Inheritance of Convexity for the $\mathcal{P}_{\min}$-Restricted Game

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Abstract

We consider restricted games on weighted graphs associated with minimum partitions. We replace in the classical definition of Myerson restricted game the connected components of any subgraph by the sub-components corresponding to a minimum partition. This minimum partition $\mathcal{P}_{\min}$ is induced by the deletion of the minimum weight edges. We provide a characterization of the graphs satisfying inheritance of convexity from the underlying game to the restricted game associated with $\mathcal{P}_{\min}$. Moreover, we prove that these graphs can be recognized in polynomial time.

Keywords: cooperative game, convexity, graph-restricted game, graph partitions.

AMS Classification: 91A12, 91A43, 90C27, 05C75.

1 Introduction

A cooperative game is a pair (N, v) where N is a finite set of players and $v : 2^N \to \mathbb{R}$ is a set function assigning a worth to each coalition of players with $v(\emptyset) = 0$. For any coalition $A \subseteq N$, $v(A)$ represents the worth that players in A can generate by cooperation. However, in many situations, the cooperation of players may be restricted by some communication or social structures. Then, the worths of coalitions have to be modified to take these restrictions into account, leading to the introduction of restricted games. Lots of restricted games considered in the literature can be described by the $\mathcal{P}$-restricted game introduced by Skoda (2017b). $\mathcal{P}$ is a correspondence associating with any subset A of N a partition $\mathcal{P}(A)$ of A. The partition restricted game $(N, \overline{v})$ associated with $\mathcal{P}$, called $\mathcal{P}$-restricted game, is defined by:

$$(1) \qquad \overline{v}(A) = \sum_{F \in \mathcal{P}(A)} v(F), \quad \text{for all } A \subseteq N.$$

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This partition restricted game appears in many different works with specific correspondences. A founding example is the graph-restricted game introduced by Myerson (1977) for communication games. Communication games are cooperative games (N, v) defined on the set of vertices N of an undirected graph $G = (N, E)$. Assuming that the members of a given coalition can cooperate if and only if they are connected in G, the Myerson graph-restricted game (N, v^M) is defined by $v^M(A) = \sum_{F \in \mathcal{P}_M(A)} v(F)$ for all $A \subseteq N$, where $\mathcal{P}_M(A)$ is the partition of A into connected components. Many other correspondences have been considered to define restricted games (see, e.g., Algaba et al. (2001); Bilbao (2000, 2003); Faigle (1989); Grabisch and Skoda (2012); Grabisch (2013)). For a given correspondence, a classical problem is to study the inheritance of convexity from the initial game (N, v) to the restricted game $(N, \overline{v})$. Inheritance of convexity is of particular interest as it implies that good properties are inherited, for instance the non-emptiness of the core, and that the Shapley value is in the core. Skoda (2017b) established an abstract characterization of inheritance of convexity for an arbitrary correspondence $\mathcal{P}$. This characterization can be used to derive several well-known results for inheritance of convexity with specific correspondences, in particular the characterization of inheritance of convexity for Myerson restricted game established by van den Nouweland and Borm (1991). Of course, due to its generality the characterization given by Skoda (2017b) does not give straightforward insights into the precise structure of a given correspondence. More direct characterizations have to be found to check inheritance of convexity in practice. In this paper, we present a characterization of inheritance of convexity for the correspondence $\mathcal{P}_{\min}$ introduced by Grabisch and Skoda (2012) for communication games on weighted graphs. The correspondence $\mathcal{P}_{\min}$ is defined on the set N of nodes of a weighted graph $G = (N, E, w)$ where w is a weight function defined on the set E of edges of G. For a subset $A \subseteq N$, $\mathcal{P}_{\min}(A)$ corresponds to the set of connected components of the subgraph $(A, E(A) \setminus \Sigma(A))$ where $\Sigma(A)$ is the set of minimum weight edges in the subgraph $G_A = (A, E(A))$. Then, the $\mathcal{P}_{\min}$-restricted game $(N, \overline{v})$ is defined by:

$$\overline{v}(A) = \sum_{F \in \mathcal{P}_{\min}(A)} v(F), \quad \text{for all } A \subseteq N. \tag{2}$$

Compared to the initial game (N, v), the $\mathcal{P}_{\min}$-restricted game $(N, \overline{v})$ conforms to the common conception that members of a coalition have to be connected to cooperate but also takes into account the weights of the links between players and therefore differents aspects of cooperation restrictions. Assuming that the edge-weights reflect the strengths of relationships between players, $\mathcal{P}_{\min}(A)$ gives a partition of a coalition A into connected coalitions where players are in privileged relationships (with respect to the minimum relationship strength in G_A). Grabisch and Skoda (2012) first established that there is always inheritance of superadditivity from (N, v) to $(N, \overline{v})$ for the correspondence $\mathcal{P}_{\min}$. In contrast, they observed that inheritance of convexity requires very restrictive conditions on the underlying graph and its edge-weights giving simple counter-examples to inheritance of convexity with graphs with only two or three different edge-weights. Grabisch and Skoda (2012) established three necessary conditions on the underlying weighted graph but they also pointed out that these conditions are not sufficient and that contradictions to inheritance of convexity are easily obtained with non-connected coalitions. Following alternative definitions of convexity in combinatorial optimization and game theory when restricted families of subsets (not necessarily closed under union and intersection) are considered (see, e.g., Edmonds and Giles (1977); Faigle (1989); Fujishige (2005)), Grabisch and Skoda (2012) introduced the $\mathcal{F}$-convexity by restricting convexity to the family $\mathcal{F}$ of connected subsets of G. Skoda (2017a) characterized inheritance of $\mathcal{F}$-convexity for $\mathcal{P}_{\min}$ by five necessary and sufficient conditions on the edge-weights of specific subgraphs of the underlying graph G. Of course, the study of inheritance of $\mathcal{F}$-convexity is a first key step to obtain a characterization of graphs satisfying inheritance of classical convexity with $\mathcal{P}_{\min}$. Inheritance of $\mathcal{F}$-convexity is also interesting in itself as it corresponds to the common restriction to connected subsets for communication games. Moreover, it is satisfied for a much larger family of graphs than inheritance of convexity. In particular, inheritance of $\mathcal{F}$-convexity allows an arbitrary number of edge-weights, in contrast to inheritance of convexity as observed in the present paper. Skoda (2017a) also highlighted that Myerson restricted game can be obtained as a restriction of the $\mathcal{P}_{\min}$-restricted game associated with graphs with only two different edge-weights, and proved that inheritance of convexity for Myerson restricted game is equivalent to inheritance of $\mathcal{F}$-convexity for the $\mathcal{P}_{\min}$-restricted game associated with these specific weighted graphs.

