that it might. First of all, we know from the work of Saari (2001, 1995) that a small set of preference profiles are behind most of the problems, and second, recent developments in mechanism design show that a seemingly innocent assumption can have a huge effect.[13]

[11] See (Fudenberg and Tirole 2000; Luce and Raiffa 1957; Myerson 1991, Osborne and Rubinspsstein (1993) or Schelling (1960) for the standard interpretation and Aumann (1987); Binmore (1990) or Rubinstein (1991) for a critical view.

[12] See Thomson (1996) for a discussion of different implementation concepts.

[13] See the work of Dutta and Sen (2012) on mechanism design with partially honest agents.

3 General results

The following condition holds a similar position in irresolute mechanism design as monotonicity does in resolute mechanism design.

Definition 3 We say that CR $f : \Theta \to A$ is *irresolute monotonic* if there exists a subset of states $\Theta^{f,R} \subseteq \Theta^D \subseteq \Theta$ such that;

(i) $f : \Theta^D \to A$ is monotonic,
(ii) for any $\theta \in \Theta \setminus \Theta^D$, and any $x \in A$, there does not exist a state $\psi \in \Theta^D$ such that $x \in f(\psi)$ and $L_i(x, \psi) \subseteq L_i(x, \theta)$ holds for all $i \in N$.

Theorem 1 *If CR $f : \Theta \to A$ is Nash implementable by an irresolute mechanism, then it must be irresolute monotonic.*

Proof Let $G = (M, g)$ be an irresolute mechanism that Nash implements f. Let Θ^D be the set of all states θ such that $g(NE(G, \theta)) = f(\theta)$ holds. By definition $\Theta^{f,R} \subseteq \Theta^D$. Then, since $f : \Theta^D \to A$ must be Nash implementable in the standard sense, it has to be monotonic. Hence (i) holds. Now take any $\theta \in \Theta \setminus \Theta^D$ and any $x \in A$. For the sake of contradiction, suppose there exists a state $\psi \in \Theta^D$, such that $x \in f(\psi)$ and $L_i(x, \psi) \subseteq L_i(x, \theta)$ holds for all $i \in N$. By definition of Θ^D there is a Nash equilibrium $m^* \in NE(G, \psi)$ such that $g(m^*) = x$. However, by nestedness of the lower contour sets, m^* is then a Nash equilibrium also at state θ. This is a contradiction with the fact that $\theta \in \Theta \backslash \Theta^D$. Hence also (ii) holds. □

Irresolute monotonicity is close to a full characterization. Just like monotonicity it becomes sufficient when combined with a NVP type condition.

Definition 4 We say that CR $f : \Theta \to A$ satisfies *strict no-veto power* (SNVP) if for all $\theta \in \Theta$, and all $x \in A$, if x is the top alternative of at least $n - 1$ agents at state θ, then $f(\theta) = \{x\}$. □

In comparison to NVP, which requires that under these conditions $x \in f(\theta)$ must hold, SNVP requires that only x is acceptable at state θ.

Theorem 2 *Let $n \geq 3$. If CR $f : \Theta \to A$ is irresolute monotonic and satisfies SNVP, then it is Nash implementable by an irresolute mechanism.*

Proof We use the Maskin mechanism (Maskin 1999) restricted to states Θ^D to prove the claim. Let the message space of agent i be $M_i = \Theta^D \times A \times \mathbb{N}_+$, denote a typical message of agent i by $m_i = (\theta^i, x^i, n^i)$, and define the outcome function $g : M \to A$ by the following three rules:

(1) If $m_i = (\theta, x, n^i)$ for all $i \in N$, and $x \in f(\theta)$, then $g(m) = x$.
(2) If $m_j = (\theta, x, n^j)$ for all $j \in N \backslash \{i\}$, $m_i = (\theta^i, x^i, n^i)$, and $x \in f(\theta)$, then

$$g(m) = \begin{cases} x^i, & \text{if } x^i \in L_i(x, \theta), \\ x, & \text{otherwise.} \end{cases}$$

(3) In all other cases, denote $k = \underset{i \in N}{\operatorname{argmax}}\, n^i$, and set $g(m) = x^k$.

