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ORIGINAL ARTICLE

# Portfolio-optimization models for small investors

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**Abstract** Since 2010, the client base of online-trading service providers has grown significantly. Such companies enable small investors to access the stock market at advantageous rates. Because small investors buy and sell stocks in moderate amounts, they should consider fixed transaction costs, integral transaction units, and dividends when selecting their portfolio. In this paper, we consider the small investor's problem of investing capital in stocks in a way that maximizes the expected portfolio return and guarantees that the portfolio risk does not exceed a prescribed risk level. Portfolio-optimization models known from the literature are in general designed for institutional investors and do not consider the specific constraints of small investors. We therefore extend four well-known portfolio-optimization models to make them applicable for small investors. We consider one nonlinear model that uses variance as a risk measure and three linear models that use the mean absolute deviation from the portfolio return, the maximum loss, and the conditional value-at-risk as risk measures. We extend all models to consider piecewise-constant transaction costs, integral transaction units, and dividends. In an out-of-sample experiment based on Swiss stock-market data and the cost structure of the online-trading service provider Swissquote, we apply both the basic models and the extended models; the former represent the perspective of an institutional investor, and the latter the perspective of a small investor. The basic models compute portfolios that yield on average a slightly higher return than the portfolios computed with the extended models. However, all generated portfolios yield on average a higher return than the Swiss performance index. There are considerable

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differences between the four risk measures with respect to the mean realized portfolio return and the standard deviation of the realized portfolio return.

**Keywords** Portfolio optimization · Transaction costs · Integral transaction units · Experimental performance analysis

## 1 Introduction

The basic portfolio-optimization problem consists of investing some capital in stocks in such a way that the expected portfolio return is maximized and the portfolio risk does not exceed a prescribed risk level. A well-known formulation of this problem as a quadratic optimization problem has been proposed by Markowitz (1952). Using alternative risk measures, Konno and Yamazaki (1991), Young (1998) and Rockafellar and Uryasev (2000) have formulated this problem as a linear optimization problem.

In recent years, the number of individuals who purchase assets for their own account has been increasing considerably. When constructing their portfolio, such individuals, whom we should call small investors, face certain constraints that institutional investors may neglect. These constraints are piecewise constant transaction costs, integral transactions units, maximum number of different stocks, maximum weight of individual stocks, and dividends. In this paper, we integrate these constraints efficiently into the above-mentioned models for portfolio optimization and discuss how the extended models differ with respect to the CPU times required and the portfolios obtained.

In the literature, it has been discussed how the above-mentioned small-investor specific constraints can be included individually in some of these basic portfolio-selection models. Speranza (1996) extends a modified version of the model of Konno and Yamazaki (1991) to account for different types of transaction costs, integral transaction units and limits on the maximum number of different stocks. Young (1998) considers piecewise-constant transaction costs and an upper bound on the number of different stocks. Konno and Wijayanayake (2001) are concerned with the model of Konno and Yamazaki (1991) under concave transaction costs and minimal transaction units and propose a model-specific branch-and-bound algorithm that involves a rounding procedure. Mansini and Speranza (2005) present an exact solution algorithm for portfolio optimization with fixed transaction costs and integral transaction units. For the experimental analysis, they use the model of Konno and Yamazaki (1991). Konno and Yamamoto (2005) show that the portfolio-optimization problem with piecewise-linear or piecewise-constant transaction costs, minimal transaction units, and constraints on the number of different stocks can be solved within reasonable CPU time when using the model of Konno and Yamazaki (1991). Bonami and Lejeune (2009) extend the model of Markowitz (1952) to consider stochastic asset returns, integral transaction units, minimum investments, and diversification requirements; for the solution of the resulting nonlinear optimization problem, a branch-and-bound algorithm and two specific branching rules are proposed.