In the present paper, we consider inheritance of classical convexity for the correspondence $\mathcal{P}_{\min}$. As convexity implies $\mathcal{F}$-convexity, the conditions established by Skoda (2017a) are necessary. Let us recall that these last conditions are also sufficient for inheritance of $\mathcal{F}$-convexity, but we will only use their necessity throughout the paper. Now dealing with disconnected subsets of N, we establish supplementary necessary conditions. As it was foreseeable by taking into account examples given by Grabisch and Skoda (2012) and Skoda (2017a), we get very strong restrictions on edge-weights and on the combinatorial structure of the underlying graph. These supplementary conditions are much more straightforward than the conditions established by Skoda (2017a) for inheritance of $\mathcal{F}$-convexity. In particular, we obtain that edge-weights can have at most three different values (Proposition 13) and that many cycles have to be complete or dominated in some sense by two specific vertices. The constraint on the number of edge-weight values implies that the family of weighted graphs satisfying inheritance of convexity is drastically smaller than the family of weighted graphs satisfying inheritance of $\mathcal{F}$-convexity. Moreover, edges have precise positions according to their weights. For example, in the case of three different values $\sigma_1 < \sigma_2 < \sigma_3$, there exists only one edge e_1 of minimum weight σ_1 and all edges of weight σ_2 are incident to the same end-vertex of e_1. Using these supplementary conditions, we obtain simple necessary and sufficient conditions. We give a complete characterization of the connected weighted graphs satisfying inheritance of convexity with $\mathcal{P}_{\min}$ in Theorems 20, 21 and 22. Though these graphs are very particular, they seem quite interesting. For instance, when there are only two values and at least two minimum weight edges, we obtain weighted graphs similar to the ones defined by Skoda (2017a) relating Myerson restricted game to the $\mathcal{P}_{\min}$-restricted game. Moreover, we prove that these graphs can be recognized in polynomial time. Theorems 20, 21 and 22 also imply that inheritance of convexity and inheritance of convexity restricted to unanimity games are equivalent for $\mathcal{P}_{\min}$. This last result was already observed as a consequence of the general characterization established by Skoda (2017b) for arbitrary correspondences.

The article is organized as follows. In Section 2, we give preliminary definitions and results established by Grabisch and Skoda (2012) and Skoda (2017b). In particular, we recall the definition of convexity, $\mathcal{F}$-convexity and general conditions on a correspondence to have inheritance of superadditivity, convexity or $\mathcal{F}$-convexity. In Section 3, we recall necessary conditions on a weighted graph established by Skoda (2017a) to ensure inheritance of $\mathcal{F}$-convexity with the correspondence $\mathcal{P}_{\min}$. In Section 4, we establish new, very restrictive conditions necessary to ensure inheritance of classical convexity with $\mathcal{P}_{\min}$. Section 5 contains the main results. We first provide characterizations of connected weighted graphs satisfying inheritance of convexity with $\mathcal{P}_{\min}$. Then, the case of disconnected graphs is considered. Finally, we prove that it can be decided in polynomial time whether a graph satisfies one of the previous characterizations.

2 Preliminary definitions and results

Let N be a given set with $|N| = n$. We denote by 2^N the set of all subsets of N. A game (N, v) is *zero-normalized* if $v(\{i\}) = 0$ for all $i \in N$. Throughout this paper, we consider only zero-normalized games. A game (N, v) is *superadditive* if, for all $A, B \in 2^N$ such that $A \cap B = \emptyset$, $v(A \cup B) \geq v(A) + v(B)$. For any given subset $\emptyset \neq S \subseteq N$, the unanimity game (N, u_S) is defined by

$$u_S(A) = \begin{cases} 1 & \text{if } A \supseteq S, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

We note that u_S is superadditive for all $S \neq \emptyset$.

Let us consider a game (N, v). For arbitrary subsets A and B of N, we define the value

$$\Delta v(A, B) := v(A \cup B) + v(A \cap B) - v(A) - v(B).$$

A game (N, v) is *convex* if its characteristic function v is supermodular, i.e., $\Delta v(A, B) \geq 0$ for all $A, B \in 2^N$. We note that u_S is supermodular for all $S \neq \emptyset$. Let $\mathcal{F}$ be a weakly union-closed family[1] of subsets of N such that

[1] $\mathcal{F}$ is weakly union-closed if $A \cup B \in \mathcal{F}$ for all A, $B \in \mathcal{F}$ such that $A \cap B \neq \emptyset$ (Faigle et al., 2010). Weakly union-closed families were introduced and analysed by Algaba (1998) (see also Algaba et al. (2000)) and called union stable systems.

