# Optimal Time to Change Premiums *† $\ddagger$ 

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#### Abstract

The claim arrival process to an insurance company is modeled by a compound Poisson process whose intensity and/or jump size distribution changes at an unobservable time with a known distribution. It is in the insurance company's interest to detect the change time as soon as possible in order to re-evaluate a new fair value for premiums to keep its profit level the same. This is equivalent to a problem in which the intensity and the jump size change at the same time but the intensity changes to a random variable with a know distribution. This problem becomes an optimal stopping problem for a Markovian sufficient statistic. Here, a special case of this problem is solved, in which the rate of the arrivals moves up to one of two possible values, and the Markovian sufficient statistic is two-dimensional.


## 1 Introduction

In insurance risk theory, the claim arrivals are modeled by a compound Poisson process. The total claim up to time $t$ is given by

$$
\begin{equation*}
X_{t}=X_{0}+\sum_{k=1}^{N_{t}} Y_{k}, \quad t \geq 0, \tag{1.1}
\end{equation*}
$$

where the number of claims up to time $t, N_{t}$, is a Poisson process with intensity $\lambda_{0}$. The claim size process $\left(Y_{k}\right)_{k \in \mathbb{N}}$ is assumed to consist of independent and identically distributed $R^{d}$ valued random variables with distribution function $\nu_{0}$. In order to compensate for the liabilities the insurance company has to pay out, it collects premiums at a such rate that it has a fair chance of survival.

In this paper, we will study the model in (1.1) with two types of regime shift. At time $\theta^{a}$ the intensity of the Poisson process changes from $\lambda_{0}$ to $\lambda_{1}$, and at time $\theta^{b}$, the distribution of the claim size changes from $\nu_{0}$ to $\nu_{1}$. (These measures are assumed to be absolutely continuous with

[^0]respect to each other.) Both $\theta^{a}$ and $\theta^{b}$ are unknown at time 0 , and they are unobservable. It is in the insurance company's interest to detect the change time or the disorder time $\theta \triangleq \theta^{a} \wedge \theta^{b}=$ $\min \left\{\theta^{a}, \theta^{b}\right\}$ as soon as possible and to re-evaluate a new fair value for premiums in order to keep the profit level the same.

We assume that the times of regime shift are independent of each other and that they have an exponential prior distribution

$$
\mathbb{P}\left\{\theta^{i}>t\right\}=\left(1-\pi^{i}\right) e^{-\lambda^{i} t}, \quad i \in\{a, b\}, \quad t \geq 0
$$

for $\lambda^{i}>0$. At time $\theta$, we do not know what the intensity is for sure: it is either $\lambda_{0}$ (a change has occurred in the distribution of the claim size) or $\lambda_{1}$ (a change occurred in the intensity). In fact at time $\theta$, the value of intensity changes from $\lambda_{0}$ to the random variable $\Lambda$ where

$$
\Lambda=\left\{\begin{array}{lll}
\lambda_{1} & \text { with probability } & \frac{\lambda^{a}}{\lambda^{a}+\lambda^{b}}  \tag{1.2}\\
\lambda_{0} & \text { with probability } & \frac{\lambda^{b}}{\lambda^{a}+\lambda^{b}}
\end{array}\right.
$$

At time $\theta$ the distribution of the claim size changes from $\nu_{0}$ to $\nu$, where

$$
\begin{equation*}
\nu=\frac{\lambda^{a}}{\lambda^{a}+\lambda^{b}} \nu_{0}+\frac{\lambda^{b}}{\lambda^{a}+\lambda^{b}} \nu_{1} \tag{1.3}
\end{equation*}
$$

Now consider a related more general problem in which at the disorder time $\theta$ the compound process introduced in (1.1) changes its intensity from $\mu \in \mathbb{R}_{+}$to a random variable $\Lambda$ (at first we will first allow the distribution of this random variable to be as general as possible) and the distribution of the claim sizes change from $\beta_{0}$ to $\beta_{1}$ (these two measures are assumed to be absolutely continuous with respect to each other). The distribution of $\theta$ is given by

$$
\begin{equation*}
\mathbb{P}\{\theta=0\}=\pi, \quad \mathbb{P}\{\theta>t \mid \theta>0\}=e^{-\lambda t}, t \geq 0 \tag{1.4}
\end{equation*}
$$

The random variables $\Lambda$ and $\theta$ are independent.
In this more general problem the aim is to detect the unknown and unobservable time $\theta$ as quickly as possible given the observations from the incoming claims. More precisely, we would like to find a stopping time $\tau$ of the observation process that minimizes the penalty function

$$
\begin{equation*}
R_{\tau}(\pi) \triangleq \mathbb{P}\{\tau<\theta\}+c \mathbb{E}[\tau-\theta]^{+} \tag{1.5}
\end{equation*}
$$

which is the sum of the frequency of $\mathbb{P}(\tau<\theta)$ false alarms and the expected cost $c \mathbb{E}\left[(\tau-\theta)^{+}\right]$ of detection delay.

We are interested in solving this more general problem for three reasons. First, setting $\pi=0$, $\lambda=\lambda^{a}+\lambda^{b}, \mu=\lambda_{0}, \beta_{0}=\nu_{0}$ and $\beta_{1}=\nu$, and the distribution of $\Lambda$ to be the Bernoulli distribution in (1.2) we see that solving this more general problem also leads to a solution of the main problem introduced in the second paragraph. Second, in the general problem if we set $\Lambda$ to be a constant, then we obtain a version of the main problem in which the rate change and change of the distribution of the claim sizes occur simultaneously. This case was analyzed by Dayanik and Sezer (2006) and Gapeev (2005). Finally, the more general problem represents a situation in which the insurance company has only some apriori information about the post disorder rate $\lambda_{1}$, but the company can not pin $\lambda_{1}$ down to a constant because it might only have very few claims after the regime change occurs. In fact, the company wants to detect the
regime change as soon as possible, so there is not really any time to collect data to estimate $\lambda_{1}$. This change detection problem when the underlying process $X$ is a (simple) Poisson process was recently analyzed by Bayraktar et al. (2006). This corresponds to setting $\beta_{0}=\nu_{0}$ and $\beta_{1}=\nu_{0}$ in the current setting.

The compound/simple Poisson disorder problem is one of the rare instances in which a stochastic control problem with partial information can be handled. The (simple) Poisson disorder problem with linear penalty for delay was partially solved by Galchuk and Rozovsky (1971), Davis (1976) and Davis and Wan (1977). This problem later was solved by Peskir and Shiryaev (2002). Bayraktar and Davanik (2006) solved the simple Poisson disorder problem for exponential penalty for delay, and Bayraktar et al. (2005) solved the standard Poisson disorder problem. These results were recently extended by Davanik and Sezer (2006) (using the results developed in Bavraktar et al. (2006)) and Gapeev (2005) for compound Poisson procesesses. On the other hand Bayraktar et al. (2006) solved the simple Poisson disorder problem when the post disorder rate is a random variable and Bayraktar and Sezer (2006) solved this problem for the case with a Phase-type disorder distribution.

We will first show that our problem is equivalent to an optimal stopping problem for a Markovian sufficient statistic. As in Bayraktar et al. (2006) it turns out that the dimension of the sufficient statistic is finite dimensional if the distribution of the random variable $\Lambda$ is discrete with finitely many atoms. We will study the case of a binary distribution in more detail. In particular, we will analyze the case when the post-disorder rate only goes up. We are able to show that the intuition that a decision would sound the alarm only at the times when it observes an arrival does not in general hold, see Remark 5.1. This intuition becomes relevant only when $\lambda$ and $c$ are small enough, i.e. when the disorder intensity and delay penalty are small. By performing a sample path analysis we are able to find the optimal stopping time exactly for most of the range of parameters. For the rest of the parameter range we provide upper and lower bounds on the optimal stopping time. To show the existence of the optimal stopping problem for the cases when we can not determine it exactly we make use of the characterization of the value function of the optimal stopping time as the fixed point of a functional operator, as in Bayraktar et al. (2006). We use this approach since the free boundary problems associated with our problem turns out to be quite difficult to manage as it involves integro-differential equations and the failure of the smooth fit principle is expected. This characterization can be used to calculate the value function through an iterative procedure. From this characterization we are able to infer that the free boundaries are decreasing convex curves located at the corner of $\mathbb{R}_{+}^{2}$. Using our sample path analysis, we are able to determine a certain subset of the free boundary exactly.

The rest of the paper is organized as follows. In Section 2, we give a more precise probabilistic description of the disorder problem and introduce a reference probability measure $\mathbb{P}_{0}$ under which the observations are coming from a compound Poisson process whose jump distribution does not change over time. In Section 3, we show that the disorder problem can be transformed into an optimal stopping problem for a Markovian sufficient statistic. The Markovian sufficient statistic may not be finite dimensional and we show in this section that it is finite dimensional when the distribution of the post disorder rate has finitely many atoms. In Section 4, we find the autonomous sufficient statistic for any Bernoulli distribution. Also we set up an optimal stopping problem for a Bernoulli sufficient statistic when the post disorder rate can only move up. Section 5 contains some of our main results in which by performing a sample path analysis we either find the optimal stopping time exactly or provide upper and lower bounds. We also
show that the optimal stopping time is finite $\mathbb{P}_{0}$-almost surely. Section 6 provides a useful characterization of the value function as a limit of a sequence of other value functions. Since the proofs of the results in this section are similar to the ones in Bayraktar et al. (2006) we omit them, except the result in which we show that the optimal stopping time we constructed is the smallest optimal stopping time and a few other that we prefer to keep for readers convenience.

## 2 A Reference Probability Measure

We will first introduce a reference probability measure $\mathbb{P}_{0}$ under which the observations have a simpler form, namely they come from a compound Poisson process whose rate and jump distribution do not change over time. Next, we will construct the model that we briefly described in the introduction in the paragraph before (1.4).

Let us start with a probability space $\left(\Omega, \mathcal{F}, \mathbb{P}_{0}\right)$ and consider a standard Poisson process $N=\left\{N_{t}: t \geq 0\right\}$ with rate $\mu$; independent and identically distributed strictly positive random variables $Y_{1}, Y_{2}, \ldots$ with a common distribution $\beta_{0}$ on $\mathbb{R}^{d}$ independent of the Poisson process; a random variable $\theta$ independent of the previously described stochastic elements on this probability space whose distribution is given by

$$
\begin{equation*}
\mathbb{P}_{0}\{\theta=0\}=\pi, \quad \mathbb{P}_{0}\{\theta>t \mid \theta>0\}=e^{-\lambda t}, t \geq 0 \tag{2.1}
\end{equation*}
$$

a random variable $\Lambda$ independent of the other stochastic elements whose distribution is $\gamma(\cdot)$. This distribution charges only the positive real numbers. We will assume that

$$
\begin{equation*}
m^{(k)} \triangleq \int_{\mathbb{R}_{+}}(v-\mu)^{k} \gamma(d v)<\infty, \quad k \in \mathbb{N}_{0} \tag{2.2}
\end{equation*}
$$

Let the process $X=\left\{X_{t}: t \geq 0\right\}$ be the compound Poisson process defined as in (1.1) and $\mathbb{F}=\left\{\mathcal{F}_{t}\right\}_{t \geq 0}$ be the natural filtration of $X$. We will also define an initial enlargement of $\mathbb{F}$, $\mathbb{G}=\left\{\mathcal{G}_{t}\right\}_{t \geq 0}$ by setting $\mathcal{G}_{t} \triangleq \mathcal{F}_{t} \vee \sigma\{\theta, \Lambda\}$. $\mathcal{G}_{t}$ is the information available to a genie at time $t$ that also observes the realizations of the disorder time $\theta$ and post-disorder rate $\Lambda$. Let $\beta_{1}(\cdot)$ be a probability measure on $\mathbb{R}^{d}$ which is absolutely continuous with respect to $\beta_{0}(\cdot)$. We will denote by $r$ the Radon-Nikodym derivative

$$
\begin{equation*}
r(y) \triangleq \frac{d \beta_{1}}{d \beta_{0}}(y), \quad y \in \mathbb{R}^{d} \tag{2.3}
\end{equation*}
$$

The process

$$
\begin{equation*}
Z_{t} \triangleq \frac{L_{t}}{L_{\theta}} 1_{\{\tau \leq t\}}+1_{\{\tau>t\}}, \quad t \geq 0 \tag{2.4}
\end{equation*}
$$

is a $\mathbb{G}$-martingale where

$$
\begin{equation*}
L_{t} \triangleq e^{-(\Lambda-\mu) t} \prod_{k=1}^{N_{t}}\left[\frac{\Lambda}{\mu} r\left(Y_{k}\right)\right] \tag{2.5}
\end{equation*}
$$

The positive martingale $Z$ defines a new probability measure $\mathbb{P}$ on every $\left(\Omega, \mathcal{G}_{t}\right), t \geq 0$ by

$$
\begin{equation*}
\left.\frac{d \mathbb{P}}{d \mathbb{P}_{0}}\right|_{\mathcal{G}_{t}}=Z_{t}, \quad t \geq 0 \tag{2.6}
\end{equation*}
$$

Note that since $Z_{0}=1, \mathbb{P}$ and $\mathbb{P}_{0}$ agree on $\mathbb{G}_{0}=\sigma\{\theta, \Lambda\}$, i.e. the random variables $\theta$ and $\Lambda$ are independent and have the same distribution under both $\mathbb{P}$ and $\mathbb{P}_{0}$. On the other hand using
the Girsanov Theorem for jump processes (see e.g Cont and Tankov (2004), Dayanik and Sezer $(2006))$ we conclude that the process $X$ is a $(\mathbb{P}, \mathbb{G})$-compound Poisson process whose arrival rate $\mu$ and jump distribution $\beta_{0}$ changes at time $\theta$ to $\Lambda$ and $\beta_{1}$, respectively. In other words, on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we have exactly the model posited in the Introduction section in the paragraph between (1.3) and (1.4).

