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A note on NTU convexity

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Abstract. For cooperative games with transferable utility, convexity has turned out to be an important and widely applicable concept. Convexity can be defined in a number of ways, each having its own specific attractions. Basically, these definitions fall into two categories, namely those based on a supermodular interpretation and those based on a marginalistic interpretation. For games with nontransferable utility, however, the literature mainly focuses on two kinds of convexity, ordinal and cardinal convexity, which both extend the supermodular interpretation. In this paper, we analyse three types of convexity for NTU games that generalise the marginalistic interpretation of convexity.

Key words: NTU games, convexity

1. Introduction

The notion of convexity for cooperative games with transferable utility (TU games) was introduced by Shapley (1971) and is one of the most analysed properties in cooperative game theory. Many economic and combinatorial situations give rise to convex (or concave) cooperative games, such as airport games (cf. Littlechild and Owen (1973)), bankruptcy games (cf. Aumann and Maschler (1985)) and sequencing games (cf. Curiel et al. (1989)).

Convexity for TU games can be defined in a number of equivalent ways. One of these is by means of the *supermodularity property*, which has its origins outside the field of game theory. Vilkov (1977) and Sharkey (1981) have extended this property towards cooperative games with nontransferable utility (NTU games) to define *ordinal* and *cardinal convexity*, respectively. The supermodular interpretation of convexity also plays an important role in the context of effectivity functions (cf. Abdou and Keiding (1991)).

Economically more appealing than this supermodular interpretation of convexity are the definitions of convexity that are based on the concept of *marginal contributions*. In cooperative games with stochastic payoffs, this marginalistic interpretation of convexity has already been successfully applied (cf. Timmer et al. (2000) and Suijs (2000)). In this paper, we build on the work originated by Ichiishi (1993) and consider three types of convexity for NTU games, which are based on three corresponding marginalistic convexity properties for TU games.

Although all five convexity properties for NTU games coincide within the subclass of TU games, this is not the case in general. In this paper we analyse the relations between these convexity concepts.

This paper is organised as follows. In Section 2, we introduce some notation and basic definitions. In Section 3, the three marginalistic types of convexity for NTU games are defined. In Section 4, we investigate how the various types of convexity are related.

2. Notation and basic definitions

The set of all real numbers is denoted by $\mathbb{R}$, the set of nonnegative reals by $\mathbb{R}_+$ and the set of nonpositive reals by $\mathbb{R}_-$. For a finite set N, we denote its power set by $2^N = \{S \mid S \subset N\}$ and its number of elements by $|N|$. By $\mathbb{R}^N$ we denote the set of all real-valued functions on N. An element of $\mathbb{R}^N$ is denoted by a vector $x = (x_i)_{i \in N}$. For $S \subset N$, $S \neq \emptyset$, we denote the restriction of x on S by $x_S = (x_i)_{i \in S}$. For $x, y \in \mathbb{R}^N$, $y \geq x$ denotes $y_i \geq x_i$ for all $i \in N$ and $y > x$ denotes $y_i > x_i$ for all $i \in N$.

A *cooperative game with transferable utility*, or *TU game*, is described by a pair (N, v), where $N = \{1, \ldots, n\}$ denotes the set of players and $v : 2^N \to \mathbb{R}$ is the *characteristic function*, assigning to every coalition $S \subset N$ of players a value $v(S)$, representing the total payoff to this group of players when they cooperate. By convention, $v(\emptyset) = 0$.

An *allocation* of $v(S)$ is a vector $x \in \mathbb{R}^S$ such that $\sum_{i \in S} x_i \leq v(S)$, with x_i representing the payoff to player $i \in S$. An allocation x of $v(S)$ is called *Pareto efficient* if $\sum_{i \in S} x_i = v(S)$. The core $C(v)$ is the set of Pareto efficient allocations of $v(N)$ for which it holds that no coalition $S \subset N$ has an incentive to split off:

$$C(v) = \left\{ x \in \mathbb{R}^N \mid \forall_{S \subset N} : \sum_{i \in S} x_i \geq v(S), \sum_{i \in N} x_i = v(N) \right\}.$$