Let us verify that $G = (M, g)$ implements $f : \Theta \to A$. First of all, since SNVP implies NVP, we know from Maskin (1999) that G Nash implements $f : \Theta^D \to A$. Therefore, we only need to consider states in $\Theta \setminus \Theta^D$. Suppose that $\psi \in \Theta \backslash \Theta^D$. There cannot exist any Nash equilibria under rule (1) at ψ since this would violate condition (ii) of irresolute monotonicity. Assume, then, that there is a Nash equilibrium under rule (2) or (3). By the definition of G together with SNVP this implies that $f(\psi)$ is a singleton, which is a contradiction, since $\Theta \setminus \Theta^D$ means that $f(\psi) = A$ by definition. This means that the mechanism has no Nash equilibria at ψ. Therefore, f is Nash implementable by an irresolute mechanism. □

As we argued in the previous section, it does not really matter how agents behave in those cases where Nash equilibrium fails to exists, since the mechanism in Theorem 2 will partially implement the CR anyway. In some specific cases it may be possible to prove a stronger result, however, if one can show that when there are no Nash equilibria, there are no rationalizable strategies either.[14] This would mean that there is no reasonable prediction to be made, and therefore, one can argue that no alternative is in a favorable position. It is important to understand that the mechanism in Theorem 2 does not have any mixed strategy equilibria at any state (Maskin 1999).[15] Therefore, at irresolute states $\Theta \setminus \Theta^D$, there really are no equilibria, pure or mixed.

4 Condorcet correspondence with three alternatives

Theorem 2 characterizes almost all CRs that are Nash implementable by an irresolute mechanism. We give an example to shows that this class includes at least some important rules. Suppose there are three agents $N = \{1, 2, 3\}$, three alternatives to choose form $A = \{x, y, z\}$, and all profiles of strict orderings are possible. Alternative $x \in A$ is a *Condorcet winner* at state θ if it beats all other alternatives in a pairwise comparison. This means that at least two individuals prefer x to y and at least two individuals prefer x to z. In the case of 3 alternative, there is either a unique Condorcet winner, or all alternatives are Condorcet winners. Thus, let us define choice rule $f^{Con} : \Theta \to A$ as:

[14] See (Osborne and Rubinstein 1994; Binmore 1990; Pearce 1984), on rationalizable strategic behavior.

[15] This is not exactly true. If all agents have the same top alternative, then there are mixed strategy equilibria where the outcome is always this top alternative. However, by SNVP, the outcome coincides with the CR.

$$f^{Con}(\theta) = \begin{cases} x, & \text{if } x \text{ is a Condorcet winner at } \theta, \\ A, & \text{otherwise.} \end{cases}$$

In the literature f^{Con} is called a *Condorcet extension* (Young 1988; Fishburn 1977). Furthermore, in this simple case of 3 agents and 3 alternatives, most, if not all, reasonable Condorcet extensions coincide with f^{Con}. Since exactly all strict rankings are possible, there are $6^3 = 216$ preference profiles in the domain of f^{Con}, only 12 of which do not have a Condorcet winner. In fact, f^{Con} is the closest thing to a function that one can hope for in this domain without violating either anonymity or neutrality.[16]

Lemma 1 *f^{Con} is not Nash implementable by a resolute mechanism.*

Proof This follows from the result of Maskin (1999) once we have shown that f^{Con} is not monotonic. Suppose that at state θ preferences are:

Agent 1: $x \succ_1^\theta y \succ_1^\theta z$

Agent 2: $z \succ_2^\theta x \succ_2^\theta y$

Agent 3: $y \succ_3^\theta z \succ_3^\theta x$

Thus, $f^{Con}(\theta) = A$ by definition. Suppose, then, that at state ψ preferences are instead:

Agent 1: $x \succ_1^\psi y \succ_1^\psi z$

Agent 2: $z \succ_2^\psi x \succ_2^\psi y$

Agent 3: $y \succ_3^\psi x \succ_3^\psi z$

We get these from the preferences at θ by propping x above z in the ranking of agent 3. Now $f^{Con}(\psi) = \{x\}$ by definition. Therefore, f^{Con} is not monotonic, since monotonicity would imply that $y \in f^{Con}(\psi)$, and as a consequence not Nash implementable by a resolute mechanism either. □

At this point it must be stressed that f^{Con} is not partially implementable either. Now consider the CR $f^{Con} : \Theta^R \rightarrow A$. As there are only 12 profiles where a unique Condorcet winner does not exist, Θ^R is almost as large as Θ.