$\emptyset \notin \mathcal{F}$. A game v on 2^N is said to be $\mathcal{F}$-convex if $\Delta v(A,B) \geq 0$, for all $A, B \in \mathcal{F}$ such that $A \cap B \in \mathcal{F}$. Let us note that a game (N,v) is convex if and only if it is superadditive and $\mathcal{F}$-convex with $\mathcal{F} = 2^N \setminus \{\emptyset\}$.

For a given graph $G = (N,E)$, we say that a subset $A \subseteq N$ is connected if the induced graph $G_A = (A, E(A))$ is connected.

A *correspondence* f with domain X and range Y is a map that associates to every element $x \in X$ a subset $f(x)$ of Y, i.e., a map from X to 2^Y. Throughout this paper, we consider correspondences $\mathcal{P}$ with domain and range 2^N, such that for every subset $\emptyset \neq A \subseteq N$, the family $\mathcal{P}(A)$ of subsets of N corresponds to a partition of A. We set $\mathcal{P}(\emptyset) = \{\emptyset\}$. For a given correspondence $\mathcal{P}$ on 2^N and subsets $A \subseteq B \subseteq N$, we denote by $\mathcal{P}(B)_{|A}$ the *restriction* of the partition $\mathcal{P}(B)$ to A. More precisely, if $\mathcal{P}(B) = \{B_1, B_2, \ldots, B_p\}$, then $\mathcal{P}(B)_{|A} = \{B_i \cap A \mid i = 1, \ldots, p, B_i \cap A \neq \emptyset\}$.

For two given subsets A and B of N, $\mathcal{P}(A)$ is a *refinement* of $\mathcal{P}(B)$ if every block of $\mathcal{P}(A)$ is a subset of some block of $\mathcal{P}(B)$.

We recall the following results established by Grabisch and Skoda (2012).

Theorem 1. *Let N be an arbitrary set and $\mathcal{P}$ a correspondence on 2^N. The following conditions are equivalent:*

1) The $\mathcal{P}$-restricted game $(N, \overline{u_S})$ is superadditive for all $\emptyset \neq S \subseteq N$.

2) $\mathcal{P}(A)$ is a refinement of $\mathcal{P}(B)_{|A}$ for all subsets $A \subseteq B \subseteq N$.

3) The $\mathcal{P}$-restricted game $(N, \overline{v})$ is superadditive for all superadditive game (N, v).

As $\mathcal{P}_{\min}(A)$ is a refinement of $\mathcal{P}_{\min}(B)_{|A}$ for all subsets $A \subseteq B \subseteq N$, Theorem 1 implies the following result.

Corollary 2. *Let $G = (N, E, w)$ be an arbitrary weighted graph. The $\mathcal{P}_{\min}$-restricted game $(N, \overline{v})$ is superadditive for every superadditive game (N, v).*

Theorem 3. *Let N be an arbitrary set and $\mathcal{P}$ a correspondence on 2^N. Let $\mathcal{F}$ be a weakly union-closed family of subsets of N with $\emptyset \notin \mathcal{F}$. If the $\mathcal{P}$-restricted game $(N, \overline{u_S})$ is superadditive for all $\emptyset \neq S \subseteq N$, then the following conditions are equivalent:*

1) The $\mathcal{P}$-restricted game $(N, \overline{u_S})$ is $\mathcal{F}$-convex for all $\emptyset \neq S \subseteq N$,

2) For all $i \in N$, for all $A \subseteq B \subseteq N \setminus \{i\}$ such that A, B, and $A \cup \{i\}$ are in $\mathcal{F}$, and for all $A' \in \mathcal{P}(A \cup \{i\})_{|A}$, $\mathcal{P}(A)_{|A'} = \mathcal{P}(B)_{|A'}$.

We also recall the following lemmas proved by Grabisch and Skoda (2012). We include the proofs as these two results are extensively used throughout the paper.

Lemma 4. *Let us consider $A, B \subseteq N$ and a partition $\{B_1, B_2, \ldots, B_p\}$ of B. Let $\mathcal{F}$ be a weakly union-closed family of subsets of N with $\emptyset \notin \mathcal{F}$. If A, B_i,* *and $A \cap B_i \in \mathcal{F}$ for all $i \in \{1, \ldots, p\}$, then for every $\mathcal{F}$-convex game (N, v) we have*

$$v(A \cup B) + \sum_{i=1}^{p} v(A \cap B_i) \geq v(A) + \sum_{i=1}^{p} v(B_i). \tag{4}$$

Proof. We prove the result by induction. (4) is obviously satisfied for $p = 1$. Let us assume it is satisfied for p and let us consider a partition $\{B_1, B_2, \ldots, B_p, B_{p+1}\}$ of B. We set $B' = B_1 \cup B_2 \cup \ldots \cup B_p$. The $\mathcal{F}$-convexity of v applied to $A \cup B'$ and B_{p+1} provides the following inequality:

$$v((A \cup B') \cup B_{p+1}) + v((A \cup B') \cap B_{p+1}) \geq v(A \cup B') + v(B_{p+1}). \tag{5}$$

By induction (4) is valid for B':

$$v(A \cup B') + \sum_{i=1}^{p} v(A \cap B_i) \geq v(A) + \sum_{i=1}^{p} v(B_i). \tag{6}$$