In this paper, we extend each of the basic portfolio-optimization models such that all of the above-mentioned constraints are considered simultaneously. In a computational experiment we investigate how the small-investor specific constraints influence the

CPU time requirements and the performance of the resulting portfolios. For this purpose we applied each basic model with and without small-investor specific constraints to 19 datasets that we generated using the stock notations of the Swiss Stock Exchange in four-year intervals from 1988 to 2009. Surprisingly, we found that all extended models can be (approximately) solved to optimality within a reasonable amount of CPU time. For the evaluation of the portfolio performance, we applied an out-of-sample approach. Our analysis demonstrates considerable differences between the four models with respect to the mean realized portfolio return, the standard deviation of the realized portfolio return, and the development of the portfolio value. Furthermore, we found that the small-investor specific constraints reduce the performance of the portfolios only to a small extent.

The remainder of this paper is structured as follows. In Sect. 2, we recapitulate the four basic portfolio-selection models using a common notation. In Sect. 3, we present the extension of the basic models by the small-investor specific constraints. In Sect. 4, we report on the design and the results of our computational experiment. In Sect. 5, we provide some concluding remarks and directions for future research.

## 2 Basic portfolio-optimization models

In this section, we briefly present the four basic portfolio-optimization models. The objective of each model is to maximize the expected return of the portfolio such that the portfolio risk does not exceed the risk level prescribed by the investor (risk constraint) and that the entire budget is invested (budget constraint). We do not account for short sales; thus, the stock weights need to be non-negative.

First, we present the mean-variance (MV) model (cf. Markowitz 1952). The portfolio risk is measured by the variance of the expected portfolio return. In general, the required covariances between the returns of the individual stocks are estimated using historical data. The MV model reads as follows.

| | | |
|---|---|---|
| | $n$ | number of stocks |
| | $\sigma_{ij}$ | covariance between return of stock $i$ and return of stock $j$ |
| | $\overline{r}_i$ | expected return of stock $i$ |
| | $\varrho$ | maximal risk level |
| * | $x_i$ | weight of stock $i$ in portfolio |

$$
\text{(MV)} \begin{cases} \text{Max. } \sum_{i=1}^{n} \overline{r}_i x_i \\ \text{s.t. } \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j \leq \varrho & (1) \\ \qquad \sum_{i=1}^{n} x_i = 1 & (2) \\ \qquad x_i \geq 0 \quad (i = 1, \ldots, n) \end{cases}
$$

Constraint (1) represents the risk constraint, and constraint (2) represents the budget constraint.

Second, we refer to the mean-absolute-deviation (MAD) model proposed by Konno and Yamazaki (1991). The portfolio risk is measured by the mean of the absolute deviations of the portfolio return from the expected portfolio return in all periods. The portfolio return is devised from the corresponding historical returns of the stocks. To obtain a linear model, the deviation in period $t$ is computed as the sum of two non-negative variables $\phi_t$ and $\psi_t$, which correspond to the positive and the negative deviations, respectively (cf. constraint (3)). The MAD model reads as follows.

| | | |
|---|---|---|
| | $T$ | number of periods |
| | $r_{it}$ | return of stock $i$ in period $t$ |
| * | $\phi_t$ | positive deviation of portfolio return |
| * | $\psi_t$ | negative deviation of portfolio return |

$$
(\mathrm{MAD}) \left\{
\begin{array}{lll}
\text{Max.} & \sum_{i=1}^{n} \bar{r}_i x_i & \\
\text{s.t.} & \phi_t - \psi_t = \frac{1}{T} \sum_{i=1}^{n} (r_{it} - \bar{r}_i) x_i \quad (t = 1, \ldots, T) & (3) \\
& \sum_{t=1}^{T} (\phi_t + \psi_t) \leq \varrho & (4) \\
& \sum_{i=1}^{n} x_i = 1 & (5) \\
& x_i \geq 0 \quad (i = 1, \ldots, n) & \\
& \phi_t, \psi_t \geq 0 \quad (t = 1, \ldots, T) &
\end{array}
\right.
$$

Constraint (4) represents the risk constraint, and constraint (5) represents the budget constraint.