## 3 Markovian Sufficient Statistics

In this section, we will show that the stopping problem posed in (1.5) can be formulated as an optimal stopping problem for a Markovian sufficient statistic, which is in general infinite dimensional. In the following sections we will see that depending on the structure of the prior of $\Lambda$ the sufficient statistic can be finite dimensional.

Let us denote all the $\mathbb{F}$-stopping times by $\mathcal{S}$ and introduce the $\mathbb{F}$-adapted processes

$$
\begin{equation*}
\Pi_{t} \triangleq \mathbb{P}\left\{\theta \leq t \mid \mathcal{F}_{t}\right\}, \quad \text { and } \quad \Phi_{t}^{(k)} \triangleq \frac{\mathbb{E}\left[(\Lambda-\mu)^{k} 1_{\{\theta \leq t\}} \mid \mathcal{F}_{t}\right]}{1-\Pi_{t}}, \quad k \in \mathbb{N}, t \geq 0 \tag{3.1}
\end{equation*}
$$

$\Pi_{t}$ is the a posteriori probability process and is the updated probability that the disorder happened at or before time $t$ given all the information up to time $t . \Phi^{(k)}$ can be read as an odds-ratio process, and in fact $\Phi^{(0)}=\frac{\Pi_{t}}{1-\Pi_{t}}$.

Using Proposition 2.1 in Bayraktar et al. (2005) we can write the Bayes error in (1.5) as

$$
\begin{equation*}
R_{\tau}(\pi)=1-\pi+c(1-\pi) \mathbb{E}_{0}\left[\int_{0}^{\tau} e^{-\lambda t}\left(\Phi_{t}^{(0)}-\frac{\lambda}{c}\right) d t\right], \quad \tau \in \mathcal{S} \tag{3.2}
\end{equation*}
$$

where the expectation $\mathbb{E}_{0}$ is taken under the reference probability measure $\mathbb{P}_{0}$. As we can see from (3.2), finding an optimal stopping time for the quickest detection problem would be considerably easier if the process $\Phi^{(0)}$ is Markovian and its natural filtration coincides with the filtration generated by the observations. In that case we would just have to solve a onedimensional optimal stopping problem. This is not true, however, unless $\Lambda$ has only one possible value to take. The following lemma shows that the whole sequence $\left\{\Phi^{(k)}\right\}_{k \in \mathbb{N}}$ is a Markovian sufficient statistic for our detection problem. This result also will help us develop sufficient conditions under which a finite dimensional sufficient statistic exists.

Lemma 3.1 Let $m^{(k)}$ be as in (2.2). Then the dynamics of $\Phi^{(k)}$ can be written as
$d \Phi_{t}^{(k)}=\left(\lambda\left(m^{(k)}+\Phi_{t}^{(k)}\right)-\Phi_{t}^{(k+1)}\right) d t+\Phi_{t-}^{(k)} \int_{y \in \mathbb{R}^{d}}(r(y)-1) p(d t d y)+\Phi_{t-}^{(k+1)} \frac{1}{\mu} \int_{y \in \mathbb{R}^{d}} r(y) p(d t d y)$,
with $\Phi_{0}^{(k)}=\frac{\pi}{1-\pi} m^{(k)}$, in which $p$ is the point process defined by

$$
\begin{equation*}
p((0, t] \times A) \triangleq \sum_{k=1}^{\infty} 1_{\left\{\sigma_{k} \leq t\right\}} 1_{\left\{Y_{k} \in A\right\}}, \quad t \geq 0, A \in \mathcal{B}\left(\mathbb{R}^{d}\right) \tag{3.4}
\end{equation*}
$$

Proof: Using Bayes' formula, and the independence of the stochastic elements $\theta, \Lambda$ and $X$ we can write

$$
\begin{equation*}
\Phi^{(k)}=\frac{\mathbb{E}_{0}\left[(\Lambda-\mu)^{k} Z_{t} 1_{\{\theta \leq t\}} \mid \mathcal{F}_{t}\right]}{\left(1-\Pi_{t}\right) \mathbb{E}_{0}\left[Z_{t} \mid \mathcal{F}_{t}\right]}=U_{t}^{(k)}+V_{t}^{(k)} \tag{3.5}
\end{equation*}
$$

in which

$$
\begin{align*}
U_{t}^{(k)} \triangleq \frac{\pi}{1-\pi} e^{\lambda t} \int_{\mathbb{R}_{+}}(\nu-\mu)^{k} L_{t}^{\nu} \gamma(d \nu), \quad \text { and }  \tag{3.6}\\
V_{t}^{(k)} \triangleq \int_{0}^{t} \int_{\mathbb{R}_{+}} \lambda e^{\lambda(t-u)} \frac{L_{t}^{\nu}}{L_{u}^{\nu}}(\nu-\mu)^{k} \gamma(d \nu) d u . \tag{3.7}
\end{align*}
$$

Here we have used the notation

$$
\begin{equation*}
L_{t}^{\nu} \triangleq e^{-(\nu-\mu) t} \prod_{k=1}^{N_{t}}\left[\frac{\nu}{\mu} r\left(Y_{k}\right)\right], \quad \nu \in \mathbb{R}_{+} \tag{3.8}
\end{equation*}
$$

To derive (3.5) we have used (2.4), (3.1) and the identity

$$
1-\Pi_{t}=\frac{(1-\pi) e^{-\lambda t}}{\mathbb{E}_{0}\left[Z_{t} \mid \mathcal{F}_{t}\right]}
$$

which we can derive using the independence of $\theta$ and $X$ under $\mathbb{P}_{0}$.
The process $L^{\nu}$ is the unique locally bounded solution of the equation (see e.g. Elliott (1982))

$$
\begin{equation*}
d L_{t}^{\nu}=L_{t-}^{\nu}\left[-(\nu-\mu) d t+\int_{y \in \mathbb{R}^{d}}\left(\frac{\nu}{\mu} r(y)-1\right) p(d t d y)\right] \tag{3.9}
\end{equation*}
$$

with $L_{0}=1$. Using (3.9) and the change of variable formula it is easy to obtain

$$
\begin{equation*}
d U_{t}^{(k)}=\left(\lambda U_{t}^{(k)}-U_{t}^{(k+1)}\right) d t+\int_{y \in \mathbb{R}^{d}}\left((r(y)-1) U_{t}^{(k)}+\frac{r(y)}{\mu} U_{t}^{(k+1)}\right) p(d t d y) \tag{3.10}
\end{equation*}
$$

with $U_{0}^{(k)}=\frac{\pi}{1-\pi} m^{(k)}$, and

$$
\begin{equation*}
d V_{t}^{(k)}=\left(\lambda m^{(k)}-V_{t}^{(k+1)}+\lambda V_{t}^{(k)}\right) d t+\int_{y \in \mathbb{R}^{d}}\left((r(y)-1) V_{t}^{(k)}+\frac{r(y)}{\mu} V_{t}^{(k+1)}\right) p(d t d y) \tag{3.11}
\end{equation*}
$$

with $V_{0}^{k}=0$. Now (3.3) follows from (3.5).
Lemma 3.1 shows that 1) $\Phi^{(0)}$ is not a Markov process, and 2) the sequence $\left\{\Phi^{(k)}\right\}_{k \in \mathbb{N}}$ has the Markovian property and its natural filtration is the same as $\mathbb{F}$. The following corollary gives a sufficient condition that the distribution of the post-disorder rate $\gamma$ must satisfy in order for the sufficient statistic to be finite dimensional.

Corollary 3.1 If $\gamma$ is a discrete distribution with only $k$ atoms then $\left\{\Phi^{(0)}, \Phi^{(1)}, \cdots, \Phi^{(k-1)}\right\}$ is a $k$-dimensional Markovian sufficient statistic.

Proof: This follows from the same line of arguments used in the proof of Corollary 3.3 in Bayraktar et al. (2006). Here, we will give it not only for readers conveneience but also because the notation we introduce here will be used later. Let us denote by $\nu_{1}, \cdots, \nu_{k}$ the atoms of the distribution $\gamma$ and define

$$
\begin{equation*}
p(v) \triangleq \prod_{k=1}^{k}\left(v-\nu_{i}+\mu\right) \equiv v^{k}+\sum_{i=0}^{k-1} c_{i} v^{i}, \quad v \in \mathbb{R} \tag{3.12}
\end{equation*}
$$

for some suitable numbers $c_{0}, \ldots, c_{k-1}$. Observe that the random variable

$$
p(\Lambda-\mu)=(\Lambda-\mu)^{k}+\sum_{i=0}^{k-1} c_{i}(\Lambda-\mu)^{i}=0, \quad \text { a.s. }
$$

The last identity together with (3.1) implies that

$$
\begin{equation*}
\Phi_{t}^{(k)}+\sum_{i=0}^{k-1} c_{i} \Phi_{t}^{(i)}=0, \quad \mathbb{P}-a . s . \tag{3.13}
\end{equation*}
$$

Now, it can be seen from the form of the penalty function in (3.2) and the dynamics in (3.3) that $\left\{\Phi^{(0)}, \Phi^{(1)}, \cdots, \Phi^{(k-1)}\right\}$ is a $k$-dimensional Markovian sufficient statistic.

In the remainder of the paper we will assume that the distribution for the post-disorder rate $\Lambda$ has Bernoulli distribution.