Lemma 2 *The CR $f^{Con} : \Theta^R \rightarrow A$ is monotonic.*

16 See (Moulin 1998) for an exact definition.

Proof The fact that Condorcet winner is always unique when the domain is Θ^R implies monotonicity. Suppose that $\{x\} = f^{Con}(\theta)$. If alternative x does not drop in the preferences of anyone when going from state θ to state ψ, in the sense that $L_i(x,\theta) \subseteq L_i(x,\psi)$ holds for all $i \in N$, then it must beat the other two alternatives in a pairwise comparison also at state ψ. Hence $\{x\} = f^{Con}(\theta)$ as required by monotonicity. □

Taken together, Lemmas 1 and 2 clearly indicate that those 12 preference profiles where Condorcet winner does not exist are behind most of the problem. But can an irresolute mechanism help us? Next we show that it can.

Lemma 3 *f^{Con} satisfies SNVP.*

Proof If at least two individuals think that alternative x is the best at state θ, then it must clearly be a unique Condorcet winner, and therefore $\{x\} = f^{Con}(\theta)$. Thus, f^{Con} satisfies SNVP. □

Theorem 3 *f^{Con} is Nash implementable by an irresolute mechanism.*

Proof By Lemma 3 f^{Con} satisfies SNVP. Therefore, by Theorem 2, we only need to show that f^{Con} is irresolute monotonic. Let us show that we can set $\Theta^D = \Theta^R$. From Lemma 2 we know that $f^{Con} : \Theta^R \to A$ is monotonic. Hence (i) in the definition of irresolute monotonicity is satisfied. To verify also (ii), suppose to the contrary; for some $\theta \in \Theta \backslash \Theta^R$, and some $x \in A$, there exists a state $\psi \in \Theta^R$ such that $x \in f(\psi)$ and $L_i(x,\psi) \subseteq L_i(x,\theta)$ holds for all $i \in N$. Since x is then a Condorcet winner at ψ, by definition, it must be Condorcet winner also at θ by the nestedness of the lower contour sets. This is a contradiction. Thus, also (ii) holds, and hence f^{Con} satisfies irresolute monotonicity. □

5 Relation to existing literature

In an important step forward, Mezzetti and Renou (2012) studied Nash implementation in mixed strategies. In this work the set of Nash implementable CRs is extended by allowing some alternatives at some states to be in a support of a non-degenerate mixed strategy equilibrium rather than an outcome of a pure strategy equilibrium as originally proposed in Maskin (1999). This means that some acceptable alternatives may not be chose for sure in any equilibrium. The necessary condition that this new definition leads to is called *set-monotonicity*.[17]

Definition 5 A choice rule $f : \Theta \to A$ is set-monotonic if for all $\theta, \psi \in \Theta$, we have $f(\theta) \subseteq f(\psi)$ whenever one of the following two conditions holds for all $i \in N$:

[17] Here $SL_i(x,\theta)$ is the *strict lower contour set* of x for agent i at state θ i.e. $SL_i(x,\theta) \equiv \{y \in A \mid x \succ_i^\theta y\}$.

either (1) $f(\theta) \subseteq \max_{\psi_i} A$ or (2) for all $x \in f(\theta)$, both (i) $L_i(x, \theta) \subseteq L_i(x, \psi)$ and (ii) $SL_i(x, \theta) \subseteq SL_i(x, \psi)$ hold.

Set-monotonicity is much weaker than monotonicity. Moreover, just like monotonicity, it becomes suffcient when combined with NVP. One result in Mezzetti and Renou (2012) is that top-cycle correspondence is implementable in mixed strategy equilibrium. This implies, in particular, that the correspondence f^{Con} studied in Sect. 4 is implementable in mixed strategy equilibrium. However, as an equilibrium concept, mixed strategies are much more controversial than pure strategies (Rubinstein 1991). Therefore, it is nice to know that there are other ways to implement f^{Con}, although it is not in general possible to implement the top-cycle correspondence using an irresolute mechanism. However, as the following example shows, implementation in mixed strategy equilibrium is not strictly stronger than Nash implementation by an irresolute mechanism.