Third, we consider the minimax (MM) model developed by Young (1998). The portfolio risk is measured by the minimum portfolio return over all periods. Thus, in contrast with the models presented by Markowitz (1952) and Konno and Yamazaki (1991), the risk measure is asymmetric; Young (1998) argues that such an asymmetric risk measure is more appropriate for skewed return distributions. For an extensive discussion of such risk measures, we refer to Lüthi and Studer (1997). Constraint (6) ensures that the portfolio return does not fall below the minimum portfolio return $\varrho$ in any period. The MM model reads as follows.

$$
(\mathrm{MM}) \left\{
\begin{array}{lll}
\text{Max.} & \sum_{i=1}^{n} \bar{r}_i x_i & \\
\text{s.t} & \sum_{i=1}^{n} x_i r_{it} \geq \varrho \quad (t = 1, \ldots, T) & (6) \\
& \sum_{i=1}^{n} x_i = 1 & (7) \\
& x_i \geq 0 \quad (i = 1, \ldots, n) &
\end{array}
\right.
$$

Constraint (6) represents the risk constraint, and constraint (7) represents the budget constraint.

Fourth, we turn to the conditional value-at-risk (CVaR) model proposed by Rockafellar and Uryasev (2000). The portfolio risk is measured by the CVaR, which is based on the value-at-risk (VaR) and also represents an asymmetric risk measure. The non-linear CVaR function is generally difficult to optimize (cf. Shapiro et al. 2009).

Therefore, Rockafellar and Uryasev (2000) propose to approximate the joint density function by a number of scenarios to obtain a linear model. Here, we use historical stock returns to represent different scenarios. It is assumed that all these scenarios have the same probability. The $\beta$-VaR is the return below which the portfolio return falls at most, with a given probability $1-\beta$; let $\alpha$ be the negative of that return. The CVaR corresponds to the mean of all portfolio returns below $-\alpha$. The CVaR model reads as follows.

| | | |
|---|---|---|
| | $\beta$ | probability level |
| $*$ | $\alpha$ | $-\beta$-VaR |
| $*$ | $u_t$ | shortfall of portfolio return with respect to $\beta$-VaR |

$$\text{(CVaR)} \begin{cases} \text{Max. } \sum_{i=1}^{n} \bar{r}_i x_i \\ \text{s.t. } \sum_{i=1}^{n} r_{it} x_i + u_t + \alpha \geq 0 \quad (t = 1, \ldots, T) & (8) \\ \alpha + \dfrac{1}{T(1-\beta)} \sum_{t=1}^{T} u_t \leq \varrho & (9) \\ \sum_{i=1}^{n} x_i = 1 & (10) \\ x_i \geq 0 \quad (i = 1, \ldots, n) \\ u_t \geq 0 \quad (t = 1, \ldots, T) \\ \alpha \geq 0 \end{cases}$$

Constraint (8) enforces for each period $t$ in which the portfolio return falls below $-\alpha$ that $u_t$ is at least equal to this shortfall. Constraint (9) represents the risk constraint: when sorting the periods by the portfolio return, the portfolio risk is equal to the mean portfolio return in the $1-\beta$ worst periods. Constraint (10) represents the budget constraint.

## 3 Model extensions

In this section, we extend the four basic portfolio-optimization models to account for integral transaction units, a maximum weight of individual stocks, piecewise constant transaction costs, a maximum number of different stocks, and dividends. To formulate some of these constraints, it is necessary to use binary or integer variables. In general, such variables strongly increase the CPU time required for solving the resulting optimization problem; thus, an efficient formulation is of particular interest. Mansini and Speranza (1999) show that, when integral transaction units are taken into account, the problem of finding a feasible solution is, independently of the risk measure, NP-complete.