## 4 Post-Disorder Rate with Bernoulli Distribution

In this section we will assume that the random variable $\Lambda$ takes either the value $\mu_{1}>0$ or $\mu_{2}>0$, i.e. $\gamma\left(\left\{\mu_{1}, \mu_{2}\right\}\right)=1$. From (3.13) it follows that $\Phi^{(2)}=\left(\mu_{1}+\mu_{2}-2 \mu\right) \Phi^{(1)}-\left(\mu_{1}-\mu\right)\left(\mu_{2}-\mu\right) \Phi^{(0)}$. According to Lemma 3.1, the pair $\left(\Phi^{(0)}, \Phi^{(1)}\right)$ satisfies

$$
\begin{align*}
& d \Phi_{t}^{(0)}=\left(\lambda\left(1+\Phi_{t}^{(0)}\right)-\Phi_{t}^{(1)}\right) d t+\Phi_{t-}^{(0)} \int_{y \in \mathbb{R}^{d}}(r(y)-1) p(d t d y)+\Phi_{t-}^{(1)} \frac{1}{\mu} \int_{y \in \mathbb{R}^{d}} r(y) p(d t d y) \\
& d \Phi_{t}^{(1)}=\left(\lambda m^{(1)}+\left(\lambda-\left(\mu_{1}+\mu_{2}-2 \mu\right)\right) \Phi_{t}^{(1)}+\left(\mu_{1}-\mu\right)\left(\mu_{2}-\mu\right) \Phi_{t}^{(0)}\right) d t \\
& +\Phi_{t-}^{(1)} \int_{y \in \mathbb{R}^{d}}(r(y)-1) p(d t d y)+\left(\left(\mu_{1}+\mu_{2}-2 \mu\right) \Phi_{t-}^{(1)}-\left(\mu_{1}-\mu\right)\left(\mu_{2}-\mu\right) \Phi_{t-}^{(0)}\right) \frac{1}{\mu} \int_{y \in \mathbb{R}^{d}} r(y) p(d t d y) \tag{4.1}
\end{align*}
$$

with initial conditions $\Phi_{0}^{(0)}=\frac{\pi}{1-\pi}$ and $\Phi_{0}^{(1)}=\frac{\pi}{1-\pi} m^{(1)}$.
Instead of the sufficient statistic $\left(\Phi^{(0)}, \Phi^{(1)}\right)$, it will be more convenient to work with

$$
\begin{equation*}
\tilde{\Phi}_{t}^{(0)} \triangleq \frac{\mathbb{P}\left\{\Lambda=\mu_{1}, \theta \leq t \mid \mathcal{F}_{t}\right\}}{\mathbb{P}\left\{\theta>t \mid \mathcal{F}_{t}\right\}} \quad \text { and } \quad \tilde{\Phi}_{t}^{(1)} \triangleq \frac{\mathbb{P}\left\{\Lambda=\mu_{2}, \theta \leq t \mid \mathcal{F}_{t}\right\}}{\mathbb{P}\left\{\theta>t \mid \mathcal{F}_{t}\right\}} . \tag{4.2}
\end{equation*}
$$

In fact the following a one-to-one relationship between these two pairs holds

$$
\begin{equation*}
\tilde{\Phi}_{t}^{(0)}=\frac{\left(\mu_{2}-\mu\right) \Phi_{t}^{(0)}-\Phi_{t}^{(1)}}{\mu_{2}-\mu_{1}} \quad \text { and } \quad \tilde{\Phi}_{t}^{(1)}=\frac{\left(\mu_{1}-\mu\right) \Phi_{t}^{(0)}-\Phi_{t}^{(1)}}{\mu_{1}-\mu_{2}} . \tag{4.3}
\end{equation*}
$$

The dynamics of this new sufficient statistic are autonomous as can be seen from

$$
\begin{align*}
& d \tilde{\Phi}_{t}^{(0)}=\left\{\frac{\lambda\left(\mu_{2}-\mu-m^{(1)}\right)}{\mu_{2}-\mu_{1}}+\left(\lambda-\mu_{1}+\mu\right) \tilde{\Phi}_{t}^{(0)}\right\} d t+\tilde{\Phi}_{t-}^{(0)} \int_{y \in \mathbb{R}^{d}}\left[\left(1+\frac{\mu_{1}-\mu}{\mu}\right) r(y)-1\right] p(d t d y) \\
& d \tilde{\Phi}_{t}^{(1)}=\left\{\frac{\lambda\left(\mu_{1}-\mu-m^{(1)}\right)}{\mu_{1}-\mu_{2}}+\left(\lambda-\mu_{2}+\mu\right) \tilde{\Phi}_{t}^{(1)}\right\} d t+\tilde{\Phi}_{t-}^{(1)} \int_{y \in \mathbb{R}^{d}}\left[\left(1+\frac{\mu_{2}-\mu}{\mu}\right) r(y)-1\right] p(d t d y) \tag{4.4}
\end{align*}
$$

When the number of atoms of the distribution $\gamma$ is more than two, we expect that sufficient statistics defined similarly will also be autonomous.

The sufficient statistic we introduced in (4.2) has a natural interpretation and is similar in flavor to particle filters: these are the normalized probabilities that are assigned to each atom $\mu_{i}$ and these are updated continuously between the times of the observations, since not having an observation in fact reveals some information about the intensity of the underlying Poisson process. Indeed from (4.4) we observe that the sufficient statistic $\left(\tilde{\Phi}^{(0)}, \tilde{\Phi}^{(1)}\right)$ solves an ordinary differential equation between the observations, and the terms that involve the counting process $p$ are inactive. When there is an observation, these normalized probabilities jump depending on the jump size of the observation. We will see the optimal alarm mechanism is to sound the alarm as soon as the sufficient statistic touches or jumps above a convex and decreasing curve in $\mathbb{R}_{+}^{2}$, if the sufficient statistic starts below this curve. Otherwise it is optimal to sound the alarm immediately. Since the jump distribution also changes at the time of disorder, not only the timing of the observations but also the magnitude of the observations is informative. Therefore, it is reasonable to expect that we are able to construct a more acute alarm in this case than the case in which the observations are coming from a simple Poisson process where the jump size does not carry any information.

In the case when the post disorder rate could go both up and down by one unit, i.e., $\mu_{1}=\mu-1$ and $\mu_{2}=\mu+1$, then the dynamics in (4.4) become

$$
\begin{align*}
& d \tilde{\Phi}_{t}^{(0)}=\left\{\frac{\lambda(1-m)}{2}+(\lambda+1) \tilde{\Phi}_{t}^{(0)}\right\} d t+\tilde{\Phi}_{t-}^{(0)} \int_{y \in \mathbb{R}^{d}}\left[\left(1-\frac{1}{\mu}\right) r(y)-1\right] p(d t d y)  \tag{4.5}\\
& d \tilde{\Phi}_{t}^{(1)}=\left\{\frac{\lambda(1+m)}{2}+(\lambda-1) \tilde{\Phi}_{t}^{(1)}\right\} d t+\tilde{\Phi}_{t-}^{(1)} \int_{y \in \mathbb{R}^{d}}\left[\left(1+\frac{1}{\mu}\right) r(y)-1\right] p(d t d y),
\end{align*}
$$

in which $m=m^{1}=\mathbb{P}\{\Lambda=\mu+1\}-\mathbb{P}\{\Lambda=\mu-1\} \in[-1,1]$. Observe that when an arrival comes, $\tilde{\Phi}_{t}^{(0)}$ jumps down and $\tilde{\Phi}^{(1)}$ jumps up. Assuming $m \in(-1,1)$ then $\tilde{\Phi}_{t}^{(0)}$ is always increasing between the observations. $\tilde{\Phi}^{(1)}$, on the other hand, can be increasing or mean reverting depending on the value of $\lambda$. Note that the values $m=-1$ or $m=1$ correspond to the degenerate cases in which the post-disorder rate is known and the sufficient statistic becomes one-dimensional.

On the other hand, in the case when the post disorder rate could only go up by one or two units, i.e., $\mu_{1}=\mu+1$ and $\mu_{2}=\mu+2$, then the dynamics in (4.4) become

$$
\begin{align*}
& d \tilde{\Phi}_{t}^{(0)}=\left\{\lambda(2-m)+(\lambda-1) \tilde{\Phi}_{t}^{(0)}\right\} d t+\tilde{\Phi}_{t-}^{(0)} \int_{y \in \mathbb{R}^{d}}\left[\left(1+\frac{1}{\mu}\right) r(y)-1\right] p(d t d y), \\
& d \tilde{\Phi}_{t}^{(1)}=\left\{\lambda(m-1)+(\lambda-2) \tilde{\Phi}_{t}^{(1)}\right\} d t+\tilde{\Phi}_{t-}^{(1)} \int_{y \in \mathbb{R}^{d}}\left[\left(1+\frac{2}{\mu}\right) r(y)-1\right] p(d t d y), \tag{4.6}
\end{align*}
$$

in which $m=2 \mathbb{P}\{\lambda=\mu+2\}+\mathbb{P}\{\lambda=\mu+1\} \in[1,2]$. Here the initial conditions are $\tilde{\Phi}_{0}^{(0)}=$ $(2-m) \frac{\pi}{1-\pi}$ and $\tilde{\Phi}_{0}^{(1)}=(m-1) \frac{\pi}{1-\pi}$. We will assume that $m \in(1,2)$ as otherwise the problem degenerates into a one-dimensional one. In the next section we will see that the intuition that a decision would sound the alarm only at the times when it observes an arrival does not in general hold; see Remark 5.1. This intuition becomes relevant only when $\lambda$ and $c$ are small enough, i.e. when the disorder intensity and delay penalty are small. If $\lambda \geq 2$ then both $\tilde{\Phi}_{t}^{(0)}$ and $\tilde{\Phi}_{t}^{(1)}$ increase between the jumps, because the rate of disorder is high enough despite the fact that there have been no arrivals. When $\lambda \in[1,2), \tilde{\Phi}_{t}^{(0)}$ increases between the jumps and $\tilde{\Phi}_{t}^{(1)}$ is mean
reverting. When $\lambda \in(0,1)$, both $\tilde{\Phi}_{t}^{(0)}$ and $\tilde{\Phi}_{t}^{(1)}$ have mean reverting paths between arrivals. Since the post disorder arrival rate can only move up, both $\tilde{\Phi}_{t}^{(0)}$ and $\tilde{\Phi}_{t}^{(1)}$ have an upward jump when there is an observation.

In the remainder of the paper we analyze the case when the sufficient statistic is of the form (4.6). Note that in this case the penalty function in (3.2) becomes

$$
\begin{equation*}
R_{\tau}(\pi)=1-\pi+c(1-\pi) \mathbb{E}_{0}\left[\int_{0}^{\tau} e^{-\lambda t}\left(\tilde{\Phi}_{t}^{(0)}+\tilde{\Phi}_{t}^{(1)}-\frac{\lambda}{c}\right) d t\right], \quad \tau \in \mathcal{S} . \tag{4.7}
\end{equation*}
$$

Let us define

$$
\begin{align*}
& x\left(t, x_{0}\right) \triangleq\left\{\begin{array}{ll}
-\frac{\lambda(2-m)}{\lambda-1}+e^{(\lambda-1) t}\left[x_{0}+\frac{\lambda(2-m)}{\lambda-1}\right], & \lambda \neq 1, \\
x_{0}+(2-m) t, & \lambda=1,
\end{array}\right. \text { and } \\
& y\left(t, y_{0}\right) \triangleq \begin{cases}-\frac{\lambda(m-1)}{\lambda-2}+e^{(\lambda-2) t}\left[y_{0}+\frac{\lambda(m-1)}{\lambda-2}\right] & \lambda \neq 2, \\
y_{0}+2(m-1) t & \lambda=2 .\end{cases} \tag{4.8}
\end{align*}
$$

Note that $x$ and $y$ satisfy the semigroup property, i.e., for every $t \in \mathbb{R}$ and $s \in \mathbb{R}$,

$$
\begin{equation*}
x\left(t+s, x_{0}\right)=x\left(s, x\left(t, x_{0}\right)\right) \quad \text { and } \quad y\left(t+s, x_{0}\right)=y\left(s, y\left(t, x_{0}\right)\right) . \tag{4.9}
\end{equation*}
$$

Let us denote by $\sigma_{n}$ the jump times of the process $X$. Then we get

$$
\begin{align*}
& \tilde{\Phi}_{t}^{(0)}=x\left(t-\sigma_{n}, \tilde{\Phi}_{\sigma_{n}}^{(0)}\right) \quad \text { and } \quad \tilde{\Phi}_{t}^{(1)}=y\left(t-\sigma_{n}, \tilde{\Phi}_{\sigma_{n}}^{(1)}\right), \quad \sigma_{n} \leq t<\sigma_{n+1}, \\
& \tilde{\Phi}_{\sigma_{n}}^{(0)}=\left(1+\frac{1}{\mu}\right) r\left(Y_{n}\right) \tilde{\Phi}_{\sigma_{n}-}^{(0)} \quad \text { and } \quad \tilde{\Phi}_{\sigma_{n}}^{(1)}=\left(1+\frac{2}{\mu}\right) r\left(Y_{n}\right) \tilde{\Phi}_{\sigma_{n}-}^{(1)} \quad n \in \mathbb{N}_{0} . \tag{4.10}
\end{align*}
$$

The minimum of the Bayes risk in (4.7) is given by;

$$
\begin{equation*}
U(\pi)=\inf _{\tau \in \mathcal{S}} R_{\tau}(\pi)=(1-\pi)+c(1-\pi) V\left((2-m) \frac{\pi}{1-\pi},(m-1) \frac{\pi}{1-\pi}\right), \tag{4.11}
\end{equation*}
$$

in which $V$ is defined as the value function of the optimal stopping problem for a two-dimensional Markov process

$$
\begin{equation*}
V\left(\phi_{0}, \phi_{1}\right) \triangleq \inf _{\tau \in \mathcal{S}} \mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\int_{0}^{\tau} e^{-\lambda t} g\left(\tilde{\Phi}_{t}\right) d t\right], \quad \tilde{\Phi}_{t} \triangleq\left(\tilde{\Phi}_{t}^{(0)}, \tilde{\Phi}_{t}^{(1)}\right), \tag{4.12}
\end{equation*}
$$

with a running cost function

$$
\begin{equation*}
g\left(\phi_{0}, \phi_{1}\right)=\phi_{0}+\phi_{1}-\frac{\lambda}{c} . \tag{4.13}
\end{equation*}
$$

Here, $\mathbb{E}_{0}^{\phi_{0}, \phi_{1}}$ is the conditional $\mathbb{P}_{0}$ expectation given that $\tilde{\Phi}_{0}^{(0)}=\phi_{0}$ and $\tilde{\Phi}_{0}^{(1)}=\phi_{1}$.

