Non-concave optimal investment and no-arbitrage: a measure theoretical approach

Laurence Carassus
LMR, Université Reims Champagne-Ardenne, France
laurence.carassus@univ-reims.fr

Romain Blanchard
LMR, Université Reims Champagne-Ardenne, France
romain.blanchard@etudiant.univ-reims.fr

Miklós Rásonyi
MTA Alfréd Rényi Institute of Mathematics, Hungary
rasonyi@renyi.mta.hu

August 29, 2016

Abstract

We consider non-concave and non-smooth random utility functions with domain of definition equal to the non-negative half-line. We use a dynamic programming framework together with measurable selection arguments to establish both the no-arbitrage condition characterization and the existence of an optimal portfolio in a (generically incomplete) discrete-time financial market model with finite time horizon.

Key words: no-arbitrage condition ; non-concave utility functions; optimal investment
AMS 2000 subject classification: Primary 93E20, 91B70, 91B16 ; secondary 91G10, 28B20

1 Introduction

We consider investors trading in a multi-asset and discrete-time financial market. We revisit two classical problems: the characterization of no arbitrage and the maximisation of the expected utility of the terminal wealth of an investor.

We consider a general random, possibly non-concave and non-smooth utility function U, defined on the non-negative half-line (that can be "S-shaped" but our results apply to a broader class of utility functions e.g. to piecewise concave ones) and we provide sufficient conditions which guarantee the existence of an optimal strategy. Similar optimization problems constitute an area of intensive study in recent years, see e.g. Bensoussan et al. (2015) , He and Zhou (2011), Jin and Zhou (2008), Carlier and Dana (2011).

We are working in the setting of Carassus et al. (2015) and remove certain restrictive hypothesis of Carassus et al. (2015). Furthermore, we use methods that are different from the ones in Rásonyi and Stettner (2005), Rásonyi and Stettner (2006), Carassus and Rásonyi (2015) and Carassus et al. (2015), where similar multistep problems were treated. In contrast to the existing literature, we propose to consider a probability space which is not necessarily complete.

We extend the paper of Carassus et al. (2015) in several directions. First, we propose an alternative integrability condition (see Assumption 4.8 and Proposition 6.1) to the rather restrictive one of Carassus et al. (2015) stipulating that $E^{-}U(\cdot,0) < \infty$. The property $U(0) = -\infty$ holds for a number of important (non-random and concave) utility functions (logarithm, $-x^{\alpha}$ for $\alpha < 0$). It is a rather natural requirement since it expresses the fear of investor for defaulting ($i.e$ reaching 0). We also introduce a new (weaker) version of the asymptotic elasticity assumption (see Assumption 4.10). In particular, Assumption 4.10 holds true for concave functions (see Remark 4.15) and therefore our result extends the one obtained in Rásonyi and Stettner (2006) to random utility function and incomplete probability spaces. Next, we do not require that the value function is finite for all initial wealth as it was postulated in Carassus et al. (2015); instead we only assumed the less restrictive and more tractable Assumption 4.7. Finally, instead of using some Carathéodory utility function U as in Carassus et al. (2015) ($i.e$ function measurable in ω and continuous in x), we consider function which is measurable in ω and upper semicontinuous (usc in the rest of the paper) in x. As U is also non-decreasing, we point out that this implies that U is jointly measurable in (ω, x). Note that in the case of complete sigma-algebra -U is then a normal integrand (see Definition 14.27 in Rockafellar and Wets (1998) or Section 3 of Chapter 5 in Molchanov (2005) as well as Corollary 14.34 in Rockafellar and Wets (1998)). This will play an important role in the dynamic programming part to obtain certain measurability properties. Allowing non-continuous U is unusual in the financial mathematics literature (though it is common in optimization). We highlight that this generalisation has a potential to model investor's behaviour which can change suddenly after reaching a desired wealth level. Such a change can be expressed by a jump of U at the given level.

To solve our optimisation problem, we use dynamic programming as in Rásonyi and Stettner (2005), Rásonyi and Stettner (2006), Carassus and Rásonyi (2015) and Carassus et al. (2015) but here we propose a different approach which provides simpler proofs. As in Nutz (2014), we consider first a one period case with strategy in $\mathbb{R}^d$. Then we use dynamic programming and measurable selection arguments, namely the Aumann Theorem (see, for example, Corollary 1 in Sainte-Beuve (1974)) to solve the multi-period problem. Our modelisation of $(\Omega, \mathcal{F}, \mathfrak{F}, P)$ is more general than in Nutz (2014) as there is only one probability measure and we don't have to postulate Borel space or analytic sets. We also use the same methodology to reprove classical results on no-arbitrage characterization (see Rásonyi and Stettner (2005) and Jacod and Shiryaev (1998)) in our context of possibly incomplete sigma-algebras.

We do not handle the case where the utility is defined on the whole real line (with a similar set of assumptions) as this would have overburdened the paper. This is left for further research.

The paper is organized as follows: in section 2 we introduce our setup; section 3 contains the main results on no-arbitrage; section 4 presents the main theorem on terminal wealth expected utility maximisation; section 5 establishes the existence of an optimal strategy for the one period case; we prove our main theorem on utility maximisation in section 6.

Finally, section 7 collects some technical results and proofs as well as elements about random sets measurability.

2 Set-up

Fix a time horizon $T \in \mathbb{N}$ and let $(\Omega_t)_{1 \leq t \leq T}$ be a sequence of spaces and $(\mathcal{G}_t)_{1 \leq t \leq T}$ be a sequence of sigma-algebra where $\mathcal{G}_t$ is a sigma-algebra on Ω_t for all $t = 1, \dots, T$. For $t = 1, \dots, T$, we denote by Ω^t the t-fold Cartesian product

$$\Omega^t = \Omega_1 \times \ldots \times \Omega_t.$$

An element of Ω^t will be denoted by $\omega^t = (\omega_1, \dots, \omega_t)$ for $(\omega_1, \dots, \omega_t) \in \Omega_1 \times \ldots \times \Omega_t$. We also denote by $\mathcal{F}_t$ the product sigma-algebra on Ω^t

$$\mathcal{F}_t = \mathcal{G}_1 \otimes \ldots \otimes \mathcal{G}_t.$$

For the sake of simplicity we consider that the state $t = 0$ is deterministic and set $\Omega^0 := \{\omega_0\}$ and $\mathcal{F}_0 = \mathcal{G}_0 = \{\emptyset, \Omega^0\}$. To avoid heavy notations we will omit the dependency in ω_0 in the rest of the paper. We denote by $\mathfrak{F}$ the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$.

Let P_1 be a probability measure on $\mathcal{F}_1$ and q_{t+1} be a stochastic kernel on $\mathcal{G}_{t+1} \times \Omega^t$ for $t = 1, \dots, T-1$. Namely we assume that for all $\omega^t \in \Omega^t$, $B \in \mathcal{G}_{t+1} \to q_{t+1}(B|\omega^t)$ is a probability measure on $\mathcal{G}_{t+1}$ and for all $B \in \mathcal{G}_{t+1}$, $\omega^t \in \Omega^t \to q_{t+1}(B|\omega^t)$ is $\mathcal{F}_t$-measurable. Here we DO NOT assume that $\mathcal{G}_1$ contains the null sets of P_1 and that $\mathcal{G}_{t+1}$ contains the null sets of $q_{t+1}(.|\omega^t)$ for all $\omega^t \in \Omega^t$. Then we define for

$A \in \mathcal{F}_t$ the probability P_t by Fubini's Theorem for stochastic kernel (see Lemma 7.1).

$$P_t(A) = \int_{\Omega_1} \int_{\Omega_2} \cdots \int_{\Omega_t} 1_A(\omega_1, \ldots, \omega_t) q_t(d\omega_t|\omega^{t-1}) \cdots q_2(d\omega_2|\omega^1) P_1(d\omega_1). \quad (1)$$

Finally $(\Omega, \mathcal{F}, \mathfrak{F}, P) := (\Omega^T, \mathcal{F}_T, \mathfrak{F}, P_T)$ will be our basic measurable space. The expectation under P_t will be denoted by E_{P_t} ; when $t = T$, we simply write E.

Remark 2.1 If we choose for Ω some Polish space, then any probability measure P can be decomposed in the form of (1) (see the measure decomposition theorem in Dellacherie and Meyer (1979) III.70-7).

From now on the positive (resp. negative) part of some number or random variable X is denoted by X^+ (resp. X^-). We will also write $f^\pm(X)$ for $(f(X))^\pm$ for any random variable X and (possibly random) function f.
In the rest of the paper we will use generalised integral: for some $f_t : \Omega^t \to \mathbb{R} \cup \{\pm\infty\}$, $\mathcal{F}_t$-measurable, such that $\int_{\Omega^t} f_t^+(\omega^t) P_t(d\omega^t) < \infty$ or $\int_{\Omega^t} f_t^-(\omega^t) P_t(d\omega^t) < \infty$, we define

$$\int_{\Omega^t} f_t(\omega^t) P_t(d\omega^t) := \int_{\Omega^t} f_t^+(\omega^t) P_t(d\omega^t) - \int_{\Omega^t} f_t^-(\omega^t) P_t(d\omega^t),$$

where the equality holds in $\mathbb{R} \cup \{\pm\infty\}$. We refer to Lemma 7.1, Definition 7.2 and Proposition 7.4 of the Appendix for more details and properties. In particular, if f_t is non-negative or if f_t is such that $\int_{\Omega^t} f_t^+(\omega^t) P_t(d\omega^t) < \infty$ (this will be the two cases of interest in the paper) we can apply Fubini's Theorem [1] and we have

$$\int_{\Omega^t} f_t(\omega^t) P_t(d\omega^t) = \int_{\Omega_1} \int_{\Omega_2} \cdots \int_{\Omega_t} f_t(\omega_1, \ldots, \omega_t) q_t(d\omega_t|\omega^{t-1}) \cdots q_2(d\omega_2|\omega^1) P_1(d\omega_1),$$

where the equality holds in $[0, \infty]$ if f_t is non-negative and in $[-\infty, \infty)$ if $\int_{\Omega^t} f_t^+(\omega^t) P_t(d\omega^t) < \infty$.
Finally, we give some notations about completion of the probability space $(\Omega^t, \mathcal{F}_t, P_t)$ for some $t \in \{1, \ldots, T\}$. We will denote by $\mathcal{N}_{P_t}$ the set of P_t negligible sets of Ω^t *i.e* $\mathcal{N}_{P_t} = \{N \subset \Omega^t,\ \exists M \in \mathcal{F}_t,\ N \subset M \text{ and } P_t(M) = 0\}$. Let $\overline{\mathcal{F}}_t = \{A \cup N, A \in \mathcal{F}_t, N \in \mathcal{N}_{P_t}\}$ and $\overline{P}_t(A \cup N) = P_t(A)$ for $A \cup N \in \overline{\mathcal{F}}_t$. Then it is well known that $\overline{P}_t$ is a measure on $\overline{\mathcal{F}}_t$ which coincides with P_t on $\mathcal{F}_t$, that $(\Omega^t, \overline{\mathcal{F}}_t, \overline{P}_t)$ is a complete probability space and that $\overline{P}_t$ restricted to $\mathcal{N}_{P_t}$ is equal to zero.

For $t = 0, \ldots, T-1$, let Ξ_t be the set of $\mathcal{F}_t$-measurable random variables mapping Ω^t to $\mathbb{R}^d$.

The following lemma makes the link between conditional expectation and kernel. To do that, we introduce $\mathcal{F}_t^T$, the filtration on Ω^T associated to $\mathcal{F}_t$, defined by

$$\mathcal{F}_t^T = \mathcal{G}_1 \otimes \ldots \otimes \mathcal{G}_t \otimes \{\emptyset, \Omega_{t+1}\} \ldots \otimes \{\emptyset, \Omega_T\}.$$

Let Ξ_t^T be the set of $\mathcal{F}_t^T$-measurable random variables from Ω^T to $\mathbb{R}^d$. Let $X_t : \Omega^T \to \Omega_t$, $X_t(\omega_1, \ldots, \omega_T) = \omega_t$ be the coordinate mapping corresponding to t. Then $\mathcal{F}_t^T = \sigma(X_1, \ldots, X_t)$. So $h \in \Xi_t^T$ if and only if there exists some $g \in \Xi_t$ such that $h = g(X_1, \ldots, X_t)$. This implies that $h(\omega^T) = g(\omega^t)$. For ease of notation we will identify h and g and also $\mathcal{F}_t$, $\mathcal{F}_t^T$, Ξ_t and Ξ_t^T.

Lemma 2.2 *Let $0 \leq s \leq t \leq T$. Let $h \in \Xi_t$ such that $\int_{\Omega^t} h^+ dP_t < \infty$ then*

$$\begin{aligned} E(h|\mathcal{F}_s) &= \varphi(X_1, \ldots, X_s)\, P_s\ a.s. \\ \varphi(\omega_1, \ldots, \omega_s) &= \int_{\Omega_{s+1} \times \ldots \times \Omega_t} h(\omega_1, \ldots, \omega_s, \omega_{s+1}, \ldots \omega_t) q_t(\omega_t|\omega^{t-1}) \ldots q_{s+1}(\omega_{s+1}|\omega^s). \end{aligned}$$

[1] From now, we call Fubini's theorem the Fubini theorem for stochastic kernel (see eg Lemma 7.1, Proposition 7.4).

Proof. For the sake of completeness, the proof is reported in Section 7.3 of the Appendix. □

Let $\{S_t,\ 0 \le t \le T\}$ be a d-dimensional $\mathcal{F}_t$-adapted process representing the price of d risky securities in the financial market in consideration. There exists also a riskless asset for which we assume a constant price equal to 1, for the sake of simplicity. Without this assumption, all the developments below could be carried out using discounted prices. The notation $\Delta S_t := S_t - S_{t-1}$ will often be used. If $x, y \in \mathbb{R}^d$ then the concatenation xy stands for their scalar product. The symbol $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$ (or on $\mathbb{R}$).

Trading strategies are represented by d-dimensional predictable processes $(\phi_t)_{1\le t\le T}$, where ϕ_t^i denotes the investor's holdings in asset i at time t; predictability means that $\phi_t \in \Xi_{t-1}$. The family of all predictable trading strategies is denoted by Φ.

We assume that trading is self-financing. As the riskless asset's price is constant 1, the value at time t of a portfolio ϕ starting from initial capital $x \in \mathbb{R}$ is given by

$$V_t^{x,\phi} = x + \sum_{i=1}^{t} \phi_i \Delta S_i.$$

3 No-arbitrage condition

The following absence of arbitrage condition or NA condition is standard, it is equivalent to the existence of a risk-neutral measure in discrete-time markets with finite horizon, see e.g. Dalang et al. (1990).

(NA) *If $V_T^{0,\phi} \ge 0$ P-a.s. for some $\phi \in \Phi$ then $V_T^{0,\phi} = 0$ P-a.s.*

Remark 3.1 It is proved in Proposition 1.1 of Rásonyi and Stettner (2006) that (NA) is equivalent to the no-arbitrage assumption which stipulates that no investor should be allowed to make a profit out of nothing and without risk, even with a budget constraint: for all $x_0 \ge 0$ if $\phi \in \Phi$ is such that with $V_T^{x_0,\phi} \ge x_0$ a.s., then $V_T^{x_0,\phi} = x_0$ a.s.

We now provide classical tools and results about the (NA) condition and its "concrete" local characterization, see Proposition 3.7, that we will use in the rest of the paper. We start with the set D^{t+1} (see Definition 3.2) where $D^{t+1}(\omega^t)$ is the smallest affine subspace of $\mathbb{R}^d$ containing the support of the distribution of $\Delta S_{t+1}(\omega^t, .)$ under $q_{t+1}(.|\omega^t)$. If $D^{t+1}(\omega^t) = \mathbb{R}^d$ then, intuitively, there are no redundant assets. Otherwise, for $\phi_{t+1} \in \Xi_t$, one may always replace $\phi_{t+1}(\omega^t, \cdot)$ by its orthogonal projection $\phi_{t+1}^{\perp}(\omega^t, \cdot)$ on $D^{t+1}(\omega^t)$ without changing the portfolio value since $\phi_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t, \cdot) = \phi_{t+1}^{\perp}(\omega^t)\Delta S_{t+1}(\omega^t, \cdot)$, $q_{t+1}(\cdot|\omega^t)$ a.s., see Remark 5.3 and Lemma 7.18 below as well as Remark 9.1 of Föllmer and Schied (2002).

Definition 3.2 Let $(\Omega, \mathcal{F})$ be a measurable space and $(T, \mathcal{T})$ a topological space. A random set R is a set valued function that assigns to each $\omega \in \Omega$ a subset $R(\omega)$ of T. We write $R : \Omega \twoheadrightarrow T$. We say that R is measurable if for any open set $O \in T$ $\{\omega \in \Omega,\ R(\omega) \cap O \neq \emptyset\} \in \mathcal{F}$.

Definition 3.3 Let $0 \le t \le T$ be fixed. We define the random set (see Definition 3.2) $\widetilde{D}^{t+1} : \Omega^t \twoheadrightarrow \mathbb{R}^d$ by

$$\widetilde{D}^{t+1}(\omega^t) := \bigcap \left\{ A \subset \mathbb{R}^d,\ \text{closed},\ q_{t+1}\left(\Delta S_{t+1}(\omega^t, .) \in A|\omega^t\right) = 1 \right\}. \tag{2}$$

For $\omega^t \in \Omega^t$, $\widetilde{D}^{t+1}(\omega^t) \subset \mathbb{R}^d$ is the support of the distribution of $\Delta S_{t+1}(\omega^t, \cdot)$ under $q_{t+1}(\cdot|\omega^t)$. We also define the random set $D^{t+1} : \Omega^t \twoheadrightarrow \mathbb{R}^d$ by

$$D^{t+1}(\omega^t) := \text{Aff}\left(\widetilde{D}^{t+1}(\omega^t)\right), \tag{3}$$

where Aff denotes the affine hull of a set.

The following lemma establishes some important properties of $\widetilde{D}^{t+1}$ and D^{t+1} and in particular $Graph(D^{t+1}) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$. This result will be central in the proof of most of our results.

Lemma 3.4 *Let $0 \leq t \leq T$ be fixed. Let $\widetilde{D}^{t+1} : \Omega^t \twoheadrightarrow \mathbb{R}^d$ and $D^{t+1} : \Omega^t \twoheadrightarrow \mathbb{R}^d$ be the random sets defined in (2) and (3) of Definition 3.3. Then $\widetilde{D}^{t+1}$ and D^{t+1} are both non-empty, closed-valued and $\mathcal{F}_t$-measurable random sets (see Definition 3.2). In particular, $Graph(D^{t+1}) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$.*

Proof. The proof is reported in Section 7.3 of the Appendix. □

In Lemma 3.5, which is used in the proof of Lemma 3.6 for projection purposes, we obtain a well-know result : for $\omega^t \in \Omega^t$ fixed and under a local version of (NA), $D^{t+1}(\omega^t)$ is a vector subspace of $\mathbb{R}^d$ (see for instance Theorem 1.48 of Föllmer and Schied (2002)). Then in Lemma 3.6 we prove that under the (NA) assumption, for P_t almost all ω^t, $D^{t+1}(\omega^t)$ is a vector subspace of $\mathbb{R}^d$. We also provide a local version of the (NA) condition (see (5)). Note that Lemma 3.6 is a direct consequence of Proposition 3.3 in Rásonyi and Stettner (2005) combined with Lemma 2.2 (see Remark 3.10). We propose alternative proofs of Lemmata 3.5 and 3.6 which are coherent with our framework and our methodology.

Lemma 3.5 *Let $0 \leq t \leq T$ and $\omega^t \in \Omega^t$ be fixed. Assume that for all $h \in D^{t+1}(\omega^t)\backslash\{0\}$*

$$q_{t+1}(h\Delta S_{t+1}(\omega^t, \cdot) \geq 0|\omega^t) < 1.$$

Then $0 \in D^{t+1}(\omega^t)$ and the set $D^{t+1}(\omega^t)$ is actually a vector subspace of $\mathbb{R}^d$.

Proof. The proof is reported in Section 7.3 of the Appendix. □

Lemma 3.6 *Assume that the (NA) condition holds true. Then for all $0 \leq t \leq T-1$, there exists a full measure set Ω^t_{NA1} such that for all $\omega^t \in \Omega^t_{NA1}$, $0 \in D^{t+1}(\omega^t)$, $i.e$ $D^{t+1}(\omega^t)$ is a vector space of $\mathbb{R}^d$. Moreover, for all $\omega^t \in \Omega^t_{NA1}$ and all $h \in \mathbb{R}^d$ we get that*

$$q_{t+1}(h\Delta S_{t+1}(\omega^t, \cdot) \geq 0|\omega^t) = 1 \Rightarrow q_{t+1}(h\Delta S_{t+1}(\omega^t, \cdot) = 0|\omega^t) = 1. \tag{4}$$

In particular, if $\omega^t \in \Omega^t_{NA1}$ and $h \in D^{t+1}(\omega^t)$ we obtain that

$$q_{t+1}(h\Delta S_{t+1}(\omega^t, \cdot) \geq 0|\omega^t) = 1 \Rightarrow h = 0. \tag{5}$$

Proof. Let $0 \leq t \leq T$ be fixed. Recall that $\overline{\mathcal{F}}_t$ is the P_t-completion of $\mathcal{F}_t$ and that $\overline{P}_t$ is the (unique) extension of P_t to $\overline{\mathcal{F}}_t$. We introduce the following random set Π^t

$$\Pi^t := \left\{\omega^t \in \Omega^t,\ \exists h \in D^{t+1}(\omega^t), h \neq 0,\ q_{t+1}(h\Delta S_{t+1}(\omega^t, \cdot) \geq 0|\omega^t) = 1\right\}.$$

Assume for a moment that $\Pi^t \in \overline{\mathcal{F}}_t$ and that $\overline{P}_t(\Pi^t) = 0$ (this will be proven below). Let $\omega^t \in \Omega^t\backslash\Pi^t$. The fact that $0 \in D^{t+1}(\omega^t)$ is a direct consequence of the definition of Π^t and of Lemma 3.5. We now prove (4). Let $h \in \mathbb{R}^d$ be fixed such that $q_{t+1}(h\Delta S_{t+1}(\omega^t, \cdot) \geq 0|\omega^t) = 1$. We prove that $q_{t+1}(h\Delta S_{t+1}(\omega^t, \cdot) = 0|\omega^t) = 1$. If $h = 0$ this is straightforward. If $h \in D^{t+1}(\omega^t) \setminus \{0\}$, $\omega^t \in \Pi^t$ which is impossible. Now if $h \notin D^{t+1}(\omega^t)$ and $h \neq 0$, let h' be the orthogonal projection of h on $D^{t+1}(\omega^t)$ (recall that since $\omega^t \notin \Pi^t$ $D^{t+1}(\omega^t)$ is a vector subspace). We first show that $q_{t+1}(h'\Delta S_{t+1}(\omega^t, \cdot) \geq 0|\omega^t) = 1$. Indeed, if it were not the case the set $B := \{\omega_{t+1} \in \Omega_{t+1},\ h'\Delta S_{t+1}(\omega^t, \omega_{t+1}) < 0\}$ would verify $q_{t+1}(B|\omega^t) > 0$. Set

$$L^{t+1}(\omega^t) := \left(D^{t+1}(\omega^t)\right)^{\perp}. \tag{6}$$

As $(h - h') \in L^{t+1}(\omega^t)$ (recall that $D^{t+1}(\omega^t)$ is a vector subspace), by Lemma 7.18 the set $A := \{\omega_{t+1} \in \Omega_{t+1},\ (h - h')\Delta S_{t+1}(\omega^t, \omega_{t+1}) = 0\}$ verify $q_{t+1}(A|\omega^t) = 1$. We would therefore obtain that $q_{t+1}(A \cap B|\omega^t) > 0$ which implies that $q_{t+1}(h\Delta S_{t+1}(\omega^t, .) \geq 0|\omega^t) < 1$, a contradiction. Thus $q_{t+1}(h'\Delta S_{t+1}(\omega^t, \cdot) \geq$

$0|\omega^t) = 1$. If $h' \neq 0$ as $h' \in D^{t+1}(\omega^t)$, $\omega^t \in \Pi^t$ which is again a contradiction. Thus $h' = 0$ and as $A \cap \{h'\Delta S_{t+1}(\omega^t,\cdot) = 0\} \subset \{h\Delta S_{t+1}(\omega^t,\cdot) = 0\}$, $q_{t+1}(h\Delta S_{t+1}(\omega^t,\cdot) = 0|\omega^t) = 1$.

As $\Omega^t \setminus \Pi^t \in \overline{\mathcal{F}_t}$ there exists $\Omega^t_{NA1} \in \mathcal{F}_t$ and $N^t \in \mathcal{N}_{P_t}$ (the collection of negligible set of (Ω^t, P_t)) such that $\Omega^t \setminus \Pi^t = \Omega^t_{NA1} \cup N^t$ and $P_t(\Omega^t_{NA1}) = \overline{P}_t(\Omega^t \backslash \Pi^t) = 1$. Since $\Omega^t_{NA1} \subset \Omega^t \setminus \Pi^t$, it follows that for all $\omega^t \in \Omega^t_{NA1}$, $0 \in D^{t+1}(\omega^t)$ and for all $h \in \mathbb{R}^d$, (4) holds true.

We prove (5). Assume now that $\omega^t \in \Omega^t_{NA1}$ and $h \in D^{t+1}(\omega^t)$ are such that $q_{t+1}(h\Delta S_{t+1}(\omega^t,\cdot) \geq 0|\omega^t) = 1$. Using (4) and Lemma 7.18 we get that $h \in L^{t+1}(\omega^t)$. So $h \in D^{t+1}(\omega^t) \cap L^{t+1}(\omega^t) = \{0\}$ and (5) holds true.

It remains to prove that $\Pi^t \in \overline{\mathcal{F}}_t$ and $\overline{P}_t(\Pi^t) = 0$. To do that we introduce the following random set $H^t : \Omega^t \twoheadrightarrow \mathbb{R}^d$

$$H^t(\omega^t) := \left\{h \in D^{t+1}(\omega^t),\ h \neq 0,\ q_{t+1}(h\Delta S_{t+1}(\omega^t,\cdot) \geq 0|\omega^t) = 1\right\}.$$

Then

$$\Pi^t \;=\; \left\{\omega^t \in \Omega^t,\ H^t(\omega^t) \neq \emptyset\right\} = \mathbf{proj}_{|\Omega^t} Graph(H^t)$$

since $Graph(H^t) = \{(\omega^t, h) \in \Omega^t \times \mathbb{R}^d,\ h \in H^t(\omega^t)\}$.