In a real-world portfolio, the number of units $z_i$ of any stock $i$ must be integral, i.e.,

$$z_i \in \mathbb{Z}_{\geq 0} \quad (i = 1, \ldots, n) \tag{11}$$

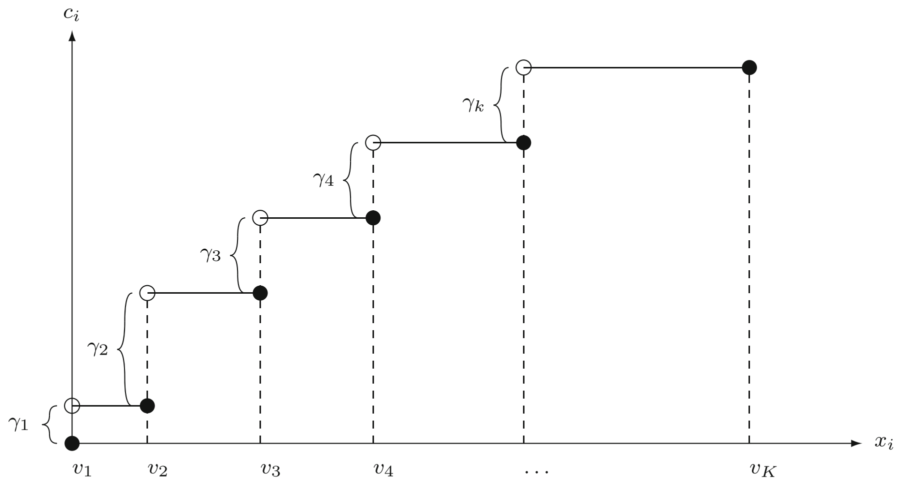


**Fig. 1** Piecewise constant transaction costs

With $P_i$ denoting the price of stock $i$ at the time of purchase and $B$ denoting the budget, let $\lambda_i := \frac{P_i}{B}$. Then, the weight of stock $i$ is defined by

$$x_i = \lambda_i z_i \quad (i = 1, \ldots, n) \tag{12}$$

Because of constraint (12), it may not be possible to meet the budget constraint $\sum_{i=1}^{n} x_i = 1$. Therefore, we replace the budget constraint by

$$1 - \delta \leq \sum_{i=1}^{n} x_i \leq 1 + \delta \tag{13}$$

with $\delta > 0$ denoting a small positive constant.

Let $\tau$ denote the maximum weight of stock $i$, i.e.,

$$x_i \leq \tau \quad (i = 1, \ldots, n) \tag{14}$$

This bound implicitly sets up a lower bound on the number of stocks in the portfolio.

Brokers charge transaction costs when their clients buy and sell stocks. In general, these costs are not proportional to the total price of the stocks. For our analysis, we model a piecewise-constant cost structure, which is applied by several banks. The computation of the transaction cost is illustrated in Fig. 1. For example, if $x_i \in ]v_2, v_3]$, then a transaction cost of $\gamma_1 + \gamma_2$ is incurred. Hence, we introduce $K$ binary variables $y_{ik}$ for each stock $i$. If the fraction of the total budget invested in stock $i$ exceeds $v_k$, then $y_{ik} = 1$:

$$y_{ik} \geq \frac{x_i - v_k}{\tau} \quad (k = 1, \ldots, K;\ i = 1, \ldots, n) \tag{15}$$

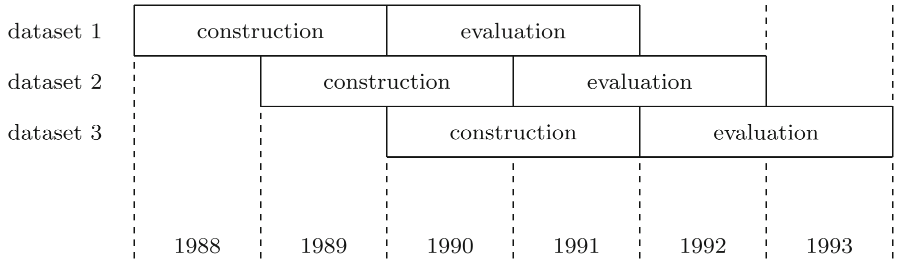


**Fig. 2** Datasets 1, 2 and 3

Thus, the transaction costs for stock $i$ are

$$c_i = \sum_{k=1}^{K} \gamma_k y_{ik} \quad (i = 1, \ldots, n) \tag{16}$$

To limit the time needed for the implementation and supervision of the portfolio, small investors generally prefer portfolios with a limited number of stocks. With $p$ denoting the corresponding upper bound, we write this constraint as