Example. Let $\Theta = \{\theta, \psi_1, \psi_2, \psi_3, \psi_4\}$, $A = \{x, y, z, v\}$, and $N = \{1, 2, 3, 4\}$. Preferences at states θ and ψ_4 are given below:

Agent 1: $x \succ_1^\theta y \succ_1^\theta z \succ_1^\theta v$,

Agent 2: $v \succ_2^\theta x \succ_2^\theta y \succ_2^\theta z$,

Agent 3: $z \succ_3^\theta v \succ_3^\theta x \succ_3^\theta y$,

Agent 4: $y \succ_3^\theta z \succ_3^\theta v \succ_3^\theta x$.

and

Agent 1: $x \succ_1^{\psi_4} y \succ_1^{\psi_4} z \succ_1^{\psi_4} v$,

Agent 2: $v \succ_2^{\psi_4} x \succ_2^{\psi_4} y \succ_2^{\psi_4} z$,

Agent 3: $z \succ_3^{\psi_4} v \succ_3^{\psi_4} x \succ_3^{\psi_4} y$,

Agent 4: $y \sim_3^{\psi_4} z \sim_3^{\psi_4} v \sim_3^{\psi_4} x$.

Preferences at states $\{\psi_1, \psi_2, \psi_3\}$ are defined similarly; at ψ_i agents $N \setminus \{i\}$ have the same preferences as at θ, while agent i is indifferent between all alternative. Now suppose that the mechanism designer wants to implement a CR such that $f(\theta) = \{x, y, z, v\} = A$, $f(\psi_1) = \{v\}$, $f(\psi_2) = \{z\}$, $f(\psi_3) = \{y\}$, and $f(\psi_4) = \{x\}$. This is arguably the most sensible goal in this particular case; alternatives hold symmetric positions at state θ, while at state ψ_4, for instance, it is alternative x that has improved the most (agents 1,2 and 3 have the same preferences at both states). This CR does not satisfy set-monotonicity, and therefore, it is not implementable in mixed strategy equilibrium. This is because set-monotonicity requires that $f(\theta) \subseteq f(\psi_4)$ must hold since, looking back at Definition 5, condition (1) holds for agent 4, while condition (2) holds for all the rest.

This CR is Nash implementable by an irresolute mechanism. This follows from Theorem 2. f satisfies SNVP and the CR $f : \Theta^R \to A$, where $\Theta^R = \{\psi_1, \psi_2, \psi_3, \psi_4\}$, is irresolute monotonic. Hence we can select $\Theta^D = \Theta^R$. This example is a formal description of the following fable. Suppose that the mechanism designer knows the preferences of all agents in some community as exemplified by state θ. However, by the time that a decision has to be made, member i may have already decided to leave the community and therefore turned indifferent on what happens as exemplified by state ψ_i. In this situation the interpretation of the CR is clear − the preferences of the member who has decided to leave should not interfere. This is one of those cases where Nash implementation by an irresolute mechanism is useful, while mixed strategy Nash implementation is not.

But this CR is not implementable in most other solution concept either. For example, it does not satisfy Condition α, which is necessary for subgame perfect implementation (Abreu and Sen 1989; Vartiainen 2007).

Definition 6 Choice rule $f : \Theta \to A$ satisfies Condition α with respect to the set $B \subseteq A$, if $f(\Theta) \subseteq B$, and if for all states $\theta, \psi \in \Theta$ and all outcomes $x \in f(\theta) - f(\psi)$, there exists a sequence of agents $j(0), j(1), \ldots, j(l)$ and a sequence of outcomes $x = x_0, x_1, \ldots, x_l, x_{l+1}$ in B, such that

(i) $x_k \succeq_j^{\theta(k)} x_{k+1}; k = 0, 1, \ldots, l$

(ii) $x_{l+1} \succ_j^{\psi(l)} x_l$

(iii) x_k is not ψ maximal for $j(k)$ in B; $k = 0, 1, \ldots, l$

(iv) if x_{l+1} is ψ maximal in B for all agents except $j(l)$, then either $l = 0$ or $j(l-1) \neq j(l)$.