A TU game (N, v) is called *superadditive* if for all coalitions $S, T \subset N$ such that $S \cap T = \emptyset$ we have

$$v(S) + v(T) \leq v(S \cup T).$$

An *ordering* of the players in N is a bijection $\sigma : \{1, \ldots, n\} \to N$, where $\sigma(i)$ denotes which player in N is at position i. The set of all $n!$ orderings of N is denoted by $\Pi(N)$. The *marginal vector* of a TU game (N, v) corresponding to the ordering $\sigma \in \Pi(N)$ is defined by

$$m^{\sigma}_{\sigma(k)}(v) = v(\{\sigma(1), \dots, \sigma(k)\}) - v(\{\sigma(1), \dots, \sigma(k-1)\})$$

for all $k \in \{1, \dots, n\}$.

A *cooperative game with nontransferable utility*, or *NTU game*, is described by a pair (N, V), where $N = \{1, \dots, n\}$ is the set of players and V is the payoff map assigning to each coalition $S \subset N$, $S \neq \emptyset$ a subset $V(S)$ of $\mathbb{R}^S$ such that, for all $i \in N$,

$$V(\{i\}) = (-\infty, 0]$$

and for all $S \subset N$, $S \neq \emptyset$ we have

$$V(S) \text{ is nonempty and closed,}$$

$$V(S) \text{ is comprehensive, ie, } x \in V(S) \text{ and } y \leq x \text{ imply } y \in V(S),$$

$$V(S) \cap \mathbb{R}^S_+ \text{ is bounded.}$$

We make the assumption of 0-normalisation for the sake of convenience; Sharkey (1981) also defined cardinal convexity only for 0-normalised NTU games. In addition, we assume that (N, V) is *monotonic*: for all $S \subset T \subset N$, $S \neq \emptyset$ and for all $x \in V(S)$ there exists a $y \in V(T)$ such that $y_S \geq x$. Note that we do not define $V(\emptyset)$. For all $S \subset N$, $S \neq \emptyset$ we define $V^{\circ}(S) = V(S) \times 0^{N \setminus S}$ and $V^{\circ}(\emptyset) = 0^N$. The class of NTU games with player set N is denoted by NTU^N. For ease of notation, we sometimes use V rather than (N, V) to denote an NTU game.

NTU games generalise TU games. Every TU game (N, v) gives rise to an NTU game (N, V) by defining $V(S) = \{x \in \mathbb{R}^S \mid \sum_{i \in S} x_i \leq v(S)\}$ for all $S \subset N$, $S \neq \emptyset$.

The set of *Pareto efficient allocations* for coalition $S \subset N$, $S \neq \emptyset$, denoted by $Par(S)$, is defined by

$$Par(S) = \{x \in V(S) \mid \nexists_{y \in V(S)} : y \geq x, y \neq x\},$$

its set of *weak Pareto efficient allocations* $W\,Par(S)$ is defined by

$$W\,Par(S) = \{x \in V(S) \mid \nexists_{y \in V(S)} : y > x\}$$

and its set of *individually rational allocations* is defined by

$$IR(S) = \{x \in V(S) \mid \forall_{i \in S} : x_i \geq 0\}.$$

The *imputation set* of an NTU game (N, V), denoted by $I(V)$, is defined by

$$I(V) = IR(N) \cap W\,Par(N).$$

The *core* of an NTU game (N, V) consists of those elements of $V(N)$ for which it holds that no coalition $S \subset N$, $S \neq \emptyset$ has an incentive to split off:

$$C(V) = \{x \in V(N) \mid \forall_{S \subset N, S \neq \emptyset} \nexists_{y \in V(S)} : y > x_S\}.$$

An NTU game (N, V) is called *superadditive* if for all coalitions $S, T \subset N$ such that $S \neq \emptyset$, $T \neq \emptyset$, $S \cap T = \emptyset$ we have

$$V(S) \times V(T) \subset V(S \cup T).$$

This definition of superadditivity is a straightforward generalisation of the concept of superadditivity for TU games. In addition, we define a weaker property concerning only the merger between individual players and coalitions rather than between two arbitrary coalitions. An NTU game (N, V) is called *individually superadditive* if for all $i \in N$ and for all $S \subset N \backslash \{i\}$, $S \neq \emptyset$ we have