## 5 Upper and Lower Bounds on the Optimal Stopping Time

Unlike the optimal stopping problem for Itô diffusions, analyzing the sample path behavior of the piece-wise deterministic Markov process $\tilde{\Phi} \triangleq\left(\tilde{\Phi}_{1}, \tilde{\Phi}_{2}\right)$, we are able to determine the optimal stopping time for most parameter values. For remaining parameter values we are able to provide some lower bound and an upper bounds on the optimal stopping time.

All the results in this section assume that an optimal stopping time exists and it is given by

$$
\begin{equation*}
\tau^{*}\left(\phi_{0}, \phi_{1}\right) \triangleq \inf \left\{t \geq 0: V\left(\widetilde{\Phi}_{t}\right)=0, \widetilde{\Phi}_{0}=\left(\phi_{0}, \phi_{1}\right)\right\} \tag{5.1}
\end{equation*}
$$

In Section 6], we verify that this assumption in fact holds. With (5.1) we will call the region

$$
\begin{equation*}
\boldsymbol{\Gamma} \triangleq\left\{\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2}: v\left(\phi_{0}, \phi_{1}\right)=0\right\}, \quad \mathbf{C} \triangleq \mathbb{R}_{+}^{2} \backslash \boldsymbol{\Gamma}, \tag{5.2}
\end{equation*}
$$

the optimal stopping region. Let us start this section with a simple observation.

Lemma 5.1 Let us define

$$
\begin{equation*}
\tau^{l} \triangleq \inf \left\{t \geq 0: \tilde{\Phi}_{t}^{(0)}+\tilde{\Phi}_{t}^{(1)} \geq \lambda / c\right\} \tag{5.3}
\end{equation*}
$$

If there is an optimal stopping time for the problem in (4.12), let us denote it by $\tau^{*}$, then $\tau^{*} \geq \tau^{l}$.
Proof: Let $\tau \in \mathcal{S}$ be any stopping rule. Then

$$
\begin{align*}
\mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\int_{0}^{\tau \vee \tau^{l}} e^{-\lambda t} g\left(\tilde{\Phi}_{t}\right) d t\right] & =\mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\int_{0}^{\tau} e^{-\lambda t} g\left(\tilde{\Phi}_{t}\right) d t\right]+\mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[1_{\{\tau \tau>\tau\}} \int_{\tau}^{\tau^{l}} e^{-\lambda t} g\left(\tilde{\Phi}_{t}\right) d t\right] \\
& \leq \mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\int_{0}^{\tau} e^{-\lambda t} g\left(\tilde{\Phi}_{t}\right) d t\right], \quad\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2} . \tag{5.4}
\end{align*}
$$

Here $\tau \vee \tau^{l}=\max \left\{\tau, \tau^{l}\right\}$.
When the rate of disorder or $c$ in (1.5) are large enough, then in fact the lower bound $\tau^{l}$ is optimal as the following proposition illustrates, i.e. the free boundary corresponding to the two-dimensional optimal stopping problem in (4.12) can be determined completely. This is a very special instance of a multi-dimensional optimal stopping problem where an explicit determination of the free boundary is possible.

Proposition 5.1 If (i) $\lambda \geq 2$, or, (ii) $\lambda \in[1,2)$ and $c \geq 2-\lambda$, or, (iii) $\lambda \in(0,1)$ and $c \geq \max (2-\lambda, 1-\lambda)$, then the stopping rule $\tau^{l}$ of (5.3) is optimal for the problem in (4.12).

Proof: (i) Let us first consider the case $\lambda \geq 2$. It is clear from the dynamics of the sufficient statistic in (4.6) that the sample paths of $\tilde{\Phi}_{t}^{(0)}$ and $\tilde{\Phi}_{t}^{(1)}$ are increasing functions of time. Therefore the process $\tilde{\Phi}$ does not return to the region $\left\{\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2}: \phi_{0}+\phi_{1} \leq \lambda / c\right\}$. Thus for every stopping time $\tau \in \mathcal{S}$

$$
\begin{align*}
& \mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\int_{0}^{\tau} e^{-\lambda t} g\left(\tilde{\Phi}_{t}\right) d t\right] \geq \mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\int_{0}^{\tau \vee \tau^{l}} e^{-\lambda t} g\left(\tilde{\Phi}_{t}\right) d t\right] \\
& =\mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\int_{0}^{\tau^{l}} e^{-\lambda t} g\left(\tilde{\Phi}_{t}\right) d t\right]+\mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[1_{\left\{\tau \geq \tau^{l}\right\}} \int_{\tau^{l}}^{\tau} e^{-\lambda t} g\left(\tilde{\Phi}_{t}\right) d t\right] \geq \mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\int_{0}^{\tau^{l}} e^{-\lambda t} g\left(\tilde{\Phi}_{t}\right) d t\right] \tag{5.5}
\end{align*}
$$

(ii) If $\lambda \in[1,2)$ then any sample path of $\tilde{\Phi}^{(0)}$ is still an increasing function of $t$, but the same is not true anymore for the sample paths of $\tilde{\Phi}^{(1)}$. The paths of $\widetilde{\Phi}^{(1)}$ increase with jumps; between
the jumps the paths are mean reverting to the level $\lambda(m-1) /(2-\lambda)$. However, since the processes $\widetilde{\Phi}^{(0)}$ and $\widetilde{\Phi}^{(1)}$ can only increase by jumps we have that

$$
\begin{equation*}
\widetilde{\Phi}_{t}^{(0)} \geq x\left(t, \phi_{0}\right) \quad \text { and } \quad \widetilde{\Phi}_{t}^{(1)} \geq y\left(t, \phi_{1}\right), \quad t \geq 0 \tag{5.6}
\end{equation*}
$$

Therefore

$$
\begin{equation*}
V\left(\phi_{0}, \phi_{1}\right) \geq \inf _{\tau \in \mathcal{S}} \mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\int_{0}^{\tau} e^{-\lambda t}\left(x\left(t, \phi_{0}\right)+y\left(t, \phi_{1}\right)-\frac{\lambda}{c}\right) d t\right] \tag{5.7}
\end{equation*}
$$

Clearly if for any $\left(\phi_{0}, \phi_{1}\right)$ if the right hand side of (5.7) is zero, then $V=0$, since we also know that $V \leq 0$. This can be used to find a superset of the continuation region. However, as we shall see shortly this superset coincides with the advantageous region

$$
\begin{equation*}
\mathbb{C}_{0} \triangleq\left\{\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2}: \phi_{0}+\phi_{1} \leq \lambda / c\right\} \tag{5.8}
\end{equation*}
$$

Observe that it is not optimal to stop before $\tilde{\Phi}$ leaves the region $\mathbb{C}_{0}$.
Let us take a look at the derivative of the integrand on the righthand side in (5.7),

$$
\begin{equation*}
\frac{d}{d t}\left[x\left(t, \phi_{0}\right)+y\left(t, \phi_{1}\right)\right]=(\lambda-1) x\left(t, \phi_{0}\right)+(\lambda-2) y\left(t, \phi_{1}\right)+\lambda \tag{5.9}
\end{equation*}
$$

The righthand side of (5.9) vanishes if the curve $t \rightarrow\left(x\left(t, \phi_{0}\right), y\left(t, \phi_{1}\right)\right)$ meets the line

$$
\begin{equation*}
l:(\lambda-1) x+(\lambda-2) y+\lambda=0 \tag{5.10}
\end{equation*}
$$

Note that since $\lambda \in[1,2)$ the $y$-intercept of the line is such that $\frac{\lambda}{2-\lambda} \geq \lambda$. Since $l$ is increasing and $c \geq 2-\lambda$, the intersection of $l$ with the set $\mathbb{C}_{0}$ is empty. Observe also that every $t \rightarrow$ $\left(x\left(t, \phi_{0}\right), y\left(t, \phi_{1}\right)\right)$ starting at $\left(\phi_{0}, \phi_{1}\right)$ is decreasing and the derivative in (5.9) is increasing. Therefore, $t \rightarrow\left(x\left(t, \phi_{0}\right), y\left(t, \phi_{1}\right)\right)$ meets the line $l$ at most once for any $\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}$.

$$
\left\{\begin{array}{l}
\text { Furthermore, if } t \rightarrow\left(x\left(t, \phi_{0}\right), y\left(t, \phi_{1}\right)\right) \text { meets } l \text { at } t_{l}=t_{l}\left(\phi_{0}, \phi_{1}\right), \text { then the function }  \tag{5.11}\\
t \rightarrow x\left(t, \phi_{0}\right)+y\left(t, \phi_{1}\right) \text { is decreasing on }\left[0, t_{l}\right] \text { and increasing on }\left[t_{l}, \infty\right) . \text { If } t \rightarrow\left(x\left(t, \phi_{0}\right)\right. \\
\left.y\left(t, \phi_{1}\right)\right) \text { does not intersect } l, \text { then the function } t \rightarrow x\left(t, \phi_{0}\right)+y\left(t, \phi_{1}\right) \text { is increasing on } \\
{[0, \infty) .}
\end{array}\right.
$$

Since the line $l$ does not meet the region $\mathbb{C}_{0}$ for every $\left(\phi_{0}, \phi_{1}\right) \in l$ we have that $\phi_{0}+\phi_{1} \geq \lambda / c$. Now (5.11) implies that $x\left(t, \phi_{0}\right)+y\left(t, \phi_{1}\right)-\frac{\lambda}{c}>0$ for $\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2}-\mathbb{C}_{0}$ and $t \geq 0$. This implies that the righthand side of (5.7) is zero, which in turn implies that $V\left(\phi_{0}, \phi_{1}\right)=0$ for all $\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2}-\mathbb{C}_{0}$.
(iii) If $\lambda \in(0,1)$, then both of the paths of $x\left(t, \phi_{0}\right)$ and $y\left(t, \phi_{1}\right)$ are mean reverting. Because of our assumption on $c$ the line $l$ in (5.10) does not intersect with $\mathbb{C}_{0}$ and lies entirely above this region. Let us denote the region between $l$ and $\mathbb{C}_{0}$ by

$$
\begin{equation*}
\mathrm{Sh} \triangleq\left\{\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2}: \phi_{0}+\phi_{1}-\lambda / c>0,(\lambda-1) \phi_{0}+(\lambda-2) \phi_{1}+\lambda<0\right\} \tag{5.12}
\end{equation*}
$$

From (5.9) it follows that $x\left(t, \phi_{0}\right)+y\left(t, \phi_{1}\right)>\lambda / c$ if $\left(\phi_{0}, \phi_{1}\right) \in$ Sh. Therefore, the path $t \rightarrow$ $\left(x\left(t, \phi_{0}\right), y\left(t, \phi_{1}\right)\right)$ never enters the region $\mathbb{C}_{0}$ if $\left(\phi_{0}, \phi_{1}\right) \notin \mathbb{C}_{0}$. Therefore, the righthand side of (5.7) is zero, which in turn implies that $V\left(\phi_{0}, \phi_{1}\right)=0$ for any $\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2}-\mathbb{C}_{0}$.