We prove now that $Graph(H^t) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$. Indeed, we can rewrite that

$$Graph(H^t) = Graph(D^{t+1}) \bigcap \left\{(\omega^t, h) \in \Omega^t \times \mathbb{R}^d,\ q_{t+1}(h\Delta S_{t+1}(\omega^t,\cdot) \geq 0|\omega^t) = 1\right\} \bigcap \left(\Omega^t \times \mathbb{R}^d \backslash \{0\}\right).$$

As from Lemma 7.9, $\left\{(\omega^t, h) \in \Omega^t \times \mathbb{R}^d,\ q_{t+1}(h\Delta S_{t+1}(\omega^t,\cdot) \geq 0|\omega^t) = 1\right\} \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ and from Lemma 3.4, $Graph(D^{t+1}) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$, we obtain that $Graph(H^t) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$. The Projection Theorem (see for example Theorem 3.23 in Castaing and Valadier (1977)) applies and $\Pi^t = \{H^t \neq \emptyset\} = \mathbf{proj}_{|\Omega^t} Graph(H^t) \in \overline{\mathcal{F}}_t$. From the Aumann Theorem (see Corollary 1 in Sainte-Beuve (1974)) there exists a $\overline{\mathcal{F}}_t$-measurable selector $\overline{h}_{t+1} : \Pi^t \to \mathbb{R}^d$ such that $\overline{h}_{t+1}(\omega^t) \in H^t(\omega^t)$ for every $\omega^t \in \Pi^t$. We now extend $\overline{h}_{t+1}$ on Ω^t by setting $\overline{h}_{t+1}(\omega^t) = 0$ for $\omega^t \in \Omega^t \backslash \Pi^t$. It is clear that $\overline{h}_{t+1}$ remains $\overline{\mathcal{F}}_t$-measurable. Applying Lemma 7.10, there exists $h_{t+1} : \Omega^t \to \mathbb{R}^d$ which is $\mathcal{F}_t$-measurable and satisfies $h_{t+1} = \overline{h}_{t+1}$ P_t-almost surely. Then if we set

$$\begin{aligned}
\varphi(\omega^t) &= q_{t+1}(h_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t,.) \geq 0|\omega^t),\\
\overline{\varphi}(\omega^t) &= q_{t+1}(\overline{h}_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t,.) \geq 0|\omega^t),
\end{aligned}$$

we get from Proposition 7.9 that φ is $\mathcal{F}_t$-measurable and from Proposition 7.6 *iii)* that $\overline{\varphi}$ is $\overline{\mathcal{F}}_t$-measurable. Furthermore as $\{\omega^t \in \Omega^t,\ \varphi(\omega^t) \neq \overline{\varphi}(\omega^t)\} \subset \{\omega^t \in \Omega^t,\ h_t(\omega^t) \neq \overline{h}_{t+1}(\omega^t)\}$, $\varphi = \overline{\varphi}$ P_t-almost surely. This implies that $\int_{\Omega^t} \overline{\varphi} d\overline{P}_t = \int_{\Omega^t} \varphi dP_t$. Now we define the predictable process $(\phi_t)_{1 \leq t \leq T}$ by $\phi_{t+1} = h_{t+1}$ and $\phi_i = 0$ for $i \neq t+1$. Then

$$\begin{aligned}
P(V_T^{0,\phi} \geq 0) &= P(h_{t+1}\Delta S_{t+1} \geq 0) = P_{t+1}(h_{t+1}\Delta S_{t+1} \geq 0)\\
&= \int_{\Omega^t} \varphi(\omega^t) P_t(d\omega^t) = \int_{\Omega^t} \overline{\varphi}(\omega^t)\overline{P}_t(d\omega^t)\\
&= \int_{\Pi^t} q_{t+1}\left(\overline{h}_t(\omega^t)\Delta S_{t+1}(\omega^t,\cdot) \geq 0|\omega^t\right)\overline{P}_t(d\omega^t) +\\
&\quad \int_{\Omega^t\backslash\Pi^t} q_{t+1}\left(0 \times \Delta S_{t+1}(\omega^t,\cdot) \geq 0|\omega^t\right)\overline{P}_t(d\omega^t)\\
&= \overline{P}_t(\Pi^t) + \overline{P}_t(\Omega^t \setminus \Pi^t) = 1,
\end{aligned}$$

where we have used that if $\omega^t \in \Pi^t$, $\overline{h}_{t+1}(\omega^t) \in H^t(\omega^t)$ and otherwise $\overline{h}_{t+1}(\omega^t) = 0$. With the same arguments we obtain that

$$\begin{aligned}
P(V_T^{0,\phi} > 0) &= P_t(h_{t+1}\Delta S_{t+1} > 0) \\
&= \int_{\Pi^t} q_{t+1}\left(\overline{h}_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t,\cdot) > 0|\omega^t\right)\overline{P}_t(d\omega^t) + \int_{\Omega^t\setminus\Pi^t} q_{t+1}\left(0 > 0|\omega^t\right)\overline{P}_t(d\omega^t) \\
&= \int_{\Pi^t} q_{t+1}\left(\overline{h}_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t,\cdot) > 0|\omega^t\right)\overline{P}_t(d\omega^t).
\end{aligned}$$

Let $\omega^t \in \Pi^t$ then $q_{t+1}\left(\overline{h}_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t,\cdot) > 0|\omega^t\right) > 0$. Indeed, if it is not the case then $q_{t+1}\left(\overline{h}_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t,\cdot) \leq 0|\omega^t\right) = 1$. As $\omega^t \in \Pi^t$, $\overline{h}_{t+1}(\omega^t) \in D^{t+1}(\omega^t)$ and $q_{t+1}\left(\overline{h}_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t,\cdot) \geq 0|\omega^t\right) = 1$, Lemma 7.18 applies and $\overline{h}_{t+1}(\omega^t) \in L^{t+1}(\omega^t)$. Thus we get that $\overline{h}_{t+1}(\omega^t) \in L^{t+1}(\omega^t) \cap D^{t+1}(\omega^t) = \{0\}$, a contradiction. So if $\overline{P}_t(\Pi^t) > 0$ we obtain that $P(V_T^{0,\phi} > 0) > 0$. This contradicts the (NA) condition and we obtain $\overline{P}_t(\Pi^t) = 0$, the required result. □

Similarly as in Rásonyi and Stettner (2005) and Jacod and Shiryaev (1998), we prove a "quantitative" characterization of (NA).

Proposition 3.7 *Assume that the* (NA) *condition holds true and let* $0 \leq t \leq T$. *Then there exists* $\Omega^t_{NA} \in \mathcal{F}_t$ *with* $P_t(\Omega^t_{NA}) = 1$ *and* $\Omega^t_{NA} \subset \Omega^t_{NA1}$ *(see Lemma 3.6 for the definition of* Ω^t_{NA1}*) such that for all* $\omega^t \in \Omega^t_{NA}$*, there exists* $\alpha_t(\omega^t) \in (0,1]$ *such that for all* $h \in D^{t+1}(\omega^t)$

$$q_{t+1}\big(h\Delta S_{t+1}(\omega^t,\cdot) \leq -\alpha_t(\omega^t)|h||\omega^t\big) \geq \alpha_t(\omega^t). \tag{7}$$

Furthermore $\omega^t \to \alpha_t(\omega^t)$ *is* $\mathcal{F}_t$*-measurable.*

Proof. Let $\omega^t \in \Omega^t_{NA1}$ be fixed (Ω^t_{NA1} is defined in Lemma 3.6).
Step 1 : Proof of (7). Introduce the following set for $n \geq 1$

$$A_n(\omega^t) := \left\{h \in D^{t+1}(\omega^t),\ |h| = 1,\ q_{t+1}\left(h\Delta S_{t+1}(\omega^t,\cdot) \leq -\frac{1}{n}|\omega^t\right) < \frac{1}{n}\right\}. \tag{8}$$

Let $\overline{n}_0(\omega^t) := \inf\{n \geq 1, A_n(\omega^t) = \emptyset\}$ with the convention that $\inf\emptyset = +\infty$. Note that if $D^{t+1}(\omega^t) = \{0\}$, then $\overline{n}_0(\omega^t) = 1 < \infty$. We assume now that $D^{t+1}(\omega^t) \neq \{0\}$ and we prove by contradiction that $\overline{n}_0(\omega^t) < \infty$. Assume that $\overline{n}_0(\omega^t) = \infty$ *i.e* for all $n \geq 1$, $A_n(\omega^t) \neq \emptyset$. We thus get $h_n(\omega^t) \in D^{t+1}(\omega^t)$ with $|h_n(\omega^t)| = 1$ and such that

$$q_{t+1}\left(h_n(\omega^t)\Delta S_{t+1}(\omega^t,\cdot) \leq -\frac{1}{n}\,|\,\omega^t\right) < \frac{1}{n}.$$

By passing to a sub-sequence we can assume that $h_n(\omega^t)$ tends to some $h^*(\omega^t) \in D^{t+1}(\omega^t)$ (recall that the set $D^{t+1}(\omega^t)$ is closed by definition) with $|h^*(\omega^t)| = 1$. Introduce

$$\begin{aligned}
B(\omega^t) &:= \{\omega_{t+1} \in \Omega_{t+1},\ h^*(\omega^t)\Delta S_{t+1}(\omega^t,\omega_{t+1}) < 0\} \\
B_n(\omega^t) &:= \{\omega_{t+1} \in \Omega_{t+1},\ h_n(\omega^t)\Delta S_{t+1}(\omega^t,\omega_{t+1}) \leq -1/n\}.
\end{aligned}$$

Then $B(\omega^t) \subset \liminf_n B_n(\omega^t)$. Furthermore as $1_{\liminf_n B_n(\omega^t)} = \liminf_n 1_{B_n(\omega^t)}$, Fatou's Lemma implies that

$$\begin{aligned}
q_{t+1}\left(h^*(\omega^t)\Delta S_{t+1}(\omega^t,\cdot) < 0|\omega^t\right) &\leq \int_{\Omega_{t+1}} 1_{\liminf_n B_n(\omega^t)}(\omega_{t+1})q_{t+1}(\omega_{t+1}|\omega^t) \\
&\leq \liminf_n \int_{\Omega_{t+1}} 1_{B_n(\omega^t)}(\omega_{t+1})q_{t+1}(\omega_{t+1}|\omega^t) = 0.
\end{aligned}$$

This implies that $q_{t+1}\left(h^*(\omega^t)\Delta S_{t+1}(\omega^t,\cdot) \geq 0|\omega^t\right) = 1$, and thus from (5) in Lemma 3.6 we get that $h^*(\omega^t) = 0$ which contradicts $|h^*(\omega^t)| = 1$. Thus $\overline{n}_0(\omega^t) < \infty$ and we can set for $\omega^t \in \Omega^t_{NA1}$

$$\overline{\alpha}_t(\omega^t) = \frac{1}{\overline{n}_0(\omega^t)}.$$

It is clear that $\overline{\alpha}_t \in (0,1]$. Then for all $\omega^t \in \Omega^t_{NA1}$, for all $h \in D^{t+1}(\omega^t)$ with $|h| = 1$, by definition of $A_{\overline{n}_0(\omega^t)}(\omega^t)$ we obtain

$$q_{t+1}\left(h\Delta S_{t+1}(\omega^t,\cdot) \le -\overline{\alpha}_t(\omega^t)|\omega^t\right) \ge \overline{\alpha}_t(\omega^t). \tag{9}$$

Step 2 : measurability issue.
We now construct a function α_t which is $\mathcal{F}_t$-measurable and satisfies (7) as well. To do that we use the Aumann Theorem again as in the proof of Lemma 3.6 but this time applied to the random set $A_n : \Omega^t \twoheadrightarrow \mathbb{R}^d$ where $A_n(\omega^t)$ is defined in (8) if $\omega^t \in \Omega^t_{NA1}$ and $A_n(\omega^t) = \emptyset$ otherwise.

We prove that $graph(A_n) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$. From Lemma 7.9, the function $(\omega^t, h) \to q_{t+1}\left(h\Delta S_{t+1}(\omega^t,\cdot) \le -\frac{1}{n}|\omega^t\right)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. From Lemma 3.4, $Graph(D^{t+1}) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ and the result follows from

$$\begin{aligned} Graph(A_n) &= Graph(D^{t+1}) \bigcap \left(\Omega^t_{NA1} \times \{h \in \mathbb{R}^d,\ |h| = 1\}\right) \\ &\bigcap \left\{(\omega^t, h) \in \Omega^t \times \mathbb{R}^d,\ q_{t+1}\left(h\Delta S_{t+1}(\omega^t,\cdot) \le -\frac{1}{n}|\omega^t\right) < \frac{1}{n}\right\}. \end{aligned}$$

Using the Projection Theorem (see for example Theorem 3.23 in Castaing and Valadier (1977)), we get that $\{\omega^t \in \Omega^t,\ A_n(\omega^t) \neq \emptyset\} \in \overline{\mathcal{F}}_t$. We now extend $\overline{n}_0$ to Ω^t by setting $\overline{n}_0(\omega^t) = 1$ if $\omega^t \notin \Omega^t_{NA1}$. Then $\{\overline{n}_0 \ge 1\} = \Omega^t \in \mathcal{F}_t \subset \overline{\mathcal{F}}_t$ and for $k > 1$

$$\{\overline{n}_0 \ge k\} = \Omega^t_{NA1} \cap \bigcap_{1 \le n \le k-1} \{A_n \neq \emptyset\} \in \overline{\mathcal{F}}_t,$$

this implies that $\overline{n}_0$ and thus $\overline{\alpha}_t$ is $\overline{\mathcal{F}}_t$-measurable. Using Lemma 7.10, we get some $\mathcal{F}_t$-measurable function α_t such that $\alpha_t = \overline{\alpha}_t$ P_t almost surely, *i.e* there exists $M^t \in \mathcal{F}_t$ such that $P_t(M^t) = 0$ and $\{\alpha_t \neq \overline{\alpha}_t\} \subset M^t$. We set $\Omega^t_{NA} := \Omega^t_{NA1} \bigcap \left(\Omega^t \setminus M_t\right)$. Then $P_t(\Omega^t_{NA}) = 1$ and as α_t is $\mathcal{F}_t$-measurable it remains to check that (7) holds true.

For $\omega^t \in \Omega^t_{NA}$, $\alpha_t(\omega^t) = \overline{\alpha}_t(\omega^t)$ (recall that $\omega^t \in \Omega^t \setminus M_t$) and since $\omega^t \in \Omega^t_{NA1}$, (9) holds true and consequently (7) as well. It is also clear that $\alpha_t(\omega^t) \in (0,1]$ and the proof is completed. □

Remark 3.8 In Definition 3.3, Lemmata 3.4, 3.5, 3.6 and Proposition 3.7 we have included the case $t = 0$. Note however that since $\Omega^0 = \{\omega_0\}$, the various statements and their respective proofs could be considerably simplified.

Remark 3.9 The characterization of (NA) given by (7) works only for $h \in D^{t+1}(\omega^t)$. This is the reason why we will have to project the strategy $\phi_{t+1} \in \Xi_t$ onto $D^{t+1}(\omega^t)$ in our proofs.

Remark 3.10 In order to obtain Proposition 3.7 we could have applied directly Proposition 3.3. of Rásonyi and Stettner (2005) (note their proof doesn't use measurable selection arguments and provides directly the $\mathcal{F}_t$ measurability of α_t) and used Lemma 2.2.

4 Utility problem and main result

We now describe the investor's risk preferences by a possibly non-concave, random utility function.

Definition 4.1 A *random utility* is any function $U : \ \Omega \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ satisfying the following conditions

- for every $x \in \mathbb{R}$, the function $U(\cdot, x) : \ \Omega \to \mathbb{R} \cup \{\pm\infty\}$ is $\mathcal{F}$-measurable,
- for all $\omega \in \Omega$, the function $U(\omega, \cdot) : \ \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ is non-decreasing and usc on $\mathbb{R}$,
- $U(\cdot, x) = -\infty$, for all $x < 0$.

We introduce the following notations.

Definition 4.2 For all $x \geq 0$, we denote by $\Phi(x)$ the set of all strategies $\phi \in \Phi$ such that $P_T(V_T^{x,\phi}(\cdot) \geq 0) = 1$ and by $\Phi(U,x)$ the set of all strategies $\phi \in \Phi(x)$ such that $EU(\cdot, V_T^{x,\phi})$ exists in a generalised sense, *i.e.* either $EU^+(\cdot, V_T^{x,\phi}(\cdot)) < \infty$ or $EU^-(\cdot, V_T^{x,\phi}(\cdot)) < \infty$.

Remark 4.3 Under (NA), if $\phi \in \Phi(x)$ then we have that $P_t(V_t^{x,\phi}(\cdot) \geq 0) = 1$ for all $1 \leq t \leq T$ see Lemma 7.19.

We now formulate the problem which is our main concern in the sequel.

Definition 4.4 Let $x \geq 0$. The *non-concave portfolio problem* on a finite horizon T with initial wealth x is

$$u(x) \quad := \quad \sup_{\phi \in \Phi(U,x)} EU(\cdot, V_T^{x,\phi}(\cdot)). \tag{10}$$

Remark 4.5 Assume that there exists some P-full measure set $\widetilde{\Omega} \in \mathcal{F}$ such that for all $\omega \in \widetilde{\Omega}$, $x \to U(\omega, x)$ is non-decreasing and usc on $[0, +\infty)$, *i.e.* $x \to U(\omega, x)$ is usc on $(0, \infty)$ and for any $(x_n)_{n \geq 1} \subset [0, +\infty)$ converging to 0, $U(\omega, 0) \geq \limsup_n U(\omega, x_n)$. We set $\overline{U} : \Omega \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$

$$\overline{U}(\omega, x) := U(\omega, x) 1_{\widetilde{\Omega} \times [0,+\infty)}(\omega, x) + (-\infty) 1_{\Omega \times (-\infty, 0)}(\omega, x).$$

Then $\overline{U}$ satisfies Definition 4.1, see Lemma 7.11 for the second item. Moreover, the value function does not change

$$u(x) = \sup_{\phi \in \Phi(U,x)} E\overline{U}(\cdot, V_T^{x,\phi}(\cdot)),$$

and if there exists some $\phi^* \in \Phi(U,x)$ such that $u(x) = E\overline{U}(\cdot, V_T^{x,\phi^*}(\cdot))$, then ϕ^* is an optimal solution for (10).

Remark 4.6 Let U be a utility function defined only on $(0, \infty)$ and verifying for every $x \in (0, \infty)$, $U(\cdot, x) : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ is $\mathcal{F}$-measurable and for all $\omega \in \Omega$, $U(\omega, \cdot) : (0, \infty) \to \mathbb{R} \cup \{\pm\infty\}$ is non-decreasing and usc on $(0, \infty)$. We may extend U on $\mathbb{R}$ by setting, for all $\omega \in \Omega$, $\overline{U}(\omega, 0) = \lim_{x \to 0} U(\omega, x)$ and for $x < 0$, $\overline{U}(\omega, x) = -\infty$. Then, as before, $\overline{U}$ verifies Definition 4.1 and the value function has not changed. Note that we could have considered a closed interval $F = [a, \infty)$ of $\mathbb{R}$ instead of $[0, \infty)$, we could have adapted our notion of upper semicontinuity and all the sequel would apply.

We now present conditions on U which allows to assert that if $\phi \in \Phi(x)$ then $EU(\cdot, V_T^{x,\phi}(\cdot))$ is well-defined and that there exists some optimal solution for (10).

Assumption 4.7 For all $\phi \in \Phi(U,1)$, $EU^+\left(\cdot, V_T^{1,\phi}(\cdot)\right) < \infty$.

Assumption 4.8 $\Phi(U,1) = \Phi(1)$.

Remark 4.9 Assumptions 4.7 and 4.8 are connected but play a different role. Assumption 4.8 guarantees that $EU\left(\cdot, V_T^{1,\phi}(\cdot)\right)$ is well-defined for all $\Phi \in \Phi(1)$ and allows us to relax Assumption 2.7 of Carassus et al. (2015) on the behavior of U around 0, namely that $EU^-(\cdot, 0) < \infty$. Then Assumption 4.7 (together with Assumption 4.10) is used to show that $u(x) < \infty$ for all $x > 0$. Note that Assumption 4.7 is much more easy to verify that the classical assumption that $u(x) < \infty$ (for all or some $x > 0$), which is usually made in the theory of maximisation of the terminal wealth utility.

In Proposition 6.1, we will show that under Assumptions 4.7, 4.8 and 4.10, $EU^+\left(\cdot, V_T^{x,\phi}(\cdot)\right) < \infty$ for all $x \geq 0$ and $\phi \in \Phi(x)$. Thus $\Phi(U,x) = \Phi(x)$. Note that if there exists some $\Phi \in \Phi(U,x)$ such that $EU^+\left(\cdot, V_T^{x,\phi}(\cdot)\right) = \infty$ and $EU^-\left(\cdot, V_T^{x,\phi}(\cdot)\right) < \infty$ then $u(x) = \infty$ and the problem is ill-posed.

We propose some examples where Assumptions 4.7 or 4.8 hold true. Example $ii)$ illustrates the distinction between Assumptions 4.7 and 4.8 and justifies we do not merge both assumptions and postulate that $EU^+\left(\cdot, V_T^{1,\phi}(\cdot)\right) < \infty$, for all $\phi \in \Phi(1)$.

i) If U is bounded above then both Assumptions are trivially true. We get directly that $\Phi(U,x) = \Phi(x)$ for all $x \geq 0$.

ii) Assume that $EU^-(\cdot, 0) < \infty$ holds true. Let $x \geq 0$ and $\phi \in \Phi(x)$ be fixed. Using that U^- is non-decreasing for all $\omega \in \Omega$ we get that

$$EU^-(\cdot, V_T^{x,\phi}(\cdot)) \leq EU^-(\cdot, 0) < +\infty,$$

Thus $EU(\cdot, V_T^{x,\phi}(\cdot))$ is well-defined, $\Phi(U,x) = \Phi(x)$ and Assumption 4.8 holds true.

iii) Assume that there exists some $\hat{x} \geq 1$ such that $U(\cdot, \hat{x} - 1) \geq 0$ P-almost surely and

$$\widehat{u}(\hat{x}) := \sup_{\phi \in \Phi(\hat{x})} EU(\cdot, V_T^{\hat{x},\phi}(\cdot)) < \infty,$$

where we set for $\phi \in \Phi(\hat{x}) \backslash \Phi(U, \hat{x})$, $EU(\cdot, V_T^{\hat{x},\phi}(\cdot)) = -\infty$. Let $\phi \in \Phi(1)$ be fixed. Then using that U is non-decreasing for all $\omega \in \Omega$, we have that P-almost surely

$$U(\cdot, V_T^{1,\phi}(\cdot) + \hat{x} - 1) \geq U(\cdot, \hat{x} - 1) \geq 0.$$

Therefore $U(\cdot, V_T^{1,\phi}(\cdot) + \hat{x} - 1) = U^+(\cdot, V_T^{1,\phi}(\cdot) + \hat{x} - 1)$ P-almost surely. Now using that U^+ is non-decreasing for all $\omega \in \Omega$ we get that for all $\phi \in \Phi(1)$

$$EU^+(\cdot, V_T^{1,\phi}(\cdot)) \leq EU^+(\cdot, V_T^{1,\phi}(\cdot) + \hat{x} - 1) = EU(\cdot, V_T^{1,\phi}(\cdot) + \hat{x} - 1) \leq \widehat{u}(\hat{x}) < +\infty$$

and Assumptions 4.7 and 4.8 are satisfied. Instead of stipulating that $\widehat{u}(\hat{x}) < \infty$ it is enough to assume that $EU(\cdot, V_T^{\hat{x},\phi}(\cdot)) < \infty$ for all $\phi \in \Phi(\hat{x})$.

iv) We will prove in Theorem 4.17 that under the (NA) condition and Assumption 4.10, Assumptions 4.7 and 4.8 hold true if $EU^+(\cdot, 1) < +\infty$ and if for all $0 \leq t \leq T$ $|\Delta S_t|$, $\frac{1}{\alpha_t} \in \mathcal{W}_t$ (see (16) for the definition of $\mathcal{W}_t$).

Assumption 4.10 We assume that there exist some constants $\overline{\gamma} \geq 0$, $K > 0$, as well as a random variable C satisfying $C(\omega) \geq 0$ for all $\omega \in \Omega$ and $E(C) < \infty$ such that for all $\omega \in \Omega$, $\lambda \geq 1$ and $x \in \mathbb{R}$, we have

$$U(\omega, \lambda x) \quad \leq \quad K\lambda^{\overline{\gamma}}\left(U\left(\omega, x + \frac{1}{2}\right) + C(\omega)\right). \tag{11}$$

Remark 4.11 First note that the constant $\frac{1}{2}$ in (11) has been chosen arbitrarily to simplify the presentation. This can be done without loss of generality. Indeed, assume there exists some constant $\overline{x} \geq 0$ such that for all $\omega \in \Omega$, $\lambda \geq 1$ and $x \in \mathbb{R}$

$$U(\omega, \lambda x) \quad \leq \quad K\lambda^{\overline{\gamma}}\left(U(\omega, x + \overline{x}) + C(\omega)\right). \tag{12}$$

Using the monotonicity of U, we can always assume $\overline{x} > 0$. Set for all $\omega \in \Omega$ and $x \in \mathbb{R}$, $\overline{U}(\omega, x) = U(\omega, 2\overline{x}x)$. Then for all $\omega \in \Omega$, $\lambda \geq 1$ and $x \in \mathbb{R}$, we have that

$$\overline{U}(\omega, \lambda x) = U(\omega, 2\lambda\overline{x}x) \leq K\lambda^{\overline{\gamma}}\left(U(\omega, 2\overline{x}x + \overline{x}) + C(\omega)\right) = K\lambda^{\overline{\gamma}}\left(\overline{U}\left(\omega, x + \frac{1}{2}\right) + C(\omega)\right),$$

and $\overline{U}$ satisfies (11). It is clear that if ϕ^* is an optimal solution for the problem $\overline{u}(x) := \sup_{\phi \in \Phi(\overline{U}, \frac{x}{2\overline{x}})} E\overline{U}(\cdot, V_T^{\frac{x}{2\overline{x}}, \phi}(\cdot))$ then $2\overline{x}\phi^*$ is an optimal solution for (10). Note as well that, since $K > 0$ and $C \geq 0$, it is immediate to see that for all $\omega \in \Omega$, $\lambda \geq 1$ and $x \in \mathbb{R}$

$$U^+(\omega, \lambda x) \quad \leq \quad K\lambda^{\overline{\gamma}} \left(U^+ \left(\omega, x + \frac{1}{2} \right) + C(\omega) \right). \tag{13}$$

Remark 4.12 We now provide some insight on Assumption 4.10. As the inequality (11) is used to control the behaviour of $U^+(\cdot, x)$ for large values of x, the usual assumption in the non-concave case (see Assumption 2.10 in Carassus et al. (2015)) is that there exists some $\hat{x} \geq 0$ such that $EU^+(\cdot, \hat{x}) < \infty$ as well as a random variable C_1 satisfying $E(C_1) < \infty$ and $C_1(\omega) \geq 0$ for all ω [2] such that for all $x \geq \hat{x}$, $\lambda \geq 1$ and $\omega \in \Omega$

$$U(\omega, \lambda x) \leq \lambda^{\overline{\gamma}} \left(U(\omega, x) + C_1(\omega) \right). \tag{14}$$

We prove now that if (14) holds true then (12) is verified with $\overline{x} = \hat{x}$, $K = 1$ and $C = C_1$. Indeed, assume that (14) is verified. For $x \geq 0$, using the monotonicity of U, we have for all $\omega \in \Omega$ and $\lambda \geq 1$ that

$$U(\omega, \lambda x) \leq U(\omega, \lambda(x + \hat{x})) \leq \lambda^{\overline{\gamma}} \left(U(\omega, x + \hat{x}) + C_1(\omega) \right).$$

And for $x < 0$ this is true as well since $U(\omega, x) = -\infty$.
Therefore (12) is a weaker assumption than (14). Note as well that if we assume that (14) holds true for all $x > 0$, then if $0 < x < 1$ and $\omega \in \Omega$ we have

$$U(\omega, 1) \leq \left(\frac{1}{x} \right)^{\overline{\gamma}} \left(U(\omega, x) + C_1(\omega) \right),$$

and $U(\omega, 0) := \lim_{x \to 0,\, x > 0} U(\omega, x) \geq -C_1(\omega)$. This excludes for instance the case where U is the logarithm. Furthermore, this also implies that $EU^-(\cdot, 0) \leq EC_1 < \infty$ and we are back to Assumption 2.7 of Carassus et al. (2015)
Alternatively, recalling the way the concave case is handled (see Lemma 2 in Rásonyi and Stettner (2005)), we could have introduced that there exists a random variable C_2 satisfying $E(C_2) < \infty$ and $C_2 \geq 0$ such that for all $x \in \mathbb{R}$, $\omega \in \Omega$

$$U^+(\omega, \lambda x) \leq \lambda^{\overline{\gamma}} \left(U^+(\omega, x) + C_2(\omega) \right). \tag{15}$$

We have not done so as it is difficult to prove that this inequality is preserved through the dynamic programming procedure when considering non-concave functions unless we assume that $EU^-(\cdot, 0) < \infty$ as in Carassus et al. (2015).