It is obvious why the CR in our Example cannot satisfy this condition. The preference reversal required by items (i) and (ii), that is $x_l \succeq_{j(l)}^{\theta} x_{l+1}$ and $x_{l+1} \succ_{j(l)}^{\psi_i} x_l$, cannot be there since only the preferences of agent i change and then this agent is indifferent between all alternatives. Furthermore, as subgame perfect implementation is one of the most powerful forms of implementation, there is not much hope for any other solution concept either. Even Nash implementation using undominated strategies does not work here (Palfrey and Srivastava 1991). Any undominated Nash equilibrium at θ is clearly an undominated Nash equilibrium at ψ_i too.

6 Concluding discussion

Instead of defining a CR $f : \Theta \to A$ to be irresolute if it selects all alternatives A at some states, we could equally well treat irresoluteness as part of the design. Suppose that CR f is not implementable. Furthermore, at some state θ, the set $f(\theta)$ of acceptable alternatives includes almost all alternative, while the alternatives $A \setminus f(\theta)$ are not that much worse either. In this case the designer could try to use an irresolute mechanism to implement f with the exception that all alternatives are acceptable at θ. In other words, the designer could implement a nearby rule rather than f itself.

The idea of this paper can be generalized much further though. We could equally well study a situation where dominant strategies are chosen whenever they exist, but otherwise Nash equilibrium is going to be played, or subgame perfect equilibrium is going to be selected whenever it exist, but otherwise Nash equilibrium is chosen. This can be a fruitful avenue for further research.

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References

Abdou J, Keiding H (1991) Effectivity functions in social choice. Kluwer Academic Publishers, Deventer, Netherlands

Abreu D, Sen A (1989) Subgame perfect implementation: a necessary and almost sufficient condition. J Econ Theory 50:285–299

Abreu D, Sen A (1990) Virtual implementation in nash equilibrium. Econometrica 59:997–1021

Arrow KJ (1963) Social choice and individual values. Wiley, New York

Artemov G (2015) Time and Nash implementation. Games Econ Behav 91:229–236

Aswal N, Chatterji S, Sen A (2003) Dictatorial domains. Econ Theory 22:45–62

Aumann R (1987) What is game theory trying to accomplish? In: Arrow KJ, Honkapohja S (eds) Frontiers of economics, Oxford Blackwell 287–324

Baliga S, Sjöström T (2007) Mechanism design: recent developments. In: Blume L, Durlauf S (eds) The New Palgrave dictionary of economics, 2nd edn

Barberá S (1977) Manipulation of social decision functions. J Econ Theory 15:266–278

Barberá S, Dutta B, Sen A (2001) Strategy-proof social choice correspondences. J Econ Theory 101:374–394

Bernheim DB (1984) Rationalizable strategic behavior. Econometrica 52:1007–1028

Binmore K (1990) Essays on the foundations of game theory. Wiley, Cambridge

Black D (1948) On the rationale of group decision-making. J Polit Econ 56:23–34

Blin JM, Satterthwaite MA (1976) Strategy-proofness and single-peakedness. Public Choice 26:51–58

Ching S, Zhou L (2002) Multi-valued strategy-proof social choice rules. Soc Choice Welf 19:569–580

Chorchón LC (2007) The theory of implementation: what did we learn? Working Paper, Universitat Carlos III, Spain

Chorchoón LC (1996) The theory of implementation of socially optimal decisions in economics. Palgrave McMillan, New York

Danilov VI, Sotskov AI (2002) Social choice mechanisms. Studies in economic design, Springer, Germany

Dufwenberg M, Stegman M (2002) Existence and uniqueness of maximal reductions under iterative strict dominance. Econometrica 70:2007–2023

Duggan J, Schwarz T (2000) Strategic manipulability without resoluteness or shared beliefs: Gibbard-Satterthwaite generalized. Soc Choice Welf 17:85–93

Dutta B (1984) Effectivity functions and acceptable game forms. Econometrica 52:1151–1166

Dutta B, Pattanaik PK (1978) On nicely consistent voting systems. Econometrica 46:163–170

Dutta B, Sen A (2012) Nash implementation with partially honest individuals. Games Econ Behav 74:154–169

Fishburn PC (1973) The theory of social choice. Princeton University Press, Princeton, New Jersey

Fishburn PC (1977) Condorcet social choice functions. SIAM J Appl Math 33:469–489

Fudenberg D, Tirole J (2000) Game theory. The MIT Press, USA

Gärdenfors P (1976) Manipulation of social choice functions. J Econ Theory 13:217–228