Proposition 5.2 Assume $\lambda \in[1,2)$ and $c \in(0,2-\lambda)$. Let us define

$$
\begin{equation*}
D \triangleq\left\{\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2}: \phi_{0} \leq \phi_{0}^{*}, \phi_{0}+\phi_{1} \leq \xi\right\} \cup\left\{\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2}: \phi_{0}>\phi_{0}^{*}, \phi_{0}+\phi_{1} \leq \lambda / c\right\}, \tag{5.13}
\end{equation*}
$$

in which $\left(\phi_{0}^{*}, \phi_{1}^{*}\right) \triangleq(\lambda(-1+(2-\lambda) / c), \lambda(1+(\lambda-1) / c))$ and

$$
\xi=y\left(-t^{*}, \lambda\left(\frac{\lambda-1}{c}+1\right)\right), \quad \text { where } \quad x\left(-t^{*}, \lambda\left(\frac{2-\lambda}{c}-1\right)\right)=0
$$

Then the region $D$ is a superset of the optimal stopping region.
Proof: Let us note that (5.7) implies that

$$
\begin{equation*}
V\left(\phi_{0}, \phi_{1}\right) \geq \inf _{t \in[0, \infty]}\left[\int_{0}^{t} e^{-\lambda s}\left(x\left(s, \phi_{0}\right)+y\left(s, \phi_{1}\right)-\frac{\lambda}{c}\right) d s\right] . \tag{5.14}
\end{equation*}
$$

Because of the assumption on $c$ the line in (5.10) intersects the region $\mathbb{C}_{0}$ defined in (5.8). Note that $l$ and the boundary $x+y-\lambda / c=0$ of the region $\mathbb{C}_{0}$ intersect at $\left(\phi_{0}^{*}, \phi_{1}^{*}\right)$. By running the time "backwards", we can find $\xi$ and $t^{*}$ such that

$$
\begin{equation*}
(0, \xi)=\left(x\left(-t^{*}, \phi_{0}^{*}\right), y\left(-t^{*}, \phi_{1}^{*}\right)\right) \tag{5.15}
\end{equation*}
$$

By the semi-group property (see (4.9)), we have

$$
x\left(t^{*}, 0\right)=x\left(t^{*}, x\left(-t^{*}, \phi_{0}^{*}\right)\right)=x\left(t^{*}+\left(-t^{*}\right), \phi_{0}^{*}\right)=x\left(0, \phi_{0}^{*}\right)=\phi_{0}^{*},
$$

and,

$$
y\left(t^{*}, \xi\right)=y\left(t^{*}, x\left(-t^{*}, \phi_{1}^{*}\right)\right)=y\left(t^{*}+\left(-t^{*}\right), \phi_{1}^{*}\right)=y\left(0, \phi_{1}^{*}\right)=\phi_{1}^{*} .
$$

So, the curve $t \rightarrow(x(t, 0), y(t, \xi)), t \geq 0$, meets line $l$ at $\left(\phi_{0}^{*}, \phi_{1}^{*}\right)$, and $t_{l}$ in (5.11) equals to $t^{*}$. This implies that

$$
x(t, 0)+y(t, \xi) \geq x\left(t^{*}, 0\right)+y\left(t^{*}, \xi\right)=\phi_{0}^{*}+\phi_{1}^{*}=\frac{\lambda}{c}
$$

and in particular $\xi \geq \lambda / c$. Now we will show that when $\lambda$ and $c$ are chosen as in the statement of the proposition it is optimal to stop outside the region $D$.

The curve $t \rightarrow(x(t, 0), y(t, \xi))$ divides $\mathbb{R}_{+}^{2}$ into two connected components containing $\mathbb{C}_{0}$ and the region

$$
\begin{equation*}
\left.M \triangleq \mathbb{R}_{+}^{2}-D\right) \cap\left\{(x, y) \in \mathbb{R}_{+}^{2}:(\lambda-1) x+(\lambda-2) y+\lambda<0\right\} \tag{5.16}
\end{equation*}
$$

respectively. Every curve $t \rightarrow\left(x\left(t, \phi_{0}\right), y\left(t, \phi_{1}\right)\right), t \geq 0$ starting at $\left(\phi_{0}, \phi_{1}\right) \in M$ will stay in $M$, since from the semi-group property (4.9) it follows that two distinct curves $t \rightarrow\left(x\left(t, \phi_{0}^{a}\right), y\left(t, \phi_{1}^{a}\right)\right)$ and $t \rightarrow\left(x\left(t, \phi_{0}^{b}\right), y\left(t, \phi_{1}^{b}\right)\right)$ do not intersect. Therefore, $t \rightarrow\left(x\left(t, \phi_{0}\right), y\left(t, \phi_{1}\right)\right), t \geq 0,\left(\phi_{0}, \phi_{1}\right) \in$ $M$ intersects the line $l$ in (5.10) away from $\mathbb{C}_{0}$ and (5.11) implies that $x\left(t, \phi_{0}\right)+y\left(t, \phi_{1}\right)>\lambda / c$ for any $\left(\phi_{0}, \phi_{1}\right) \in M$. Now, from (5.14) we conclude that $V=0$ since the infimum on the right-hand-side is equal to 0 from the arguments above and we already know that $V \leq 0$.

On the other hand, if $\left(\phi_{0}, \phi_{1}\right) \in\left(\mathbb{R}_{+}^{2}-D\right) \cap\left\{(x, y) \in \mathbb{R}_{+}^{2}:(\lambda-1) x+(\lambda-2) y+\lambda \geq 0\right\}$, the curve $t \rightarrow\left(x\left(t, \phi_{0}\right), y\left(t, \phi_{1}\right)\right), t \geq 0$ does not intersect the line $l$; therefore, the function $t \rightarrow x\left(t, \phi_{0}\right)+y\left(t, \phi_{1}\right)$ is increasing and

$$
x\left(t, \phi_{0}\right)+y\left(t, \phi_{1}\right)>x\left(0, \phi_{0}\right)+y\left(0, \phi_{1}\right) \geq \phi_{0}+\phi_{1} \geq \xi \geq \frac{\lambda}{c}, \quad 0<t<\infty
$$

Again, the infimum on the right-hand-side of (5.14) is equal to zero, which implies that $V=0$.

Remark 5.1 If $\lambda \in[1,2)$ and $c \in(0,2-\lambda)$, then the following line segment is a subset of the free boundary

$$
\begin{equation*}
H \triangleq\left\{\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2}: \phi_{0}+\phi_{1}-\frac{\lambda}{c}=0, \quad \phi_{1} \leq \phi_{1}^{*}\right\} . \tag{5.17}
\end{equation*}
$$

This set in fact in the entrance boundary of the stopping region (the boundary through which the path $t \rightarrow\left(x\left(t, \phi_{0}\right), y\left(t, \phi_{1}\right)\right)$ enters the stopping region $)$.

Proposition 5.3 Assume that $\lambda \in(0,1)$ and that $0<c \leq \frac{(2-\lambda)(1-\lambda)}{3-\lambda-m}$. If furthermore $c \geq 2 \frac{1-\lambda}{3-m}$, then

$$
\begin{equation*}
P \triangleq\left\{\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2}: \phi_{0}+\frac{1}{2} \phi_{1}+\frac{3}{2}-\frac{1}{2} m-\frac{1}{c} \geq 0, \phi_{0}+\phi_{1}-\frac{\lambda}{c} \geq 0\right\} \tag{5.18}
\end{equation*}
$$

is a subset of the optimal stopping region.
If on the other hand, $0<c<2 \frac{1-\lambda}{3-m}$, then the first time time $\left(\widetilde{\Phi}^{(0)}, \widetilde{\Phi}^{(1)}\right)$ reaches the set,

$$
\begin{equation*}
R \triangleq\left\{\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2}: \phi_{0}+\frac{1}{2} \phi_{1}+\frac{3}{2}-\frac{1}{2} m-\frac{1}{c} \geq 0\right\} \tag{5.19}
\end{equation*}
$$

is an upper bound on the optimal stopping time.
Proof: When $0<c \leq \frac{(2-\lambda)(1-\lambda)}{3-\lambda-m}$, then the line $l \cap \mathbb{R}_{+}^{2}$ lies entirely in $\mathbb{C}_{0}$. The paths, $t \rightarrow x\left(t, \phi_{0}, \phi_{1}\right), t \geq 0$, that do not originate in $\mathbb{C}_{0}$ enter into this region through the boundary $\left\{\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2}: \phi_{0}+\phi_{1}=\lambda / c\right\}$ and once they cross into $\mathbb{C}_{0}$ they never leave it again since $x\left(t, \phi_{0}\right)+y\left(t, \phi_{0}\right)<\phi_{0}+\phi_{1}<\lambda / c$ for any point $\left(\phi_{0}, \phi_{1}\right) \in \mathbb{C}_{0} \cap\left\{\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2}:(\lambda-1) \phi_{0}+\right.$ $\left.(\lambda-2) \phi_{1}+\lambda<0\right\}$, which follows from (5.9). Therefore the infimum on the right-hand-side of (5.14) is attained by either $t=0$ or $t=\infty$ if $\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2}-\mathbb{C}_{0}$. Either one never stops, pays a penalty for being outside $\mathbb{C}_{0}$ for a while and then enjoys being in this region ad infinitum, or stops immediately because the cost of the initial penalty is deterrent enough. Since

$$
\begin{equation*}
\int_{0}^{\infty}\left(x\left(t, \phi_{0}\right)+y\left(t, \phi_{1}\right)-\frac{\lambda}{c}\right) d t=\phi_{0}+\frac{1}{2} \phi_{1}+\frac{3}{2}-\frac{1}{2} m-\frac{1}{c}, \tag{5.20}
\end{equation*}
$$

the infimum on the right-hand-side of (5.14) is attained by $t=0$ if $\left(\phi_{0}, \phi_{1}\right) \in P$, which in turn implies that $V\left(\phi_{0}, \phi_{1}\right)=0$ for any $\left(\phi_{0}, \phi_{1}\right) \in P$. Observe that, if $0<c<2 \frac{1-\lambda}{3-m}$ then $P=R$.

Remark 5.2 Observe that if $\lambda \in(0,1)$ and $2 \frac{1-\lambda}{3-m} \leq c \leq \frac{(2-\lambda)(1-\lambda)}{3-\lambda-m}$, then the following line segment is a subset of the free boundary;

$$
\begin{equation*}
F \triangleq\left\{\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2}: \phi_{0}+\phi_{1}-\frac{\lambda}{c}=0, \quad \phi_{1} \leq 2\left(-\frac{1-\lambda}{c}+\frac{3-m}{2}\right)\right\} . \tag{5.21}
\end{equation*}
$$

This region is in the exit boundary of the stopping region (i.e., the boundary through which the path $t \rightarrow\left(x\left(t, \phi_{0}\right), y\left(t, \phi_{1}\right)\right)$ exits from the stopping region $)$.

Proposition 5.4 Assume that $\lambda \in(0,1)$ and that

$$
\begin{equation*}
\frac{(2-\lambda)(1-\lambda)}{3-\lambda-m}<c<\max (2-\lambda, 1-\lambda) . \tag{5.22}
\end{equation*}
$$

Then the region $D$ defined in Proposition 5.2 is a superset of the optimal stopping region.

Proof: From the assumption on the parameters $\lambda$ and $c$ it follows that the mean reversion level $M=\left(\frac{\lambda(2-m)}{1-\lambda}, \frac{\lambda(m-1)}{2-\lambda}\right)$ of the path $t \rightarrow\left(x\left(t, \phi_{0}\right), y\left(t, \phi_{1}\right)\right), t \geq 0$, is in the region $[0, \lambda / c] \times$ $[0, \lambda / c]-\mathbb{C}_{0}$. Also, one can easily check that $M \in l$, in which $l$ is as in (5.10). Line $l$ and the boundary of the region $\mathbb{C}_{0}$ intersect at $\left(\phi_{0}^{*}, \phi_{1}^{*}\right)$. Because $c>(2-\lambda)(1-\lambda) /(3-\lambda-m)$, the equation (as an equation in the $t$-variable) $x(t, 0)=\phi_{0}^{*}$ has a positive solution, $t^{*}$ and $y_{0}=y\left(-t^{*}, \phi_{1}^{*}\right)>0$. The rest of the proof follows by using the same arguments as in the proof of Proposition 5.2.