Adding (5) and (6) we obtain the result for $p + 1$. □

Lemma 5. *Let us consider a correspondence $\mathcal{P}$ on 2^N and subsets $A \subseteq B \subseteq N$ such that $\mathcal{P}(A) = \mathcal{P}(B)_{|A}$. Let $\mathcal{F}$ be a weakly union-closed family of subsets of N with $\emptyset \notin \mathcal{F}$. If $A \in \mathcal{F}$ and if all elements of $\mathcal{P}(A)$ and $\mathcal{P}(B)$ are in $\mathcal{F}$, then for every $\mathcal{F}$-convex game (N, v) we have*

$$v(B) - \overline{v}(B) \geq v(A) - \overline{v}(A). \tag{7}$$

Proof. If $\mathcal{P}(B) = \{B_1, B_2, \ldots, B_p\}$, then $\mathcal{P}(A) = \{B_i \cap A \mid i = 1, \ldots, p, B_i \cap A \neq \emptyset\}$ and Lemma 4 implies (7). □

Skoda (2017b) established that inheritance of convexity and inheritance of convexity restricted to unanimity games are equivalent for $\mathcal{P}_M$ and $\mathcal{P}_{\min}$.

Theorem 6. *Let $G = (N, E)$ (resp. $G = (N, E, w)$) be a graph (resp. weighted graph). The following statements are equivalent:*

1) *The Myerson restricted game (N, u_S^M) (resp. $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$) is convex for all $\emptyset \neq S \subseteq N$.*

2) *There is inheritance of convexity for $\mathcal{P}_M$ (resp. $\mathcal{P}_{\min}$).*

Finally, we recall a characterization of inheritance of convexity for Myerson's correspondence $\mathcal{P}_M$. A graph $G = (N, E)$ is cycle-complete if for any cycle $C = \{v_1, e_1, v_2, e_2, \ldots, e_m, v_1\}$ in G the subset $\{v_1, v_2, \ldots, v_m\} \subseteq N$ of vertices of C induces a complete subgraph in G.

Theorem 7. *(van den Nouweland and Borm, 1991). Let $G = (N, E)$ be a graph. There is inheritance of convexity for $\mathcal{P}_M$ if and only if G is cycle-complete.*

3 Inheritance of $\mathcal{F}$-convexity

Let $G = (N, E, w)$ be a connected weighted graph and let $\mathcal{F}$ be the family of connected subsets of N. In this section we recall necessary conditions on the weight vector w established by Skoda (2017a) for inheritance of $\mathcal{F}$-convexity from the original communication game (N, v) to the $\mathcal{P}_{\min}$-restricted game $(N, \overline{v})$. We assume that all weights are strictly positive and denote by w_k or w_{ij} the weight of an edge $e_k = \{i, j\}$ in E.

A star S_k corresponds to a tree with one internal vertex and k leaves. We consider a star S_3 with vertices $1, 2, 3, 4$ and edges $e_1 = \{1, 2\}$, $e_2 = \{1, 3\}$ and $e_3 = \{1, 4\}$.

Star Condition. *For every star in G of type S_3, the edge-weights satisfy*

$$w_1 \leq w_2 = w_3,$$

after renumbering the edges if necessary.

Path Condition. *For every elementary path $\gamma = \{1, e_1, 2, e_2, 3, \ldots, m, e_m, m+1\}$ in G and for all i, j, k such that $1 \leq i < j < k \leq m$, the edge-weights satisfy*

$$w_j \leq \max(w_i, w_k).$$

For a given cycle $C = \{1, e_1, 2, e_2, \ldots, m, e_m, 1\}$ with $m \geq 3$, we denote by $E(C)$ the set of edges $\{e_1, e_2, \ldots, e_m\}$ of C and by $\hat{E}(C)$ the set composed of $E(C)$ and of the chords of C in G.

Cycle Condition. *For every simple cycle $C = \{1, e_1, 2, e_2, \ldots, m, e_m, 1\}$ in G with $m \geq 3$, the edge-weights satisfy*

$$w_1 \leq w_2 \leq w_3 = \cdots = w_m = \hat{M},$$

after renumbering the edges if necessary, where $\hat{M} = \max_{e \in \hat{E}(C)} w(e)$. Moreover, $w(e) = w_2$ for all chord incident to 2, and $w(e) = \hat{M}$ for all $e \in \hat{E}(C)$ non-incident to 2.

For a given cycle C, an edge e in $\hat{E}(C)$ is a maximum weight edge of C if $w(e) = \max_{e \in \hat{E}(C)} w(e)$. Otherwise, e is a non-maximum weight edge of C. Moreover, we call maximum (resp. non-maximum) weight chord of C a maximum (resp. non-maximum) weight edge in $\hat{E}(C) \setminus E(C)$.

Pan Condition. *For all connected subgraphs of G corresponding to the union of a simple cycle $C = \{e_1, e_2, \dots, e_m\}$ with $m \geq 3$, and an elementary path P such that there is an edge e in P with $w(e) \leq \min_{1 \leq k \leq m} w_k$ and $|V(C) \cap V(P)| = 1$, the edge-weights satisfy*

(a) either $w_1 = w_2 = w_3 = \cdots = w_m = \hat{M}$,

(b) or $w_1 = w_2 < w_3 = \cdots = w_m = \hat{M}$,

where $\hat{M} = \max_{e \in \hat{E}(C)} w(e)$. If Condition (b) is satisfied, then $V(C) \cap V(P) = \{2\}$, and if moreover $w(e) < w_1$, then $\{1, 3\}$ is a maximum weight chord of C.

Two cycles are said *adjacent* if they share at least one common edge.

Adjacent Cycles Condition. *For all pairs $\{C, C'\}$ of adjacent simple cycles in G such that*

(a) $V(C) \setminus V(C') \neq \emptyset$ and $V(C') \setminus V(C) \neq \emptyset$,

(b) C has at most one non-maximum weight chord,

(c) C and C' have no maximum weight chord,

(d) C and C' have no common chord.