Remark 4.13 If there exists some set $\Omega_{AE} \in \mathcal{F}$ with $P(\Omega_{AE}) = 1$ such that (11) holds true only for $\omega \in \Omega_{AE}$, then setting as in Remark 4.5, $\overline{U}(\omega, x) := U(\omega, x) 1_{\Omega_{AE} \times \mathbb{R}}(\omega, x)$, $\overline{U}$ satisfies (11) and the value function in (10) does not change. We also assume without loss of generality that $C(\omega) \geq 0$ for all ω in (11). Indeed, if $C \geq 0$ P-a.s, we could consider $\widetilde{C} := C \mathbb{I}_{\overline{C} \geq 0}$. Then Assumption 4.10 would hold true with $\widetilde{C}$ instead of C.

Remark 4.14 In the case where (14) holds true, we refer to remark 2.5 of Carassus and Rásonyi (2015) and remark 2.10 of Carassus et al. (2015) for the interpretation of $\overline{\gamma}$: for $C_1 = 0$, it can be seen as a generalization of the "asymptotic elasticity" of U at $+\infty$ (see Kramkov and Schachermayer (1999)). So (14) requires that the (generalized) asymptotic elasticity at $+\infty$ is finite. In this case and if U is differentiable there is a nice economic interpretation of the "asymptotic elasticity" as the ratio of "marginal utility": $U'(x)$ and the "average utility": $\frac{U(x)}{x}$, see again Section 6 of Kramkov and Schachermayer

[2] In the cited paper $C_1 \geq 0$ a.s but this is not an issue, see Remark 4.13 below

(1999) for further discussions. The case $C_1 > 0$ allows bounded utilities. In Carassus et al. (2015) it is proved that unlike in the concave case, the fact that U is bounded from above (and therefore satisfies (12)) does not implies that the asymptotic elasticity is bounded.
We propose now an example of an unbounded utility function satisfying (12) and such that $\limsup_{x\to\infty} \frac{xU'(x)}{U(x)} = +\infty$. This shows (as the counterexample of Carassus et al. (2015)), that Assumption 4.10 is less strong that the usual "asymptotic elasticity". Let $U : \mathbb{R} \to \mathbb{R}$ be defined by

$$U(x) = -\infty 1_{(-\infty,0)}(x) + \sum_{p\geq 0} p 1_{[p,p+1-\frac{1}{2^{p+1}})}(x) + f_p(x) 1_{[p+1-\frac{1}{2^{p+1}},p+1)}(x)$$

where $f_p(x) = 2^{p+1}x + (p+1)\left(1 - 2^{p+1}\right)$ for $p \in \mathbb{N}$. Then U satisfies Definition 4.1 and we have

$$U'(x) = \sum_{p\geq 0} 2^{p+1} 1_{[p+1-\frac{1}{2^{p+1}},p+1)}(x).$$

We prove that (12) holds true. Note that for all $x \geq 0$ we have $x - 1 \leq U(x) \leq x + 1$. Let $x \geq 0$ and $\lambda \geq 1$ be fixed. Then we get that

$$U(\lambda x) \leq \lambda x + 1 \leq \lambda \left(U(x+1) + 1\right) + 1 \leq \lambda \left(U(x+1) + 2\right),$$

and (12) is true with $K = \overline{x} = 1$ and $C = 2$. Now for $k \geq 0$, let $x_k = k + 1 - \frac{1}{2^{k+2}}$. We have $U(x_k) = f_k(x_k) = k + \frac{1}{2}$ and

$$\frac{x_k U'(x_k)}{U(x_k)} = 2^{k+1} \frac{\left(k + 1 - \frac{1}{2^{k+2}}\right)}{k + \frac{1}{2}} \to_{k\to\infty} +\infty.$$

Remark 4.15 We propose further examples where Assumption 4.10 holds true.

i) Assume that U is bounded from above by some integrable random constant $C_1 \geq 0$ and that $EU^-(\cdot, \frac{1}{2}) < \infty$. Then for all $x \geq 0$, $\lambda \geq 1$, $\omega \in \Omega$ we have

$$\begin{aligned} U(\omega, \lambda x) \leq C_1(\omega) &\leq \lambda U\left(\omega, x + \frac{1}{2}\right) + \lambda\left(C_1(\omega) - U\left(\omega, x + \frac{1}{2}\right)\right) \\ &\leq \lambda U\left(\omega, x + \frac{1}{2}\right) + \lambda\left(C_1(\omega) + U^-\left(\omega, \frac{1}{2}\right)\right), \end{aligned}$$

and (11) holds true for $x \geq 0$ with $K = 1$, $\overline{\gamma} = 1$ and $C(\cdot) = C_1(\cdot) + U^-(\cdot, \frac{1}{2})$. As $U(\cdot, x) = -\infty$ for $x < 0$, (11) is true for all $x \in \mathbb{R}$.

ii) Assume that U satisfies Definition 4.1 and that the restriction of U to $[0, \infty)$ is concave and non-decreasing and that $EU^-(\cdot, 1) < \infty$. We use similar arguments as in Lemma 2 in Rásonyi and Stettner (2006). Indeed, let $x \geq 2$, $\lambda \geq 1$ be fixed we have

$$\begin{aligned} U(\omega, \lambda x) \leq U(\omega, x) + U'(\omega, x)(\lambda x - x) &\leq U(\omega, x) + \frac{U(\omega, x) - U(\omega, 1)}{x - 1}(\lambda - 1)x \\ &\leq U(\omega, x) + 2(\lambda - 1)\left(U(\omega, x) - U(\omega, 1)\right) \\ &\leq U(\omega, x) + 3(\lambda - \frac{1}{3})\left(U(\omega, x) - U(\omega, 1)\right) \\ &\leq 3\lambda\left(U(\omega, x) + U^-(\omega, 1)\right), \end{aligned}$$

where we have used the concavity of U for the first two inequalities and the fact that $x \geq 2$ and U is non-decreasing for the other ones. Thus from the proof that (14) implies (12), we obtain that (12) holds true with $K = 3$, $\overline{\gamma} = 1$, $\overline{x} = 2$ and $C(\cdot) = U^-(\cdot, 1)$.

We can now state our main result.

Theorem 4.16 *Assume the (NA) condition and that Assumptions 4.7, 4.8 and 4.10 hold true. Let* $x \geq 0$*. Then,* $u(x) < \infty$ *and there exists some optimal strategy* $\phi^* \in \Phi(U, x)$ *such that*

$$u(x) = EU(\cdot, V_T^{x,\phi^*}(\cdot)).$$

Moreover $\phi_t^*(\cdot) \in D^t(\cdot)$ *a.s. for all* $0 \leq t \leq T$.

We will use dynamic programming in order to prove our main result. We will combine the approach of Rásonyi and Stettner (2005), Rásonyi and Stettner (2006), Carassus and Rásonyi (2015), Carassus et al. (2015) and Nutz (2014). As in Nutz (2014), we will consider a one period case where the initial filtration is trivial (so that strategies are in $\mathbb{R}^d$) and thus the proofs are much simpler than the ones of Rásonyi and Stettner (2005), Rásonyi and Stettner (2006), Carassus and Rásonyi (2015) and Carassus et al. (2015). The price to pay is that in the multi-period case where we use intensively measurable selection arguments (as in Nutz (2014)) in order to obtain Theorem 4.16. In our model, there is only one probability measure, so we don't have to introduce Borel spaces and analytic sets. Thus our modelisation of $(\Omega, \mathcal{F}, \mathfrak{F}, P)$ is more general than the one of Nutz (2014) restricted to one probability measure. As we are in a non concave setting we use similar ideas to theses of Carassus and Rásonyi (2015) and Carassus et al. (2015).

Finally, as in Rásonyi and Stettner (2005), Rásonyi and Stettner (2006), Carassus and Rásonyi (2015) and Carassus et al. (2015), we propose the following result as a simpler but still general setting where Theorem 4.16 applies. We introduce for all $0 \leq t \leq T$

$$\mathcal{W}_t := \left\{X : \Omega^t \to \mathbb{R} \cup \{\pm\infty\},\ \mathcal{F}_t\text{-measurable},\ E|X|^p < \infty \ \text{ for all } p > 0\right\} \tag{16}$$

Theorem 4.17 *Assume the (NA) condition and that Assumption 4.10 hold true. Assume furthermore that* $EU^+(\cdot, 1) < +\infty$ *and that for all* $0 \leq t \leq T$ $|\Delta S_t|, \frac{1}{\alpha_t} \in \mathcal{W}_t$*. Let* $x \geq 0$*. Then, for all* $\phi \in \Phi(x)$ *and all* $0 \leq t \leq T$*,* $V_t^{x,\phi} \in \mathcal{W}_t$*. Moreover, there exists some optimal strategy* $\phi^* \in \Phi(U, x)$ *such that*

$$u(x) = EU(\cdot, V_T^{x,\phi^*}(\cdot)) < \infty$$

5 One period case

Let $(\overline{\Omega}, \mathcal{H}, Q)$ be a probability space (we denote by E the expectation under Q) and $Y(\cdot)$ a $\mathcal{H}$-measurable $\mathbb{R}^d$-valued random variable. $Y(\cdot)$ could represent the change of value of the price process. Let $D \subset \mathbb{R}^d$ be the smallest affine subspace of $\mathbb{R}^d$ containing the support of the distribution of $Y(\cdot)$. We assume that D contains 0, so that D is in fact a non-empty vector subspace of $\mathbb{R}^d$. The condition corresponding to (NA) in the present setting is

Assumption 5.1 There exists some constant $0 < \alpha \leq 1$ such that for all $h \in D$

$$Q(hY(\cdot) \leq -\alpha|h|) \geq \alpha. \tag{17}$$

Remark 5.2 If $D = \{0\}$ then (17) is trivially true.

Remark 5.3 below is exactly Remark 8 of Carassus and Rásonyi (2015) (see also Lemma 2.6 of Nutz (2014)).

Remark 5.3 Let $h \in \mathbb{R}^d$ and let $h' \in \mathbb{R}^d$ be the orthogonal projection of h on D. Then $h - h' \perp D$ hence $\{Y(\cdot) \in D\} \subset \{(h - h')Y(\cdot) = 0\}$. It follows that

$$Q(hY(\cdot) = h'Y(\cdot)) = Q((h - h')Y(\cdot) = 0) \geq Q(Y(\cdot) \in D) = 1$$

by the definition of D. Hence $Q(hY(\cdot) = h'Y(\cdot)) = 1$.

Assumption 5.4 We consider a *random utility* $V: \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfying the following two conditions

- for every $x \in \mathbb{R}$, the function $V(\cdot, x) : \overline{\Omega} \to \mathbb{R}$ is $\mathcal{H}$-measurable,
- for every $\omega \in \overline{\Omega}$, the function $V(\omega, \cdot) : \mathbb{R} \to \mathbb{R}$ is non-decreasing and usc on $\mathbb{R}$,
- $V(\cdot, x) = -\infty$, for all $x < 0$.

Let $x \geq 0$ be fixed. We define

$$\mathcal{H}_x := \left\{ h \in \mathbb{R}^d,\ Q(x + hY(\cdot) \geq 0) = 1 \right\}, \tag{18}$$
$$D_x := \mathcal{H}_x \cap D. \tag{19}$$

It is clear that $\mathcal{H}_x$ and D_x are closed subsets of $\mathbb{R}^d$. We now define the function which is our main concern in the one period case

$$v(x) = (-\infty)1_{(-\infty,0)}(x) + 1_{[0,+\infty)}(x) \sup_{h \in \mathcal{H}_x} EV\left(\cdot, x + hY(\cdot)\right). \tag{20}$$

Remark 5.5 First note that, from Remark 5.3,

$$v(x) = (-\infty)1_{(-\infty,0)}(x) + 1_{[0,+\infty)}(x) \sup_{h \in D_x} EV(\cdot, x + hY(\cdot)). \tag{21}$$

Remark 5.6 It will be shown in Lemma 5.11 that under Assumptions 5.1, 5.4, 5.7 and 5.9, for all $h \in \mathcal{H}_x$, $E(V(\cdot, x + hY(\cdot))$ is well-defined and more precisely that $EV^+(\cdot, x + hY(\cdot)) < +\infty$. So, under this set of assumptions, $\Phi(V, x)$, the set of $h \in \mathcal{H}_x$ such that $EV(\cdot, x+hY(\cdot))$ is well-defined, equals $\mathcal{H}_x$.

We present now the assumptions which allow to assert that there exists some optimal solution for (20). First we introduce the "asymptotic elasticity" assumption.

Assumption 5.7 There exist some constants $\overline{\gamma} \geq 0$, $K > 0$, as well as some $\mathcal{H}$-measurable C with $C(\omega) \geq 0$ for all $\omega \in \overline{\Omega}$ and $E(C) < \infty$, such that for all $\omega \in \overline{\Omega}$, for all $\lambda \geq 1$, $x \in \mathbb{R}$ we have

$$V(\omega, \lambda x) \leq K\lambda^{\overline{\gamma}} \left(V\left(\omega, x + \frac{1}{2}\right) + C(\omega) \right). \tag{22}$$

Remark 5.8 The same comments as in Remark 4.13 apply. Furthermore, note that since $K > 0$ and $C \geq 0$ we also have that for all $\omega \in \overline{\Omega}$, all $\lambda \geq 1$ and $x \in \mathbb{R}$

$$V^+(\omega, \lambda x) \leq K\lambda^{\overline{\gamma}} \left(V^+\left(\omega, x + \frac{1}{2}\right) + C(\omega) \right). \tag{23}$$

We introduce now some integrability assumption on V^+.

Assumption 5.9 For every $h \in \mathcal{H}_1$,

$$EV^+(\cdot, 1 + hY(\cdot)) < \infty. \tag{24}$$

The following lemma corresponds to Lemma 2.1 of Rásonyi and Stettner (2006) in the deterministic case.

Lemma 5.10 *Assume that Assumption 5.1 holds true. Let $x \geq 0$ be fixed. Then $D_x \subset B(0, \frac{x}{\alpha})$ (see* (19) *for the definition of D_x), where $B(0, \frac{x}{\alpha}) = \{h \in \mathbb{R}^d,\ |h| \leq \frac{x}{\alpha}\}$ and D_x is a convex, compact subspace of $\mathbb{R}^d$.*

Note that if $x = 0$, it follows that $D_x = \{0\}$.

Proof. Let $h \in D_x$. Assume that $|h| > \frac{x}{\alpha}$ and let $\omega \in \{hY(\cdot) \leq -\alpha|h|\}$. Then $x + hY(\omega) < x - \alpha|h| < 0$ and from Assumption 5.1 $Q(x+hY(\cdot) < 0) \geq Q(hY(\cdot) \leq -\alpha|h|) \geq \alpha > 0$, a contradiction. The convexity and the closedness of D_x are clear and the compactness follows from the boundness property. □

This lemma corresponds in the deterministic case to Lemma 4.8 of Carassus et al. (2015) (see also Lemma 2.3 of Rásonyi and Stettner (2006) and Lemma 2.8 of Nutz (2014)).

Lemma 5.11 *Assume that Assumptions 5.1, 5.4, 5.7 and 5.9 hold true. Then there exists a $\mathcal{H}$-measurable $L \geq 0$ satisfying $E(L) < \infty$ and such that for all $x \geq 0$ and $h \in \mathcal{H}_x$*

$$V^+(\cdot, x + hY(\cdot)) \leq \left((2x)^{\overline{\gamma}}K + 1\right) L(\cdot) \; Q - a.s. \tag{25}$$

Proof. The proof is reported in Section 7.3 of the Appendix □

Lemma 5.12 *Assume that Assumptions 5.1, 5.4, 5.7 and 5.9 hold true. Let $\mathcal{D}$ be the set valued function that assigns to each $x \geq 0$ the set D_x. Then $Graph(\mathcal{D}) := \{(x,h) \in [0,+\infty) \times \mathbb{R}^d, \; h \in D_x\}$ is a closed subset of $\mathbb{R} \times \mathbb{R}^d$. Let $\psi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$ be defined by*

$$\psi(x,h) := \begin{cases} EV(\cdot, x + hY(\cdot)), \textit{if } (x,h) \in Graph(\mathcal{D}) \\ -\infty, \textit{otherwise.} \end{cases} \tag{26}$$

Then ψ is usc on $\mathbb{R} \times \mathbb{R}^d$ and $\psi < +\infty$ on $Graph(\mathcal{D})$.

Proof. Let $(x_n, h_n)_{n \geq 1} \in Graph(\mathcal{D})$ be a sequence converging to some $(x^*, h^*) \in \mathbb{R} \times \mathbb{R}^d$. We prove first that $(x^*, h^*) \in Graph(\mathcal{D})$, *i.e* that $Graph(\mathcal{D})$ is a closed set. It is clear that $x^* \geq 0$. Set for $n \geq 1$ $E_n := \{\omega \in \overline{\Omega}, \; x_n + h_n Y(\omega) \geq 0\}$ and $E^* := \{\omega \in \overline{\Omega}, \; x^* + h^* Y(\omega) \geq 0\}$. It is clear that $\limsup_n E_n \subset E^*$ and applying the Fatou Lemma (the *limsup* version) we get

$$Q\left(x^* + h^*Y(\cdot) \geq 0\right) = E1_{E^*}(\cdot) \geq E \limsup_n 1_{E_n}(\cdot) \geq \limsup_n E1_{E_n}(\cdot) = 1,$$

and $h^* \in \mathcal{H}_{x^*}$. Since D is closed by definition we have $h^* \in D_{x^*}$ and $(x^*, h^*) \in Graph(\mathcal{D})$.
We prove now that ψ is usc on $Graph(\mathcal{D})$. The upper semicontinuity on $\mathbb{R} \times \mathbb{R}^d$ will follow immediately from Lemma 7.11. By Assumption 5.4 $x \in \mathbb{R} \to V(x, \omega)$ is usc on $\mathbb{R}$ for all $\omega \in \overline{\Omega}$ and thus

$$\limsup_n V(\omega, x_n + h_n Y(\omega)) \leq V(\omega, x^* + h^* Y(\omega)).$$

By Lemma 5.11 for all $\omega \in \overline{\Omega}$

$$V(\omega, x_n + h_n Y(\cdot)) \leq V^+(\omega, x_n + h_n Y(\cdot)) \leq (|2x_n|^{\overline{\gamma}}K + 1)L(\omega) \leq (|2x^*|^{\overline{\gamma}}K + 2)L(\omega)$$

for n big enough. We can apply Fatou's Lemma (the *limsup* version) and ψ is usc on $Graph(\mathcal{D})$. From Lemma 5.11 it is also clear that $\psi < +\infty$ on $Graph(\mathcal{D})$. □
We are now able to state our main result.

Theorem 5.13 *Assume that Assumptions 5.1, 5.4, 5.7 and 5.9 hold true. Then for all $x \geq 0$, $v(x) < \infty$ and there exists some optimal strategy $\widehat{h} \in D_x$ such that*

$$v(x) = E(V(\cdot, x + \widehat{h}Y(\cdot))).$$

Moreover, $v : \; \mathbb{R} \to [-\infty, \; \infty)$ is non-decreasing and usc on $\mathbb{R}$.

Proof. Let $x \geq 0$ be fixed. We show first that $v(x) < \infty$. Indeed, using Lemma 5.11,

$$E(V(\cdot, x + hY(\cdot))) \leq E(V^+(\cdot, x + hY(\cdot))) \leq \left((2x)^{\overline{\gamma}} K + 1\right) EL(\cdot),$$

for all $h \in D_x$. Thus, recalling (21), $v(x) \leq \left((2x)^{\overline{\gamma}} + 1\right) EL(\cdot) < \infty$.
From Lemma 5.12, $h \in \mathbb{R}^d \to E(V(\cdot, x + hY(\cdot)))$ is usc on $\mathbb{R}^d$ and thus on D_x (recall that D_x is closed and see Lemma 7.11). Since by (21), $v(x) = \sup_{h \in D_x} E(\cdot, V(x + hY(\cdot)))$ and D_x is compact (see Lemma 5.10), applying Theorem 2.43 of Aliprantis and Border (2006) there exists some $\widehat{h} \in D_x$ such that

$$v(x) = E(V(\cdot, x + \widehat{h}Y(\cdot))). \tag{27}$$

We show that v is usc on $[0, +\infty)$. As previously, the upper semicontinuity on $\mathbb{R}$ will follow immediately from Lemma 7.11. Let $(x_n)_{n \geq 0}$ be a sequence of non-negative numbers converging to some $x^* \in [0, +\infty)$. Let $\widehat{h}_n \in D_{x_n}$ be the associated optimal strategies to x_n in (27). Let $(n_k)_{k \geq 1}$ be a subsequence such that $\limsup_n v(x_n) = \lim_k v(x_{n_k})$. By Lemma 5.10 $|\widehat{h}_{n_k}| \leq x_{n_k}/\beta \leq (x^* + 1)/\beta$ for k big enough. So we can extract a subsequence (that we still denote by $(n_k)_{k \geq 1}$) such that there exists some $\underline{h}^*$ with $\widehat{h}_{n_k} \to \underline{h}^*$. As the sequence $(x_{n_k}, \hat{h}_{n_k})_{k \geq 1} \in Graph(\mathcal{D})$ converges to $(x^*, \underline{h}^*)$ and $Graph(\mathcal{D})$ is closed (see Lemma 5.12), we get that $\underline{h}^* \in \mathcal{D}_{x^*}$. Using Lemma 5.12

$$\limsup_n v(x_n) = \lim_k v(x_{n_k}) = \lim_k EV(\cdot, x_{n_k} + \widehat{h}_{n_k} Y(\cdot)) \leq EV(\cdot, x^* + \underline{h}^* Y(\cdot)) \leq v(x^*),$$

where the last inequality holds true because $\underline{h}^* \in D_{x^*}$ and therefore v is usc on $[0, +\infty)$. Now as, by Assumption 5.4, $V(\omega, \cdot)$ is non-decreasing for all $\omega \in \overline{\Omega}$, v is also non-decreasing on $[0, +\infty)$ and since $v(x) = -\infty$ on $(-\infty, 0)$, v is non-decreasing on $\mathbb{R}$. □

6 Multi-period case

We first prove the following proposition.

Proposition 6.1 *Let Assumptions 4.7, 4.8 and 4.10 hold true. Then* $EU^+\left(\cdot, V_T^{x,\phi}(\cdot)\right) < \infty$ *for all* $x \geq 0$ *and* $\phi \in \Phi(x)$. *This implies that* $\Phi(U, x) = \Phi(x)$.

Proof. Fix $0 \leq x \leq 1$ and let $\phi \in \Phi(x)$. Then $V_T^{x,\phi} \leq V_T^{1,\phi}$ and $\phi \in \Phi(1) = \Phi(1, U)$ (recall Assumption 4.8). For any $\omega \in \Omega$, the function $y \to U(\omega, y)$ is non-decreasing on $\mathbb{R}$, so that $EU^+\left(\cdot, V_T^{x,\phi}(\cdot)\right) \leq EU^+\left(\cdot, V_T^{1,\phi}(\cdot)\right) < \infty$ by Assumption 4.7. Now, if $x \geq 1$, let $\phi \in \Phi(x)$ be fixed. From Assumption 4.10 we get that for all $\omega \in \Omega$

$$U(\omega, V_T^{x,\phi}(\omega)) = U\left(\omega, 2x\left(\frac{1}{2} + \sum_{t=1}^{T} \frac{\phi_t(\omega^{t-1})}{2x} \Delta S_t(\omega^t)\right)\right) \leq (2x)^{\overline{\gamma}} K \left(U(\omega, V_T^{1, \frac{\phi}{2x}}(\omega)) + C(\omega)\right).$$

By Assumption 4.8, $\frac{\phi}{2x} \in \Phi(\frac{1}{2}) \subset \Phi(1) = \Phi(1, U)$. Thus

$$EU^+\left(\cdot, V_T^{x,\phi}(\cdot)\right) \leq (2x)^{\overline{\gamma}} K \left(EU^+\left(\cdot, V_T^{1, \frac{\phi}{2x}}(\cdot)\right) + E(C)\right) < \infty$$

using Assumption 4.7 and the fact that C is integrable (see Assumption 4.10). In both cases, we conclude that $\Phi(x) = \Phi(U, x)$. □

We introduce now the dynamic programming procedure. First we set for all $t \in \{0, \dots, T-1\}$, $\omega^t \in \Omega^t$ and $x \geq 0$

$$\mathcal{H}_x^{t+1}(\omega^t) := \left\{ h \in \mathbb{R}^d,\ q_{t+1}(x + h\Delta S_{t+1}(\omega^t, \cdot) \geq 0 | \omega^t) = 1 \right\}, \tag{28}$$

$$\mathcal{D}_x^{t+1}(\omega^t) := \mathcal{H}_x^{t+1}(\omega^t) \cap D^{t+1}(\omega^t), \tag{29}$$

where D^{t+1} was introduced in Definition 3.3. For $x < 0$ we set $\mathcal{H}_x^{t+1}(\omega^t) = \emptyset$.
We define for all $t \in \{0, \dots, T\}$ the following functions U_t from $\Omega^t \times \mathbb{R} \to \mathbb{R}$. Starting with $t = T$, we set for all $x \in \mathbb{R}$, all $\omega^T \in \Omega$

$$U_T(\omega^T) := U(\omega^T). \tag{30}$$

Recall that $U(\omega^T, x) = -\infty$ for all $(\omega^T, x) \in \Omega \times (-\infty, 0)$.
Using for $t \geq 1$ the full-measure set $\widetilde{\Omega}^t \in \mathcal{F}_t$ that will be defined by induction in Propositions 6.9 and 6.10, we set for all $x \in \mathbb{R}$ and $\omega^t \in \Omega^t$

$$U_t(\omega^t, x) := (-\infty)1_{(-\infty,0)}(x) + 1_{\widetilde{\Omega}^t \times [0,+\infty)}(\omega^t, x) \sup_{h \in \mathcal{H}_x^{t+1}(\omega^t)} \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, x + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) q_{t+1}(d\omega_{t+1}|\omega^t). \tag{31}$$

Finally for $t = 0$

$$U_0(x) := (-\infty)1_{(-\infty,0)}(x) + 1_{[0,+\infty)}(x) \sup_{h \in \mathcal{H}_x^1} \int_{\Omega_1} U_1(\omega_1, x + h\Delta S_1(\omega_1)) P_1(d\omega_1). \tag{32}$$

Remark 6.2 We will prove by induction that U_t is well-defined (see (34)), $i.e$ the integrals in (31) and (32) are well-defined in the generalised sense.

Remark 6.3 Before going further we provide some explanations on the choice of U_t. The natural definition of U_t should have been

$$\mathcal{U}_t(\omega^t, x) := (-\infty)1_{(-\infty,0)}(x) + 1_{[0,+\infty)}(x) \sup_{h \in \mathcal{H}_x^{t+1}(\omega^t)} \int_{\Omega_{t+1}} \mathcal{U}_{t+1}(\omega^t, \omega_{t+1}, x + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) q_{t+1}(d\omega_{t+1}|\omega^t).$$

Introducing the P_t full measure set $\widetilde{\Omega}^t$ in (31) is related to measurability issues that will be tackled in Proposition 6.11. This is not a surprise as this is related to the use of conditional expectations which are defined only almost everywhere.

Lemma 6.4 *Let $0 \leq t \leq T-1$ and H be a fixed $\mathbb{R}$-valued and $\mathcal{F}_t$-measurable random variable. Consider the following random sets*

$$\mathcal{H}_H^{t+1} : \omega^t \in \Omega^t \twoheadrightarrow \mathcal{H}_{H(\omega^t)}^{t+1}(\omega^t),$$
$$\mathcal{D}_H^{t+1} : \omega^t \in \Omega^t \twoheadrightarrow \mathcal{D}_{H(\omega^t)}^{t+1}(\omega^t).$$

Then those random sets are all closed-valued and with graph valued in $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$.