Remark 5.3 Observe that if $\lambda \in(0,1)$ and $c$ satisfies (5.22), then the following line segment is a subset of the free boundary

$$
\begin{equation*}
A \triangleq\left\{\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2}: \phi_{0}+\phi_{1}-\frac{\lambda}{c}=0, \quad \phi_{1} \leq \lambda\left(1-\frac{1-\lambda}{c}\right)\right\} . \tag{5.23}
\end{equation*}
$$

Moreover, this set is a subset of entrance boundary of the stopping region.

Remark 5.4 If the assumptions of Proposition 5.3 are satisfied, then it is optimal to sound the alarm only at arrival times of the observation. This corresponds to the case when the mean reversion level of the paths $t \rightarrow\left(x\left(t, \phi_{0}\right), y\left(t, \phi_{1}\right)\right)$ is inside the advantageous region $\mathbb{C}_{0}$, which is defined in (5.8). Otherwise, since the paths of the sufficient statistic, $t \rightarrow \widetilde{\Phi}_{t}$, may reach the stopping region continuously or via jumps, it might be optimal to declare the alarm between two observations.

We will close this section by proving that the optimal stopping time $\tau^{*}$ is finite almost surely.

Proposition 5.5 Let $\eta$ be a positive number such that the region $\left\{\left(\phi_{0}, \phi_{1}\right): \phi_{0}+\phi_{1} \geq \eta\right\}$ is a subset of the stopping region. (The existence of $\eta$ is guaranteed by Propositions 5.1.5.4). Let us denote the hitting time of this region by $\tau^{u}$. Then $\mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\tau^{u}\right] \leq \eta(2+1 / \mu)$. This implies that $\tau^{*}$ is finite $\mathbb{P}_{0}$ almost surely.

Proof: Since the compensator of $p(d t d y)$ (defined in (3.4)) is equal to $\mu \beta_{0}(y)$ we can write the dynamics of $\widetilde{\Phi}^{(0)}$ in (4.6) as

$$
\begin{align*}
\widetilde{\Phi}_{t \wedge \tau^{u}}^{(0)} & =\widetilde{\Phi}_{0}^{(0)}+\int_{0}^{t \wedge \tau^{u}}\left\{\lambda(2-m)+(\lambda-1) \tilde{\Phi}_{t}^{(0)}\right\} d t+\int_{0}^{t \wedge \tau^{u}} \mu \tilde{\Phi}_{t-}^{(0)} \int_{y \in \mathbb{R}^{d}}\left[\left(1+\frac{1}{\mu}\right) r(y)-1\right] \beta_{0}(d y) d s \\
& +\int_{0}^{t \wedge \tau^{u}} \tilde{\Phi}_{t-}^{(0)} \int_{y \in \mathbb{R}^{d}}\left[\left(1+\frac{1}{\mu}\right) r(y)-1\right] q(d s d y) \\
& =\widetilde{\Phi}_{0}^{(0)}+\int_{0}^{t \wedge \tau^{u}}\left\{\lambda(2-m)+\lambda \tilde{\Phi}_{t}^{(0)}\right\} d s+\int_{0}^{t \wedge \tau^{u}} \tilde{\Phi}_{t-}^{(0)} \int_{y \in \mathbb{R}^{d}}\left[\left(1+\frac{1}{\mu}\right) r(y)-1\right] q(d s d y), \tag{5.24}
\end{align*}
$$

in which $q(d t d y) \triangleq p(d t d y)-\mu \beta_{0}(y)$ Here, we have used the fact that $\int_{y \in \mathbb{R}_{+}^{d}} r(y) \beta_{0}(y)=1$. The
integral with respect to $q(d t d y)$ is an $\mathbb{F}$ martingale under the measure $\mathbb{P}_{0}$, since

$$
\begin{aligned}
\mathbb{E}^{\phi_{0}, \phi_{1}}\left[\int_{0}^{t \wedge \tau^{u}} \mu \tilde{\Phi}_{t-}^{(0)} \int_{y \in \mathbb{R}^{d}}\left|\left(1+\frac{1}{\mu}\right) r(y)-1\right| \beta_{0}(d y) d s\right] & \leq \mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\int_{0}^{t \wedge \tau^{u}}\left(2+\frac{1}{\mu}\right) \widetilde{\Phi}_{s-}^{(0)} d s\right] \\
& \leq t\left(2+\frac{1}{\mu} \eta\right)
\end{aligned}
$$

Therefore

$$
\mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\widetilde{\Phi}_{t \wedge \tau^{u}}^{(0)}\right]=\phi_{0}+\mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\int_{0}^{t \wedge \tau^{u}}\left\{\lambda(2-m)+\lambda \tilde{\Phi}_{t}^{(0)}\right\} d s\right] \geq \lambda(2-m) \mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[t \wedge \tau^{u}\right]
$$

On the other hand,

$$
\widetilde{\Phi}_{t \wedge \tau^{u}}^{(0)} \leq \max \left(\eta,\left(1+\frac{1}{\mu}\right) r\left(Y_{N_{t \wedge \tau^{u}}}\right) \widetilde{\Phi}_{t \wedge \tau^{u}-}^{(0)}\right) \leq \eta\left(1+\left(1+\frac{1}{\mu}\right) r\left(Y_{N_{t \wedge \tau^{u}}}\right)\right),
$$

almost surely; therefore

$$
\begin{aligned}
\mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[t \wedge \tau^{u}\right] \leq \frac{1}{\lambda(2-m)} \mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\widetilde{\Phi}_{t \wedge \tau^{u}}^{(0)}\right] & \leq \mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\eta\left(1+\left(1+\frac{1}{\mu}\right) r\left(Y_{1}\right)\right)\right] \\
& =\eta\left(2+\frac{1}{\mu}\right) .
\end{aligned}
$$

The result follows after an application of the monotone convergence theorem.
In what follows we will consider the cases in which the parameters do not satisfy the hypothesis of Proposition 5.1 and construct a sequence of functions iteratively, using an appropriately defined functional operator, that converges to the value function exponentially fast.

## 6 Optimal Stopping Time and Properties of the Value Function and the Stopping Boundary

The usual starting point to calculate the value function in (4.12) and find the optimal stopping time would be to try to characterize the value function as the unique solution of the free boundary problem

$$
\begin{equation*}
\min \{(\mathcal{A}-\lambda) v(\varphi)+g(\varphi),-v(\varphi)\}=0 \tag{6.1}
\end{equation*}
$$

in which the differential operator is the inifinitesimal generator of the Markov process $\left(\tilde{\Phi}^{(0)}, \tilde{\Phi}^{(1)}\right)$ and whose action on a test function $f$ is given by

$$
\begin{align*}
\mathcal{A} f\left(\phi_{0}, \phi_{1}\right) & =\frac{\partial f}{\partial \phi_{0}}\left(\phi_{0}, \phi_{1}\right)\left[\lambda(2-m)+(\lambda-1) \phi_{0}\right]+\frac{\partial f}{\partial \phi_{1}}\left(\phi_{0}, \phi_{1}\right)\left[\lambda(m-1)+(\lambda-2) \phi_{1}\right] \\
& +\mu \int_{y \in \mathbb{R}^{d}}\left[f\left(\left(1+\frac{1}{\mu}\right) r(y) \phi_{0},\left(1+\frac{2}{\mu}\right) r(y) \phi_{1}\right)-f\left(\phi_{0}, \phi_{1}\right)\right] \beta_{0}(d y) . \tag{6.2}
\end{align*}
$$

The solution of the free boundary problem (6.1) may be identified by using certain boundary conditions (the smooth fit principle). The smooth fit is expected to fail for (6.1) at the exit boundary of the stopping region. See e.g. Bayraktar et al. (2005), Bayraktar and Dayanik (2006) for failure of the smooth fit principle when the infinitesimal generator $\mathcal{A}$ is a differential
delay operator (these papers consider one dimensional free boundary problems). Instead of the characterization of the value function as a solution of quasi-variational inequalities, we will use a new characterization of the value function of the optimal stopping problem in (4.12). Specifically, we will construct a sequence of functions iteratively, using an appropriately defined functional operator, that converges to the value function exponentially fast. This will let us show that $\tau^{*}$ in (5.1) is the optimal stopping time. We will also be able to show the concavity of the value function and the convexity of the free boundary.

### 6.1 Optimal Stopping with Time Horizon $\sigma_{n}$

In this section, we will approximate the value function $V$ with a sequence of optimal stopping problems. Let us denote

$$
\begin{equation*}
V_{n}\left(\phi_{0}, \phi_{1}\right) \triangleq \inf _{\tau \in \mathcal{S}} \mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\int_{0}^{\tau \wedge \sigma_{n}} e^{-\lambda t} g\left(\widetilde{\Phi}_{t}^{(0)}, \widetilde{\Phi}_{t}^{(1)}\right) d t\right] \tag{6.3}
\end{equation*}
$$

where $\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2}, n \in \mathbb{N}$, and $\sigma_{n}$ is the $n^{\text {th }}$ jump time of the process $X$. Observe that $\left(V_{n}\right)_{n \in \mathbb{N}}$ is decreasing and satisfies $-1 / c<V_{n}<0$. Since $\left(\sigma_{n}\right)_{n \geq 1}$ is an almost surely increasing sequence, $\left(V_{n}\right)_{n \geq 1}$ is decreasing. Therefore $\lim _{n} V_{n}$ exists. It is also immediate that $V_{n} \geq V$. In fact we can say more about the limit of the sequence $\left(V_{n}\right)_{n \geq 1}$ as the next proposition illustrates.

Proposition $6.1 V_{n}\left(\phi_{0}, \phi_{1}\right)$ converges to $V$ uniformly in $\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2}$. In fact the rate of convergence is exponential as the following equation illustrates:

$$
\begin{equation*}
-\frac{1}{c}\left(\frac{\mu}{\mu+\lambda}\right)^{n} \geq V_{n}\left(\phi_{0}, \phi_{1}\right)-V\left(\phi_{0}, \phi_{1}\right) \geq 0 \tag{6.4}
\end{equation*}
$$

Proof:

$$
\begin{equation*}
\mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\int_{0}^{\tau} e^{-\lambda t} g\left(\tilde{\Phi}_{t}\right) d t\right]=\mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\int_{0}^{\tau \wedge \sigma_{n}} e^{-\lambda t} g\left(\tilde{\Phi}_{t}\right) d t\right]+\mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[1_{\left\{\tau \geq \sigma_{n}\right\}} \int_{\sigma_{n}}^{\tau} e^{-\lambda t} g\left(\tilde{\Phi}_{t}\right) d t\right] \tag{6.5}
\end{equation*}
$$

The first term on the right-hand-side of (6.5) is greater than $V_{n}$. Since $g(\cdot, \cdot)>-\lambda / c$ we can show that the second term is greater than

$$
\begin{equation*}
-\frac{\lambda}{c} \mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[1_{\left\{\tau \geq \sigma_{n}\right\}} \int_{\sigma_{n}}^{\tau} e^{-\lambda s} d s\right] \geq-\frac{1}{c} \mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[e^{-\lambda \sigma_{n}}\right] \geq-\frac{1}{c}\left(\frac{\mu}{\lambda+\mu}\right)^{n} \tag{6.6}
\end{equation*}
$$

To show the last inequality we have used the fact that $\sigma_{n}$ is a sum of $n$ independent and identically distributed exponential random variables with rate $\mu$ (i.e. $\sigma_{n}$ has the Erlang distribution).