C and C' cannot have two common non-maximum weight edges. Moreover, C and C' have a unique common non-maximum weight edge e_1 if and only if there are non-maximum weight edges $e_2 \in E(C) \setminus E(C')$ and $e'_2 \in E(C') \setminus E(C)$ such that e_1, e_2, e'_2 are adjacent and

- *$w_1 = w_2 = w'_2$ if $|E(C)| \geq 4$ and $|E(C')| \geq 4$,*
- *$w_1 = w_2 \geq w'_2$ or $w_1 = w'_2 \geq w_2$ if $|E(C)| = 3$ or $|E(C')| = 3$.*

Proposition 8. *Let $\mathcal{F}$ be the family of connected subsets of N. If for all $\emptyset \neq S \subseteq N$, the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is $\mathcal{F}$-convex, then the Star, Path, Cycle, Pan and Adjacent cycles conditions are satisfied.*

Let us note that Proposition 8 only requires inheritance of $\mathcal{F}$-convexity for the unanimity games to obtain the necessity of the five previous conditions. Skoda (2017a) also proved that these necessary conditions are sufficient for inheritance of $\mathcal{F}$-convexity if we consider superadditive games.

Theorem 9. *Let $\mathcal{F}$ be the family of connected subsets of N. The $\mathcal{P}_{\min}$-restricted game $(N, \overline{v})$ is $\mathcal{F}$-convex for every superadditive and $\mathcal{F}$-convex game (N, v) if and only if the Star, Path, Cycle, Pan, and Adjacent cycles conditions are satisfied.*

4 Inheritance of convexity

We consider in this section inheritance of convexity. As convexity implies superadditivity and $\mathcal{F}$-convexity (where $\mathcal{F}$ is the family of connected subsets of N), the conditions stated in Section 3 are necessary. We now have to deal with disconnected subsets of N. We establish supplementary necessary conditions implying strong restrictions on edge-weights. In particular, we obtain that edge-weights can have at most three different values. We first need the following lemmas.

Lemma 10. *Let us assume that for all $\emptyset \neq S \subseteq N$ the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is convex. Let $e_1 = \{1,2\}$ and $e_2 = \{2,3\}$ be two adjacent edges, and e be an edge such that*

$$\max(w_1, w_2) < w(e). \tag{8}$$

Then, there exists an edge $e' \in E$ linking e to vertex 2. Moreover, if $e' \neq e_1$ and e_2, then $w(e') = \max(w_1, w_2)$ if $w_1 \neq w_2$ and $w(e') \leq w_1 = w_2$ otherwise.

We represent in Figure 1 the possible situations corresponding to Lemma 10.

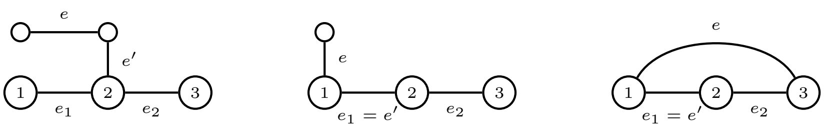


Figure 1: e' linking e to vertex 2.

Proof. We set $e = \{j,k\}$. Star condition implies $j \neq 2$ and $k \neq 2$ (otherwise it contradicts (8)). By contradiction, let us assume that there is no edge linking e to 2. We can assume $w_1 \leq w_2 < w(e)$. Let us consider $i = 3$, $A_1 = \{2\}$, $A_2 = \{j,k\}$, $A = A_1 \cup A_2$, and $B = A \cup \{1\}$ as represented in Figure 2. As no edge

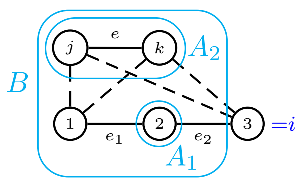


Figure 2: $w_1 \leq w_2 < w(e)$.

links e to 2, we have $\mathcal{P}_{\min}(A) = \{\{2\},\{j\},\{k\}\}$. Let us note that, as $w_2 < w(e)$ (resp. $w_1 < w(e)$), there is a component A' (resp. B') of $\mathcal{P}_{\min}(A \cup \{i\})$ (resp. $\mathcal{P}_{\min}(B)$) containing A_2. Then, $\mathcal{P}_{\min}(B)_{|A' \cap A} \neq \mathcal{P}_{\min}(A)_{|A' \cap A}$ as $\mathcal{P}_{\min}(A)$ corresponds to a singleton partition but $B' \cap A'$ contains A_2. It contradicts Theorem 3 applied with $\mathcal{F} = 2^N \setminus \{\emptyset\}$. Therefore, there exists an edge e' linking e to 2. Finally, if $e' \neq e_1$ and e_2, then Star condition applied to $\{e_1, e_2, e'\}$ implies the result. $\square$

Lemma 11. *Let us assume that for all $\emptyset \neq S \subseteq N$ the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is convex. Let $e_1 = \{1, 2\}$ and $e_2 = \{2, 3\}$ be two adjacent edges and let e and e' be two edges in E such that*

$$\max(w_1, w_2) < \min(w(e), w(e')). \tag{9}$$

Then, $w(e) = w(e')$.