Proof. First it is clear that $\mathcal{H}_H^{t+1}$ is closed-valued. As D^{t+1} is closed-valued (see Lemma 3.4) it follows that $\mathcal{D}_H^{t+1}$ is closed-valued as well. The fact that $Graph(\mathcal{H}_H^{t+1}) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ follows immediately from

$$Graph(\mathcal{H}_H^{t+1}) = \left\{ (\omega^t, h) \in \Omega^t \times \mathbb{R}^d, H(\omega^t) \geq 0,\ q_{t+1}\left(\left\{ H(\omega^t) + h\Delta S_{t+1}(\omega^t, .) \geq 0 \right\} = 1 | \omega^t \right) \right\},$$

and Lemma 7.9 (recall that H is $\mathcal{F}_t$-measurable). We know from Lemma 3.4 that $Graph(D^{t+1}) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ and it follows that

$$Graph(\mathcal{D}_H^{t+1}) = Graph(D^{t+1}) \cap Graph(\mathcal{H}_H^{t+1}) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$$

. $\square$

Finally we introduce

$$\begin{aligned} C_T(\omega^T) &:= C(\omega^T), \text{ for } \omega^T \in \Omega^T, \text{ where } C \text{ is defined in Assumption 4.10} \\ C_t(\omega^t) &:= \int_{\Omega_{t+1}} C_{t+1}(\omega^t, \omega_{t+1}) q_{t+1}(d\omega_{t+1}|\omega^t) \text{ for } t \in \{0, \dots, T-1\}, \omega^t \in \Omega^t. \end{aligned} \tag{33}$$

Lemma 6.5 *The functions $\omega^t \in \Omega^t \to C_t(\omega^t)$ are well-defined, non-negative (for all ω^t), $\mathcal{F}_t$-measurable and satisfy $E(C_t) = E(C_T) < \infty$. Furthermore, for all $t \in \{1, \dots, T\}$, there exists $\Omega_C^t \in \mathcal{F}_t$ and with $P_t(\Omega_C^t) = 1$ and such that $C_t < \infty$ on Ω_C^t. For $t = 0$ we have $C_0 < \infty$.*

Proof. We proceed by induction. For $t = T$ by Assumption 4.10 $C_T = C$ is $\mathcal{F}_T$-measurable, $C_T \geq 0$ and $E(C_T) < \infty$. Assume now that C_{t+1} is $\mathcal{F}_{t+1}$-measurable, $C_{t+1} \geq 0$ and $E(C_{t+1}) = E(C_T) < \infty$. From Proposition 7.6 $i)$ applied to $f = C_{t+1}$ we get that $\omega^t \to C_t(\omega^t) = \int_{\Omega_{t+1}} C_{t+1}(\omega^t, \omega_{t+1}) q_{t+1}(d\omega_{t+1}|\omega^t)$ is $\mathcal{F}_t$-measurable. As $C_{t+1}(\omega^{t+1}) \geq 0$ for all ω^{t+1}, it is clear that $C_t(\omega^t) \geq 0$ for all ω^t. Applying the Fubini theorem (see Lemma 7.1) we get that

$$\begin{aligned} E(C_t) &= \int_{\Omega^t} \int_{\Omega_{t+1}} C_{t+1}(\omega^t, \omega_{t+1}) q_{t+1}(d\omega_{t+1}|\omega^t) P_t(d\omega^t) \\ &= \int_{\Omega^{t+1}} C_{t+1}(\omega^{t+1}) P_{t+1}(d\omega^{t+1}) = E(C_{t+1}) = E(C_T) < \infty. \end{aligned}$$

and the induction step is complete. For the second part of the lemma, we apply Lemma 7.7 to $f = C_{t+1}$ and we obtain that $\Omega_C^t := \{\omega^t \in \Omega^t,\ C_t(\omega^t) < \infty\} \in \mathcal{F}_t$ and $P_t(\Omega_C^t) = 1$. $\square$

Propositions 6.7 to 6.11 below solve the dynamic programming procedure and hold true under the following set of conditions. Let $1 \leq t \leq T$ be fixed.

$$U_t\left(\omega^t, \cdot\right) : \mathbb{R} \to \mathbb{R} \text{ is well-defined, non-decreasing and usc on } \mathbb{R} \text{ for all } \omega^t \in \Omega^t, \tag{34}$$

$$U_t\left(\cdot, \cdot\right) : \Omega^t \times \mathbb{R} \to \mathbb{R}\{\pm\infty\} \text{ is } \mathcal{F}_t \otimes \mathbb{B}(\mathbb{R})\text{-measurable}, \tag{35}$$

$$\int_{\Omega^t} U_t^+(\omega^t, H(\omega^{t-1}) + \xi(\omega^{t-1})\Delta S_t(\omega^t)) P_t(d\omega^t) < \infty, \tag{36}$$

for all $\xi \in \Xi_{t-1}$ and $H = x + \sum_{s=1}^{t-1} \phi_s \Delta S_s$ where $x \geq 0$, $\phi_1 \in \Xi_0, \dots, \phi_{t-1} \in \Xi_{t-2}$ and $P_t(H(\cdot) + \xi(\cdot)\Delta S_t(\cdot) \geq 0) = 1$,

$$U_t(\omega^t, \lambda x) \leq \lambda^{\overline{\gamma}} K \left(U_t\left(\omega^t, x + \frac{1}{2}\right) + C_t(\omega^t) \right), \text{ for all } \omega^t \in \Omega^t,\ \lambda \geq 1,\ x \in \mathbb{R}. \tag{37}$$

Remark 6.6 Note that from (34) and (35) we have that $-U_t$ is a $\overline{\mathcal{F}}_t$-normal integrand (see Definition 14.27 in Rockafellar and Wets (1998) or Section 3 of Chapter 5 in Molchanov (2005) and Corollary 14.34 of Rockafellar and Wets (1998)). However to prove that this property is preserved in the dynamic programming procedure we need to show separately that (34) and (35) are true. Furthermore, as our sigma-algebras are not assumed to be complete, obtaining some $\mathcal{F}_t$-normal integrand from $-U_t$ would introduce yet another layer of difficulty. For these reasons we choose to prove (34) and (35) instead of some normal integrand property. Nevertheless we will use again the properties of normal integrands in the proof of Lemma 6.11.

The next proposition is a first step in the construction of $\widetilde{\Omega}^t$.

Proposition 6.7 *Let $0 \leq t \leq T-1$ be fixed. Assume that (NA) condition holds true and that (34), (35), (36) and (37) hold true at stage $t+1$. Then there exists $\widetilde{\Omega}_1^t \in \mathcal{F}_t$ such that $P_t(\widetilde{\Omega}_1^t) = 1$ and such that for all $\omega^t \in \widetilde{\Omega}_1^t$ the function $(\omega_{t+1}, x) \to U_{t+1}(\omega^t, \omega_{t+1}, x)$ satisfies the assumptions of Theorem 5.13 with $\overline{\Omega} = \Omega_{t+1}$, $\mathcal{H} = \mathcal{G}_{t+1}$, $Q(\cdot) = q_{t+1}(\cdot|\omega^t)$, $Y(\cdot) = \Delta S_{t+1}(\omega^t, \cdot)$, $V(\cdot, y) = U_{t+1}(\omega^t, \cdot, y)$ where V is defined on $\Omega_{t+1} \times \mathbb{R}$.*

Remark 6.8 Note that Lemmata 5.11, 5.12 and Theorem 5.13 hold true under the same set of assumptions. Therefore we can replace Theorem 5.13 by either Lemmata 5.11 or 5.12 in the above proposition.

Proof. To prove the proposition we will review one by one the assumptions needed to apply Theorem 5.13 in the context $\overline{\Omega} = \Omega_{t+1}$, $\mathcal{H} = \mathcal{G}_{t+1}$, $Q(\cdot) = q_{t+1}(\cdot|\omega^t)$, $Y(\cdot) = \Delta S_{t+1}(\omega^t, \cdot)$, $V(\cdot, y) = U_{t+1}(\omega^t, \cdot, y)$ where V is defined on $\Omega_{t+1} \times \mathbb{R}$. In the sequel we shortly call this the context $t+1$.
From (34) at $t+1$ for all $\omega^t \in \Omega^t$ and $\omega_{t+1} \in \Omega_{t+1}$, the function $x \in \mathbb{R} \to U_{t+1}(\omega^t, \omega_{t+1}, x)$ is non-decreasing and usc on $\mathbb{R}$. From (35) at $t+1$ for all fixed $\omega^t \in \Omega^t$ and $x \in \mathbb{R}$, the function $\omega_{t+1} \in \Omega_{t+1} \to U_{t+1}(\omega^t, \omega_{t+1}, x)$ is $\mathcal{G}_{t+1}$-measurable and thus Assumption 5.4 is satisfied in the context $t+1$ (recall that $U_{t+1}(\omega^t, \omega_{t+1}, x) = -\infty$ for all $x < 0$ by assumption).
We move now to the assumptions that are verified for ω^t chosen in some specific P_t-full measure set. First from Lemma 3.6 for all $\omega^t \in \Omega^t_{NA1}$ we have $0 \in D^{t+1}(\omega^t)$ (recall that in Section 5 we have assume that D contains 0). From Proposition 3.7, Assumption 5.1 holds true for all $\omega^t \in \Omega^t_{NA}$ in the context $t+1$.
We handle now Assumption 5.7 on asymptotic elasticity in context $t+1$. Let $\omega^t \in \Omega^t_C$ be fixed where Ω^t_C is defined in Lemma 6.5. From (37) at $t+1$ we have that for all $\omega_{t+1} \in \Omega_{t+1}$, $\lambda \geq 1$ and $x \in \mathbb{R}$

$$U_{t+1}(\omega^t, \omega_{t+1}, \lambda x) \leq \lambda^{\overline{\gamma}} K \left(U_{t+1}\left(\omega^t, \omega_{t+1}, x + \frac{1}{2}\right) + C_{t+1}(\omega^t, \omega_{t+1}) \right).$$

Now from Lemma 6.5 since $\omega^t \in \Omega^t_C$, we get that

$$\int_{\Omega_{t+1}} C_{t+1}(\omega^t, \omega_{t+1}) q_{t+1}(\omega_{t+1}|d\omega^t) = C_t(\omega^t) < \infty$$

and thus Assumption 5.7 in context $t+1$ is verified for all $\omega^t \in \Omega^t_C$. want to show that for ω^t in some P_t full measure set to be determined and for all $h \in \mathcal{H}^{t+1}_1(\omega^t)$ we have that

$$\int_{\Omega_{t+1}} U^+_{t+1}(\omega^t, \omega_{t+1}, 1 + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) q_{t+1}(d\omega_{t+1}|\omega^t) < \infty.$$

We introduce the following random set $I_1 : \Omega^t \twoheadrightarrow \mathbb{R}^d$

$$I_1(\omega^t) := \left\{ h \in \mathcal{H}^{t+1}_1(\omega^t),\ \int_{\Omega_{t+1}} U^+_{t+1}(\omega^t, \omega_{t+1}, 1 + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) q_{t+1}(d\omega_{t+1}|\omega^t) = \infty \right\}. \tag{38}$$

Arguing by contradiction and using measurable selection arguments we will prove that $I_1(\omega^t) = \emptyset$ for P_t-almost all $\omega^t \in \Omega^t$. We show first that $Graph(I_1) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$. It is clear from (35) at $t+1$ that $(\omega^t, \omega_{t+1}, h) \to U^+_{t+1}(\omega^t, \omega_{t+1}, 1 + h\Delta S_{t+1}(\omega^t, \omega_{t+1}))$ is $\mathcal{F}_t \otimes \mathcal{G}_{t+1} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. Using Proposition 7.6 *ii)* we get that $(\omega^t, h) \to \int_{\Omega_{t+1}} U^+_{t+1}(\omega^t, \omega_{t+1}, 1 + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) q_{t+1}(d\omega_{t+1}|\omega^t)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$-measurable (taking potentially the value $+\infty$). From Lemma 6.4, we obtain $Graph(\mathcal{H}^{t+1}_1) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ and $Graph(I_1) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ follows.
Applying the Projection Theorem (see for example Theorem 3.23 in Castaing and Valadier (1977)) we obtain that $\{I_1 \neq \emptyset\} \in \overline{\mathcal{F}}_t$ and using the Aumann Theorem (see Corollary 1 in Sainte-Beuve (1974)) there exists some $\overline{\mathcal{F}}_t$-measurable $\overline{h}_1 : \{I_1 \neq \emptyset\} \to \mathbb{R}^d$ such that for all $\omega^t \in \{I_1 \neq \emptyset\}$, $\overline{h}_1(\omega^t) \in I_1(\omega^t)$. We extend $\overline{h}_1$ on all Ω^t by setting $\overline{h}_1(\omega^t) = 0$ on $\Omega^t \setminus \{I_1 \neq \emptyset\}$. As $\{I_1 \neq \emptyset\} \in \overline{\mathcal{F}}_t$ it is clear that $\overline{h}_1$ remains $\overline{\mathcal{F}}_t$-measurable. Using Lemma 7.10 we get some $\mathcal{F}_t$-measurable $h_1 : \Omega^t \to \mathbb{R}^d$ and $\Omega^t_{I_1} \in \mathcal{F}_t$ such that $P_t(\Omega^t_{I_1}) = 1$ and $\Omega^t_{I_1} \subset \{\omega^t \in \Omega^t, h_1(\omega^t) = \overline{h}_1(\omega^t)\}$. Arguing as in the proof of Lemma 3.6 and using the Fubini Theorem (see Lemma 7.1) we get that

$$\begin{aligned} P_{t+1}\left(1 + h_1(\cdot)\Delta S_{t+1}(\cdot) \geq 0\right) &= \int_{\Omega^t} q_{t+1}(1 + h_1(\omega^t)\Delta S_{t+1}(\omega^t, \cdot) \geq 0|\omega^t) P_t(d\omega^t) \\ &= \int_{\Omega^t} q_{t+1}(1 + \overline{h}_1(\omega^t)\Delta S_{t+1}(\omega^t, \cdot) \geq 0|\omega^t) \overline{P}_t(d\omega^t) \\ &= 1. \end{aligned}$$

Now assume that $\overline{P}_t(\{I_1 \neq \emptyset\}) > 0$. Since $h_1 \in \Xi_t$ and $P_{t+1}(1 + h_1(\cdot)\Delta S_{t+1}(\cdot) \geq 0) = 1$ from (36) at $t+1$ applied to $H = 1$

$$\int_{\Omega^{t+1}} U^+_{t+1}(\omega^{t+1}, 1 + h_1(\omega^t)\Delta S_{t+1}(\omega^{t+1}))P_{t+1}(d\omega^{t+1}) < \infty.$$

We argue as in Lemma 3.6 again. Let

$$\varphi_1(\omega^t) = \int_{\Omega_{t+1}} U^+_{t+1}(\omega^t, \omega_{t+1}, 1 + h_1(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t),$$
$$\overline{\varphi}_1(\omega^t) = \int_{\Omega_{t+1}} U^+_{t+1}(\omega^t, \omega_{t+1}, 1 + \overline{h}_1(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t).$$

We have already seen that $(\omega^t, h) \in \Omega^t \times \mathbb{R}^d \to \int_{\Omega_{t+1}} U^+_{t+1}(\omega^t, \omega_{t+1}, 1 + h\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$-measurable (taking potentially value $+\infty$). By composition it is clear that φ_1 is $\mathcal{F}_t$-measurable and that $\overline{\varphi}_1$ is $\overline{\mathcal{F}}_t$-measurable. Furthermore as $\{\omega^t \in \Omega^t,\ \varphi_1(\omega^t) \neq \overline{\varphi}_1(\omega^t)\} \subset \{\omega^t \in \Omega^t,\ h_1(\omega^t) \neq \overline{h}_1(\omega^t)\}$, $\varphi_1 = \overline{\varphi}_1$ P_t-almost surely. This implies that $\int_{\Omega^t} \overline{\varphi}_1 d\overline{P}_t = \int_{\Omega^t} \varphi_1 dP_t$ and using again the Fubini Theorem (see Lemma 7.1) we get that

$$\begin{aligned}
&\int_{\Omega^{t+1}} U^+_{t+1}(\omega^{t+1}, x + h_1(\omega^t)\Delta S_{t+1}(\omega^{t+1})P_{t+1}(d\omega^{t+1})\\
&= \int_{\Omega^t}\int_{\Omega_{t+1}} U^+_{t+1}(\omega^t, \omega_{t+1}, 1 + h_1(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t)P_t(d\omega^t)\\
&= \int_{\Omega^t}\int_{\Omega_{t+1}} U^+_{t+1}(\omega^t, \omega_{t+1}, 1 + \overline{h}_1(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t)\overline{P}_t(d\omega^t)\\
&\geq \int_{\{I_1 \neq \emptyset\}} (+\infty)\overline{P}_t(d\omega^t) = +\infty.
\end{aligned}$$

Therefore we must have $\overline{P}_t(\{I_1 \neq \emptyset\}) = 0$ *i.e* $\overline{P}_t(\{I_1 = \emptyset\}) = 1$. Now since $\{I_1 = \emptyset\} \in \overline{\mathcal{F}}_t$ there exists $\Omega^t_{int} \subset \{I_1 = \emptyset\}$ such that $\Omega^t_{int} \in \mathcal{F}_t$ and $P_t(\Omega^t_{int}) = \overline{P}_t(\{I_1 = \emptyset\}) = 1$. For all $\omega^t \in \Omega^t_{int}$, Assumption 5.9 in the context $t+1$ is true and we can now define $\widetilde{\Omega}^t_1 \subset \Omega^t$

$$\widetilde{\Omega}^t_1 := \Omega^t_{NA} \cap \Omega^t_{int} \cap \Omega^t_C. \tag{39}$$

It is clear that $\widetilde{\Omega}^t_1 \in \mathcal{F}_t$, $P_t(\widetilde{\Omega}^t_1) = 1$ and the proof is complete. □

The next proposition enables us to initialize the induction argument that will be carried on in Proposition 6.11.

Proposition 6.9 *Assume that the (NA) condition and Assumptions 4.7, 4.8 and 4.10 hold true. Then U_T satisfies* (34)*,* (35)*,* (36) *and* (37) *for $t = T$. We set $\widetilde{\Omega}^T = \Omega$.*

Proof. We start with (34) for $t = T$. As $U_T = U$ (see (30)), using Definition 4.1, $x \in \mathbb{R} \to U_T(\omega^T, x)$ is well-defined, non-decreasing and usc on $\mathbb{R}$ and (34) for $t = T$ is true. We prove now (35) for $t = T$ *i.e* that $U_T = U$ is $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R})$-measurable. To do that we show that for all $\omega^T \in \Omega^T$, $x \in \mathbb{R} \to U_T(\omega^T, x)$ is right-continuous and for all $x \in \mathbb{R}$, $\omega^T \in \Omega^T \to U_T(x, \omega^T)$ is $\mathcal{F}_T$-measurable (this is just the second point of Definition 4.1) so that we can use Lemma 7.16 and establish (35) for $t = T$. Let $\omega^T \in \Omega^T$ be fixed. From (34) at T that we have just proved, $x \in \mathbb{R} \to U_T(\omega^T, x)$ is non-decreasing and usc on $\mathbb{R}$, thus applying Lemma 7.12 we get that $x \in \mathbb{R} \to U_T(\omega^T, x)$ is right-continuous on $\mathbb{R}$.
We prove now that (36) is true for $t = T$. Let $\xi \in \Xi_{T-1}$ and $H = x + \sum_{t=1}^{T-1} \phi_t \Delta S_t$ where $x \geq 0$, $\phi_1 \in \Xi_0$, $\ldots$,$\phi_{T-1} \in \Xi_{T-2}$ and $P_T(H(\cdot) + \xi(\cdot)\Delta S_T(\cdot) \geq 0) = 1$. Let $(\phi^\xi_i)_{1 \leq i \leq T} \in \Phi$ be defined by $\phi^\xi_T = \xi$ and $\phi^\xi_i = \phi_i$ for $1 \leq i \leq T-1$ then $V^{x,\phi^\xi}_T = H + \xi\Delta S_T$ and thus $\phi^\xi \in \Phi(x)$. Using Proposition 6.1 we get that $EU^+(\cdot, V^{x,\phi^\xi}_T(\cdot)) = EU^+_T(\cdot, H(\cdot) + \xi(\cdot)\Delta S_T(\cdot)) < \infty$ (recall that $U = U_T$). Therefore (36) is verified for

$t = T$. Finally, from Assumption 4.10, (37) for $t = T$ is true. $\square$

The next proposition proves that if (34), (35), (36) and (37) hold true at $t+1$ then they are also true at U_t for some well chosen $\widetilde{\Omega}^t$.

Proposition 6.10 *Let $0 \leq t \leq T-1$ be fixed. Assume that the (NA) condition holds true and that* (34)*,* (35)*,* (36) *and* (37) *are true at $t+1$ (where U_{t+1} is defined from a given $\widetilde{\Omega}^{t+1}$ see* (31)*). Then there exists some $\widetilde{\Omega}^t \in \mathcal{F}_t$ with $P_t(\widetilde{\Omega}^t) = 1$ such that* (34)*,* (35)*,* (36) *and* (37) *are true for t.*
Moreover for all $H = x + \sum_{s=1}^t \phi_s \Delta S_s$, with $x \geq 0$ and $\phi_1 \in \Xi_0, \dots, \phi_t \in \Xi_{t-1}$, such that $P_t(H \geq 0) = 1$ there exists some $\widetilde{\Omega}^t_H \in \mathcal{F}_t$ such that $P(\widetilde{\Omega}^t_H) = 1$, $\widetilde{\Omega}^t_H \subset \widetilde{\Omega}^t$ and some $\widehat{h}^H_{t+1} \in \Xi_t$ such that for all $\omega^t \in \widetilde{\Omega}^t_H$, $\widehat{h}^H_{t+1}(\omega^t) \in \mathcal{D}^{t+1}_{H(\omega^t)}(\omega^t)$ and [3]

$$U_t(\omega^t, H(\omega^t)) = \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, H(\omega^t) + \widehat{h}^H_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t). \tag{40}$$

Proof. First we define $\widetilde{\Omega}^t$ and prove that (34) and (35) are true for U_t. Applying Proposition 6.7, we get that for all $\omega^t \in \widetilde{\Omega}^t_1$, the function $(\omega_{t+1}, x) \to U_{t+1}(\omega^t, \omega_{t+1}, x)$ satisfies the assumptions of Lemma 5.11 and Theorem 5.13 with $\overline{\Omega} = \Omega_{t+1}$, $\mathcal{H} = \mathcal{G}_{t+1}$, $Q = q_{t+1}(\cdot|\omega^t)$, $Y(\cdot) = \Delta S_{t+1}(\omega^t, \cdot)$, $V(\cdot, y) = U_{t+1}(\omega^t, \cdot, y)$ where V is defined on $\Omega_{t+1} \times \mathbb{R}$. In particular, for $\omega^t \in \widetilde{\Omega}^t_1$ and all $h \in \mathcal{H}^{t+1}_x(\omega^t)$, recalling (25) we have

$$\int_{\Omega_{t+1}} U^+_{t+1}(\omega^t, \omega_{t+1}, x + h\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t) < \infty. \tag{41}$$

Now, we introduce $\overline{U}_t : \Omega^t \times \mathbb{R}$ defined by

$$\overline{U}_t(\omega^t, x) := (-\infty)1_{(-\infty,0)}(x) + 1_{[0,\infty)}(x)1_{\widetilde{\Omega}^t_1}(\omega^t) \sup_{h \in \mathcal{D}^{t+1}_x(\omega^t)} \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, x + h\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t).$$

From (41), $\overline{U}_t$ is well-defined (in the generalised sense). First, we prove that $\overline{U}_t$ is $\overline{\mathcal{F}}_t \otimes \mathbb{R}$-measurable and then we will show that this implies that U_t is $\mathcal{F}_t \otimes \mathbb{R}$-measurable for a well chosen $\widetilde{\Omega}^t$. To show that $\overline{U}_t$ is $\overline{\mathcal{F}}_t \otimes \mathcal{B}(\mathbb{R})$-measurable, we use Lemma 7.16 (and Remark 7.17) after having proved that it is an extended Carathéodory function (see Definition 7.15). Applying Theorem 5.13, we get that for all $\omega^t \in \widetilde{\Omega}^t_1$, the function $x \in \mathbb{R} \to \overline{U}_t(\omega^t, x)$ is non-decreasing and usc on $\mathbb{R}$. Actually, this is true for all $\omega^t \in \Omega^t$ since outside $\widetilde{\Omega}^t_1$, $x \in \mathbb{R} \to U_t(\omega^t, x)$ is constant equal to zero on $[0, \infty)$ and to $-\infty$ on $(-\infty, 0)$. Let now $\omega^t \in \Omega^t$ be fixed. As $x \in \mathbb{R} \to \overline{U}_t(\omega^t, x)$ is non-decreasing and usc on $\mathbb{R}$ we can apply Lemma 7.12 and we get that $x \in \mathbb{R} \to \overline{U}_t(\omega^t, x)$ is right-continuous on $\mathbb{R}$. For $x \geq 0$ fixed, applying Lemma 6.11 with $H = x$ (here $\Omega^t_H = \widetilde{\Omega}^t_1$) we obtain that $\omega^t \in \Omega^t \to \sup_{h \in \mathbb{R}^d} u_x(\omega^t, h)$ is $\overline{\mathcal{F}}_t$-measurable. Finally, from the definitions of $\overline{U}_t$ and u_x, we get that

$$\overline{U}_t(\omega^t, x) = (-\infty)1_{(-\infty,0)} + 1_{[0,\infty)}(x)1_{\widetilde{\Omega}^t_1}(\omega^t) \sup_{h \in \mathbb{R}^d} u_x(\omega^t, h),$$

and this implies that $\omega^t \in \Omega^t \to \overline{U}_t(\omega^t, x)$ is $\overline{\mathcal{F}}_t$-measurable for all $x \in \mathbb{R}$ and thus that $\overline{U}_t$ is an extended Carathéodory function as claimed
Finally, we prove the $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R})$-measurability of U_t. To do that we apply Lemma 7.13 and we obtain some $\Omega^t_{mes} \in \mathcal{F}_t$ such that $P_t(\Omega^t_{mes}) = 1$ and some $\mathcal{F}_t \otimes \mathbb{R}$-measurable $\widetilde{U}_t : \Omega^t \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ such that for all $x \in \mathbb{R}$, $\{\omega^t \in \Omega^t,\ \overline{U}_t(\omega^t, x) \neq \widetilde{U}_t(\omega^t, x)\} \subset \Omega^t \backslash \Omega^t_{mes}$. We are now in a position to define $\widetilde{\Omega}^t$ and set

$$\widetilde{\Omega}^t := \widetilde{\Omega}^t_1 \cap \Omega^t_{mes}. \tag{42}$$

[3] Recall that the integral on the right hand side is defined in the generalised sense.