$$V(S) \times V(\{i\}) \subset V(S \cup \{i\}).$$

Note that individual superadditivity is stronger than monotonicity. We define the *marginal vector* M^σ corresponding to the ordering $\sigma \in \Pi(N)$ by

$$M^\sigma_{\sigma(k)}(V) = \max\{x_{\sigma(k)} \mid x \in V(\{\sigma(1), \ldots, \sigma(k)\}), \\ \forall_{i \in \{1, \ldots, k-1\}} : x_{\sigma(i)} = M^\sigma_{\sigma(i)}(V)\}$$

for all $k \in \{1, \ldots, n\}$. We use the assumption of monotonicity to ensure that the sets over which the maximums are taken are nonempty. By construction, $M^\sigma(V) \in W\,Par(N)$. If a game is individually superadditive, then all marginal vectors belong to $IR(N)$.

3. Convexity

A TU game (N, v) is called *convex* if it satisfies the following four equivalent conditions (cf. Shapley (1971) and Ichiishi (1981)):

$$\forall_{S, T \subset N} : v(S) + v(T) \leq v(S \cap T) + v(S \cup T), \tag{3.1}$$

$$\forall_{U \subset N} \forall_{S \subset T \subset N \backslash U} : v(S \cup U) - v(S) \leq v(T \cup U) - v(T), \tag{3.2}$$

$$\forall_{i \in N} \forall_{S \subset T \subset N \backslash \{i\}} : v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T), \tag{3.3}$$

$$\forall_{\sigma \in \Pi(N)} : m^\sigma(v) \in C(v). \tag{3.4}$$

Condition (3.1), which is called the *supermodularity property*, was originally stated in Shapley (1971) as the definition of convexity for TU games. Subsequently, Vilkov (1977) and Sharkey (1981) generalised this property to ordinal and cardinal convexity for NTU games, respectively. An NTU game (N, V) is called *ordinally convex* if for all coalitions $S, T \subset N$ such that $S \neq \emptyset$, $T \neq \emptyset$ and for all $x \in \mathbb{R}^N$ such that $x_S \in V(S)$ and $x_T \in V(T)$ we have

$$x_{S \cap T} \in V(S \cap T) \quad \text{or} \quad x_{S \cup T} \in V(S \cup T). \tag{3.5}$$

A game is called *cardinally convex* if for all coalitions $S, T \subset N$ such that $S \neq \emptyset$, $T \neq \emptyset$ we have

$$V^\circ(S) + V^\circ(T) \subset V^\circ(S \cap T) + V^\circ(S \cup T).$$

In contrast to these *supermodular* definitions of convexity by Vilkov (1977) and Sharkey (1981), Ichiishi (1993) considers the *marginalistic* interpretation of convexity. We analyse three types of convexity for NTU games, based on the marginalistic properties (3.2)–(3.4). First of all, we have *coalition merge convexity*[1], which generalises property (3.2). For $U = \emptyset$ and $S = T$, (3.2) is trivial and these cases can therefore be ignored when defining an analogous property for NTU games. If $S = \emptyset$, (3.2) is equivalent to superadditivity. Because we do not define $V(\emptyset)$ for NTU games, we require superadditivity as a separate condition. For $S \neq \emptyset$, (3.2) states that for any coalition U, the marginal contribution to the larger coalition T is larger than the marginal contribution to the smaller coalition S. In terms of allocations, this can be interpreted as follows: given the situation in which coalitions S and T have agreed upon a weak Pareto efficient and individually rational allocation of $v(S)$ and $v(T)$ (say, p and q, resp.), if coalition U joins the smaller coalition S, then for any allocation r of $v(S \cup U)$ such that the players in S get at least their previous amount ($r_S \geq p$), it is possible for U to join the larger coalition T using allocation s of $v(T \cup U)$, which gives the players in T at least their previous amount ($s_T \geq q$) and makes all players in U better off than in case they join S ($s_U \geq r_U$). Using this interpretation of (3.2), we can now define an analogous property for NTU games.