Gibbard A (1973) Manipulation of voting schemes: a general result. Econometrica 41:587–601

Gibbard A (1977) Manipulation of voting schemes that mix voting with chance. Econometrica 45:665–681

Hurwicz L (1972) On informationally decentralized systems. In: Radner R, McGuire CB (eds) Decision and organization, North-Holland, Amsterdam, pp 297–336

Hurwicz L (1960) Optimality and informational efficiency in resource allocation processes. In: Arrow KJ (ed) Mathematical methods in the social sciences. Stanford University Press, S. Karlin and P. Suppes, pp 27–46

Jackson MO (2001) A crash course in implementation theory. Soc Choice Welf 18:655–708

Kelly JS (1977) Strategy-proofness and social choice functions without single- valuedness. Econometrica 45:439–446

Luce D, Raiffa H (1957) Games and decisions. Wiley, New York

Maskin E (1999) Nash equilibrium and welfare optimality. Rev Econ Stud 66:23–38

Maskin E, Sjöström T (2002) Implementation theory. In: Arrow KJ, Sen AK (eds) Handbook of social choice and welfare, Elsevier

Mezzetti C, Renou L (2012) Implementation in mixed Nash equilibrium. J Econ Theory 147:2357–2375

Moore J (1992) Implementation, Contracts, and Renegotiation in Environments with Complete Information. In *Advances in Economic Theory: Sixth World Congress Vol. I*, ed. J.-J. Laffont, Cambridge, UK, Cambridge University Press

Moore J, Repullo R (1988) Subgame perfect implementation. Econometrica 56:1191–1220

Moore J, Repullo R (1990) Nash implementation: a full characterization. Econometrica 58:1083–1099

Moulin H (1980) On strategy-proofness and single peakedness. Public Choice 35:437–455

Moulin H (1988) Axioms of cooperative decision making. Cambridge University Press, Cambridge

Muller E, Satterthwaite M (1977) The equivalence of strong positive association and strategy-proofness. J Econ Theory 14:412–418

Muller E, Satterthwaite M (1977) The equivalence of strong positive association and strategy-proofnes. J Econ Theory 14:412–418

Myerson R (1991) Game theory: analysis of conflict. Harvard University Press, Harvard

Nash J (1950) Equilibrium points in n-person games. Proc Natl Acad Sci 36:48–49

Nash J (1951) Non-cooperative games. Ann Math 54:286–295

Osborne M, Rubinstein A (1994) A course in game theory. The MIT Press, Cambridge, England

Palfrey T (1992) Implementation in Bayesian equilibrium: the multiple equilibrium problem in mechanism design. In: Laffont J-J (ed) Advances in economic theory: sixth world congress, vol I. Cambridge University Press, Cambridge

Palfrey T, Srivastava S (1991) Nash implementation using undominated strategies. Econometrica 59:479–501

Pearce D (1984) Rationalizable strategic behavior and the problem of perfection. Econometrica 52:1029–1050

Peleg B (1978) Consistent voting systems. Econometrica 46:153–161

Peleg B (1984) Game theoretic analysis of voting in committees. Cambridge University Press, Cambridge

Rubinstein A (1991) Comments on the interpretation of game theory. Econometrica 59:909–924

Saari DG (1995) Basic geometry of voting. Springer, Germany

Saari DG (2001) Decisions and elections. Cambridge University Press, New York, Explaining the Unexpected

Sato S (2008) On strategy-proof social choice correspondences. Soc Choice Welf 31:331–343

Satterthwaite MA (1975) Strategy proofness and arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions. J Econ Theory 10:187–217
Schelling T (1960) The strategy of conflict. Harvard University Press, Cambridge
Sen AK (1970) Collective choice and social welfare. Oliver and Boyd, Edinburgh
Serrano R (2004) The theory of implementation of social choice rules. SIAM Rev 46:377–414
Sjöström T (1991) On the necessary and sufficient conditions for nash implementation. Soc Choice Welf 8:333–340
Thomson W (1996) Concepts of implementation. Jpn Econ Rev 47:133–143
Vartiainen H (2007) Subgame perfect implementation: a full characterization. J Econ Theory 133:111–126
Young PH (1988) Condorcet's theory of voting. Am Polit Sci Rev 82:1231–1244