It is clear that $\widetilde{\Omega}^t \in \mathcal{F}_t$ and that $P_t(\widetilde{\Omega}^t) = 1$ Furthermore, recalling (31), Remark 5.5 (see (21)) and the definition of $\overline{U}_t$ we have that for all $x \in \mathbb{R}$, $\omega^t \in \Omega^t$

$$\begin{aligned}
U_t(\omega^t, x) &= (-\infty)1_{(-\infty,0)}(x) + 1_{[0,\infty)}(x)1_{\Omega^t_{mes}}(\omega^t)1_{\widetilde{\Omega}^t_1}(\omega^t) \sup_{h \in \mathcal{H}^{t+1}_x(\omega^t)} \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, x + h\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t) \\
&= (-\infty)1_{(-\infty,0)}(x) + 1_{[0,\infty)}(x)1_{\Omega^t_{mes}}(\omega^t)1_{\widetilde{\Omega}^t_1}(\omega^t) \sup_{h \in \mathcal{D}^{t+1}_x(\omega^t)} \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, x + h\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t) \\
&= 1_{\Omega^t_{mes}}(\omega^t)\overline{U}_t(\omega^t, x) + (-\infty)1_{\Omega^t \setminus \Omega^t_{mes}}(\omega^t)1_{(-\infty,0)}(x) \\
&= 1_{\Omega^t_{mes}}(\omega^t)\widetilde{U}_t(\omega^t, x) + (-\infty)1_{\Omega^t \setminus \Omega^t_{mes}}(\omega^t)1_{(-\infty,0)}(x),
\end{aligned}$$

and the $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R})$-measurability of U_t follows immediately, $i.e$ (35) is true at t. It is clear as well from the third equality that (34) is true for t since we have proven that for all $\omega^t \in \Omega^t$, $x \in \mathbb{R} \to \overline{U}_t(\omega^t, x)$ is well-defined, non-decreasing and usc on $\mathbb{R}$.
We turn now to the assumption on asymptotic elasticity $i.e$ (37) for t. If $\omega^t \notin \widetilde{\Omega}^t$, then (37) is true since $C_t(\omega^t) \geq 0$ for all ω^t. Let $\omega^t \in \widetilde{\Omega}^t$ be fixed. Let $x \geq 0$, $\lambda \geq 1$, $h \in \mathbb{R}^d$ such that $q_{t+1}(\lambda x + h\Delta S_{t+1}(\omega^t, .) \geq 0|\omega^t) = 1$ be fixed. By (37) for $t+1$ for all $\omega_{t+1} \in \Omega_{t+1}$, we have that

$$U_{t+1}\left(\omega^t, \omega_{t+1}, \lambda x + h\Delta S_{t+1}(\omega^t, \omega_{t+1})\right) \leq \lambda^{\overline{\gamma}} K U_{t+1}\left(\omega^t, \omega_{t+1}, x + \frac{1}{2} + \frac{h}{\lambda}\Delta S_{t+1}(\omega^t, \omega_{t+1})\right) + \lambda^{\overline{\gamma}} C_{t+1}(\omega^t, \omega_{t+1}).$$

By integrating both sides (recall (41)) we get that

$$\begin{aligned}
&\int_{\Omega_{t+1}} U_{t+1}\left(\omega^t, \omega_{t+1}, \lambda x + h\Delta S_{t+1}(\omega^t, \omega_{t+1})\right) q_{t+1}(d\omega_{t+1}|\omega^t) \leq \\
&\lambda^{\overline{\gamma}} K \int_{\Omega_{t+1}} U_{t+1}\left(\omega^t, \omega_{t+1}, x + \frac{1}{2} + \frac{h}{\lambda}\Delta S_{t+1}(\omega^t, \omega_{t+1})\right) q_{t+1}(d\omega_{t+1}|\omega^t) + \lambda^{\overline{\gamma}} K \int_{\Omega_{t+1}} C_{t+1}(\omega^t, \omega_{t+1}) q_{t+1}(d\omega_{t+1}|\omega^t).
\end{aligned}$$

Since $C_t(\omega^t) = \int_{\Omega_{t+1}} C_{t+1}(\omega^t, \omega_{t+1})q_{t+1}(d\omega_{t+1}|\omega^t)$ (see Lemma 6.5) and $h \in \mathcal{H}^{t+1}_{\lambda x}(\omega^t)$ implies that $\frac{h}{\lambda} \in \mathcal{H}^{t+1}_x(\omega^t) \subset \mathcal{H}^{t+1}_{x+\frac{1}{2}}(\omega^t)$, we obtain by definition of U_t (see (31)) that

$$\int_{\Omega_{t+1}} U_{t+1}\left(\omega^t, \omega_{t+1}, \lambda x + h\Delta S_{t+1}(\omega^t, \omega_{t+1})\right) q_{t+1}(d\omega_{t+1}|\omega^t) \leq \quad \lambda^{\overline{\gamma}} K U_t\left(\omega^t, x + \frac{1}{2}\right) + \lambda^{\overline{\gamma}} K C_t(\omega^t).$$

Taking the supremum over all $h \in \mathcal{H}^{t+1}_{\lambda x}(\omega^t)$ we conclude that (37) is true for t for $x \geq 0$. If $x < 0$, then (37) is true by definition of U_t. Note that we might have $\omega^t \in \Omega^t \setminus \Omega^t_C$ and $C_t(\omega^t) = +\infty$ since (37) does not require that $C_t(\omega^t) < +\infty$.

We now prove (40) for U_t. First, from Proposition 6.7 and Theorem 5.13 and since $\widetilde{\Omega}^t \subset \widetilde{\Omega}^t_1$, we have for all $\omega^t \in \widetilde{\Omega}^t$ and $x \geq 0$ that there exists some $\xi^* \in \mathcal{D}^{t+1}_x(\omega^t)$ such that

$$U_t(\omega^t, x) \quad = \quad \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, x + \xi^* \Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t), \tag{43}$$

where the integral on the right hand side is defined in the generalised sense (recall (41) and Lemma 5.11). Let $H = x + \sum_{s=1}^{t-1} \phi_s \Delta S_s$, with $x \geq 0$ and $\phi_s \in \Xi_s$ for $s \in \{1, \dots, t-1\}$, be fixed such that $P(H \geq 0) = 1$. Let $\widetilde{\Omega}^t_H := \widetilde{\Omega}^t \cap \{\omega^t \in \Omega^t, H(\omega) \geq 0\}$. Then $\widetilde{\Omega}^t_H \in \mathcal{F}_t$ and $P(\widetilde{\Omega}^t_H) = 1$. We introduce the following random set $\psi : \Omega^t \twoheadrightarrow \mathbb{R}^d$

$$\psi_H(\omega^t) := \left\{h \in \mathcal{D}^{t+1}_{H(\omega^t)}(\omega^t),\ U_t(\omega^t, H(\omega^t)) = \int_{\Omega^{t+1}} U_{t+1}\left(\omega^t, \omega_{t+1}, H(\omega^t) + h\Delta S_{t+1}(\omega^t, \omega_{t+1})\right) q_{t+1}(d\omega_{t+1}|\omega^t)\right\},$$

for $\omega^t \in \widetilde{\Omega}^t_H$ and $\psi_H(\omega^t) = \emptyset$ otherwise. To prove (40) it is enough to find a $\mathcal{F}_t$-measurable selector for ψ_H. From the definitions of ψ_H and u_H (see (45)) we obtain that (recall that $\widetilde{\Omega}^t_H \subset \widetilde{\Omega}^t$ and $\widetilde{\Omega}^t_H \subset \Omega^t_H$, see (42) and the definition of Ω^t_H in Lemma 6.11).

$$Graph(\psi_H) = \left\{(\omega^t, h) \in \left(\widetilde{\Omega}^t_H \times \mathbb{R}^d\right) \cap Graph(\mathcal{D}^{t+1}_H),\ U_t(\omega^t, H(\omega^t)) = u_H(\omega^t, h)\right\}.$$

From Lemma 6.4 we have that $Graph(\mathcal{D}_H^{t+1}) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$. We have already proved that $(\omega^t, y) \to U_t(\omega^t, y)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R})$-measurable and, as H is $\mathcal{F}_t$-measurable, we obtain that $\omega^t \to U_t(\omega^t, H(\omega^t))$ is $\mathcal{F}_t$-measurable. Now applying Lemma 6.11 we obtain that u_H is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. The fact that $Graph(\psi_H) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ follows immediately.

So we can apply the Projection Theorem (see for example Theorem 3.23 in Castaing and Valadier (1977)) and we get that $\{\psi_H \neq \emptyset\} \in \overline{\mathcal{F}}_t$ and using the Aumann Theorem (see Corollary 1 in Sainte-Beuve (1974)) that there exists some $\overline{\mathcal{F}}_t$-measurable $\overline{h}_{t+1}^H : \{\psi_H \neq \emptyset\} \to \mathbb{R}^d$ such that for all $\omega^t \in \{\psi_H \neq \emptyset\}$, $\overline{h}_{t+1}^H(\omega^t) \in \psi_H(\omega^t)$. Then we extend $\overline{h}_{t+1}^H$ on all Ω^t by setting $\overline{h}_{t+1}^H = 0$ on $\Omega^t \setminus \{\psi_H \neq \emptyset\}$. Now applying Lemma 7.10 we get some $\mathcal{F}_t$-measurable $\widehat{h}_{t+1}^H : \Omega^t \to \mathbb{R}^d$ and some $\overline{\Omega}_H^t \in \mathcal{F}_t$ such that $P(\overline{\Omega}_H^t) = 1$ and $\overline{\Omega}_H^t \subset \{\overline{h}_{t+1}^H = \widehat{h}_{t+1}^H\}$. We prove now that the set $\{\psi_H \neq \emptyset\}$ is of full measure. Indeed, let $\omega^t \in \widetilde{\Omega}_H^t$ be fixed. Using (43) for $x = H(\omega^t) \geq 0$, there exists $h^*(\omega^t) \in \psi_H(\omega^t)$. Therefore $\widetilde{\Omega}_H^t \subset \{\psi_H \neq \emptyset\}$ and $\overline{P}_t(\{\psi_H \neq \emptyset\}) = 1$. So for all $\omega^t \in \overline{\Omega}_H^t \cap \widetilde{\Omega}_H^t$ we have

$$
\begin{aligned}
U_t(\omega^t, H(\omega^t)) &= \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, H(\omega^t) + \overline{h}_{t+1}^H(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t) \\
&= \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, H(\omega^t) + \widehat{h}_{t+1}^H(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t).
\end{aligned}
$$

So setting

$$
\widetilde{\Omega}_H^t = \widetilde{\Omega}_H^t \cap \overline{\Omega}_H^t \subset \widetilde{\Omega}^t \tag{44}
$$

(40) is proved for t.

We are now left with the proof of (36) for U_t. Let $\xi \in \Xi_{t-1}$ and $H = x + \sum_{s=1}^{t-1} \phi_s \Delta S_s$ where $x \geq 0$ and $\phi_1 \in \Xi_0, \ldots, \phi_{t-1} \in \Xi_{t-2}$ and such that $P_t(H(\cdot) + \xi(\cdot)\Delta S_t(\cdot) \geq 0) = 1$. We fix some $\omega^t \in \widetilde{\Omega}^t$. Let $X(\omega^t) = H(\omega^{t-1}) + \xi(\omega^{t-1})\Delta S_t(\omega^t)$ then X is $\mathcal{F}_t$-measurable. We apply (40) to $X(\omega^t)$ (and $\mathcal{D}_{X(\omega^t)}^{t+1}(\omega^t)$), and we get some $\omega^t \in \Omega^t \to \widehat{h}_{t+1}(\omega^t)$ which is $\mathcal{F}_t$-measurable and $\widetilde{\Omega}_X^t \in \mathcal{F}_t$ such that $P_t(\widetilde{\Omega}_X^t) = 1$ and such that for all $\omega^t \in \widetilde{\Omega}_X^t$, $q_{t+1}\left(X(\omega^t) + \widehat{h}_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t, \cdot) \geq 0|\omega^t\right) = 1$ and

$$
U_t(\omega^t, X(\omega^t)) = \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, X(\omega^t) + \widehat{h}_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t).
$$

Using Jensen's Inequality

$$
U_t^+(\omega^t, X(\omega^t)) \leq \int_{\Omega_{t+1}} U_{t+1}^+(\omega^t, \omega_{t+1}, X(\omega^t) + \widehat{h}_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t).
$$

Thus as $P_t(\widetilde{\Omega}_X^t) = 1$

$$
\begin{aligned}
\int_{\widetilde{\Omega}_X^t} U_t^+(\omega^t, X(\omega^t))P_t(d\omega^t) &= \int_{\Omega^t} U_t^+(\omega^t, X(\omega^t))P_t(d\omega^t) \\
&\leq \int_{\Omega^{t+1}} U_{t+1}^+(\omega^{t+1}, X(\omega^t) + \widehat{h}_{t+1}(\omega^t)\Delta S_{t+1}(\omega^{t+1}))P_{t+1}(d\omega^{t+1}) < \infty,
\end{aligned}
$$

because of (36) for $t+1$ which applies since $X = x + \sum_{s=1}^{t-1} \phi_s \Delta S_s + \xi \Delta S_t$ where $x \geq 0$, $\phi_1 \in \Xi_1, \ldots, \phi_{t-1} \in \Xi_{t-2}$, $\xi \in \Xi_{t-1}$ and $\widehat{h}_{t+1} \in \Xi_t$: (36) for t is proved. □

The following lemma was essential to obtain measurability issues in the proof of Lemma 6.10.

Lemma 6.11 *Fix some $0 \leq t \leq T-1$ and $x \geq 0$. Let $H := x + \sum_{s=1}^{t-1} \phi_s \Delta S_s$, where $\phi_1 \in \Xi_0, \dots, \phi_{t-1} \in \Xi_{t-2}$ and $P_t(H \geq 0) = 1$. Assume that the (NA) condition holds true and that (34), (35), (36) and (37) are true at $t+1$. Let $u_H : \Omega^t \times \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$ be defined by*

$$u_H(\omega^t, h) := \begin{cases} \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, H(\omega^t) + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) q_{t+1}(d\omega_{t+1}|\omega^t), \\ \quad \textit{if } (\omega^t, h) \in \left(\Omega_H^t \times \mathbb{R}^d\right) \cap Graph(\mathcal{D}_H^{t+1}), \\ -\infty \ \textit{ if } (\omega^t, h) \notin Graph(\mathcal{D}_H^{t+1}), \\ 0 \ \textit{ otherwise}. \end{cases} \tag{45}$$

where $\mathcal{D}_H^{t+1}$ is defined in Lemma 6.4 and $\Omega_H^t := \widetilde{\Omega}_1^t \bigcap \{\omega^t \in \Omega^t, \ H(\omega^t) \geq 0\}$ (see (39) for the definition of $\widetilde{\Omega}_1^t$). Then u_H is well-defined, $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$-measurable and for all $\omega^t \in \Omega^t$, $h \in \mathbb{R}^d \to u_H(\omega^t, h)$ is usc. Morevover, $\omega^t \in \Omega^t \to \sup_{h \in \mathbb{R}^d} u_H(\omega^t, h)$ is $\overline{\mathcal{F}}_t$-measurable.

Remark 6.12 In the proof below we will show that for $(\omega^t, h) \in \left(\Omega_H^t \times \mathbb{R}^d\right) \cap Graph(\mathcal{D}_H^{t+1})$ the integral in (45) is well-defined. Note that this is not the case for all $(\omega^t, h) \in \Omega^t \times \mathbb{R}^d$. Indeed, let (ω^t, h) be fixed such that $q_{t+1}(H(\omega^t) + h\Delta S_{t+1}(\omega^t, \cdot) < 0|\omega^t) > 0$. Then it is clear that $\int_{\Omega_{t+1}} U_{t+1}^-(\omega^t, \omega_{t+1}, H(\omega^t) + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) q_{t+1}(d\omega_{t+1}|\omega^t) = \infty$ and as without further assumption we cannot prove that $\int_{\Omega_{t+1}} U_{t+1}^+(\omega^t, \omega_{t+1}, H(\omega^t) + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) q_{t+1}(d\omega_{t+1}|\omega^t) < \infty$ (it is easy to find some counterexamples), the integral in (45) may fail to be well-defined. We could have circumvented this issue by using the convention $\infty - \infty = -\infty$ but we prefer to refrain from doing so.

Proof. From (35) at $t+1$, U_{t+1} is $\mathcal{F}_t \otimes \mathcal{G}_{t+1} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable and since H and ΔS_{t+1} are respectively $\mathcal{F}_t$ and $\mathcal{F}_{t+1}$-measurable, we obtain that $(\omega^t, \omega_{t+1}, h) \in \Omega^t \times \Omega_{t+1} \times \mathbb{R}^d \to U_{t+1}(\omega^t, \omega_{t+1}, H(\omega^t) + h\Delta S_{t+1}(\omega^t, \omega_{t+1}))$ is also $\mathcal{F}_t \otimes \mathcal{G}_{t+1} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. In order to prove that for $(\omega^t, h) \in \left(\Omega_H^t \times \mathbb{R}^d\right) \cap Graph(\mathcal{D}_H^{t+1})$ the integral in (45) is well-defined, we introduce

$$\widetilde{u}_H : (\omega^t, h) \in \left(\Omega_H^t \times \mathbb{R}^d\right) \cap Graph(\mathcal{D}_H^{t+1}) \to \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, H(\omega^t) + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) q_{t+1}(d\omega_{t+1}|\omega^t).$$

First we show that $\widetilde{u}_H$ is well-defined in the generalised sense. Indeed, let $(\omega^t, h) \in \left(\Omega_H^t \times \mathbb{R}^d\right) \cap Graph(\mathcal{D}_H^{t+1})$ be fixed. As ω^t is fixed in Ω_H^t, we can show as in Proposition 6.10 that (41) holds true (here $H(\omega^t)$ is a fixed number as ω^t is fixed) and thus

$$\int_{\Omega_{t+1}} U_{t+1}^+(\omega^t, \omega_{t+1}, H(\omega^t) + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) q_{t+1}(d\omega_{t+1}|\omega^t) < \infty,$$

So $\widetilde{u}_H$ is well-defined (but may be infinite-valued).

We now prove that u_H is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. We can apply Proposition 7.6 *iv*) to $\mathcal{S} = \left(\Omega_H^t \times \mathbb{R}^d\right) \cap Graph(\mathcal{D}_H^{t+1})$, with $f(\omega^t, h, \omega_{t+1})$ equal to both $U_{t+1}^{\pm}(\omega^t, \omega_{t+1}, H(\omega^t) + h\Delta S_{t+1}(\omega^t, \omega_{t+1}))$, since $\left(\Omega_H^t \times \mathbb{R}^d\right) \cap Graph(\mathcal{D}_H^{t+1}) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ (see Lemma 6.4), and both $(\omega^t, h, \omega_{t+1}) \in \Omega^t \times \mathbb{R}^d \times \Omega_{t+1} \to U_{t+1}^{\pm}(\omega^t, \omega_{t+1}, H(\omega^t) + h\Delta S_{t+1}(\omega^t, \omega_{t+1}))$ are $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{G}_{t+1}$-measurable. So we obtain that $\widetilde{u}_H$ is $\left[\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)\right]_{\mathcal{S}}$-measurable, where $\left[\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)\right]_{\mathcal{S}}$ denotes the trace sigma algebra of $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ on $\mathcal{S}$. Now we extend $\widetilde{u}_H$ to $\Omega^t \times \mathbb{R}^d$ by setting $\widetilde{u}_H(\omega^t, h) = -\infty$ if $(\omega^t, h) \notin Graph(\mathcal{D}_H^{t+1})$ and $\widetilde{u}_H(\omega^t, h) = 0$ if $(\omega^t, h) \in Graph(\mathcal{D}_H^{t+1})$ and $\omega^t \notin \Omega_H^t$. Since $\left[\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)\right]_{\mathcal{S}} \subset \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$, $\Omega_H^t \in \mathcal{F}_t$ and $Graph(\mathcal{D}_H^{t+1}) \in \mathcal{F}_t \times \mathcal{B}(\mathbb{R}^d)$, this extension of $\widetilde{u}_H$ is again $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. As it is clear that this extension of $\widetilde{u}_H$ and u_H coincide, the measurability of u_H is proved.

We turn now to the usc property. Let $\omega^t \in \Omega_H^t \subset \widetilde{\Omega}_1^t$ be fixed. We apply Proposition 6.7 to U_{t+1} and we get, as $\omega^t \in \widetilde{\Omega}_1^t$, that the function $(\omega_{t+1}, x) \to U_{t+1}(\omega^t, \omega_{t+1}, x)$ satisfies the assumptions of Lemma 5.12 (see Remark 6.8) with $\overline{\Omega} = \Omega_{t+1}$, $\mathcal{H} = \mathcal{G}_{t+1}$, $Q = q_{t+1}(\cdot|\omega^t)$, $Y(\cdot) = \Delta S_{t+1}(\omega^t, \cdot)$, $V(\cdot, y) = U_{t+1}(\omega^t, \cdot, y)$ where V is defined on $\Omega_{t+1} \times \mathbb{R}$. Therefore the function $\phi_{\omega^t}(\cdot, \cdot)$ defined on $\mathbb{R} \times \mathbb{R}^d$ by

$$\phi_{\omega^t}(x, h) = \begin{cases} \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, x + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) q_{t+1}(d\omega_{t+1}|\omega^t) \text{ if } x \geq 0 \text{ and } h \in D_x^{t+1}(\omega^t) \\ -\infty \text{ otherwise/}. \end{cases}$$

is usc on $\mathbb{R} \times \mathbb{R}^d$ (see (26)). In particular, for $x = H(\omega^t) \geq 0$ fixed, the function $h \in \mathbb{R}^d \to u_H(\omega^t, h) = \phi_{\omega^t}(H(\omega^t), h)$ is usc on $\mathbb{R}^d$. Now for $\omega^t \notin \Omega^t_H$, as u_H is equal to 0 if $h \in \mathcal{D}^{t+1}_{H(\omega^t)}(\omega^t)$ and to $-\infty$ otherwise, Lemma 7.11 applies (recall that the random set $\mathcal{D}^{t+1}_H$ is closed-valued) and $h \in \mathbb{R}^d \to u_H(\omega^t, h)$ is usc on all $\mathbb{R}^d$.
Finally, we apply Corollary 14.34 in Rockafellar and Wets (1998) and find that $-u_H$ is a $\overline{\mathcal{F}}_t$- normal integrand [4]. Now from Theorem 14.37 of Rockafellar and Wets (1998), we obtain that $\omega^t \in \Omega^t \to \sup_{h \in \mathbb{R}^d} u_H(\omega^t, h)$ is $\overline{\mathcal{F}}_t$-measurable and this concludes the proof. $\square$

Proof. of Theorem 4.16. We proceed in three steps. First, we handle some integrability issues that are essential to the proof. Then, we build by induction a candidate for the optimal strategy and finally we establish its optimality.

Integrability Issues

We fix some $\phi \in \Phi(x) = \Phi(U, x)$ (recall Proposition 6.1). Since Proposition 6.9 holds true, we can apply Proposition 6.10 for $t = T-1$, and by backward induction, we can therefore apply Proposition 6.10 for all $t = T-2, \dots, 0$. In particular, we get that (36) holds true for all $0 \leq t \leq T$. So choosing $H = V^{x,\phi}_{t-1}$ and $\xi = \phi_t$ we get that (recall Remark 4.3, from $\phi \in \Phi(x)$ we get that $P_t(V^{x,\phi}_t(\cdot) \geq 0) = 1$)

$$\int_{\Omega^t} U_t^+ \left(\omega^t, V^{x,\phi}_t(\omega^t)\right) P_t(d\omega^t) < \infty. \tag{46}$$

This implies that $\int_{\Omega^t} U_t \left(\omega^t, V^{x,\phi}_t(\omega^t)\right) P_t(d\omega^t)$ is defined in the generalised sense and that we can apply the Fubini Theorem for generalised integral (see Proposition 7.4)

$$\int_{\Omega^t} U_t \left(\omega^t, V^{x,\phi}_t(\omega^t)\right) P_t(d\omega^t) = \int_{\Omega^{t-1}} \int_{\Omega_t} U_t \left(\omega^{t-1}, \omega_t, V^{x,\phi}_t(\omega^{t-1}, \omega_t)\right) q_{t-1}(d\omega_t|\omega^{t-1}) P_{t-1}(d\omega^{t-1}). \tag{47}$$

Construction of ϕ^*

We fix some $x \geq 0$ and build our candidate for the optimal strategy by induction. We start at $t = 0$ and use (40) in Proposition 6.10 with $H = x \geq 0$. We set $\phi^*_1 := \widehat{h}^x_1$ and we obtain that (recall that $\mathcal{F}_0 = \{\emptyset, \Omega^0\}$)

$$P_1(x + \phi^*_1 \Delta S_1(.) \geq 0) = 1.$$

$$U_0(x) = \int_{\Omega_1} U_1\left(\omega_1, x + \phi^*_1 \Delta S_1(\omega_1)\right) P_1(d\omega_1).$$

Recall from (46) that the above integral is well-defined in the generalised sense. Assume that until some $t \geq 1$ we have found some $\phi^*_1 \in \Xi_0, \dots, \phi^*_t \in \Xi_{t-1}$ and some $\overline{\Omega}^1 \in \mathcal{F}_1, \dots, \overline{\Omega}^{t-1} \in \mathcal{F}_{t-1}$ such that for all $i = 1, \dots, t-1$, $\overline{\Omega}^i \subset \widetilde{\Omega}^i$, $P_i(\overline{\Omega}^i) = 1$, for all $i = 0, \dots, t-1$, $\phi^*_{i+1}(\omega^i) \in D^{i+1}(\omega^i)$ and

$$P_t\left(x + \phi^*_1 \Delta S_1(\omega_1) + \dots + \phi^*_t(\omega^{t-1}) \Delta S_t(\omega^{t-1}, \omega_t) \geq 0\right) = 1,$$

and finally, for all $\omega^t \in \overline{\Omega}^t$

$$U_{t-1}\left(\omega^{t-1}, V^{x,\phi^*}_{t-1}(\omega^{t-1})\right) = \int_{\Omega_t} U_t\left(\omega^{t-1}, \omega_t, V^{x,\phi^*}_{t-1}(\omega^{t-1}) + \phi^*_t(\omega^{t-1}) \Delta S_t(\omega^{t-1}, \cdot)\right) q_t(d\omega_t|\omega^{t-1}),$$

where again the integral is well-defined in the generalised sense (see (46)). We apply Proposition 6.10 with $H(\cdot) = V^{x,\phi^*}_t(\cdot) = V^{x,\phi^*}_{t-1}(\cdot) + \phi^*_t(\cdot)\Delta S_t(\cdot)$ (recall that $P_t(V^{x,\phi^*}_t \geq 0 = 1)$ and there exists $\overline{\Omega}^t := \widetilde{\Omega}^t_{V^{x,\phi^*}_t} \in \mathcal{F}_t$ such that $\overline{\Omega}^t \subset \widetilde{\Omega}^t$, $P_t(\overline{\Omega}^t) = 1$ and some some $\mathcal{F}_t$-measurable $\omega^t \to \phi^*_{t+1}(\omega^t) := \widehat{h}^{V^{x,\phi^*}_t}_{t+1}(\omega^t)$ such that for all $\omega^t \in \overline{\Omega}^t$, $\phi^*_{t+1}(\omega^t) \in D^{t+1}(\omega^t)$

$$q_{t+1}(V^{x,\phi^*}_t(\omega^t) + \phi^*_{t+1}(\omega^t) \Delta S_{t+1}(\omega^t, \cdot) \geq 0 | \omega^t) = 1,$$

[4] Corollary 14.34 of Rockafellar and Wets (1998) holds true only for complete σ-algebra. That is the reason why $-u_H$ is a $\overline{\mathcal{F}}_t$- normal integrand and not a $\mathcal{F}_t$- normal integrand.

$$U_t\left(\omega^t, V_t^{x,\phi^*}(\omega^t)\right) = \int_{\Omega_{t+1}} U_{t+1}\left(\omega^t, \omega_{t+1}, V_t^{x,\phi^*}(\omega^t) + \phi_{t+1}^*(\omega^t)\Delta S_{t+1}(\omega^t, \cdot)\right) q_{t+1}(d\omega_{t+1}|\omega^t). \tag{48}$$

Now since $P_t(\overline{\Omega}^t) = 1$, we obtain by the Fubini Theorem that

$$P_{t+1}(V_{t+1}^{x,\phi^*} \geq 0) = \int_{\Omega^t} q_{t+1}(V_t^{x,\phi^*}(\omega^t) + \phi_{t+1}^*(\omega^t)\Delta S_{t+1}(\omega^t, \cdot) \geq 0|\omega^t) P_t(d\omega^t) = 1$$

and we can continue the recursion.
Thus, we have found $\phi^* = (\phi_t^*)_{1 \leq t \leq T}$ such that for all $t = 0, \dots, T$, $P_t(V_t^{x,\phi^*} \geq 0) = 1$, $i.e$ $\phi^* \in \Phi(x)$. We have also found some $\overline{\Omega}^t \in \mathcal{F}_t$, such that $\overline{\Omega}^t \subset \widetilde{\Omega}^t$, $P_t(\overline{\Omega}^t) = 1$ and for all $\omega^t \in \overline{\Omega}^t$, (48) holds true for all $t = 0, \dots, T-1$. Moreover, from Proposition 6.1, $\phi^* \in \Phi(U, x)$ and we have that $E(U(V_T^{x,\phi^*})) < \infty$.

Optimality of ϕ^*

We prove that ϕ^* is optimal in two steps.