$$\sum_{i=1}^{n} y_{i1} \leq p \tag{17}$$

Usually, small investors do not reinvest dividends because of the relatively high transaction costs. Therefore, the dividends $D_i$ must be treated explicitly in the optimization model. The same holds for any transaction costs that are proportional to the invested amounts, e.g. stamp duties $s$ or stock exchange fees $b$. Using the notation introduced above, the modified objective function reads

$$\sum_{i=1}^{n} [(\overline{r}_i - s - b)x_i + D_i z_i - c_i]$$

Note that the parameters $s$, $b$ and $\gamma_k$ must be adjusted to the budget and the investment horizon; moreover, the parameter $D_i$ must be adjusted to the price of stock $i$ and the investment horizon.

## 4 Computational analysis

In this section, we describe the design of the experimental analysis (cf. Sect. 4.1) and report the numerical results (cf. Sect. 4.2).

### 4.1 Experimental design

The aim of the experimental analysis is twofold. First, we want to evaluate the practical performance of the extended models in a real-world situation of a small investor. In

**Table 1** Portfolio-optimization models

| Model | Constraints |
|---|---|
| MV | (1), (2) |
| MAD | (3)–(5) |
| MM | (6), (7) |
| CVaR | (8)–(10) |
| MV$^+$ | (1), (11)–(17) |
| MAD$^+$ | (3), (4), (11)–(17) |
| MM$^+$ | (6), (11)–(17) |
| CVaR$^+$ | (8), (9), (11)–(17) |

particular, we are interested in the CPU time required to compute the portfolios and the out-of-sample portfolio return. Second, we want to analyse to what extend the small-investor specific constraints affect the composition and performance of the portfolios. To compare the investment situation of a small investor to the investment situation of an institutional investor, we use the basic models to compute portfolios for institutional investors and evaluate their composition and out-of-sample performance.

We created 19 datasets using weekly stock returns from 1988 to 2009. Each dataset contains stock returns over four years (cf. Fig. 2). Depending on the dataset, the number of different stocks varies from 171 to 239. We used the first two years of each dataset to construct the portfolios with the basic/extended models (cf. Table 1) and the last two years to study the out-of-sample performance of the portfolios. To define the model-specific maximal risk level $\varrho$, we computed the variance (MV-model), the mean absolute deviation (MAD-model), the minimum return (Minimax-model) and the conditional value-at-risk (CVaR-model), based on the returns of the Swiss performance index (SPI) over the first two years of each dataset.

We assume that both the small investor and the institutional investor want to invest in stocks listed on the Swiss performance index (SPI) for two years. Both investors do not have any special knowledge that allows them to select undervalued stocks. Therefore, they follow a buy-and-hold strategy.

Table 2 shows the different investment settings that we consider. For the small investor's situation, we distinguish among six investment scenarios regarding the budget. The other problem parameters listed in Table 2 are the same for all scenarios. We implemented the transaction-cost structure of the online-trading service provider Swissquote. For the institutional investor's situation, we assume that the invested amount of money is large enough that broker fees and integral transaction units are negligible. Further, we do not impose a limit on the number of different stocks in the portfolio. For the sake of comparability, we adjust the expected returns for proportional transaction costs such as stamp duty and stock exchange fees, and we enforce that the maximum weight of a stock does not exceed 10 %.

In total, we apply all extended models to 114 problem instances and all basic models to 19 problem instances. The computations were performed on an HP Z600 workstation with 2 Intel Xeon X5650 CPUs and 24GB RAM using AMPL together with CPLEX 12.4.