Next, we will show that $V_{n}$ can be determined using an iterative algorithm. To this end we introduce the following operators acting on bounded Borel functions $f: \mathbb{R}_{+}^{2} \rightarrow \mathbb{R}$

$$
\begin{align*}
& J f\left(t, \phi_{0}, \phi_{1}\right) \triangleq \mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\int_{0}^{t \wedge \sigma_{1}} e^{-\lambda s} g\left(\widetilde{\Phi}_{s}^{(0)}, \widetilde{\Phi}_{s}^{(1)}\right) d s+1_{\left\{t \geq \sigma_{1}\right\}} e^{-\lambda \sigma_{1}} f\left(\widetilde{\Phi}_{\sigma_{1}}^{(0)}, \widetilde{\Phi}_{\sigma_{1}}^{(1)}\right)\right], t \in[0, \infty]  \tag{6.7}\\
& J_{t} f\left(\phi_{0}, \phi_{1}\right) \triangleq \inf _{s \in[t, \infty]} J f\left(s, \phi_{0}, \phi_{1}\right), \quad t \in[0, \infty] \tag{6.8}
\end{align*}
$$

Recall that under $\mathbb{P}_{0}, \sigma_{1}$ (the first time an observation arrives) has the exponential distribution with rate $\mu$. Using Fubini's theorem we can write (6.7) as

$$
\begin{equation*}
J f\left(t, \phi_{0}, \phi_{1}\right)=\int_{0}^{t} e^{-(\lambda+\mu) s}(g+\mu \cdot S f)\left(x\left(s, \phi_{0}\right), y\left(s, \phi_{1}\right)\right) d s, \quad t \in[0, \infty] \tag{6.9}
\end{equation*}
$$

in which $x$ and $y$ are the functions defined in (4.8) and $S$ is the linear operator

$$
\begin{equation*}
S f\left(\phi_{0}, \phi_{1}\right)=\int_{\mathbb{R}^{d}} f\left(\left(1+\frac{1}{\mu}\right) r(y) \phi_{0},\left(1+\frac{2}{\mu}\right) r(y) \phi_{1}\right) \beta_{0}(d y) \tag{6.10}
\end{equation*}
$$

Below we list a few useful properties of the operator $J_{0}$.

Lemma 6.1 For every bounded Borel function $f: \mathbb{R}_{+}^{2} \rightarrow \mathbb{R}$, the mapping $J_{0} f$ is bounded. If $f$ is a concave function, then $J_{0} f$ is also a concave function. If $f_{1} \leq f_{2}$ are real value bounded Borel functions, then $J_{0} f_{1} \leq J_{0} f_{2}$. That is, the operator $J_{0}$ preserves boundedness, concavity and ordering.

Proof: Let us define $\|f\| \triangleq \sup _{\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2}}\left|f\left(\phi_{0}, \phi_{1}\right)\right|<\infty$. Since $g(\cdot) \geq g(0,0)=\lambda / c$ and $\|S(f)\| \leq\|f\|$ we can write (6.9) as

$$
J f\left(t, \phi_{0}, \phi_{1}\right) \geq-\left(\frac{\lambda}{c}+\mu\|f\|\right) \int_{0}^{\infty} e^{-(\lambda+\mu) s} d s=-\left(\frac{\lambda}{c}+\mu\|f\|\right) \frac{1}{\lambda+\mu}
$$

Since we also have $J_{0} f\left(\phi_{0}, \phi_{1}\right) \leq J\left(0, \phi_{0}, \phi_{1}\right)=0$, we obtain

$$
\begin{equation*}
-\left(\frac{\lambda}{c}+\mu\|f\|\right) \frac{1}{\lambda+\mu} \leq J_{0} f\left(\phi_{0}, \phi_{1}\right) \leq 0 \tag{6.11}
\end{equation*}
$$

which proves the first assertion.
The second assertion follows since $S(f)(\cdot, \cdot)$ defined in (6.10) is concave if $f$ is concave, and the functions $\phi_{0} \rightarrow x\left(t, \phi_{0}\right)$ and $\phi_{1} \rightarrow y\left(t, \phi_{1}\right)$ are linear for every $t \geq 0$. The preservation of ordering follows immediately from (6.9).

Corollary 6.1 Let us define $v_{n}: \mathbb{R}_{+}^{2} \rightarrow \mathbb{R}$ by

$$
\begin{equation*}
v_{0}=0 \quad \text { and } v_{n}=J_{0} v_{n-1} \tag{6.12}
\end{equation*}
$$

Then, for every $n \in \mathbb{N}$, $v_{n}$ is bounded and concave, and $-1 / c \leq v_{n+1} \leq v_{n} \leq 0$. Therefore $v=\lim _{n \rightarrow \infty} v_{n}$, exists, and is bounded and concave. Both $v_{n}$ and $v$ are continuous (not only in the interior of $\mathbb{R}_{+}^{2}$ ), they are increasing in each of their arguments, and their left and right partial derivatives are bounded on every compact subset of $\mathbb{R}_{+}^{2}$.

Proposition 6.2 For every $n \in \mathbb{N}$, $v_{n}$ defined in Corollary 6.1 is equal to $V_{n}$ of (6.3). For $\varepsilon>0$, let us denote

$$
\begin{equation*}
r_{n}^{\varepsilon}\left(\phi_{0}, \phi_{1}\right) \triangleq \inf \left\{t \in(0, \infty]: J v_{n}\left(t,\left(\phi_{0}, \phi_{1}\right)\right) \leq J_{0} v_{n}\left(\phi_{0}, \phi_{1}\right)+\varepsilon\right\} \tag{6.13}
\end{equation*}
$$

And let us define a sequence of stopping times by $S_{1}^{\varepsilon} \triangleq r_{0}^{\varepsilon}(\widetilde{\Phi}) \wedge \sigma_{1}$ and

$$
S_{n+1}^{\varepsilon} \triangleq \begin{cases}r_{n}^{\varepsilon / 2}(\widetilde{\Phi}) & \text { if } \sigma_{1} \geq r_{n}^{\varepsilon / 2}(\widetilde{\Phi})  \tag{6.14}\\ \sigma_{1}+S_{n}^{\varepsilon / 2} \circ \theta_{\sigma_{1}} & \text { otherwise. }\end{cases}
$$

Here $\theta_{s}$ is the shift operator on $\Omega$, i.e., $X_{t} \circ \theta_{s}=X_{s+t}$. Then $S_{n}^{\varepsilon}$ is $\varepsilon$ optimal, i.e.,

$$
\begin{equation*}
\mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\int_{0}^{S_{n}^{\varepsilon}} e^{-\lambda t} g\left(\widetilde{\Phi}_{t}\right) d t\right] \leq v_{n}\left(\phi_{0}, \phi_{1}\right)+\varepsilon . \tag{6.15}
\end{equation*}
$$

### 6.2 Optimal Stopping Time

Proposition $6.3 \tau^{*}$ defined in (5.1) the smallest optimal stopping time for (4.12).

We will divide the proof of this theorem into several lemmas. The following lemma shows that if there exists an optimal stopping time it is necessarily greater than or equal to $\tau^{*}$.

## Lemma 6.2

$$
\begin{equation*}
V\left(\phi_{0}, \phi_{1}\right)=\inf _{\tau \geq \tau^{*}} \mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\int_{0}^{\tau} e^{-\lambda s} g\left(\widetilde{\Phi}_{s}\right) d s\right] \tag{6.16}
\end{equation*}
$$

Proposition 6.4 We have $v\left(\phi_{0}, \phi_{1}\right)=V\left(\phi_{0}, \phi_{1}\right)$ for every $\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2}$. Moreover, $V$ is the largest nonpositive solution $U$ of the equation $U=J_{0} U$.

As an immediate corollary to Propositions 6.1 and 6.4 and Propositions 5.155 .4 which construct bounds on the optimal stopping region, we can state the following:

Corollary 6.2 Let us define the optimal stopping regions

$$
\begin{equation*}
\boldsymbol{\Gamma}_{n} \triangleq\left\{\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2}: v_{n}\left(\phi_{0}, \phi_{1}\right)=0\right\}, \quad \mathbf{C}_{n} \triangleq \mathbb{R}_{+}^{2} \backslash \boldsymbol{\Gamma}_{n}, \quad n \in \mathbb{N} \tag{6.17}
\end{equation*}
$$

and recall that

$$
\begin{equation*}
\boldsymbol{\Gamma}=\left\{\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2}: v\left(\phi_{0}, \phi_{1}\right)=0\right\}, \quad \mathbf{C} \triangleq \mathbb{R}_{+}^{2} \backslash \boldsymbol{\Gamma} . \tag{6.18}
\end{equation*}
$$

There are decreasing, convex and continuous mappings $\gamma_{n}: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}, n \in \mathbb{N}$, and $\gamma: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$ such that

$$
\begin{equation*}
\Gamma_{n}=\left\{\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}: \phi_{1} \geq \gamma_{n}\left(\phi_{0}\right)\right\}, \in \mathbb{N} \quad \text { and } \quad \Gamma=\left\{\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}: \phi_{1} \geq \gamma\left(\phi_{0}\right)\right\} . \tag{6.19}
\end{equation*}
$$

The sequence $\left\{\gamma_{n}\left(\phi_{0}\right)\right\}_{n \in \mathbb{N}}$ is increasing and $\gamma\left(\phi_{0}\right)=\lim \uparrow \gamma_{n}\left(\phi_{0}\right)$ for every $\phi_{0} \in \mathbb{R}_{+}$. If there are paths $t \rightarrow\left(x\left(t, \phi_{0}\right), y\left(t, \phi_{1}\right)\right), t \geq 0,\left(\phi_{0}, \phi_{1}\right) \in \mathbb{C}_{0}$, that exit $\mathbb{C}_{0}$, then there exists $\xi \in[0, \lambda / c)$ (the value of $\xi$ depends on the parameter values) such that $\gamma_{n}\left(\phi_{0}\right)=\gamma\left(\phi_{0}\right)=\lambda / c-\phi_{0}$ for $\phi_{0} \geq \xi$, i.e., the free boundary coincides with the boundary of the region $\mathbb{C}_{0}$ defined in (5.8). In fact if (i) $\lambda \geq 2$, or, (ii) $\lambda \in[1,2)$ and $c \geq 2-\lambda$, or, (iii) $\lambda \in(0,1)$ and $c \geq \max (2-\lambda, 1-\lambda)$ then $\xi=0$. If (iv) $\lambda \in[1,2)$ and $c \in(0,2-\lambda)$, (v) $\lambda \in(0,1)$ and $(2-\lambda)(1-\lambda) /(3-\lambda-m)<$ $c<\max (2-\lambda, 1-\lambda)$, then $\xi=\lambda(-1+(2-\lambda) / c)$. If on the other hand, $\lambda \in(0,1)$ and $c \geq 2(1-\lambda) /(3-m)<c \leq(2-\lambda)(1-\lambda) /(3-\lambda-m)$, then $\xi=(2-\lambda) / c+m-3$.

Lemma 6.3 Let $f: \mathbb{R}_{+}^{2} \mapsto \mathbb{R}$ be a bounded function. For every $t \in \mathbb{R}_{+}$and $\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2}$,

$$
\begin{equation*}
\left.J_{t} f\left(\phi_{0}, \phi_{1}\right)=J f\left(t,\left(\phi_{0}, \phi_{1}\right)\right)+e^{-(\lambda+\mu) t} J_{0} f\left(x\left(t, \phi_{0}\right), y\left(t, \phi_{1}\right)\right)\right) . \tag{6.20}
\end{equation*}
$$

Remark 6.1 Since $V$ is bounded, and $V=J_{0} V$ by Proposition 6.4, we have

$$
\begin{equation*}
\left.J_{t} V\left(\phi_{0}, \phi_{1}\right)=J V\left(t,\left(\phi_{0}, \phi_{1}\right)\right)+e^{-(\lambda+\mu) t} V\left(x\left(t, \phi_{0}\right), y\left(t, \phi_{1}\right)\right)\right), \quad t \in \mathbb{R}_{+} \tag{6.21}
\end{equation*}
$$

for every $\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2}$.
Let us define the $\mathbb{F}$-stopping times

$$
\begin{equation*}
U_{\varepsilon} \triangleq \inf \left\{t \geq 0: V\left(\widetilde{\Phi}_{t}\right) \geq-\varepsilon\right\}, \quad \varepsilon \geq 0 \tag{6.22}
\end{equation*}
$$

By Remark 6.1, we have

$$
\begin{equation*}
V\left(\widetilde{\Phi}_{U_{\varepsilon}}\right) \geq-\varepsilon \quad \text { on the event } \quad\left\{U_{\varepsilon}<\infty\right\} . \tag{6.23}
\end{equation*}
$$