Step 1: Using (47) with $\phi = \phi^*$ and the fact that $P_{T-1}(\overline{\Omega^{T-1}}) = 1$, we get that

$$\begin{aligned}
E(U(V_T^{x,\phi^*})) &= \int_{\Omega^{T-1}} \int_{\Omega_T} U\left(\omega^{T-1}, \omega_T, V_{T-1}^{x,\phi^*}(\omega^{T-1}) + \phi_T^*(\omega^{T-1})\Delta S_T(\omega^{T-1}, \omega_T)\right) q_T(d\omega_T|\omega^{T-1}) P_{T-1}(d\omega^{T-1}) \\
&= \int_{\overline{\Omega}^{T-1}} \int_{\Omega_T} U_T\left(\omega^{T-1}, \omega_T, V_{T-1}^{x,\phi^*}(\omega^{T-1}) + \phi_T^*(\omega^{T-1})\Delta S_T(\omega^{T-1}, \omega_T)\right) q_T(d\omega_T|\omega^{T-1}) P_{T-1}(d\omega^{T-1}).
\end{aligned}$$

Using (48) for $t = T-1$ and again the fact that $P_{T-1}(\overline{\Omega}^{T-1}) = 1$, we have that

$$E(U(V_T^{x,\phi^*})) = \int_{\Omega^{T-1}} U_{T-1}\left(\omega^{T-1}, V_{T-1}^{x,\phi^*}(\omega^{T-1})\right) P_{T-1}(d\omega^{T-1}).$$

We iterate the process for $T-1$: using the Fubini Theorem (see (47)), $P_{T-2}(\overline{\Omega}^{T-2}) = 1$ and (48), we obtain that

$$E(U(V_T^{x,\phi^*})) = \int_{\Omega^{T-2}} U_{T-2}\left(\omega^{T-2}, V_{T-2}^{x,\phi^*}(\omega^{T-2})\right) P_{T-2}(d\omega^{T-2}).$$

By backward induction, we therefore obtain that (recall $\Omega^0 := \{\omega_0\}$)

$$E(U(V_T^{x,\phi^*})) = U_0(x).$$

As $\phi^* \in \Phi(U, x)$, we get that $U_0(x) \leq u(x)$. So ϕ^* will be optimal if $U_0(x) \geq u(x)$.

Step 2: We fix again some $\phi \in \Phi(U, x)$ (recall Proposition 6.1). We get that $V_t^{x,\phi} \geq 0$ P_t-a.s. for all $t = 1, \dots, T$ (recall Remark 4.3). As $\phi_1 \in \mathcal{H}_x^1$ we obtain that

$$U_0(x) \geq \int_{\Omega_1} U_1(\omega_1, x + \phi_1 \Delta S_1(\omega_1)) P_1(d\omega_1).$$

As $P_2(V_1^{x,\phi} + \phi_2 \Delta S_2 \geq 0) = 1$, there exists some P_1-full measure set $\widehat{\Omega}^1 \in \mathcal{F}_1$ such that for all $\omega_1 \in \widehat{\Omega}^1$, $q_2\left(V_1^{x,\phi}(\omega_1) + \phi_2(\omega_1)\Delta S_2(\omega_1, \cdot)) \geq 0|\omega_1\right) = 1$ $i.e$ $q_2\left(\phi_2(\omega_1) \in \mathcal{H}^2_{V_1^{x,\phi}(\omega_1)}(\omega_1)|\omega_1\right) = 1$ (see Lemma 7.9). So for $\omega_1 \in \widehat{\Omega}^1$, we have that

$$U_1(\omega_1, V_1^{x,\phi}(\omega_1)) \geq \int_{\Omega_2} U_2\left(\omega_1, \omega_2, V_1^{x,\phi}(\omega_1) + \phi_2(\omega_1)\Delta S_1(\omega_1, \omega_2)\right) q_2(d\omega_2|\omega^1). \tag{49}$$

From (46), $\int_{\Omega^2} U_2^+\left(\omega^2, V_2^{x,\phi}(\omega^2)\right) P_2(d\omega^2) < \infty$ and we can apply the Fubini Theorem (see (47)) and

$$\begin{aligned}
\int_{\Omega^2} U_2\left(\omega^2, V_2^{x,\phi}(\omega^2)\right) P_2(d\omega^2) &= \int_{\Omega^1} \int_{\Omega_2} U_2\left(\omega_1, \omega_2, V_1^{x,\phi}(\omega_1) + \phi_2 \Delta S_1(\omega_1, \omega_2)\right) q_2(d\omega_2|\omega_1) P_1(d\omega_1) \\
&= \int_{\widehat{\Omega}^1} \int_{\Omega_2} U_2\left(\omega_1, \omega_2, V_1^{x,\phi}(\omega_1) + \phi_2 \Delta S_1(\omega_1, \omega_2)\right) q_2(d\omega_2|\omega_1) P_1(d\omega_1).
\end{aligned}$$

Using again (46), $\int_{\Omega^1} U_1^+ \left(\omega^1, V_1^{x,\phi}(\omega^1)\right) P_1(d\omega^1) < \infty$ and integrating (in the generalised sense) both side of (49) we obtain

$$\begin{aligned}
\int_{\Omega^1} U_1(\omega_1, V_1^{x,\phi}(\omega_1))P_1(d\omega_1) &= \int_{\widehat{\Omega}^1} U_1(\omega_1, V_1^{x,\phi}(\omega_1))P_1(d\omega_1) \\
&\geq \int_{\widehat{\Omega}^1} \int_{\Omega_2} U_2\left(\omega_1, \omega_2, V_1^{x,\phi}(\omega_1) + \phi_2 \Delta S_1(\omega_1, \omega_2)\right) q_2(d\omega_2|\omega_1)P_1(d\omega_1) \\
&= \int_{\Omega^2} U_2\left(\omega^2, V_2^{x,\phi}(\omega^2)\right) P_2(d\omega^2).
\end{aligned}$$

Therefore

$$U_0(x) \geq \int_{\Omega^2} U_2\left(\omega^2, V_2^{x,\phi}(\omega^2)\right) P_2(d\omega^2).$$

We can go forward since for P_2-almost all ω^2 we have that $q_3\left(\phi_3(\omega^2) \in \mathcal{H}^3_{V_2^{x,\phi}(\omega^2)}(\omega^2)|\omega^2\right) = 1, \dots,$ for P_{T-1} almost all ω^{T-1} we have that $q_T\left(\phi_T(\omega^{T-1}) \in \mathcal{H}^T_{V_{T-1}^{x,\phi}(\omega^{T-1})}(\omega^{T-1})|\omega^{T-1}\right) = 1$, we obtain using again (46) and the Fubini Theorem (see (47)) that

$$U_0(x) \geq \int_{\Omega_1}\int_{\Omega_2}\cdots\int_{\Omega_T} U\left(\omega^T, V_T^{x,\phi}(\omega^T)\right) q_T(d\omega_T|\omega^{T-1})\cdots q_2(d\omega_2|\omega^1)P_1(d\omega_1). \tag{50}$$

So we have that $U_0(x) \geq E(U(\cdot, V_T^{x,\phi}(\cdot)))$ for any $\phi \in \Phi(U,x)$ and the proof is complete since $u(x) = E(U(\cdot, V_T^{x,\phi^*}(\cdot))) < \infty$. □

Proof. of Theorem 4.17. To prove Theorem 4.17, we want to apply Theorem 4.16 and thus we need to establish that Assumptions 4.7 and 4.8 hold true. To do so we will prove (53) below. First we show that for all $x \geq 0$, $\phi \in \Phi(x)$ and $0 \leq t \leq T$, we have for P_t-almost all $\omega^t \in \Omega^t$

$$|V_t^{x,\phi}(\omega^t)| \leq x \prod_{s=1}^{t}\left(1 + \frac{|\Delta S_s(\omega^s)|}{\alpha_{s-1}(\omega^{s-1})}\right). \tag{51}$$

To do so we first fix $x \geq 0$, some $\phi = (\phi_t)_{t=1,\dots T} \in \Phi(x)$ and $1 \leq t \leq T$. For $\omega^{t-1} \in \Omega^{t-1}$ fixed, we denote by $\phi_t^\perp(\omega^{t-1})$ the orthogonal projection of $\phi_t(\omega^{t-1})$ on $D^t(\omega^t)$. Recalling Remark 5.3 we have

$$q_t\left(\phi_t^\perp(\omega^{t-1})\Delta S_t(\omega^{t-1}, \cdot) = \phi_t(\omega^{t-1})\Delta S_t(\omega^{t-1}, \cdot)|\omega^{t-1}\right) = 1,$$

and thus $\phi_t^\perp(\omega^{t-1}) \in \mathcal{D}^t_{V_{t-1}^{x,\phi}(\omega^{t-1})}(\omega^{t-1})$ (see (29) for the definition of $\mathcal{D}^t_x$). As the NA condition holds true, Lemma 3.6 applies and $0 \in D^t(\omega^{t+1})$. We can then apply Lemma 5.10 and we obtain that

$$|\phi_t^\perp(\omega^{t-1})| \leq \frac{V_{t-1}^{x,\phi}(\omega^{t-1})}{\alpha_{t-1}(\omega^{t-1})}. \tag{52}$$

Furthermore, as it is well-know that $\omega^{t-1} \in \Omega^{t-1} \to \phi_t^\perp(\omega^{t-1})$ is $\mathcal{F}_{t-1}$-measurable we obtain, applying the Fubini Theorem (see Lemma 7.1), that $P_t\left(\phi_t^\perp \Delta S_t = \phi_t \Delta S_t\right) = 1$ and we denote by Ω^t_{EQ} the P_t-full measure set on which this equality is verified. We need to slightly modify the set Ω^t_{EQ} to use it for different periods. We proceed by induction. We start at $t = 1$ (recall that $\Omega^0 := \{\omega_0\}$) with Ω^1_{EQ}. For $t = 2$ we reset, with an abuse of notation, $\Omega^2_{EQ} = \Omega^2_{EQ} \cap \left(\Omega^1_{EQ} \times \Omega_2\right)$ and we reiterate the process until

T. To prove (51) we proceed by induction. It is clear at $t=0$. Fix some $t \geq 0$ and assume that (51) holds true at t. Let $\omega^{t+1} \in \Omega^{t+1}_{EQ}$, using (51) at t and (52) we get that

$$\begin{aligned}|V^{x,\phi}_{t+1}(\omega^{t+1})| &= \left|V^{x,\phi}_t(\omega^t) + \phi_{t+1}(\omega^t)\Delta S_{t+1}(\omega^{t+1})\right| = \left|V^{x,\phi}_t(\omega^t) + \phi^{\perp}_{t+1}(\omega^t)\Delta S_{t+1}(\omega^{t+1})\right| \\ &\leq \left|V^{x,\phi}_t(\omega^t)\right| \left(1 + \frac{|\Delta S_{t+1}(\omega^{t+1})|}{\alpha_t(\omega^t)}\right) \leq x \prod_{s=1}^{t+1} \left(1 + \frac{|\Delta S_s(\omega^s)|}{\alpha_{s-1}(\omega^{s-1})}\right)\end{aligned}$$

and (51) is proven for $t+1$. It follows since for all $0 \leq s \leq t$, $|\Delta S_s| \in \mathcal{W}_s$ and $\frac{1}{\alpha_s} \in \mathcal{W}_s$ that $V^{x,\phi}_t \in \mathcal{W}_t$. We will prove that for all $\Phi \in \Phi(x)$ and ω^T in a full measure set

$$U^+(\omega^T, V^{x,\phi}_T(\omega^T)) \leq 2^{\overline{\gamma}} K \max(x,1)^{\overline{\gamma}} \left(\prod_{s=1}^{T}\left(1 + \frac{|\Delta S_s(\omega^s)|}{\alpha_{s-1}(\omega^{s-1})}\right)\right)^{\overline{\gamma}} \left(U^+(\omega^T,1) + C_T(\omega^T)\right). \tag{53}$$

Since by assumptions $EU^+(\cdot,1) < \infty$, $EC_T < \infty$ and since for all $0 \leq t \leq T$, $|\Delta S_t| \in \mathcal{W}_t$ and $\frac{1}{\alpha_t} \in \mathcal{W}_t$, we get that $EU^+(\cdot, V^{x,\phi}_T(\cdot)) < \infty$ for all $\Phi \in \Phi(x)$ and both Assumptions 4.7 and 4.8 hold true. We prove now (53). We fix some $x \geq 0$ and $\phi \in \Phi(x)$. Then from the monotonicity of U^+, (51), Assumption 4.10, the fact that $\prod_{s=1}^T \left(1 + \frac{|\Delta S_s(\omega^s)|}{\alpha_{s-1}(\omega^{s-1})}\right) \geq 1$, we have for all $\omega^T \in \Omega^T_{EQ} \bigcap \widetilde{\Omega}_T$ that

$$\begin{aligned}U^+\left(\omega^T, V^{x,\phi}_T(\omega^T)\right) &\leq U^+\left(\omega^T, \max(x,1)\prod_{s=1}^{T}\left(1 + \frac{|\Delta S_s(\omega^s)|}{\alpha_{s-1}(\omega^{s-1})}\right)\right) \\ &\leq K\left(2\max(x,1)\prod_{s=1}^{T}\left(1 + \frac{|\Delta S_s(\omega^s)|}{\alpha_{s-1}(\omega^{s-1})}\right)\right)^{\overline{\gamma}} \left(U^+(\omega^T,1) + C_T(\omega^T)\right).\end{aligned}$$

□

7 Appendix

In this appendix we report basic facts about measure theory, measurable selection theorems and random sets. We also provide the proof of some technical results.

7.1 Generalised integral and Fubini's Theorem

For ease of the reader we provide some well know results on measure theory, stochastic kernels and integrals. The first lemma provides a version of the Fubini Theorem for non-negative functions (see for instance to Theorem 10.7.2 in Bogachev (2007)). We then present our definition of generalised integral and provide another version of the Fubini Theorem for generalised integral (see Proposition 7.4), which is essential throughout the paper.

Let $(H,\mathcal{H})$ and $(K,\mathcal{K})$ be two measurable spaces, p be a probabilty measure on $(H,\mathcal{H})$ and q a stochastic kernel on $(K,\mathcal{K})$ given $(H,\mathcal{H})$, *i.e* such that for any $h \in H$, $C \in \mathcal{K} \to q(C|h)$ is a probability measure on $(K,\mathcal{K})$ and for any $C \in \mathcal{K}$, $h \in H \to q(C|h)$ is $\mathcal{H}$-measurable. Furthermore, for any $A \in \mathcal{H} \otimes \mathcal{K}$ and any $h \in H$, the section of A along h is defined by

$$(A)_h := \{k \in K,\ (h,k) \in A\}. \tag{54}$$

Lemma 7.1 *Let $A \in \mathcal{H} \otimes \mathcal{K}$ be fixed. For any $h \in H$ we have $(A)_h \in \mathcal{K}$ and we define P by*

$$P(A) := \int_H \int_K 1_A(h,k) q(dk|h) p(dh) = \int_H q((A)_h\,|h) p(dh). \tag{55}$$

Then P is a probability measure on $(H \times K, \mathcal{H} \otimes \mathcal{H})$.
Furthermore, if $f : H \times K \to \mathbb{R}_+ \cup \{+\infty\}$ is non-negative and $\mathcal{H} \otimes \mathcal{K}$-measurable then $h \in H \to \int_K f(h,k) q(dk|h)$ is $\mathcal{H}$-measurable with value in $\mathbb{R}_+ \cup \{\infty\}$ and we have

$$\int_{H\times K} f dP := \int_{H\times K} f(h,k) P(dh,dk) = \int_H \int_K f(h,k) q(dk|h) p(dh). \tag{56}$$

Proof. Let $h \in H$ be fixed. Let $\mathcal{T} = \{A \in \mathcal{H} \otimes \mathcal{K} \,|\, (A)_h \in \mathcal{K}\}$. It is easy to see that $\mathcal{T}$ is a sigma algebra on $H \times K$ and is included in $\mathcal{H} \otimes \mathcal{K}$. Let $A = B \times C \in \mathcal{H} \times \mathcal{K}$ then $(A)_h = \emptyset$ if $h \notin B$ and $(A)_h = C$ if $h \in B$. Thus $(A)_h \in \mathcal{K}$ and $\mathcal{H} \times \mathcal{K} \subset \mathcal{T}$. As $\mathcal{T}$ is a sigma-algebra, $\mathcal{H} \otimes \mathcal{K} \subset \mathcal{T}$ and $\mathcal{T} = \mathcal{H} \otimes \mathcal{K}$ follows.
We show now that

$$h \to \int_K 1_A(h,k) q(dk|h) = \int_K 1_{(A)_h}(k) q(dk|h) = q\left((A)_h\,|h\right)$$

is $\mathcal{H}$-measurable for any $A \in \mathcal{H} \otimes \mathcal{K}$.
Let $\mathcal{E} = \{A \in \mathcal{H} \otimes \mathcal{K} \,|\, h \in H \to q\left((A)_h\,|h\right)$ is $\mathcal{H}$-measurable$\}$. It is easy to see that $\mathcal{E}$ is a sigma algebra on $H \times K$ and is included in $\mathcal{H} \otimes \mathcal{K}$. Let $A = B \times C \in \mathcal{H} \times \mathcal{K}$ then $q\left((A)_h)|h\right)$ equals to 0 if $h \notin B$ and to $q(C|h)$ if $h \in B$. So by definition of $q(\cdot|\cdot)$, $\mathcal{H} \times \mathcal{K} \subset \mathcal{E}$. As $\mathcal{E}$ is a sigma-algebra, $\mathcal{H} \otimes \mathcal{K} \subset \mathcal{E}$ and $\mathcal{E} = \mathcal{H} \otimes \mathcal{K}$ follows. Thus the last integral in (55) is well-defined. We verify that P defines a probability measure on $(H \times K, \mathcal{H} \otimes \mathcal{H})$. It is clear that $P(\emptyset) = 0$ and $P(H \times K) = 1$. The sigma-additivity property follows from the monotone convergence theorem.
We prove now that for $f : H \times K \to \mathbb{R}_+ \cup \{+\infty\}$ non-negative and $\mathcal{H} \otimes \mathcal{K}$-measurable, $h \in H \to \int_K f(h,k) q(dk|h)$ is $\mathcal{H}$-measurable and (56) holds true. If $f = 1_A$ for $A \in \mathcal{H} \otimes \mathcal{K}$ the claim is proved. By taking linear combinations, it is proved for $\mathcal{H} \otimes \mathcal{K}$-measurable step functions. Then if $f : H \times K \to \mathbb{R} \cup \{+\infty\}$ is non-negative and $\mathcal{H} \otimes \mathcal{K}$-measurable, then there exists some increasing sequence $(f_n)_{n\geq 1}$ such that $f_n : H \times K \to \mathbb{R}$ is a $\mathcal{H} \otimes \mathcal{K}$-measurable step function and $(f_n)_{n\geq 1}$ converge to f. Using the monotone convergence theorem and (56) for steps functions, we conclude that (56) holds true for f. □

Definition 7.2 Let $f : H \times K \to \mathbb{R} \cup \{\pm\infty\}$ be a $\mathcal{H} \otimes \mathcal{K}$-measurable function. If $\int_{H\times K} f^+ dP < \infty$ or $\int_{H\times K} f^- dP < \infty$, we define the generalised integral of f by

$$\int_{H\times K} f dP := \int_{H\times K} f^+ dP - \int_{H\times K} f^- dP.$$

Remark 7.3 Note that if both $\int_{H\times K} f^+ dP = \infty$ and $\int_{H\times K} f^- dP = \infty$, the integral above is not defined. We could have introduced some convention to handle this situation, however, as in most of the cases we treat we have $\int_{H\times K} f^+ dP < \infty$, we refrain from doing so.

Proposition 7.4 *Let $f : H \times K \to \mathbb{R} \cup \{\pm\infty\}$ be a $\mathcal{H} \otimes \mathcal{K}$-measurable function such that $\int_{H\times K} f^+ dP < \infty$. Then, we have*

$$\int_{H\times K} f dP = \int_H \int_K f(h,k) q(dk|h) p(dh). \tag{57}$$

Remark 7.5 Note that we can assume instead that $\int_{H\times K} f^- dP < \infty$ and the result holds as well. We will use this in the proof of Lemma 2.2 later in the Appendix.

Proof. Using Definition 7.2 and applying Lemma 7.1 to f^+ and f^- we obtain that

$$\begin{aligned}\int_{H\times K} f dP &= \int_{H\times K} f^+ dP - \int_{H\times K} f^- dP \\ &= \int_H \int_K f^+ q(dk|h)p(dh) + \int_H \int_K f^- q(dk|h)p(dh).\end{aligned}$$

To establish (57), assume for a moment that the followng linearity result have been proved: let $g_i : H \times K \to \mathbb{R} \cup \{\pm\infty\}$ be some $\mathcal{H} \otimes \mathcal{K}$-measurable functions such that $\int_{H\times K} g_i^+ dP < \infty$ for $i = 1, 2$. Then

$$\int_H (g_1 + g_2)\, dp = \int_H g_1 dp + \int_H g_2 dp. \tag{58}$$

We apply (58) with $g_1(h) = \int_K f^+(h,k)q(dh|k)$ and $g_2 = -\int_K f^-(h,k)q(dh|k)$ since by Lemma 7.1,

$$\begin{aligned}\int_H g_1^+ dp &= \int_H \left(\int_K f^+(h,k)q(dh|k)\right) p(dh) \\ &= \int_{H\times K} f^+(h,k)q(dh|k)p(dh) = \int_{H\times K} f^+ dP < \infty\end{aligned}$$

and clearly $\int_H g_2^+ dp = 0 < \infty$. So we obtain that

$$\begin{aligned}&\int_H \int_K f^+(h,k)q(dk|h)p(dh) - \int_H \int_K f^-(h,k)q(dk|h)p(dh) \\ &= \int_H \left(\int_K f^+(h,k)q(dk|h) - \int_K f^-(h,k)q(dk|h)\right) p(dh) \\ &= \int_H \int_K f(h,k)q(dk|h)p(dh),\end{aligned}$$

where the second equality comes from the definition of the generalised integral of $f(h,\cdot)$ with respect to $q(\cdot|h)$ and (57) is proven.
We prove now (58). If $\int_H g_i^- dp < \infty$ for $i = 1, 2$ this is trivial. From $\int_H g_i^+ dp < \infty$ we get that $g_i^+ < \infty$ p-almost surely for $i = 1, 2$, so the sum $g_1 + g_2$ is p-almost surely well-defined, taking its value in $[-\infty, \infty)$. As $(g_1 + g_2)^+ \le g_1^+ + g_2^+$, using the linearity of the integral for non-negative functions we get that

$$\int_H (g_1 + g_2)^+ (h)p(dh) \le \int_H g_1^+ dp + \int_H g_2^+ dp < \infty.$$

Now from

$$g_1^+ + g_2^+ - g_1^- - g_2^- = g_1 + g_2 = (g_1 + g_2)^+ - (g_1 + g_2)^-,$$

using again the linearity of the integral for non-negative functions we get that

$$\int_H (g_1 + g_2)^+ dp + \int_H g_1^- dp + \int_H g_2^- dp = \int_H (g_1 + g_2)^- dp + \int_H g_1^+ dp + \int_H g_2^+ dp.$$

Checking the different cases, *i.e* $\int_H g_1^- dp = \infty$ and $\int_H g_2^- dp < \infty$ (and the opposite case) as well as $\int_H g_i^- dp = \infty$ for $i = 1, 2$ we get that (58) is true. □

7.2 Further measure theory issues

We present now specific applications or results that are used throughout the paper. We start with four extensions of the Fubini results presented previously. As noted in Remark 6.12, the introduction of the trace sigma-algebra is the price to pay in order to avoid using the convention $\infty - \infty = -\infty$.

Proposition 7.6 *Fix some* $t \in \{1, \dots, T\}$.

i) Let $f : \Omega^t \to \mathbb{R}_+ \cup \{+\infty\}$ *be a non-negative* $\mathcal{F}_t$*-measurable function. Then* $\omega^{t-1} \in \Omega^{t-1} \to \int_{\Omega_t} f(\omega^{t-1}, \omega_t) q_t(d\omega_t|\omega^{t-1})$ *is* $\mathcal{F}_{t-1}$*-measurable with values in* $\mathbb{R}_+ \cup \{+\infty\}$.

ii) Let $f : \Omega^t \times \mathbb{R}^d \to \mathbb{R}_+ \cup \{+\infty\}$ *be a non-negative* $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$*-measurable function. Then* $(\omega^{t-1}, h) \in \Omega^{t-1} \times \mathbb{R}^d \to \int_{\Omega_t} f(\omega^{t-1}, \omega_t, h) q_t(d\omega_t|\omega^{t-1})$ *is* $\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)$*-measurable with values in* $\mathbb{R}_+ \cup \{+\infty\}$

iii) Let $f : \Omega^t \to \mathbb{R}_+ \cup \{+\infty\}$ *be a non-negative* $\overline{\mathcal{F}}_{t-1} \otimes \mathcal{G}_t$*-measurable function. Then* $\omega^{t-1} \in \Omega^{t-1} \to \int_{\Omega_t} f(\omega^{t-1}, \omega_t) q_t(d\omega_t|\omega^{t-1})$ *is* $\overline{\mathcal{F}}_{t-1}$*-measurable with values in* $\mathbb{R}_+ \cup \{+\infty\}$.

iv) Let $\mathcal{S} \in \mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)$. *Introduce* $\left[\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)\right]_{\mathcal{S}} := \left\{A \cap \mathcal{S},\ A \in \mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)\right\}$ *the trace sigma-algebra of* $\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)$ *on* $\mathcal{S}$. *Let* $f : \Omega^{t-1} \times \mathbb{R}^d \times \Omega_t \to \mathbb{R}_+ \cup \{+\infty\}$ *be a non-negative* $\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{G}_t$*-measurable function. Then* $(\omega^{t-1}, h) \in \mathcal{S} \to \int_{\Omega_t} f(\omega^{t-1}, h, \omega_t) q_t(d\omega_t|\omega^{t-1})$ *is* $\left[\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)\right]_{\mathcal{S}}$*-measurable with values in* $\mathbb{R}_+ \cup \{+\infty\}$.