**Table 2** Investment settings

| Parameters | Small investor | Institutional investor |
|---|---|---|
| Budget ($B$) in 1,000 CHF | 5, 10, 20, 50, 75, 100 | – |
| Model specific maximal risk level ($\varrho$) | Risk of SPI | Risk of SPI |
| Maximum weight of a stock ($\tau$) | 10 % | 10 % |
| Allowed deviation from budget ($\delta$) | 1 % | – |
| Maximum number of different stocks ($p$) | 30 | $\infty$ |
| Number of kinks in transaction-cost function ($K$) | 4 | – |
| Stamp duty | 0.075 % | 0.075 % |
| Stock exchange fee | 0.01 % | 0.01 % |
| Probability level for CVaR-model ($\beta$) | 90 % | 90 % |

**Table 3** Comparison of CPU times ($t^{CPU}$) and MIP gaps (G), when $B = 100{,}000$ CHF

| DS | $MM^+$ | | $CVaR^+_{90\,\%}$ | | $MAD^+$ | | $MV^+$ | |
|---|---|---|---|---|---|---|---|---|
| | $t^{CPU}$ (s) | G (%) | $t^{CPU}$ (s) | G (%) | $t^{CPU}$ (s) | G (%) | $t^{CPU}$ (s) | G (%) |
| 1 | <1 | 0.0 | <1 | 0.0 | <1 | 0.0 | <1 | 0.0 |
| 2 | <1 | 0.0 | <1 | 0.0 | 2 | 0.0 | Lim | 0.2 |
| 3 | <1 | 0.0 | 21 | 0.0 | <1 | 0.0 | <1 | 0.0 |
| 4 | 3 | 0.0 | Lim | 0.1 | Lim | 0.1 | Lim | 0.1 |
| 5 | 1 | 0.0 | 8 | 0.0 | 136 | 0.0 | Lim | 0.2 |
| 6 | 1 | 0.0 | 2 | 0.0 | Lim | 0.1 | Lim | 0.4 |
| 7 | <1 | 0.0 | 2 | 0.0 | Lim | 0.0 | Lim | 0.4 |
| 8 | <1 | 0.0 | 845 | 0.0 | Lim | 0.3 | Lim | 0.1 |
| 9 | <1 | 0.0 | 135 | 0.0 | Lim | 0.2 | Lim | 0.2 |
| 10 | <1 | 0.0 | <1 | 0.0 | Lim | 0.2 | Lim | 0.1 |
| 11 | <1 | 0.0 | <1 | 0.0 | 3 | 0.0 | 1 | 0.0 |
| 12 | <1 | 0.0 | 142 | 0.0 | Lim | 0.1 | Lim | 0.4 |
| 13 | <1 | 0.0 | 17 | 0.0 | Lim | 0.2 | Lim | 0.3 |
| 14 | <1 | 0.0 | 1 | 0.0 | 1 | 0.0 | 1 | 0.0 |
| 15 | <1 | 0.0 | <1 | 0.0 | 1 | 0.0 | 1 | 0.0 |
| 16 | <1 | 0.0 | 2 | 0.0 | Lim | 0.1 | Lim | 0.2 |
| 17 | <1 | 0.0 | 13 | 0.0 | Lim | 0.2 | Lim | 0.3 |
| 18 | <1 | 0.0 | 204 | 0.0 | Lim | 0.2 | Lim | 0.4 |
| 19 | <1 | 0.0 | 1 | 0.0 | Lim | 0.0 | Lim | 0.0 |
| ∅ | <1 | 0.0 | 168 | 0.0 | 1,145 | 0.1 | 1,327 | 0.2 |

## 4.2 Numerical results

Table 3 lists for each extended model and for each dataset the CPU time required to compute an optimal portfolio, when the budget is set to 100,000 Swiss Francs. We imposed a time limit of 1,800 s, which was reached 12 times by the MAD$^+$-model and 14 times by the MV$^+$-model. In all cases, when the time limit was reached, the MIP gap did not exceed 0.4 %. The quadratic MV$^+$-model and the linear MAD$^+$-model require significantly more CPU time to solve the problem instances than do the CVaR$^+$-model and the MAD$^+$-model. However, both the MV$^+$- and the MAD$^+$-model are able to find near-optimal solutions within a reasonable amount of CPU time. Thus, all extended models are applicable to the considered real-world investment situation.