Proposition 6.5 Let $M_{t} \triangleq e^{-\lambda t} V\left(\widetilde{\Phi}_{t}\right)+\int_{0}^{t} e^{-\lambda s} g\left(\widetilde{\Phi}_{s}\right) d s, t \geq 0$. For every $n \in \mathbb{N}, \varepsilon \geq 0$, and $\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2}$, we have $\mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[M_{0}\right]=\mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[M_{U_{\varepsilon} \wedge \sigma_{n}}\right]$, i.e.,

$$
\begin{equation*}
V\left(\phi_{0}, \phi_{1}\right)=\mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[e^{-\lambda\left(U_{\varepsilon} \wedge \sigma_{n}\right)} V\left(\widetilde{\Phi}_{U_{\varepsilon} \wedge \sigma_{n}}\right)+\int_{0}^{U_{\varepsilon} \wedge \sigma_{n}} e^{-\lambda s} g\left(\widetilde{\Phi}_{s}\right) d s\right] . \tag{6.24}
\end{equation*}
$$

Proof of Proposition 6.3 First we will show that $\tau^{*}$ is an optimal stopping time. It is enough to show that for every $\varepsilon \geq 0$, the stopping time $U_{\varepsilon}$ in (6.22) is an $\varepsilon$-optimal stopping time for the optimal stopping problem (4.12), i.e.,

$$
\mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\int_{0}^{U_{\varepsilon}} e^{-\lambda s} g\left(\widetilde{\Phi}_{s}\right) d s\right] \leq V\left(\phi_{0}, \phi_{1}\right)+\varepsilon, \quad \text { for every } \quad\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2} .
$$

Note that the sequence of random variables

$$
\int_{0}^{U_{\varepsilon} \wedge \sigma_{n}} e^{-\lambda s} g\left(\widetilde{\Phi}_{s}\right) d s+e^{-\lambda\left(U_{\varepsilon} \wedge \sigma_{n}\right)} V\left(\widetilde{\Phi}_{U_{\varepsilon} \wedge \sigma_{n}}\right) \geq-\int_{0}^{\infty} e^{-\lambda s} \frac{\lambda}{c} d s-\frac{1}{c}=-\frac{2}{c}
$$

is bounded from below. By (6.24) and Fatou's Lemma, we have

$$
\begin{aligned}
V\left(\phi_{0}, \phi_{1}\right) & =\liminf _{n \rightarrow \infty} \mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\int_{0}^{U_{\varepsilon} \wedge \sigma_{n}} e^{-\lambda s} g\left(\widetilde{\Phi}_{s}\right) d s+e^{-\lambda\left(U_{\varepsilon} \wedge \sigma_{n}\right)} V\left(\widetilde{\Phi}_{U_{\varepsilon} \wedge \sigma_{n}}\right)\right] \\
& \geq \mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\liminf _{n \rightarrow \infty}\left(\int_{0}^{U_{\varepsilon} \wedge \sigma_{n}} e^{-\lambda s} g\left(\widetilde{\Phi}_{s}\right) d s+e^{-\lambda\left(U_{\varepsilon} \wedge \sigma_{n}\right)} V\left(\widetilde{\Phi}_{U_{\varepsilon} \wedge \sigma_{n}}\right)\right)\right] \\
& =\mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\int_{0}^{U_{\varepsilon}} e^{-\lambda s} g\left(\widetilde{\Phi}_{s}\right) d s+1_{\left\{U_{\varepsilon}<\infty\right\}} e^{-\lambda U_{\varepsilon}} V\left(\widetilde{\Phi}_{U_{\varepsilon}}\right)\right] \\
& \geq \mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\int_{0}^{U_{\varepsilon}} e^{-\lambda s} g\left(\widetilde{\Phi}_{s}\right) d s\right]-\varepsilon \mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[1_{\left\{U_{\varepsilon}<\infty\right\}} e^{-\lambda U_{\varepsilon}}\right] \quad \text { by (6.23) } \\
& \geq \mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\int_{0}^{U_{\varepsilon}} e^{-\lambda s} g\left(\widetilde{\Phi}_{s}\right) d s\right]-\varepsilon
\end{aligned}
$$

for every $\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2}$. This shows that $U_{\varepsilon}$ is an $\varepsilon$-optimal stopping time.
Now we will show that $\tau^{*}$ is the smallest optimal stopping time. Let us define

$$
\tilde{\tau} \triangleq \begin{cases}\tau, & \text { if } \tau \geq \tau^{*},  \tag{6.25}\\ \tau+\tau^{*} \circ \theta_{\tau}, & \text { if } \tau<\tau^{*} .\end{cases}
$$

Then the stopping time $\tilde{\tau}$ satisfies

$$
\begin{align*}
\mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\int_{0}^{\tilde{\tau}} e^{-\lambda s} g\left(\widetilde{\Phi}_{s}\right) d s\right] & =\mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\int_{0}^{\tau} e^{-\lambda s} g\left(\widetilde{\Phi}_{s}\right) d s+\int_{\tau}^{\tilde{\tau}} e^{-\lambda s} g\left(\widetilde{\Phi}_{s}\right) d s\right] \\
& =\mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\int_{0}^{\tau} e^{-\lambda s} g\left(\widetilde{\Phi}_{s}\right) d s+e^{-\lambda \tau} \int_{0}^{\tau^{*} \circ \theta_{\tau}} e^{-\lambda s} g\left(\widetilde{\Phi}_{s+\tau}\right) d s\right]  \tag{6.26}\\
& =\mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\int_{0}^{\tau} e^{-\lambda s} g\left(\widetilde{\Phi}_{s}\right) d s+e^{-\lambda \tau} V\left(\widetilde{\Phi}_{\tau}\right)\right] \\
& \leq \mathbb{E}_{0}^{\phi_{0}, \phi_{1}}\left[\int_{0}^{\tau} e^{-\lambda s} g\left(\widetilde{\Phi}_{s}\right) d s\right] .
\end{align*}
$$

Here the third equality follows from the strong Markov property of the process $\widetilde{\Phi}$. Now the proof immediately follows.

### 6.3 Structure of the Optimal Stopping Times

Finally, let us describe here the structure of the optimal stopping times. For this purpose we will need the following lemma.

## Lemma 6.4 Let

$$
\begin{equation*}
r_{n}\left(\phi_{0}, \phi_{1}\right)=\inf \left\{s \in(0, \infty]: J v_{n}\left(s,\left(\phi_{0}, \phi_{1}\right)\right)=J_{0} v_{n}\left(\phi_{0}, \phi_{1}\right)\right\} \tag{6.27}
\end{equation*}
$$

be the same as $r_{n}^{\varepsilon}\left(\phi_{0}, \phi_{1}\right)$ in Proposition 6.2 with $\varepsilon=0$. Then

$$
\begin{equation*}
r_{n}\left(\phi_{0}, \phi_{1}\right)=\inf \left\{t>0: v_{n+1}\left(x\left(t, \phi_{0}\right), y\left(t, \phi_{1}\right)\right)=0\right\} \quad(\inf \emptyset \equiv \infty) \tag{6.28}
\end{equation*}
$$

Proof: Let us fix $\left(\phi_{0}, \phi_{1}\right) \in \mathbb{R}_{+}^{2}$, and denote $r_{n}\left(\phi_{0}, \phi_{1}\right)$ by $r_{n}$. We have $J v_{n}\left(r_{n},\left(\phi_{0}, \phi_{1}\right)\right)=$ $J_{0} v_{n}\left(\phi_{0}, \phi_{1}\right)=J_{r_{n}} v_{n}\left(\phi_{0}, \phi_{1}\right)$.

Suppose first that $r_{n}<\infty$. Since $J_{0} v_{n}=v_{n+1}$, taking $t=r_{n}$ and $w=v_{n}$ in (6.20) gives

$$
J v_{n}\left(r_{n},\left(\phi_{0}, \phi_{1}\right)\right)=J_{r_{n}} v_{n}\left(\phi_{0}, \phi_{1}\right)=J v_{n}\left(r_{n},\left(\phi_{0}, \phi_{1}\right)\right)+e^{-(\lambda+\mu) r_{n}} v_{n+1}\left(x\left(r_{n}, \phi_{0}\right), y\left(r_{n}, \phi_{1}\right)\right) .
$$

Therefore, $v_{n+1}\left(x\left(r_{n}, \phi_{0}\right), y\left(r_{n}, \phi_{1}\right)\right)=0$.
If $0<t<r_{n}$, then $J v_{n}\left(t,\left(\phi_{0}, \phi_{1}\right)\right)>J_{0} v_{n}\left(\phi_{0}, \phi_{1}\right)=J_{r_{n}} v_{n}\left(\phi_{0}, \phi_{1}\right)=J_{t} v_{n}\left(\phi_{0}, \phi_{1}\right)$ since $u \mapsto J_{u} v_{n}\left(\phi_{0}, \phi_{1}\right)$ is nondecreasing. Taking $t \in\left(0, r_{n}\right)$ and $w=v_{n}$ in (6.20) imply

$$
J_{0} v_{n}\left(\phi_{0}, \phi_{1}\right)=J_{t} v_{n}\left(\phi_{0}, \phi_{1}\right)=J v_{n}\left(t,\left(\phi_{0}, \phi_{1}\right)\right)+e^{-(\lambda+\mu) t} v_{n+1}\left(x\left(t, \phi_{0}\right), y\left(t, \phi_{1}\right)\right) .
$$

Therefore, $v_{n+1}\left(x\left(t, \phi_{0}\right), y\left(t, \phi_{1}\right)\right)<0$ for every $t \in\left(0, r_{n}\right)$, and (6.28) follows.

Suppose now that $r_{n}=\infty$. Then we have $v_{n+1}\left(x\left(t, \phi_{0}\right), y\left(t, \phi_{1}\right)\right)<0$ for every $t \in(0, \infty)$ by the same argument in the last paragraph above. Hence, $\left\{t>0: v_{n+1}\left(x\left(t, \phi_{0}\right), y\left(t, \phi_{1}\right)\right)=0\right\}=\emptyset$, and (6.28) still holds.

By Proposition 6.3, the set $\boldsymbol{\Gamma}$ is the optimal stopping region for the optimal stopping problem (4.12). Namely, stopping at the first hitting time $U_{0}=\inf \left\{t \in \mathbb{R}_{+}: \widetilde{\Phi}_{t} \in \boldsymbol{\Gamma}\right\}$ of the process $\widetilde{\Phi}=\left(\widetilde{\Phi}^{(0)}, \widetilde{\Phi}^{(1)}\right)$ to the set $\boldsymbol{\Gamma}$ is optimal for (4.12).

Similarly, we shall call each set $\boldsymbol{\Gamma}_{n}, n \in \mathbb{N}$ a stopping region for the family of the optimal stopping problems in (6.3). However, unlike the case above, we need the first $n$ stopping regions, $\boldsymbol{\Gamma}_{1}, \ldots, \boldsymbol{\Gamma}_{n}$, in order to describe an optimal stopping time for the optimal stopping problem in (6.3) (the optimal stopping times are not hitting times of a certain set). Using Corollary 6.4, the optimal stopping time $S_{n} \equiv S_{n}^{0}$ in Proposition 6.2 for $V_{n}$ of (6.3) may be described as follows: Stop if the process $\widetilde{\Phi}$ hits $\boldsymbol{\Gamma}_{n}$ before $X$ jumps. If $X$ jumps before $\widetilde{\Phi}$ reaches $\boldsymbol{\Gamma}_{n}$, then wait, and stop if $\widetilde{\Phi}$ hits $\boldsymbol{\Gamma}_{n-1}$ before the next jump of $X$, and so on. If the rule is not met before ( $n-1$ )st jump of $X$, then stop at the earliest of the hitting time of $\boldsymbol{\Gamma}_{1}$ and the next jump time of $X$.

## 7 Conclusion

We have solved a change detection problem for a compound Poisson process in which the intensity and the jump size change at the same time but the intensity changes to a random variable with a known distribution. This problem becomes an optimal stopping problem for a Markovian sufficient statistic. We have analyzed a special case of this problem, in which the rate of the arrivals moves up to one of two possible values, and the Markovian sufficient statistic is twodimensional, in more detail. We have shown that the intuition that a decision would sound the alarm only at the times when it observes an arrival does not in general hold, see Remark 5.1. This intuition becomes relevant only when the disorder intensity and delay penalty are small. Performing a sample path analysis we have been able to find the optimal stopping time exactly for most of the range of parameters, and tight upper and lower bounds for the rest of the parameter range. This work has applications in insurance risk, in which the subject Poisson process can be viewed as the claim arrivals process for an insurance company.

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