Proof. Statement $i)$ is a direct application of Lemma 7.1 for $H = \Omega^{t-1}$, $\mathcal{H} = \mathcal{F}_{t-1}$, $K = \Omega_t$, $\mathcal{K} = \mathcal{G}_t$ and $q(\cdot|\cdot) = q_t(\cdot|\cdot)$. To prove statement $ii)$, let $\bar{q}_t$ be defined by

$$\bar{q}_t : (G, \omega^{t-1}, h) \in \mathcal{G}_t \times \Omega^{t-1} \times \mathbb{R}^d \to \bar{q}_t(G|\omega^{t-1}, h) := q_t(G|\omega^{t-1}). \tag{59}$$

We first prove that $\bar{q}_t$ is a stochastic kernel on $\mathcal{G}_t$ given $\Omega^{t-1} \times \mathbb{R}^d$ where measurability is with respect to $\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)$. Let $(\omega^{t-1}, h) \in \Omega^{t-1} \times \mathbb{R}^d$ be fixed, $B \in \mathcal{G}_t \to \bar{q}_t(B|\omega^{t-1}, h) = q_t(B|\omega^{t-1})$ is a probability measure on $(\Omega_t, \mathcal{G}_t)$ by definition of q_t. Let $B \in \mathcal{G}_t$ be fixed, then $(\omega^{t-1}, h) \in \Omega^{t-1} \times \mathbb{R} \to \bar{q}_t(B|\omega^{t-1}, h) = q_t(B|\omega^{t-1})$ is $\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable since for any $B' \in \mathcal{B}(\mathbb{R})$, we have, by definition of q_t,

$$\left\{(\omega^{t-1}, h) \in \Omega^{t-1} \times \mathbb{R}^d, \bar{q}_t(B|\omega^{t-1}, h) \in B'\right\} = \left\{\omega^{t-1} \in \Omega^{t-1}, q_t(B|\omega^{t-1}) \in B'\right\} \times \mathbb{R}^d \in \mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d).$$

Statement $ii)$ follows by an application of Lemma 7.1 for $H = \Omega^{t-1} \times \mathbb{R}^d$, $\mathcal{H} = \mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)$, $K = \Omega_t$, $\mathcal{K} = \mathcal{G}_t$ and $q(\cdot|\cdot) = \bar{q}_t(\cdot|\cdot)$. To prove statement $iii)$ note that since $\mathcal{F}_{t-1} \subset \overline{\mathcal{F}}_{t-1}$ it is clear that q_t is a stochastic kernel on $(\Omega_t, \mathcal{G}_t)$ given $(\Omega^{t-1}, \overline{\mathcal{F}}_{t-1})$ ($i.e$ measurability is with respect to $\overline{\mathcal{F}}_{t-1}$). And statement $iii)$ follows immediately from an application of Lemma 7.1 for $H = \Omega^{t-1}$, $\mathcal{H} = \overline{\mathcal{F}}_{t-1}$, $K = \Omega_t$, $\mathcal{K} = \mathcal{G}_t$ and $q(\cdot|\cdot) = q_t(\cdot|\cdot)$. We prove now the last statement. It is well known that $(S, \left[\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)\right]_{\mathcal{S}})$ is a measurable space. Let $\tilde{q}_t$ be defined by

$$\tilde{q}_t : (G, \omega^{t-1}, h) \in \mathcal{G}_t \times S \to \tilde{q}_t(G|\omega^{t-1}, h) := q_t(G|\omega^{t-1}). \tag{60}$$

We prove that $\tilde{q}_t$ is a stochastic kernel on $(\Omega_t, \mathcal{G}_t)$ given $\left(\mathcal{S}, \left[\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)\right]_{\mathcal{S}}\right)$. Indeed, let $(\omega^{t-1}, h) \in S$ be fixed, $B \in \mathcal{G}_t \to \tilde{q}_t(B|\omega^{t-1}, h) = q_t(B|\omega^{t-1})$ is a probability measure on $(\Omega_t, \mathcal{G}_t)$, by definition of q_t. Let $B \in \mathcal{G}_t$ be fixed, then $(\omega^{t-1}, h) \in S \to \tilde{q}_t(B|\omega^{t-1}, h) = q_t(B|\omega^{t-1})$ is $\left[\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)\right]_{\mathcal{S}}$-measurable since for any $B' \in \mathcal{B}(\mathbb{R})$, we have, by definition of q_t

$$\begin{aligned}\left\{(\omega^{t-1}, h) \in S,\ \tilde{q}_t(B|\omega^{t-1}, h) \in B'\right\} &= \left(\left\{\omega^{t-1} \in \Omega^{t-1}, q_t(B|\omega^{t-1}) \in B'\right\} \times \mathbb{R}^d\right) \bigcap S \\ &\in \left[\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)\right]_{\mathcal{S}}.\end{aligned}$$

Now let f_S be the restriction of f to $\mathcal{S} \times \Omega_t$. Using similar arguments and the fact that

$$\left[\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{G}_t\right]_{\mathcal{S} \times \Omega_t} = \left[\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)\right]_{\mathcal{S}} \otimes \mathcal{G}_t, \tag{61}$$

we obtain that f_S is $\left[\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)\right]_{\mathcal{S}} \otimes \mathcal{G}_t$-measurable. Finally, statement $iv)$ follows from another application of Lemma 7.1 for $H = S$, $\mathcal{H} = \left[\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)\right]_{\mathcal{S}}$, $K = \Omega_t$, $\mathcal{K} = \mathcal{G}_t$ and $q(\cdot|\cdot) = \tilde{q}_t(\cdot|\cdot)$. $\square$

Lemma 7.7 *Let $f : \Omega^{t+1} \to \mathbb{R}_+ \cup \{\infty\}$ be $\mathcal{F}_{t+1}$-measurable, non-negative and such that $\int_{\Omega^{t+1}} f(\omega^{t+1}) P_{t+1}(d\omega^{t+1}) < \infty$. Then $\omega^t \in \Omega^t \to \int_{\Omega_{t+1}} f(\omega^t, \omega_{t+1}) q_{t+1}(d\omega_{t+1}|\omega^t)$ is $\mathcal{F}_t$-measurable. Furthermore, let*

$$N^t := \{\omega^t \in \Omega^t, \int_{\Omega_{t+1}} f(\omega^t, \omega_{t+1}) q_{t+1}(d\omega_{t+1}|\omega^t) = \infty\}.$$

Then $N_t \in \mathcal{F}_t$ and $P_t(N^t) = 0$

Proof. The first assertion of the lemma is a direct application of $i)$ of Proposition 7.6. So it is clear that $N^t \in \mathcal{F}_t$. Furthermore, applying the Fubini Theorem (see Lemma 7.1) we get that

$$\int_{\Omega^t} \int_{\Omega_{t+1}} f(\omega^t, \omega_{t+1}) q_{t+1}(d\omega_{t+1}|\omega^t) P_t(d\omega^t) = \int_{\Omega^{t+1}} f(\omega^{t+1}) P_{t+1}(d\omega^{t+1}) < \infty.$$

Assume that $P_t(N^t) > 0$. Then

$$\int_{\Omega^{t+1}} f(\omega^{t+1}) P_{t+1}(d\omega^{t+1}) \geq \int_{N^t} \int_{\Omega_{t+1}} f(\omega^t, \omega_{t+1}) q_{t+1}(d\omega_{t+1}|\omega^t) P_t(d\omega^t) = \infty.$$

We get a contradiction : $P_t(N^t) = 0$. □

The next lemma, loosely speaking, allows to obtain "nice" sections ($i.e$ set of full measure for a certain probability measure). We use it in the proofs of Theorem 4.17 and Lemma 7.9.

Lemma 7.8 *Fix some $t \in \{1, \dots, T\}$. Let $\widetilde{\Omega}^t \in \mathcal{F}_t$ such that $P_t(\widetilde{\Omega}^t) = 1$ and $\widetilde{\Omega}^{t-1} \in \mathcal{F}_{t-1}$ such that $P_{t-1}(\widetilde{\Omega}^{t-1}) = 1$ and set*

$$\overline{\Omega}^{t-1} := \left\{\omega^{t-1} \in \widetilde{\Omega}^{t-1},\ q_t\left(\left(\widetilde{\Omega}^t\right)_{\omega^{t-1}} |\omega^{t-1}\right) = 1\right\}$$

see Lemma 7.1 for the definition of $\left(\widetilde{\Omega}^t\right)_{\omega^{t-1}}$. Then $\overline{\Omega}^{t-1} \in \mathcal{F}_{t-1}$ and $P_t(\overline{\Omega}^{t-1}) = 1$.

Proof. From Lemma 7.1 we know $\omega^{t-1} \to q_t\left(\left(\widetilde{\Omega}^t\right)_{\omega^{t-1}} |\omega^{t-1}\right)$ is $\mathcal{F}_{t-1}$-measurable and the fact that $\overline{\Omega}^{t-1} \in \mathcal{F}_{t-1}$ follows immediately.
Furthermore, using the Fubini Theorem (see Lemma 7.1) we have that

$$\begin{aligned}
1 = P_t(\widetilde{\Omega}^t) &= \int_{\Omega^{t-1}} \int_{\Omega_t} 1_{\widetilde{\Omega}^t}(\omega^{t-1}, \omega_t) q_t(d\omega_t|\omega^{t-1}) P_{t-1}(d\omega^{t-1}) \\
&= \int_{\Omega^{t-1}} \int_{\Omega_t} 1_{\left(\widetilde{\Omega}^t\right)_{\omega^{t-1}}}(\omega_t) q_t(d\omega_t|\omega^{t-1}) P_{t-1}(d\omega^{t-1}) \\
&= \int_{\widetilde{\Omega}^{t-1}} \int_{\Omega_t} 1_{\left(\widetilde{\Omega}^t\right)_{\omega^{t-1}}}(\omega_t) q_t(d\omega_t|\omega^{t-1}) P_{t-1}(d\omega^{t-1}) \\
&= \int_{\widetilde{\Omega}^{t-1}} q_t\left(\left(\widetilde{\Omega}^t\right)_{\omega^{t-1}} |\omega^{t-1}\right) P_{t-1}(d\omega^{t-1}) \\
&= \int_{\overline{\Omega}^{t-1}} 1 \times P_{t-1}(d\omega^{t-1}) + \int_{\widetilde{\Omega}^{t-1} \setminus \overline{\Omega}^{t-1}} q_t\left(\left(\widetilde{\Omega}^t\right)_{\omega^{t-1}} |\omega^{t-1}\right) P_{t-1}(d\omega^{t-1}),
\end{aligned}$$

where we have used for the third line the fact that $P(\widetilde{\Omega}^{t-1}) = 1$.
But if $P(\widetilde{\Omega}^{t-1} \setminus \overline{\Omega}^{t-1}) > 0$ then we have that by definition of $\overline{\Omega}^{t-1}$ that

$$\int_{\widetilde{\Omega}^{t-1} \setminus \overline{\Omega}^{t-1}} q_t\left(\left(\widetilde{\Omega}^t\right)_{\omega^{t-1}} |\omega^{t-1}\right) P_{t-1}(d\omega^{t-1}) < P_{t-1}(\widetilde{\Omega}^{t-1} \setminus \overline{\Omega}^{t-1}),$$

and thus

$$1 < P_{t-1}(\overline{\Omega}^{t-1}) + P_{t-1}(\widetilde{\Omega}^{t-1} \setminus \overline{\Omega}^{t-1}) = 1,$$

which is absurd and thus $P_{t-1}(\widetilde{\Omega}^{t-1}\backslash\overline{\Omega}^{t-1}) = 0$. We conclude using again that $P_{t-1}(\widetilde{\Omega}^{t-1}) = 1$. $\square$

The following lemma is used throughout the paper. In particular, the last statement is used in the proof of the main theorem

Lemma 7.9 *Let $0 \le t \le T-1$, $B \in \mathcal{B}(\mathbb{R})$, $H : \Omega^t \to \mathbb{R}$ and $h_t : \Omega^t \to \mathbb{R}^d$ be $\mathcal{F}_t$-measurable be fixed. Then the functions*

$$(\omega^t, h) \in \Omega^t \times \mathbb{R}^d \to q_{t+1}(H(\omega^t) + h\Delta S_{t+1}(\omega^t, \cdot) \in B|\omega^t), \tag{62}$$
$$\omega^t \in \Omega^t \to q_{t+1}(H(\omega^t) + h_t(\omega^t)\Delta S_{t+1}(\omega^t, \cdot) \in B|\omega^t), \tag{63}$$

are respectively $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$-measurable and $\mathcal{F}_t$-measurable. Furthermore, assume that $P_{t+1}\left(H(\cdot) + h_t(\cdot)\Delta S_{t+1}(\cdot) \in B\right) = 1$, then there exists some P_t-full measure set $\overline{\Omega}^t$ such that for all $\omega^t \in \overline{\Omega}^t$, $q_{t+1}(H(\omega^t) + h_t(\omega^t)\Delta S_{t+1}(\omega^t, \cdot) \in B|\omega^t) = 1$.

Proof. As $h \in \mathbb{R}^d \to h\Delta S_{t+1}(\omega^t, \omega_{t+1})$ is continuous for all $(\omega^t, \omega_{t+1}) \in \Omega^t \times \Omega_{t+1}$ and $(\omega^t, \omega_{t+1}) \in \Omega^t \times \Omega_{t+1} \to h\Delta S_{t+1}(\omega^t, \omega_{t+1})$ is $\mathcal{F}_{t+1} = \mathcal{F}_t \otimes \mathcal{G}_{t+1}$-measurable for all $h \in \mathbb{R}^d$ (recall that S_t and S_{t+1} are respectively $\mathcal{F}_t$ and $\mathcal{F}_{t+1}$ measurable by assumption), $(\omega^t, \omega_{t+1}, h) \in \Omega^t \times \Omega_{t+1} \times \mathbb{R}^d \to h\Delta S_{t+1}(\omega^t, \omega_{t+1})$ is $\mathcal{F}_t \otimes \mathcal{G}_{t+1} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable as a Carathéodory function. As H is $\mathcal{F}_t$-measurable we obtain that $\psi : (\omega^t, \omega_{t+1}, h) \in \Omega^t \times \Omega_{t+1} \times \mathbb{R}^d \to H(\omega^t) + h\Delta S_{t+1}(\omega^t, \omega_{t+1})$ is also $\mathcal{F}_t \otimes \mathcal{G}_{t+1} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. Therefore, for any $B \in \mathcal{B}(\mathbb{R})$, $f_B : (\omega^t, \omega_{t+1}, h) \in \Omega^t \times \Omega_{t+1} \times \mathbb{R}^d \to 1_{\psi(\cdot,\cdot,\cdot) \in B}(\omega^t, \omega_{t+1}, h)$ is $\mathcal{F}_t \otimes \mathcal{G}_{t+1} \otimes \mathcal{B}(\mathbb{R}^d)$. We conclude using statement $i)$ of Proposition 7.6 applied to f_B and (62) is proved. We prove (63) using similar arguments. Since h_t is $\mathcal{F}_t$-measurable, it is clear that $\psi_{h_t} : (\omega^t, \omega_{t+1}) \in \Omega^t \times \Omega_{t+1} \to H(\omega^t) + h_t(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1})$ is $\mathcal{F}_t \otimes \mathcal{G}_{t+1}$-measurable. Therefore, for any $B \in \mathcal{B}(\mathbb{R})$, $f_{B,h_t} : (\omega^t, \omega_{t+1}) \in \Omega^t \times \Omega_{t+1} \to 1_{\psi_{h_t}(\cdot,\cdot) \in B}(\omega^t, \omega_{t+1})$ is $\mathcal{F}_t \otimes \mathcal{G}_{t+1}$-measurable. We conclude applying $i)$ of Proposition 7.6 to f_{B,h_t}.

For the last statement, we set

$$\widetilde{\Omega}^{t+1} := \left\{\omega^{t+1} = (\omega^t, \omega_{t+1}) \in \Omega^t \times \Omega_{t+1},\ H(\omega^t) + h_t(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}) \in B\right\}.$$

It is clear that $\widetilde{\Omega}^{t+1} \in \mathcal{F}_{t+1}$ and that $P_{t+1}(\widetilde{\Omega}^{t+1}) = 1$. We can then apply Lemma 7.8 and we obtain some P_t-full measure set $\overline{\Omega}^t$ such that for all $\omega^t \in \overline{\Omega}^t$, $q_{t+1}(H(\omega^t) + h_t(\omega^t)\Delta S_{t+1}(\omega^t, \cdot) \in B|\omega^t) = 1$. $\square$

Lemma 7.10 is often used in conjunction with the Aumann Theorem (see Corollary 1 in Sainte-Beuve (1974)) to obtain a $\mathcal{F}_t$-measurable selector.

Lemma 7.10 *Let $f : \Omega^t \to \mathbb{R}$ be $\overline{\mathcal{F}}_t$-measurable. Then there exists $g : \Omega^t \to \mathbb{R}$ that is $\mathcal{F}_t$-measurable and such that $f = g$ P_t-almost surely, $i.e$ there exists $\Omega^t_{fg} \in \mathcal{F}_t$ with $P_t\left(\Omega^t_{fg}\right) = 1$ and $\Omega^t_{fg} \subset \{f = g\}$.*

Proof. Let $f = 1_B$ with $B \in \overline{\mathcal{F}}_t$ then $B = A \cup N$, with $A \in \mathcal{F}_t$ and $N \in \mathcal{N}_{P_t}$. Let $g = 1_A$. Then g is $\mathcal{F}_t$-measurable. Clearly, $\{f \neq g\} = N \in \mathcal{N}_{P_t}$, thus $f = g$ P_t a.s. By taking linear combinations, the lemma is proven for step functions using the same argument for each indicator function. Then it is always possible to approximate some $\overline{\mathcal{F}}_t$-measurable function f by a sequence of step function $(f_n)_{n\ge1}$. From the preceding step for all $n \ge 1$, we get some $\mathcal{F}_t$-measurable step functions g_n such that $f_n = g_n$ P_t-almost surely. Let $g = \limsup g_n$, g is $\mathcal{F}_t$-measurable and we conclude since $\{f \neq g\} \subset \cup_{n\ge1}\{f_n \neq g_n\}$ which is again in $\mathcal{N}_{P_t}$. $\square$

Next we provide some simple but useful results on usc functions.

Lemma 7.11 *Let C be a closed subset of $\mathbb{R}^m$ for some $m \ge 1$. Let $g : \mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}$ be such that $g = -\infty$ on $\mathbb{R}^m \backslash C$. Then g is usc on $\mathbb{R}^m$ if and only if g is usc on C.*

Proof. We prove that if g is usc on C then it is usc on $\mathbb{R}^m$ as the reverse implication is trivial. Let $\alpha \in \mathbb{R}$ be fixed. We prove that $S_\alpha := \{x \in \mathbb{R}^m,\ g(x) \geq \alpha\}$ is closed in $\mathbb{R}^m$. Let $(x_n)_{n\geq 1} \subset S_\alpha$ converge to $x \in \mathbb{R}^m$. Then $x_n \in C$ for all $n \geq 1$ and as C is a closed set, $x \in C$. As g is usc on C, ($i.e$ the set $\{x \in C,\ g(x) \geq \alpha\}$ is closed for the induced topology of $\mathbb{R}^m$ on C) we get that $g(x) \geq \alpha$, $i.e$ $x \in S_\alpha$ and g is usc on $\mathbb{R}^m$. □

Lemma 7.12 *Let $S \subset \mathbb{R}$ be a closed subset of $\mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ be such that f is usc and non-decreasing on S. Then f is right-continuous on S.*

Proof. Let $(x_n)_{n\geq 1} \subset S$ be a sequence converging to some x^* from above. Then $x^* \in S$ since S is closed. As $x \in S \to f(x)$ is non-decreasing, for all $n \geq 1$ we have that $f(x_n) \geq f(x^*)$ and thus $\liminf_n f(x_n) \geq f(x^*)$. Now as f is usc on S, we get that $\limsup_n f(x_n) \leq f(x^*)$. The right continuity of f on S follows immediately. □

We now establish a useful extension of Lemma 7.10.

Lemma 7.13 *Let $f : \Omega^t \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ be an $\overline{\mathcal{F}}_t \otimes \mathcal{B}(\mathbb{R})$-measurable function such that for all $\omega^t \in \Omega^t$, $x \in \mathbb{R} \to f(\omega^t, x)$ is usc and non-decreasing. Then, there exists some $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R})$-measurable function g from $\Omega^t \times \mathbb{R}$ to $\mathbb{R} \cup \{\pm\infty\}$ and some $\Omega^t_{mes} \in \mathcal{F}_t$ such that $P_t(\Omega^t_{mes}) = 1$ and $f(\omega^t, x) = g(\omega^t, x)$ for all $(\omega^t, x) \in \Omega^t_{mes} \times \mathbb{R}$.*

Remark 7.14 In particular, for all $\omega^t \in \Omega^t_{mes}$, $x \in \mathbb{R} \to g(\omega^t, x)$ is usc and non-decreasing.

Proof. Let $n \geq 1$ and $k \in \mathbb{Z}$ be fixed. We apply Lemma 7.10 to $f(\cdot) = f(\cdot, \frac{k}{2^n})$ that is $\overline{\mathcal{F}}_t$-measurable by assumption and we get some $\mathcal{F}_t$-measurable $g_{n,k} : \Omega^t \to \mathbb{R} \cup \{\pm\infty\}$ and some $\Omega^t_{n,k} \in \mathcal{F}_t$ such that $P_t(\Omega^t_{n,k}) = 1$ and $\Omega^t_{n,k} \subset \left\{\omega^t \in \Omega^t,\ f(\omega^t, \frac{k}{2^n}) = g_{n,k}(\omega^t)\right\}$. We set

$$\Omega^t_{mes} := \bigcap_{n\geq 1, k\in\mathbb{Z}} \Omega^t_{n,k}. \tag{64}$$

It is clear that $\Omega^t_{mes} \in \mathcal{F}_t$ and that $P_t(\Omega^t_{mes}) = 1$.
Now, we define for all $n \geq 1$, $g_n : \Omega^t \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ by

$$g_n(\omega^t, x) := \sum_{k\in\mathbb{Z}} 1_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right]}(x) g_{n,k}(\omega^t).$$

It is clear that g_n is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R})$-measurable for all $n \geq 1$. Finally, we define $g : \Omega^t \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ by

$$g(\omega^t, x) := \lim_n g_n(\omega^t, x). \tag{65}$$

Then g is again $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R})$-measurable and it remains to prove that $f(\omega^t, x) = g(\omega^t, x)$ for all $(\omega^t, x) \in \Omega^t_{mes} \times \mathbb{R}$. Let $(\omega^t, x) \in \Omega^t_{mes} \times \mathbb{R}$ be fixed. For all $n \geq 1$, there exists $k_n \in \mathbb{Z}$ such that $\frac{k_n - 1}{2^n} < x \leq \frac{k_n}{2^n}$ and such that $g_n(\omega^t, x) = g_{n,k_n}(\omega^t) = f(\omega^t, \frac{k_n}{2^n})$. Applying Lemma 7.12 to $f(\cdot) = f(\omega^t, \cdot)$ (and $S = \mathbb{R}$), we get that $x \in \mathbb{R} \to f(\omega^t, x)$ is right-continuous on $\mathbb{R}$. As $\left(\frac{k_n}{2^n}\right)_{n\geq 1}$ converges to x from above, it follows that $g(\omega^t, x) = \lim_n f(\omega^t, \frac{k_n}{2^n}) = f(\omega^t, x)$ and this concludes the proof. □

Finally, we introduce the following definition.

Definition 7.15 Let S be a closed interval of $\mathbb{R}$. A function $f : \Omega^t \times S \to \mathbb{R}$ is an extended Carathéodory function if

i) for all $\omega^t \in \Omega^t$, $x \in S \to f(\omega^t, x)$ is right-continuous,

ii) for all $x \in S$, $\omega^t \in \Omega^t \to f(\omega^t, x)$ is $\mathcal{F}_t$-measurable.

And we prove the following lemma that is an extension of a well-know result on Carathéodory functions (see for example 4.10 in Aliprantis and Border (2006))

Lemma 7.16 *Let $S \subset \mathbb{R}$ be a closed interval of $\mathbb{R}$ and $f : \Omega^t \times S \to \mathbb{R}$ be an extended Carathéodory function. Then f is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R})$-measurable.*

Proof. We define for all $n \geq 1$, $f_n : \Omega^t \times \mathbb{R} \to \mathbb{R}$ by

$$f_n(\omega^t, x) := \sum_{k \in \mathbb{Z}} 1_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right]}(x) 1_S(\frac{k}{2^n}) f(\omega^t, \frac{k}{2^n}).$$

It is clear that f_n is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R})$-measurable. From the right continuity of f, we can show as in the proof of Lemma 7.13 that $f(\omega^t, x) = \lim_n f_n(\omega^t, x)$ for all $(\omega^t, x) \in \Omega^t \times S$ and the proof is complete (recall that $\Omega \times S \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R})$ as S is a closed subset of $\mathbb{R}$). □

Remark 7.17 Note that we have the same result if we replace $\mathcal{F}_t$ with $\overline{\mathcal{F}_t}$.

7.3 Proof of technical results

Finally, we provide the missing results and proofs of the paper. We start with the following results from Section 2.

Proof of Lemma 2.2. We refer to Section 6.1 of Carassus and Rásonyi (2015) for the definition and various properties of generalized conditional expectations. In particular since $E(h^+) = \int_{\Omega^t} h^+ dP_t < \infty$, $E(h|\mathcal{F}_s)$ is well-defined (in the generalised sense) for all $0 \leq s \leq t$ (see Lemma 6.2 of Carassus and Rásonyi (2015)). Similarly, from Proposition 7.4 we have that $\varphi : \Omega^s \to \mathbb{R} \cup \{\pm\infty\}$ is well-defined (in the generalised sense) and $\mathcal{F}_s$-measurable.
As $\varphi(X_1, \ldots, X_s)$ is $\mathcal{F}_s$-measurable, it remains to prove that $E(gh) = E(g\varphi(X_1, \ldots, X_s))$ for all $g : \Omega^s \to \mathbb{R}_+$ non-negative, $\mathcal{F}_s$-measurable and such that $E(gh)$ is well-defined in the generalised sense, $i.e$ such that $E\,(gh)^+ < \infty$ or $E\,(gh)^- < \infty$. Recalling the notations of the beginning of Section 2 and using the Fubini Theorem for the third and fourth equality (see Proposition 7.4 and Remark 7.5), we get that

$$\begin{aligned}
E(gh) &= E(g(X_1, \ldots, X_s)h(X_1, \ldots, X_t)) = \int_{\Omega^T} g(\omega_1, \ldots, \omega_s)h(\omega_1, \ldots, \omega_t)P(d\omega^T) \\
&= \int_{\Omega^t} g(\omega_1, \ldots, \omega_s)h(\omega_1, \ldots, \omega_t)q_t(\omega_t|\omega^{t-1}) \ldots q_{s+1}(\omega_{s+1}|\omega^s)P_s(d\omega^s) \\
&= \int_{\Omega^s} g(\omega_1, \ldots, \omega_s)\left(\int_{\Omega_{s+1}\times\ldots\times\Omega_t} h(\omega_1, \ldots, \omega_s, \omega_{s+1}, \ldots, \omega_t)q_t(\omega_t|\omega^{t-1}) \ldots q_{s+1}(\omega_{s+1}|\omega^s)\right) P_s(d\omega^s) \\
&= \int_{\Omega^s} g(\omega_1, \ldots, \omega_s)\varphi(\omega_1, \ldots, \omega_s)P_s(d\omega^s) \\
&= E(g(X_1, \ldots, X_s)\varphi(X_1, \ldots, X_t)),
\end{aligned}$$

which concludes the proof. □

We give now the proof of results of Section 3.
Proof of Lemma 3.4. We first prove that $\widetilde{D}^{t+1}$ is a non-empty, closed-valued and $\mathcal{F}_t$-measurable random set. It is clear from its definition (see (2)) that for all $\omega^t \in \Omega^t$, $\widetilde{D}^{t+1}(\omega^t)$ is a non-empty and closed subset of $\mathbb{R}^d$. We now show that $\widetilde{D}^{t+1}$ is measurable. Let O be a fixed open set in $\mathbb{R}^d$ and introduce

$$\begin{aligned}
\mu_O : \omega^t \in \Omega^t \to \mu_O(\omega^t) &:= q_{t+1}\left(\Delta S_{t+1}(\omega^t, .) \in O|\omega^t\right) \\
&= \int_{\Omega_{t+1}} 1_{\Delta S_{t+1}(\cdot,\cdot)\in O}(\omega^t, \omega_{t+1})q_{t+1}(d\omega_{t+1}|\omega^t).
\end{aligned}$$

We prove that μ_O is $\mathcal{F}_t$-measurable. As $(\omega^t, \omega_{t+1}) \in \Omega^t \times \Omega_{t+1} \to \Delta S_{t+1}(\omega^t, \omega_{t+1})$ is $\mathcal{F}_t \otimes \mathcal{G}_{t+1}$-measurable and $O \in \mathcal{B}(\mathbb{R}^d)$, $(\omega^t, \omega_{t+1}) \to 1_{\Delta S_{t+1}(\cdot,\cdot) \in O}(\omega^t, \omega_{t+1})$ is $\mathcal{F}_t \otimes \mathcal{G}_{t+1}$-measurable and the result follows from Proposition 7.9.
By definition of $\widetilde{D}^{t+1}(\omega^t)$ we get that

$$\{\omega^t \in \Omega^t, \ \widetilde{D}^{t+1}(\omega^t) \cap O \neq \emptyset\} = \{\omega^t \in \Omega^t, \ \mu_O(\omega^t) > 0\} \in \mathcal{F}_t.$$