Table 4 reports the two-year portfolio return for each basic/extended model and each dataset. For all extended models we report the results when the budget is set to 100,000 Swiss Francs. The portfolio return for smaller budgets does not differ significantly from the values indicated in Table 4. The portfolios yield on average a higher return than the SPI, independent of the model used. However, the standard deviation of the portfolio returns is considerably higher compared with that of the SPI. In terms

**Table 4** Two-year return of the portfolios (for all extended models $B = 100{,}000$ CHF)

| Dataset | MV | MV$^+$ | MAD | MAD$^+$ | MM | MM$^+$ | CVaR | CVaR$^+_{90\,\%}$ | SPI |
|---|---|---|---|---|---|---|---|---|---|
| 1 | −26.0 | −26.3 | −26.5 | −27.5 | −28.1 | −29.4 | −28.2 | −28.2 | −11.2 |
| 2 | −7.3 | −16.8 | −7.3 | −16.8 | −7.3 | −11.1 | −19.0 | −28.2 | 30.5 |
| 3 | 20.4 | 38.5 | 25.9 | 37.7 | 17.1 | 15.2 | 29.0 | 20.4 | 73.6 |
| 4 | 66.8 | 71.5 | 55.4 | 55.6 | 37.0 | 43.2 | 52.6 | 54.5 | 38.1 |
| 5 | 7.2 | 10.5 | 3.9 | 7.3 | 4.4 | 10.7 | 6.8 | 6.6 | 11 |
| 6 | 10.7 | 10.1 | 16.3 | 14.6 | 14.4 | 13.7 | −5.4 | −8.6 | 41.8 |
| 7 | 60.4 | 57.4 | 60.2 | 58.1 | 39.4 | 39.4 | 40.5 | 39.4 | 77.5 |
| 8 | 192.4 | 197.7 | 130.1 | 138.1 | 114.5 | 153.6 | 88.6 | 103.1 | 77 |
| 9 | 133.0 | 142.7 | 114.7 | 116.1 | 283.4 | 285.4 | 130.5 | 132.2 | 27.3 |
| 10 | 141.5 | 142.2 | 84.9 | 87.4 | 141.0 | 148.4 | 150.0 | 136.7 | 20.2 |
| 11 | −40.6 | −39.3 | −24.5 | −27.0 | −37.2 | −38.1 | −27.8 | −23.9 | −15.6 |
| 12 | −19.8 | −56.2 | 12.1 | −56.7 | 100.2 | −44.8 | −6.0 | −59.1 | −41.5 |
| 13 | 29.6 | 26.8 | 29.9 | 26.9 | −10.6 | −18.2 | 22.7 | 24.3 | −11.7 |
| 14 | 79.7 | 75.8 | 80.3 | 74.1 | 64.5 | 58.5 | 64.0 | 57.7 | 25.3 |
| 15 | 38.7 | 39.9 | 35.4 | 37.1 | 34.8 | 36.5 | 39.4 | 39.2 | 42.6 |
| 16 | 106.2 | 110.5 | 115.5 | 124.1 | 74.3 | 82.8 | 94.3 | 100.0 | 61.4 |
| 17 | 79.8 | 82.5 | 76.2 | 75.9 | 79.1 | 77.2 | 79.7 | 81.2 | 17.3 |
| 18 | −25.2 | −25.1 | −31.1 | −32.1 | −33.8 | −35.5 | −25.8 | −25.0 | −35.4 |
| 19 | −32.1 | −32.4 | −34.3 | −35.0 | −43.2 | −45.8 | −42.0 | −43.8 | −34.2 |
| ∅ | 42.9 | 42.6 | 37.7 | 34.6 | 44.4 | 39.1 | 33.9 | 30.4 | 21.4 |
| SD | 64.6 | 68.7 | 50.9 | 56.9 | 76.7 | 82.0 | 54.6 | 58.3 | 37.83 |
| ∅/SD | 0.66 | 0.62 | 0.74 | 0.61 | 0.58 | 0.48 | 0.62 | 0.52 | 0.57 |