An NTU game (N, V) is called *coalition merge convex*, if it is superadditive and it satisfies the coalition merge property, ie, for all $U \subset N$ such that $U \neq \emptyset$ and all $S \subsetneq T \subset N \backslash U$ such that $S \neq \emptyset$ the following statement is true: for all $p \in W\,Par(S) \cap IR(S)$, all $q \in V(T)$ and all $r \in V(S \cup U)$ such that $r_S \geq p$, there exists an $s \in V(T \cup U)$ such that

$$\begin{cases} \forall_{i \in T} : s_i \geq q_i \\ \forall_{i \in U} : s_i \geq r_i. \end{cases}$$

Note that it makes no differences whether we require the coalition merge property for all $q \in V(T)$ or only for $q \in W\,Par(T) \cap IR(T)$. The extension

[1] This notion is introduced for stochastic cooperative games in Suijs and Borm (1999). The name *coalition merge convexity* and the subsequent names *individual merge* and *marginal convexity* are from Timmer et al. (2000).

of (3.3) towards NTU games goes in a similar manner: an NTU game (N, V) is called *individual merge convex* if it is individually superadditive and it satisfies the individual merge property, ie, for all $k \in N$ and all $S \subsetneq T \subset N \backslash \{k\}$ such that $S \neq \emptyset$, the following statement is true: for all $p \in W\,Par(S) \cap IR(S)$, all $q \in V(T)$ and all $r \in V(S \cup \{k\})$ such that $r_S \geq p$ there exists an $s \in V(T \cup \{k\})$ such that

$$\begin{cases} \forall_{i \in T} : s_i \geq q_i \\ s_k \geq r_k. \end{cases}$$

And finally, an NTU game (N, V) is called *marginal convex* if for all $\sigma \in \Pi(N)$ we have

$$M^{\sigma}(V) \in C(V).$$

One important aspect of the five convexity properties defined in this section is that within the class of NTU games that correspond to TU games, they are all equivalent and coincide with TU convexity.

Another property of these concepts is the following: if an NTU game (N, V) satisfies some form of convexity, then all its subgames do, where the subgame of (N, V) with respect to coalition $S \subset N$, $S \neq \emptyset$ is defined as the NTU game (S, V^S) with $V^S(T) = V(T)$ for all $T \subset S$, $T \neq \emptyset$.

4. Relations between the five types of convexity

In this section we investigate the relations between the five types of convexity for NTU games that were presented in the previous section. For 2-player NTU games, all five types are equivalent to (individual) superadditivity. For general n-player NTU games, equivalence between the five types of convexity does not hold. The remainder of this section shows which relations do exist between these properties.

It follows immediately from the definitions that coalition merge convexity implies individual merge convexity. In the following theorem, we show that individual merge convexity implies marginal convexity.

Theorem 4.1. *Let $(N, V) \in NTU^N$. If (N, V) is individual merge convex, then it is marginal convex.*

Proof: Assume (N, V) is individual merge convex and let $\sigma \in \Pi(N)$. To simplify notation, assume without loss of generality that $\sigma(i) = i$ for all $i \in N$. We prove that $M^{\sigma}(V) \in C(V)$ by induction on the player set. For this, we define for $k \in \{1, \ldots, n\}$ the subgame (N^k, V^k) where $N^k = \{1, \ldots, k\}$ and $V^k(S) = V(S)$ for all $S \subset N^k$, $S \neq \emptyset$. $M^{\sigma,k}(V^k)$ denotes the marginal vector in (N^k, V^k) that corresponds to the ordering σ restricted to the first k positions. For $k = 1$, $M^{\sigma,k}(V^k) \in C(V^k)$ by construction. Next, let $k \in \{2, \ldots, n\}$ and assume $M^{\sigma,k-1}(V^{k-1}) \in C(V^{k-1})$. We show that $M^{\sigma,k}(V^k) \in C(V^k)$, ie, no coalition has an incentive to leave the "grand" coalition N^k. Define $T =$

$\{1, \ldots, k-1\}$ and let $S \subsetneq T$, $S \neq \emptyset$. Then it is sufficient to show that coalitions S, T, $\{k\}$, $T \cup \{k\}$ and $S \cup \{k\}$ have no incentive to split off:

- Because $M^{\sigma,k-1}(V^{k-1}) \in C(V^{k-1})$, by definition there does not exist an $y \in V(S)$ such that $y > M_S^{\sigma,k-1}(V^{k-1})$. By construction, $M_S^{\sigma,k}(V^k) = M_S^{\sigma,k-1}(V^{k-1})$, so there does not exist an $y \in V(S)$ such that $y > M_S^{\sigma,k}(V^k)$. Hence, coalition S has no incentive to leave N^k when the payoff is $M^{\sigma,k}(V^k)$. The same argument holds for coalition T.
- Player k will not deviate on his own, because individual merge convexity implies individual superadditivity and hence, $M^{\sigma,k}(V^k) \in IR(V^k)$.
- Because $M^{\sigma,k}(V^k) \in W\,Par(N^k)$, there exists no $y \in V^k(N^k)$ such that $y > M^{\sigma,k}(V^k)$ and hence, the "grand" coalition $T \cup \{k\}$ has no incentive to deviate.
- Finally, we show that coalition $S \cup \{k\}$ has no incentive to split off. Define $R = \{r \in V(S \cup \{k\}) \mid r_S \geq M_S^{\sigma,k}(V^k)\}$ to be the set of allocations in $V(S \cup \{k\})$ according to which the players in S get at least the amount they get according to the marginal vector $M^{\sigma,k}(V^k)$. If $R = \emptyset$, then $S \cup \{k\}$ will be satisfied with the allocation $M^{\sigma,k}(V^k)$. Because $M^{\sigma,k}(V^k) \in IR(N^k)$, it follows from the basic assumptions of an NTU game that R is closed and bounded, so if $R \neq \emptyset$, we can compute $\max\{r_k \mid r = (r_S, r_k) \in R\}$. Let $r \in R$ be a point in which this maximum is reached. Because $M^{\sigma,k-1}(V^{k-1}) \in C(V^{k-1})$, we must have $M_S^{\sigma,k}(V^k) \notin V(S)$ or $M_S^{\sigma,k}(V^k) \in W\,Par(S)$. Let p be the intersection point of the line segment between 0 and $M_S^{\sigma,k}(V^k)$ and the set $W\,Par(S) \cap IR(S)$. By construction, $r \in V(S \cup \{k\})$ is such that $r_S \geq p$.

  Next, take $q = M^{\sigma,k-1}(V^{k-1}) \in V(T)$. As a result of individual individual merge convexity and comprehensiveness, there exists an $s \in V(T \cup \{k\})$ such that $s_T = q$ and $s_k \geq r_k$. Because $s_T = M^{\sigma,k-1}(V^{k-1})$, it follows from the construction of $M^{\sigma,k}(V^k)$ that $M_k^{\sigma,k}(V^k) \geq s_k$. But then, $M_k^{\sigma,k}(V^k) \geq r_k$. We constructed r_k as the maximum amount player k can obtain by co-operating with coalition S, while giving each player $i \in S$ at least $M_i^{\sigma,k}(V^k)$. We conclude that there does not exist a $y \in V(S \cup \{k\})$ such that $y_i > M_i^{\sigma,k}(V^k)$ for all $i \in S \cup \{k\}$.

From these four cases we conclude $M^{\sigma,k}(V^k) \in C(V^k)$ and by induction on k, we obtain $M^{\sigma}(V) \in C(V)$. □

In the following example, we prove that ordinal convexity does not imply any of the other four types of convexity. Note that this example disproves Theorem 2.2.3 in Ichiishi (1993), which states that in an ordinally convex NTU game, all marginal vectors are in the core.

Example 4.1. Consider the following NTU game with player set $N = \{1, 2, 3\}$:

$$V(\{i\}) = (-\infty, 0] \quad \text{for all } i \in N,$$

$$V(\{1,2\}) = \{x \in \mathbb{R}^{\{1,2\}} \mid x_1 \leq 0, x_2 \leq 2\},$$

$$V(\{1,3\}) = \{x \in \mathbb{R}^{\{1,3\}} \mid x_1 + x_3 \leq 1\},$$

$$V(\{2,3\}) = \{x \in \mathbb{R}^{\{2,3\}} \mid x_2 \leq 0, x_3 \leq 0\},$$

$$V(N) = \left\{ x \in \mathbb{R}^N \mid \sum_{i \in N} x_i \leq 2 \right\}.$$

This game (N, V) is ordinally convex: let $S, T \subset N$ such that $S \neq \emptyset$, $T \neq \emptyset$ and let $x \in \mathbb{R}^N$ such that $x_S \in V(S)$ and $x_T \in V(T)$. We distinguish between four cases: if $S \subset T$ or $T \subset S$, (3.5) is trivially satisfied. If $S \cap T = \emptyset$, (3.5) is equivalent to superadditivity, which is satisfied by this game. If $S = \{1,2\}$ and $T = \{1,3\}$, then $x_1 \leq 0$ and hence, $x_{S \cap T} \in V(S \cap T)$. Otherwise, $\sum_{i \in N} x_i \leq 2$ and hence, $x_{S \cup T} \in V(S \cup T)$. From these four cases we conclude that (3.5) is satisfied and (N, V) is ordinally convex. However, this game is not marginal convex, because the marginal vector corresponding to $\sigma = (1,2,3)$, $M^\sigma(V) = (0,2,0)$, does not belong to the core, because player 1 and 3 have an incentive to leave the grand coalition. Using Theorem 4.1, we conclude that (N, V) is neither coalition merge nor individual merge convex. Furthermore, this game is not cardinally convex: $(0,2,0) \in V^\circ(\{1,2\})$ and $(0,0,1) \in V^\circ(\{1,3\})$, but $(0,2,0) + (0,0,1) = (0,2,1) \notin V^\circ(\{1\}) + V^\circ(N)$. $\lhd$

The five types of convexity for NTU games are related as is depicted in Diagram 1. An arrow from one type of convexity to another indicates that the first one implies the second one. Where an arrow is absent, such an implication does not hold in general (cf. Sharkey (1981) and Ichiishi (1993)).

$$\begin{tikzcd}
& \text{cm-convex} \arrow[dd] \arrow[ddr] & \text{ord-convex} \\
\text{card-convex} & & \\
& \text{im-convex} \arrow[r] & \text{m-convex}
\end{tikzcd}$$

Diagram 1

The results in Diagram 1 hold for general n-player NTU games. The relations between the five types of convexity for 3-player NTU games are depicted in Diagram 2 (without proof). To keep the picture clear, the arrows from cardinal convexity to ordinal and marginal convexity have been omitted.[2]

$$\begin{tikzcd}
& \text{cm-convex} \arrow[r] \arrow[dd, leftrightarrow] \arrow[ddr] & \text{ord-convex} \\
\text{card-convex} \arrow[ur] \arrow[dr] & & \\
& \text{im-convex} \arrow[r] \arrow[uur] & \text{m-convex}
\end{tikzcd}$$

Diagram 2

[2] The proofs relating to Diagram 2 can be found in Hendrickx et al. (2000).

References

Abdou J and Keiding H (1991) *Effectivity functions in social choice*. Dordrecht: Kluwer Academic Press

Aumann R, and Maschler M (1985) Game theoretic analysis of a bankruptcy problem from the Talmud. *Journal of Economic Theory*, **36**, 195–213

Curiel I, Pederzoli G, and Tijs S (1989) Sequencing games. *European Journal of Operational Research*, **40**, 344–351

Hendrickx R, Borm P, and Timmer J (2000) On convexity for NTU-games. CentER Discussion Paper 2000-108, Tilburg University, Tilburg, The Netherlands

Ichiishi T (1981) Supermodularity: applications to convex games and the greedy algorithm for LP. *Journal of Economic Theory*, **25**, 283–286

Ichiishi T (1993) *The cooperative nature of the firm*. Cambridge: Cambridge University Press

Littlechild S and Owen G (1973) A simple expression for the Shapley value in a special case. *Management Science*, **20**, 370–372

Shapley L (1971) Cores of convex games. *International Journal of Game Theory*, **1**, 11–26

Sharkey W (1981) Convex games without side payments. *International Journal of Game Theory*, **10**, 101–106

Suijs J (2000) *Cooperative decision-making under risk*. Boston: Kluwer Academic Publishers

Suijs J and Borm P (1999) Stochastic cooperative games: superadditivity, convexity and certainty equivalents. *Games and Economic Behavior*, **27**, 331–345

Timmer J, Borm P, and Tijs S (2000) Convexity in stochastic cooperative situations. CentER Discussion Paper 2000-04, Tilburg University, Tilburg, The Netherlands

Vilkov V (1977) Convex games without side payments. *Vestnik Leningradskiva Universitata*, **7**, 21–24. (In Russian)