Next we prove that D^{t+1} is a non-empty, closed-valued and $\mathcal{F}_t$-measurable random set. Using (3), D^{t+1} is a non-empty and closed-valued random set. It remains to prove that D^{t+1} is $\mathcal{F}_t$-measurable. As $\widetilde{D}^{t+1}$ is $\mathcal{F}_t$-measurable, applying the Castaing representation (see Theorem 2.3 in Chapter 1 of Molchanov (2005) or Theorem 14.5 of Rockafellar and Wets (1998)), we obtain a countable family of $\mathcal{F}_t$-measurable functions $(f_n)_{n \geq 1} : \Omega^t \to \mathbb{R}^d$ such that for all $\omega^t \in \Omega^t$, $\widetilde{D}^{t+1}(\omega^t) = \overline{\{f_n(\omega^t), \ n \geq 1\}}$ (where the closure is taken in $\mathbb{R}^d$ with respect to the usual topology). Let $\omega^t \in \Omega^t$ be fixed. It can be easily shown that

$$D^{t+1}(\omega^t) = \mathrm{Aff}(\widetilde{D}^{t+1}(\omega^t)) = \overline{\left\{ f_1(\omega^t) + \sum_{i=2}^{p} \lambda_i (f_i(\omega^t) - f_1(\omega^t)), \ (\lambda_2, \dots, \lambda_p) \in \mathbb{Q}^{p-1}, \ p \geq 2 \right\}}. \tag{66}$$

So, using again the Castaing representation (see Theorem 14.5 of Rockafellar and Wets (1998)), we obtain that $D^{t+1}(\omega^t)$ is $\mathcal{F}_t$-measurable. From Theorem 14.8 of Rockafellar and Wets (1998), $Graph(D^{t+1}) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ (recall that D^{t+1} is closed-valued). $\square$

Proof of Lemma 3.5. Introduce $C^{t+1}(\omega^t) := \overline{\mathrm{Conv}}(\widetilde{D}^{t+1}(\omega^t))$ the closed convex hull generated by $\widetilde{D}^{t+1}(\omega^t)$. As $C^{t+1}(\omega^t) \subset D^{t+1}(\omega^t)$ we will prove that $0 \in C^{t+1}(\omega^t)$. Since $C^{t+1}(\omega^t) \subset D^{t+1}(\omega^t)$ by assumption, for all $h \in C^{t+1}(\omega^t) \backslash \{0\}$

$$q_{t+1}(h \Delta S_{t+1}(\omega^t, \cdot) \geq 0 | \omega^t) < 1. \tag{67}$$

Thus if we find some $h_0 \in C^{t+1}(\omega^t)$ such that $q_{t+1}(h_0 \Delta S_{t+1}(\omega^t, \cdot) \geq 0 | \omega^t) = 1$ then $h_0 = 0$. We distinguish two cases. First assume that for all $h \in \mathbb{R}^d$, $h \neq 0$, $q_{t+1}(h \Delta S_{t+1}(\omega^t, .) \geq 0 | \omega^t) < 1$. Then the polar cone of $C^{t+1}(\omega^t)$, *i.e* the set

$$\left(C^{t+1}(\omega^t)\right)^\circ := \{y \in \mathbb{R}^d, \ yx \leq 0, \forall \, x \in C^{t+1}(\omega^t)\}$$

is reduced to $\{0\}$. Indeed if this is not the case there exists $y_0 \in \mathbb{R}^d$ such that $-y_0 x \geq 0$ for all $x \in C^{t+1}(\omega^t)$. As $A := \{\omega_{t+1} \in \Omega_{t+1}, \ \Delta S_{t+1}(\omega^t, \omega_{t+1}) \in \widetilde{D}^{t+1}(\omega^t)\} \subset \{\omega_{t+1} \in \Omega_{t+1}, \ -y_0 \Delta S_{t+1}(\omega^t, \omega_{t+1}) \geq 0\}$ and $q_{t+1}(A|\omega^t) = 1$ we obtain that $q_{t+1}(-y_0 \Delta S_{t+1}(\omega^t, \cdot) \geq 0 | \omega^t) = 1$ a contradiction. As $\left(\left(C^{t+1}(\omega^t)\right)^\circ\right)^\circ =$ cone $\left(C^{t+1}(\omega^t)\right)$ where cone $\left(C^{t+1}(\omega^t)\right)$ denote the cone generated by $C^{t+1}(\omega^t)$ we get that cone $\left(C^{t+1}(\omega^t)\right) = \mathbb{R}^d$. Let $u \neq 0 \in$ cone $\left(C^{t+1}(\omega^t)\right)$ then $-u \in$ cone $\left(C^{t+1}(\omega^t)\right)$ and there exist $\lambda_1 > 0$, $\lambda_2 > 0$ and $v_1, \ v_2 \in C^{t+1}(\omega^t)$ such that $u = \lambda_1 v_1$ and $-u = \lambda_2 v_2$. Thus $0 = \frac{\lambda_1}{\lambda_1 + \lambda_2} v_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} v_2 \in C^{t+1}(\omega^t)$ by convexity of $C^{t+1}(\omega^t)$.
Now we assume that there exists some $h_0 \in \mathbb{R}^d$, $h_0 \neq 0$ such that $q_{t+1}(h_0 \Delta S_{t+1}(\omega^t, .) \geq 0 | \omega^t) = 1$. Note that since $h_0 \in \mathbb{R}^d$ we cannot use (67). Introduce the orthogonal projection on $C^{t+1}(\omega^t)$ (recall that $C^{t+1}(\omega^t)$ is a closed convex subset of $\mathbb{R}^d$)

$$p : h \in \mathbb{R}^d \to p(h) \in C^{t+1}(\omega^t).$$

Then p is continuous and we have $(h - p(h))\,(x - p(h)) \leq 0$ for all $x \in C^{t+1}(\omega^t)$. Fix $\omega_{t+1} \in \{\omega_{t+1} \in \Omega_{t+1}, \ \Delta S_{t+1}(\omega^t, \omega_{t+1}) \in \widetilde{D}^{t+1}(\omega^t)\} \cap \{\omega_{t+1} \in \Omega_{t+1}, \ h_0 \Delta S_{t+1}(\omega^t, \omega_{t+1}) \geq 0\}$ and $\lambda \geq 0$. Let $h = \lambda h_0$ and $x = \Delta S_{t+1}(\omega^t, \omega_{t+1}) \in C^{t+1}(\omega^t)$ in the previous equation, we obtain (recall that $\widetilde{D}^{t+1}(\omega^t) \subset C^{t+1}(\omega^t)$)

$$\begin{aligned} 0 \leq \lambda h_0 \Delta S_{t+1}(\omega^t, \omega_{t+1}) &= (\lambda h_0 - p(\lambda h_0))\, \Delta S_{t+1}(\omega^t, \omega_{t+1}) + p(\lambda h_0) \Delta S_{t+1}(\omega^t, \omega_{t+1}) \\ &\leq (\lambda h_0 - p(\lambda h_0))\, p(\lambda h_0) + p(\lambda h_0) \Delta S_{t+1}(\omega^t, \omega_{t+1}). \end{aligned}$$

As this is true for all $\lambda \geq 0$ we may take the limit when λ goes to zero and use the continuity of p

$$p(0)\Delta S_{t+1}(\omega^t, \omega_{t+1}) \geq |p(0)|^2 \geq 0$$

As $q_{t+1}\left(\left\{\omega_{t+1} \in \Omega_{t+1},\ \Delta S_{t+1}(\omega^t, \omega_{t+1}) \in \widetilde{D}^{t+1}(\omega^t)\right\} |\omega^t\right) = 1$ by definition of $\widetilde{D}^{t+1}(\omega^t)$ and as $q_{t+1}(h_0 \Delta S_{t+1}(\omega^t, .) \geq 0|\omega^t) = 1$ as well we have obtained that

$$q_{t+1}(p(0)\Delta S_{t+1}(\omega^t, \cdot) \geq 0|\omega^t) = 1.$$

The fact that $p(0) \in C^{t+1}(\omega^t)$ together with (67) implies that $p(0) = 0$ and $0 \in C^{t+1}(\omega^t)$ follows.
□

The following lemma has been used in the proof of Lemma 3.6. It corresponds to Lemma 2.5 of Nutz (2014)

Lemma 7.18 *Let $\omega^t \in \Omega^t$ be fixed. Recall that $L^{t+1}(\omega^t) := \left(D^{t+1}(\omega^t)\right)^{\perp}$ is the orthogonal space of $D^{t+1}(\omega^t)$ (see (6)). Then for $h \in \mathbb{R}^d$ we have that*

$$q_{t+1}(h\Delta S_{t+1}(\omega^t, \cdot) = 0|\omega^t) = 1 \iff h \in L^{t+1}(\omega^t).$$

Proof. Assume that $h \in L^{t+1}(\omega^t)$. Then $\{\omega \in \Omega_t,\ \Delta S_{t+1}(\omega^t, \omega) \in D^{t+1}(\omega^t)\} \subset \{\omega \in \Omega_t, h\Delta S_{t+1}(\omega^t, \omega) = 0\}$. As by definition of $D^{t+1}(\omega^t)$, $q_{t+1}(\Delta S_{t+1}(\omega^t, .) \in D^{t+1}(\omega^t)|\omega^t) = 1$, we conclude that $q_{t+1}(h\Delta S_{t+1}(\omega^t, .) = 0|\omega^t) = 1$. Conversely, we assume that $h \notin L^{t+1}(\omega^t)$ and we show that $q_{t+1}(h\Delta S_{t+1}(\omega^t, .) = 0|\omega^t) < 1$. We first show that there exists $v \in \widetilde{D}^{t+1}(\omega^t)$ such that $hv \neq 0$. If not, for all $v \in \widetilde{D}^{t+1}(\omega^t)$, $hv = 0$ and for any $w \in D^{t+1}(\omega^t)$ with $w = \sum_{i=1}^m \lambda_i v_i$ where $\lambda_i \in \mathbb{R}$, $\sum_{i=1}^m \lambda_i = 1$ and $v_i \in \widetilde{D}^{t+1}(\omega^t)$, we get that $hw = 0$, a contradiction. Furthermore there exists an open ball centered in v with radius $\varepsilon > 0$, $B(v, \varepsilon)$, such that $hv' \neq 0$ for all $v' \in B(v, \varepsilon)$. Assume that $q_{t+1}(\Delta S_{t+1}(\omega^t, .) \in B(v, \varepsilon)|\omega^t) = 0$ or equivalently that $q_{t+1}(\Delta S_{t+1}(\omega^t, .) \in \mathbb{R}^d \setminus B(v, \varepsilon)|\omega^t) = 1$. By definition of the support, $\widetilde{D}^{t+1}(\omega^t) \subset \mathbb{R}^d \setminus B(v, \varepsilon)$: this contradicts $v \in \widetilde{D}^{t+1}(\omega^t)$. Therefore $q_{t+1}(\Delta S_{t+1}(\omega^t, .) \in B(v, \varepsilon)|\omega^t) > 0$. Let $\omega \in \{\Delta S_{t+1}(\omega^t, .) \in B(v, \varepsilon)\}$, then $h\Delta S_{t+1}(\omega^t, \omega) \neq 0$ *i.e* $q_{t+1}(h\Delta S_{t+1}(\omega^t, .) = 0|\omega^t)) < 1$. □

We prove now the following result of Section 5.
Proof of Proposition 5.11. We start with the proof of (25) when $h \in D_x$. Since D is a vectorial subspace of $\mathbb{R}^d$ and $0 \in \mathcal{H}_x$, the affine hull of D_x is also a vector space that we denote by Aff(D_x). If $x \leq 1$ we have by Assumption 5.4 that for all $\omega \in \overline{\Omega}$, $h \in D_x$,

$$V^+(\omega, x + hY(\omega)) \leq V^+\left(\omega, 1 + hY(\omega)\right). \tag{68}$$

If $x > 1$ using Assumption 5.7 (see (23) in Remark 5.8) we get that for all $\omega \in \overline{\Omega}$, $h \in D_x$

$$V^+(\omega, x + hY(\omega)) = V^+\left(2x\left(\frac{1}{2} + \frac{h}{2x}Y(\omega)\right)\right) \leq (2x)^{\overline{\gamma}} K\left(V^+\left(\omega, 1 + \frac{h}{2x}Y(\omega)\right) + C(\omega)\right). \tag{69}$$

First we treat the case of $Dim(\text{Aff}(D_x)) = 0$, *i.e* $D_x = \{0\}$. For all $\omega \in \overline{\Omega}$, $h \in D_x = \{0\}$, using (68) and (69), we obtain that

$$V^+(\omega, x + hY(\omega)) \leq V^+(\omega, 1) + (2x)^{\overline{\gamma}} K\left(V^+(\omega, 1) + C(\omega)\right) \leq ((2x)^{\overline{\gamma}} K + 1)(V^+(\omega, 1) + C(\omega)). \tag{70}$$

We assume now that $Dim(\text{Aff}(D_x)) > 0$. If $x = 0$ then $Y = 0$ Q-a.s. If this is not the case then we should have $D_0 = \{0\}$ a contradiction. Indeed if there exists some $h \in D_0$ with $h \neq 0$, then $Q\left(\frac{h}{|h|}Y(\cdot) < 0\right) > 0$ by Assumption 5.1 which contradicts $h \in D_0$. So for $x = 0$, $Y = 0$ Q-a.s and by Assumption 5.4 we get that for all $\omega \in \overline{\Omega}$, $h \in D_0$,

$$V^+(\omega, 0 + hY(\omega)) \leq V^+(\omega, 1).$$

From now we assume that $x > 0$. Then as for $g \in \mathbb{R}^d$, $g \in D_x$ if and only if $\frac{g}{x} \in D_1$, we have that $\mathrm{Aff}(D_x) = \mathrm{Aff}(D_1)$. We set $d' := Dim(\mathrm{Aff}(D_1))$. Let $(e_1, \dots, e_{d'})$ be an orthonormal basis of $\mathrm{Aff}(D_1)$ (which is a sub-vector space of $\mathbb{R}^d$) and $\varphi : (\lambda_1, \dots, \lambda_{d'}) \in \mathbb{R}^{d'} \to \Sigma_{i=1}^{d'} \lambda_i e_i \in \mathrm{Aff}(D_1)$. Then φ is an isomorphism (recall that $(e_1, \dots, e_{d'})$ is a basis of $\mathrm{Aff}(D_1)$). As φ is linear and the spaces considered are of finite dimension, it is also an homeomorphism between $\mathbb{R}^{d'}$ and $\mathrm{Aff}(D_1)$. Since D_1 is compact by Lemma 5.10, $\varphi^{-1}(D_1)$ is a compact subspace of $\mathbb{R}^{d'}$. So there exists some $c \geq 0$ such that for all $h = \Sigma_{i=1}^{d'} \lambda_i e_i \in D_1$, $|\lambda_i| \leq c$ for all $i = 1, \dots, d'$. We complete the family of vector $(e_1, \dots, e_{d'})$ in order to obtain an orthonormal basis of $\mathbb{R}^d$, denoted by $(e_1, \dots, e_{d'}, e_{d'+1}, \dots e_d)$. For all $\omega \in \Omega$, let $(y_i(\omega))_{i=1,\dots,d}$ be the coordinate of $Y(\omega)$ in this basis.
Now let $h \in D_x$ be fixed. Then $\frac{h}{2x} \in D_{\frac{1}{2}} \subset D_1$ and $\frac{h}{2x} = \Sigma_{i=1}^{d'} \lambda_i e_i$ for some $(\lambda_1, \dots \lambda_{d'}) \in \mathbb{R}^{d'}$ with $|\lambda_i| \leq c$ for all $i = 1, \dots, d'$. Note that as $\frac{h}{2x} \in D_1$, $\lambda_i = 0$ for $i \geq d' + 1$. Then as $(e_1, \dots, e_d)$ is an orthonormal basis of $\mathbb{R}^d$, we obtain for all $\omega \in \overline{\Omega}$

$$\begin{aligned} 1 + \frac{h}{2x} Y(\omega) &= 1 + \Sigma_{i=1}^{d'} \lambda_i y_i(\omega) \\ &\leq 1 + \Sigma_{i=1}^{d'} |\lambda_i||y_i(\omega)| \\ &\leq 1 + c\Sigma_{i=1}^{d'} |y_i(\omega)|. \end{aligned}$$

Thus from Assumption 5.4 for all $\omega \in \overline{\Omega}$ we get that

$$V^+ \left(\omega, 1 + \frac{h}{2x} Y(\omega) \right) \leq V^+ \left(\omega, 1 + c\Sigma_{i=1}^{d'} |y_i(\omega)| \right).$$

We set

$$L(\cdot) := V^+ \left(\omega, 1 + c\Sigma_{i=1}^{d'} |y_i(\omega)| \right) 1_{d'>0} + V^+(\cdot, 1) + C(\cdot).$$

As $d' = Dim(\mathrm{Aff}(D_1))$ it is clear that L does not depend on x. It is also clear that L is $\mathcal{H}$-measurable. Then using (68), (69) and (70) we obtain that for all $\omega \in \overline{\Omega}$

$$V^+(\omega, x + hY(\omega)) \leq ((2x)^{\overline{\gamma}} K + 1) L(\omega).$$

Note that the first term in L is used in the above inequality if $x \neq 0$ and $Dim(\mathrm{Aff}(D_x)) > 0$. The second and the third one are there for both the case of $Dim(\mathrm{Aff}(D_x)) = 0$ and the case of $x = 0$ and $Dim(\mathrm{Aff}(D_x)) > 0$. As by Assumptions 5.7 and 5.9, $E(V^+(\cdot, 1) + C(\cdot)) < \infty$, it remains to prove that $d' > 0$ implies $E\left(V^+\left(\cdot, 1 + c\Sigma_{i=1}^{d'} |y_i(\cdot)|\right)\right) < \infty$.
Introduce W, the finite set of $\mathbb{R}^d$ whose coordinates on $(e_1, \dots, e_{d'})$ are 1 or -1 and 0 on $(e_{d'+1}, \dots e_d)$. Then $W \subset \mathrm{Aff}(D_1)$ and the vectors of W will be denoted by θ^j for $j \in \{1, \dots, 2^{d'}\}$. Let θ^ω be the vector whose coordinates on $(e_1, \dots, e_{d'})$ are $(sign(y_i(\omega)))_{i=1\dots d'}$ and 0 on $(e_{d'+1}, \dots e_d)$. Then $\theta^\omega \in W$ and we get that

$$V^+ \left(\omega, 1 + c\Sigma_{i=1}^{d'} |y_i(\omega)| \right) = V^+(\omega, 1 + c\theta^\omega Y(\omega)) \leq \sum_{j=1}^{2^{d'}} V^+(\omega, 1 + c\theta^j Y(\omega)).$$

So to prove that $EL < \infty$ it is sufficient to prove that if $d' > 0$ for all $1 \leq j \leq 2^{d'}$, $EV^+(\cdot, 1 + c\theta^j Y(\cdot)) < \infty$. Recall that $\theta^j \in \mathrm{Aff}(D_1)$. Let $ri(D_1) = \{y \in D_1,\ \exists \alpha > 0\, s.t\ \mathrm{Aff}(D_1) \cap B(y, \alpha) \subset D_1\}$ [5] denote the relative interior of D_1. As D_1 is convex and non-empty (recall $d' > 0$), $ri(D_1)$ is also non-empty and convex and we fix some $e^* \in ri(D_1)$. We prove that $\frac{e^*}{2} \in ri(D_1)$. Let $\alpha > 0$ be such that $\mathrm{Aff}(D_1) \cap B(e^*, \alpha) \subset D_1$ and $g \in \mathrm{Aff}(D_1) \cap B(\frac{e^*}{2}, \frac{\alpha}{2})$. Then $2g \in \mathrm{Aff}(D_1) \cap B(e^*, \alpha)$ (recall that $\mathrm{Aff}(D_1)$ is actually a vector space) and thus $2g \in D_1$. As D_1 is convex and $0 \in D_1$, we get that $g \in D_1$ and $\mathrm{Aff}(D_1) \cap B(\frac{e^*}{2}, \frac{\alpha}{2}) \subset D_1$ which proves that $\frac{e^*}{2} \in ri(D_1)$. Now let ε_j be such that $\varepsilon_j(\frac{c}{2}\theta^j - \frac{e^*}{2}) \in B(0, \frac{\alpha}{2})$. It is easy to see that one can

[5] Here $B(y, \alpha)$ is the ball of $\mathbb{R}^d$ centered at y and with radius α.

chose $\varepsilon_j \in (0,1)$. Then as $\bar{e}^j := \frac{e^*}{2} + \frac{\varepsilon_j}{2}(c\theta^j - e^*) \in \text{Aff}(D_1) \cap B(\frac{e^*}{2}, \frac{\alpha}{2})$ (recall that $\theta^j \in W \subset \text{Aff}(D_1)$), we deduce that $\bar{e}^j \in D_1$. Using (23) we obtain that for Q-almost all ω

$$\begin{aligned}
V^+(\omega, 1 + c\theta^j Y(\omega)) &= V^+(\omega, 1 + e^* Y(\omega) + (c\theta^j - e^*)Y(\omega)) \\
&\leq \left(\frac{2}{\varepsilon_j}\right)^{\overline{\gamma}} K \left[V^+\left(\omega, \frac{\varepsilon_j}{2}(1 + e^* Y(\omega)) + \frac{\varepsilon_j}{2}(c\theta^j - e^*)Y(\omega) + \frac{1}{2}\right) + C(\omega)\right] \\
&\leq \left(\frac{2}{\varepsilon_j}\right)^{\overline{\gamma}} K \left[V^+\left(\omega, \frac{1}{2} + \frac{e^*}{2}Y(\omega) + \frac{\varepsilon_j}{2}(c\theta^j - e^*)Y(\omega) + \frac{1}{2}\right) + C(\omega)\right] \\
&\leq \left(\frac{2}{\varepsilon_j}\right)^{\overline{\gamma}} K \left[V^+(\omega, 1 + \bar{e}^j Y(\omega)) + C(\omega)\right],
\end{aligned}$$

where the second inequality follows from the fact that $1 + e^* Y(\cdot) \geq 0$ Q-a.s (recall that $e^* \in ri(D_1)$) and the monotonicity property of V in Assumption 4.1. Note that the above inequalities are true even if $1 + c\theta^j Y(\omega) < 0$ since (23) (see remark 5.8) and the monotonicity property of V hold true for all $x \in \mathbb{R}$. From Assumption 5.9 we get that $EV^+(\cdot, 1 + \bar{e}^j Y(\cdot)) < \infty$ (recall that $\bar{e}_j \in D_1$) and Assumption 5.7 implies $EC < \infty$, therefore $EV^+(\cdot, 1 + c\theta^j Y(\cdot)) < \infty$ and (25) is proven for $h \in D_x$. Now let $h \in \mathcal{H}_x$ and h' its orthogonal projection on D, then $hY(\cdot) = h'Y(\cdot)$ Q-a.s (see Remark 5.3). It is clear that $h' \in D_x$ thus $V^+(\cdot, x + hY(\cdot)) = V^+(\cdot, x + h'Y(\cdot))$ Q-a.s and (25) is true also for $h \in \mathcal{H}_x$. □

To conclude, the following lemma was used in the proof of Theorem 4.16.

Lemma 7.19 *Assume that (NA) holds true. Let $\phi \in \Phi$ such that $V_T^{x,\phi} \geq 0$ P-a.s, then $V_t^{x,\phi} \geq 0$ P_t-a.s.*

Proof. Assume that there is some t such that $P_t(V_t^{x,\phi} \geq 0) < 1$ or equivalently $P_t(V_t^{x,\phi} < 0) > 0$ and let $n = \sup\{t | P_t(V_t^{x,\phi} < 0) > 0\}$. Then $P_n(V_n^{x,\phi} < 0) > 0$ and for all $s \geq n+1$, $P_s(V_s^{x,\phi} \geq 0) = 1$. Let $\Psi_s(\omega) = 0$ if $s \leq n$ and $\Psi_s(\omega) = 1_A \phi_s(\omega)$ if $s \geq n+1$ with $A = \{V_n^\Phi < 0\}$. Then

$$V_s^{0,\Psi} = \sum_{k=1}^{s} \Psi_s \Delta S_s = \sum_{k=n+1}^{s} \Psi_s \Delta S_s = 1_A \left(V_s^{x,\phi} - V_n^{x,\phi}\right)$$

If $s \geq n+1$ $P_s(V_s^{x,\phi} \geq 0) = 1$ and on A, $-V_n^\Phi > 0$ thus $P_T(V_T^{0,\Psi} \geq 0) = 1$ and $V_T^{0,\Psi} > 0$ on A. As by the (usual) Fubini Theorem $P_T(A) = P_n(V_n^{x,\phi} < 0) > 0$, we get an arbitrage opportunity. Thus for all $t \leq T$, $P_t(V_t^{x,\phi} \geq 0) = 1$. □

Acknowledgments.

L. Carassus thanks LPMA (UMR 7599) for support. M. Rásonyi was supported by the “Lendület” grant LP2015-6 of the Hungarian Academy of Sciences.

References

C. D. Aliprantis and K. C. Border. *Infinite Dimensional Analysis : A Hitchhiker's Guide*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 3rd edition, 2006.

Alain Bensoussan, Abel Cadenillas, and Hyeng Keun Koo. Entrepreneurial decisions on effort and project with a nonconcave objective function. *Math. Oper. Res.*, 40(4):902–914, 2015. doi: 10.1287/moor.2014.0702. URL `http://dx.doi.org/10.1287/moor.2014.0702`.

V.I. Bogachev. *Measure Theory, vol 2*. Springer-Verlag, Berlin, 2007.

L. Carassus and M. Rásonyi. Maximization of non-concave utility functions in discrete-time financial market. *Mathematics of Operations Research, Published Online ISSN 1526-5471*, 2015.

L. Carassus, Rásonyi M., and A. M. Rodrigues. Non-concave utility maximisation on the positive real axis in discrete time. *Mathematics and Financial Economics*, 9(4):325–349, 2015.

G. Carlier and R.-A. Dana. Optimal demand for contingent claims when agents have law invariant utilities. *Math. Finance*, 21:169–201, 2011.

C. Castaing and M. Valadier. *Convex Analysis and Measurable Multifunctions*, volume 580. Springer, Berlin, 1977.

R.C. Dalang, A. Morton, and W. Willinger. Equivalent martingale measures and no-arbitrage in stochastic securities market models. *Stochastics Stochastics Rep.*, 29:185–201, 1990.

C. Dellacherie and P.-A. Meyer. *Probability and potential*. North-Holland, Amsterdam, 1979.

H. Föllmer and A. Schied. *Stochastic Finance: An Introduction in Discrete Time*. Walter de Gruyter & Co., Berlin, 2002.

X. He and X.Y. Zhou. Portfolio choice under cumulative prospect theory: An analytical treatment. *Management Science*, 57:315–331, 2011.

J. Jacod and A. N. Shiryaev. Local martingales and the fundamental asset pricing theorems in the discrete-time case. *Finance Stoch.*, 2:259–273, 1998.

H. Jin and X.Y. Zhou. Behavioural portfolio selection in continuous time. *Math. Finance*, 18:385–426, 2008.

D. O. Kramkov and W. Schachermayer. The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Ann. Appl. Probab.*, 9:904–950, 1999.

I. Molchanov. *Theory of random sets*. Springer-Verlag, London, 2005.

M. Nutz. Utility maximisation under model uncertainty in discrete time. *Math. Finance. Published Online DOI: 10.1111/mafi.12068*, 2014.

M. Rásonyi and L. Stettner. On the utility maximization problem in discrete-time financial market models. *Ann. Appl. Probab.*, 15:1367–1395, 2005.

M. Rásonyi and L. Stettner. On the existence of optimal portfolios for the utility maximization problem in discrete time financial models. *In: Kabanov, Y.; Lipster, R.; Stoyanov,J. (Eds), From Stochastic Calculus to Mathematical Finance, Springer.*, pages 589–608, 2006.

R. T. Rockafellar and R. J.-B. Wets. *Variational analysis*, volume 317 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1998. ISBN 3-540-62772-3.

M.-F. Sainte-Beuve. On the extension of von Neumann-Aumann's theorem. *J. Functional Analysis*, 17(1):112–129, 1974.