**Table 5** Number of different stocks in the portfolios (for all extended models $B = 100{,}000$ CHF)

| Dataset | MV | MV$^+$ | MAD | MAD$^+$ | MM | MM$^+$ | CVaR | CVaR$^+_{90\,\%}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 171 | 16 | 14 | 14 | 10 | 11 | 11 | 12 |
| 2 | 185 | 11 | 10 | 11 | 10 | 11 | 11 | 11 |
| 3 | 219 | 11 | 12 | 12 | 10 | 11 | 18 | 16 |
| 4 | 216 | 28 | 28 | 28 | 19 | 21 | 26 | 25 |
| 5 | 212 | 18 | 23 | 19 | 15 | 14 | 16 | 16 |
| 6 | 206 | 23 | 23 | 21 | 14 | 15 | 14 | 14 |
| 7 | 205 | 19 | 20 | 18 | 11 | 11 | 15 | 15 |
| 8 | 196 | 19 | 25 | 24 | 14 | 15 | 24 | 23 |
| 9 | 194 | 19 | 26 | 22 | 11 | 11 | 23 | 24 |
| 10 | 195 | 16 | 27 | 24 | 10 | 12 | 12 | 11 |
| 11 | 199 | 12 | 16 | 15 | 10 | 11 | 13 | 13 |
| 12 | 215 | 23 | 22 | 22 | 13 | 12 | 21 | 21 |
| 13 | 226 | 17 | 20 | 19 | 11 | 12 | 18 | 16 |
| 14 | 239 | 12 | 12 | 11 | 11 | 12 | 13 | 14 |
| 15 | 238 | 12 | 11 | 11 | 11 | 11 | 12 | 11 |
| 16 | 234 | 22 | 21 | 20 | 13 | 12 | 16 | 16 |
| 17 | 228 | 20 | 24 | 24 | 11 | 11 | 16 | 14 |
| 18 | 222 | 23 | 24 | 24 | 11 | 11 | 18 | 17 |
| 19 | 226 | 14 | 16 | 16 | 11 | 13 | 13 | 14 |
| ∅ | 211.9 | 17.6 | 19.7 | 18.7 | 11.9 | 12.5 | 16.3 | 15.9 |

of return per unit of standard deviation, all basic models outperform the SPI. From all extended models, the MV$^+$-model achieves the best result, followed by the MAD$^+$-model; the SPI is third, followed by the CVaR$^+_{90\,\%}$- and the MM$^+$-model. The high standard deviation of the MM$^+$-portfolios can be explained by the low average number of stocks included (cf. Table 5). Table 3 shows that portfolios that were computed with small-investor specific constraints perform slightly worse than the corresponding portfolios that were computed without these constraints. This finding is in line with the value-of-flexibility concept discussed in Lüthi and Doege (2005). However, the relative performance loss for small investors is surprisingly small when using the proposed model extensions.

## 5 Conclusions

We investigated four well-known portfolio-selection models from the perspective of a small investor. We have shown how these models can be extended to account for constraints that are vital to small investors, i.e., integral transactions units, maximum weight of individual stocks, piecewise constant transaction costs, maximum number of different stocks, and dividends. For our computational analysis, we constructed

19 instances using real data from the Swiss Stock Exchange in four-year intervals from 1988 to 2009 and the cost structure of the largest Swiss online broker Swissquote.

We found that all of the extended models are able to compute portfolios within a reasonable amount of CPU time. All models yield on average a higher return than the SPI. The best risk-return ratio is achieved by the extended MV model of Markowitz (1952). The influence of the budget on the portfolio performance, however, can be neglected.

In the future, the influence of a periodic rebalancing on the portfolio performance should be studied. Moreover, alternative asset types, e.g. bonds, could be included in the study. Another interesting direction of future research is to analyze whether there are subcases for which the extended portfolio-selection models can be solved in polynomial time; for general integer optimization problems, such subcases have been devised in De Loera et al. (2008a,b).

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