Proof. We can assume $w_1 \leq w_2$. By contradiction, let us assume $w(e) < w(e')$. Applying Lemma 10 to e (resp. e'), there exists an edge e_2' (resp. e_2'') linking e (resp. e') to 2 such that $w_2' \leq \max(w_1, w_2) < w(e)$ (resp. $w_2'' \leq \max(w_1, w_2) < w(e')$) ($e_2'$ (resp. e_2'') may coincide with e_1 or e_2). We set $e_2' = \{2, 2'\}$ (resp. $e_2'' = \{2, 2''\}$) where $2'$ (resp. $2''$) is an end-vertex of e (resp. e') as represented in Figure 3. If $2' = 2''$, then $e_2' = e_2''$ and as $w_2' < w(e) < w(e')$ it contradicts

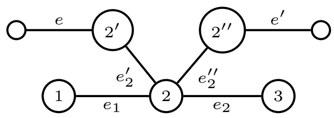
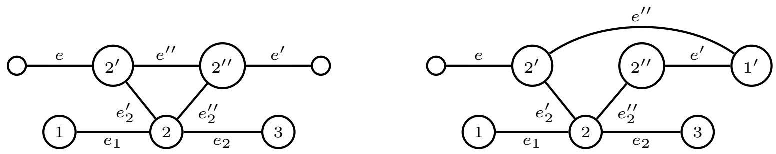


Figure 3: $w_2' \leq \max(w_1, w_2) < w(e)$ and $w_2'' \leq \max(w_1, w_2) < w(e')$.

the Star condition applied to $\{e_2', e, e'\}$. Otherwise, as $w_2' < w(e) < w(e')$, Lemma 10 applied to e' and the pair of adjacent edges $\{e_2', e\}$ implies the existence of an edge $e'' \in E$ linking e' to $2'$ (e'' can coincide with e). Let us first assume $e'' = \{2', 2''\}$ as represented in Figure 4a. As $w_2' < w(e)$ (resp.



(a) $e'' = \{2', 2''\}$ (b) $e'' = \{1', 2'\}$

Figure 4: e'' linking e' to $2'$.

$w_2'' < w(e')$), Star condition applied to $\{e_2', e, e''\}$ (resp. $\{e_2'', e', e''\}$) implies $w(e'') = w(e)$ (resp. $w(e'') = w(e')$) and then $w(e) = w(e')$, a contradiction. Let us now assume $e' = \{1', 2''\}$ and $e'' = \{1', 2'\}$ as represented in Figure 4b. Then, there is a cycle $C = \{2, e_2', 2', e'', 1', e', 2'', e_2'', 2\}$. As $w_2' < w(e)$, Star condition applied to $\{e_2', e, e''\}$ implies $w(e'') = w(e)$. As $w(e) < w(e')$, we get $w(e'') < w(e')$. Therefore, C has three non-maximum weight edges e_2', e_2'', e'', contradicting the Cycle condition. □

For a given weighted graph $G = (N, E, w)$, let $\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ be the set of its edge-weights such that $\sigma_1 < \sigma_2 < \ldots < \sigma_k$ with $1 \leq k \leq |E|$. We denote by E_i the set of edges in E with weight σ_i, and by N_i the set of end-vertices of edges in E_i.

Lemma 12. *Let $G = (N, E, w)$ be a connected weighted graph satisfying the Path condition and with at least two different edge-weights σ_1, σ_2. Then*

1. *There exists a pair of adjacent edges* $\{e_1, e_2\}$ *with* $w_1 = \sigma_1$ *and* $w_2 = \sigma_2$.
2. G_{N_1} *is connected.*

Proof. 1. If there is no such pair, then it contradicts the Path condition.

2. Let C', C'' be two distinct connected components of G_{N_1}. By definition of N_1, C' (resp. C'') contains at least one edge e' (resp. e'') of weight σ_1. Let γ be a shortest path in G linking e' to e''. Path condition applied to $\gamma' = e' \cup \gamma \cup e''$ implies $w(e) = \sigma_1$ for all edge e in γ. Then, γ' is a path in G_{N_1} linking C' to C'', a contradiction. □

The following proposition is a direct consequence of Proposition 8 and Lemmas 11 and 12.

Proposition 13. *Let* $G = (N, E, w)$ *be a connected weighted graph. Let us assume that for all* $\emptyset \neq S \subseteq N$ *the* $\mathcal{P}_{\min}$*-restricted game* $(N, \overline{u_S})$ *is convex. Then, the edge-weights have at most three different values* $\sigma_1 < \sigma_2 < \sigma_3$. *Moreover, if* $|E_1| \geq 2$, *then they have at most two different values* $\sigma_1 < \sigma_2$.

Of course, Proposition 13 implies that if the edge-weights have three different values, then there is only one edge with minimum weight σ_1. We will now establish necessary conditions on adjacency and incidence of edges in E_1, E_2, E_3. We first need the two following lemmas.

Lemma 14. *Let* $G = (N, E, w)$ *be a connected weighted graph. Let us assume that for all* $\emptyset \neq S \subseteq N$, *the* $\mathcal{P}_{\min}$*-restricted game* $(N, \overline{u_S})$ *is convex and that the edge-weights have exactly three different values* $\sigma_1 < \sigma_2 < \sigma_3$. *Then, there exist three edges* e_1, e_2, e_3 *with respective weights* σ_1, σ_2, σ_3 *such that* e_1 *and* e_2 *are incident to a vertex* v *and* e_3 *is adjacent to* e_1 *or* e_2 *but not incident to* v.

Three edges e_1, e_2, e_3 satisfying Lemma 14 correspond to three possible situations represented in Figure 5.

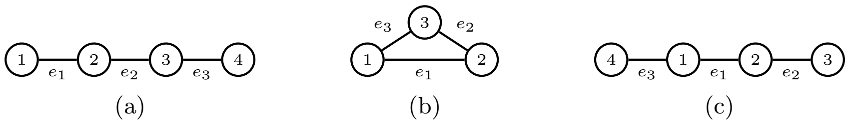


Figure 5: $w_1 = \sigma_1 < w_2 = \sigma_2 < w_3 = \sigma_3$.

Proof. Claim 1 of Lemma 12 implies the existence of e_1 and e_2. Proposition 13 implies the uniqueness of e_1. Let us assume $e_1 = \{1, 2\}$ and $e_2 = \{2, 3\}$, and let e_3 be an edge of weight σ_3. By Lemma 10 there exists an edge e' linking e_3 to 2 and if $e' \neq e_1$ and $e' \neq e_2$, then $w(e') = \max(w_1, w_2) = \sigma_2$. In this last case, we can substitute e' for e_2 as represented in Figure 6, and then the 3-tuple $\{e_1, e', e_3\}$ satisfies the conclusion of the lemma. □

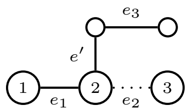
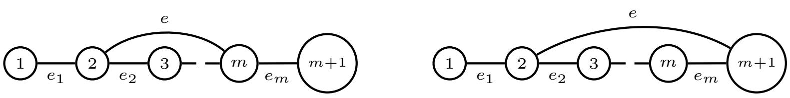


Figure 6: e' linking e_3 to 2 with $w_1 = \sigma_1 < w_2 = \sigma_2 < w_3 = \sigma_3$.

Lemma 15. *Let us assume that for all $\emptyset \neq S \subseteq N$, the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is convex. Then, for all elementary path $\gamma = \{1, e_1, 2, e_2, \ldots, m, e_m, m+1\}$ with $w_1 < w_m$, we have*

$$\max(w_1, w_2) \leq w_3 = w_4 = \cdots = w_m. \tag{10}$$

Proof. The Path condition implies $w_j \leq \max(w_1, w_m) = w_m$ for all j, $1 \leq j \leq m$. Then, (10) is satisfied if $m = 3$. Let us assume $m \geq 4$ and by contradiction $w_{m-1} < w_m$. The Path condition implies $w_2 \leq \max(w_1, w_{m-1}) < w_m$. Therefore, $\max(w_1, w_2) < w_m$. Then, by Lemma 10, there exists an edge e linking e_m to 2 with $w(e) \leq \max(w_1, w_2)$. Hence, we have $w(e) < w_m$. Let us first assume $e = \{2, m\}$ as represented in Figure 7a. Then, Star condition



(a) (b)

Figure 7: $e = \{2, m\}$ or $e = \{2, m+1\}$.

applied to $\{e_{m-1}, e_m, e\}$ implies $w_{m-1} = w_m$, a contradiction. Let us now assume $e = \{2, m+1\}$ as represented in Figure 7b. Then, the cycle $C = \{2, e_2, 3 \ldots, m, e_m, m+1, e, 2\}$ contains at least three non-maximum weight edges (e_2, e_{m-1}, e), contradicting the Cycle condition. Hence, we have $w_{m-1} = w_m$ and therefore $w_{m-1} > w_1$. We can iterate to get (10). $\square$

Proposition 16. *Let $G = (N, E, w)$ be a connected weighted graph. Let us assume that for all $\emptyset \neq S \subseteq N$, the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is convex and that the edge-weights have exactly two different values $\sigma_1 < \sigma_2$. Then, either $|E_1| = 1$ or all edges in E_1 are incident to the same vertex v and no edge in E_2 is incident to v.*

Graphs corresponding to the two possible situations described in Proposition 16 are represented in Figure 8.

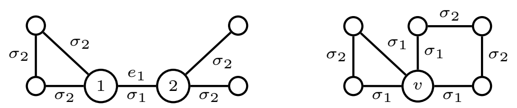


Figure 8: Edge-weights with only two values $\sigma_1 < \sigma_2$.

Proof. As G is connected there is at least one pair of adjacent edges $e_1 = \{1, 2\}$, and $e_2 = \{2, 3\}$ with weights $\sigma_1 < \sigma_2$. Let e be an edge in $E_1 \setminus \{e_1\}$ non-incident to 1. Then, e cannot be incident to 2 otherwise it contradicts the

Star condition. As G is connected, there exists a shortest path γ linking e to 1 or 2. All edges in γ have weight σ_1 by the Path condition applied to $\{e\} \cup \gamma \cup \{e_1\}$. Let e' be the edge of γ incident to 1 or 2. As $w(e') = \sigma_1$, the first part of the proof implies that e' is necessarily incident to 1. Then, Lemma 15 applied to $\gamma' = \{e\} \cup \gamma \cup \{e_1, e_2\}$ as represented in Figure 9 implies $\sigma_1 = \sigma_2$, a contradiction. Therefore, e is incident to 1, and any edge of weight

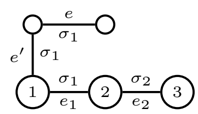


Figure 9: $\gamma' = \{e\} \cup \gamma \cup \{e_1, e_2\}$.

σ_2 incident to 1 would contradict the Star condition. $\square$

Proposition 17. *Let $G = (N, E, w)$ be a connected weighted graph. Let us assume that for all $\emptyset \neq S \subseteq N$, the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is convex and that the edge-weights have exactly three different values $\sigma_1 < \sigma_2 < \sigma_3$. Then, there is only one edge e_1 in E_1, every edge in E_2 is incident to the same end-vertex v of e_1, and every edge in E_3 is non-incident to v but linked to v by e_1 or by an edge in E_2.*

We give in Figure 10 an example of a graph satisfying the conditions of Proposition 17.

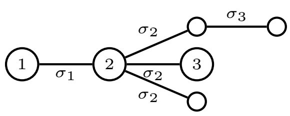


Figure 10: $\sigma_1 < \sigma_2 < \sigma_3$.

Proof. $|E_1| = 1$ by Proposition 13. By Lemma 14 there exist e_1, e_2, e_3 in E with $w_i = \sigma_i$ for $i \in \{1, 2, 3\}$ and such that one of the three situations represented in Figure 5 holds. Let e be an edge in $E_2 \setminus \{e_2\}$ non-incident to 2. Let us first assume $e_3 = \{3, 4\}$ (Case a in Figure 5). Then, e cannot be incident to 3 (resp. 4) otherwise $\{e_2, e_3, e\}$ contradicts the Star (resp. Path) condition. Finally, e cannot be incident to 1 otherwise Lemma 15 applied to $\{e, e_1, e_2, e_3\}$ implies $\sigma_2 = \sigma_3$, a contradiction. Let us now assume $e_3 = \{3, 1\}$ (Case b in Figure 5). Then, e cannot be incident to 1 (resp. 3) otherwise $\{e, e_1, e_3\}$ (resp. $\{e, e_2, e_3\}$) contradicts the Star condition. Finally, if $e_3 = \{4, 1\}$ (Case c in Figure 5), then we can establish as before with $e_3 = \{3, 4\}$ that e cannot be incident to $1, 3$, and 4. As G is connected, there exists a shortest path γ linking e to $1, 2$ or 3. The Path condition applied to $\{e\} \cup \gamma \cup \{e_1\}$ or $\{e\} \cup \gamma \cup \{e_2\}$ implies that all edges in γ have weight σ_1 or σ_2. As e_1 is the unique edge with weight σ_1, we get $w(e) = \sigma_2$ for any edge e in γ. Let e' be the edge of γ incident to 1, 2, or 3. As $w(e') = \sigma_2$, the first part of the proof implies that e' is necessarily incident to 2. If $e_3 = \{3, 4\}$ (resp. $e_3 = \{3, 1\}$ or $\{4, 1\}$), we consider $\gamma' = \{e\} \cup \gamma \cup \{e_2, e_3\}$ (resp. $\gamma' = \{e\} \cup \gamma \cup \{e_1, e_3\}$) as represented in Figure 11 (resp. Figure 12) with γ reduced to e'. Then, Lemma 15 applied to γ' implies $\sigma_2 = \sigma_3$ (resp. $\sigma_1 = \sigma_3$), a contradiction.

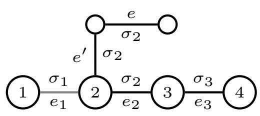


Figure 11: $\gamma' = \{e\} \cup \gamma \cup \{e_2, e_3\}$.

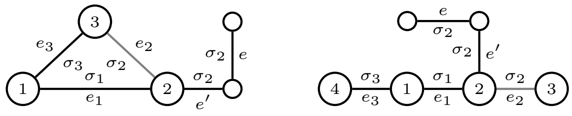


Figure 12: $\gamma' = \{e\} \cup \gamma \cup \{e_1, e_3\}$.

Finally, if an edge in E_3 is incident to v, then it contradicts the Star condition. Then, Lemma 10 implies the result. ☐

A *chordless cycle* in G is an induced cycle, i.e., a cycle corresponding to an induced subgraph of G.

Remark 1. By Proposition 17 any cycle containing e_1 has length at most 4, and there are only two possible chordless cycles containing e_1 represented in Figure 13. Moreover, by the Adjacent cycles condition such a chordless cycle

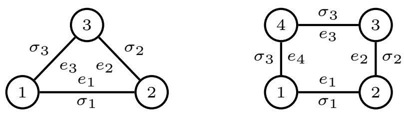


Figure 13: $\tilde{C}_3$ and $\tilde{C}_4$ containing the edge of weight σ_1.

is necessarily unique (the existence of two such chordless cycles would imply $\sigma_1 = \sigma_2$).

We end this section with supplementary necessary conditions corresponding to refinements of the Pan condition. A cycle C is constant if all edges in $E(C)$ have the same weight, and non-constant otherwise.

Non-Constant Cycle Refined Pan Condition. *For all connected subgraphs corresponding to the union of a non-constant simple cycle $C_m = \{1, e_1, 2, e_2, \ldots, m, e_m, 1\}$ with $m \geq 3$ and an elementary path P containing an edge e with $w(e) < \min_{1\leq j\leq m} w_j$, e is incident to 2 but not a chord of C_m, C_m is a complete cycle, and the edge-weights satisfy*

$$(11) \qquad w(e) < w_1 = w_2 < w_3 = \cdots = w_m = \hat{M} = \max_{e \in \hat{E}(C_m)} w(e).$$

If P is reduced to e, then a pan satisfying the previous condition is represented in Figure 14.

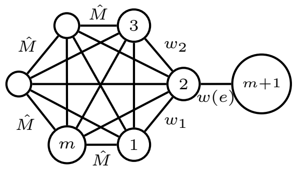


Figure 14: $w(e) < w_1 = w_2 < w_3 = \cdots = w_m = \hat{M}$.

Proposition 18. *If for all $\emptyset \neq S \subseteq N$ the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is convex, then the Non-Constant Cycle Refined Pan Condition is satisfied.*

Proof. By the Cycle condition any chord of C_m has weight w_2 or $\hat{M}$. Therefore, e cannot be a chord of C_m. Let us assume e non-incident to C_m. Let P' be the shortest path induced by P linking e to C_m. As C_m is a non-constant cycle, the Pan condition applied to C_m and P' implies (11) and $V(C_m) \cap V(P') = \{2\}$. If e is not incident to 2, then Lemma 15 implies that C_m is a constant cycle, a contradiction. Let us set $e = \{2, m+1\}$.

Let us first prove $e'_j = \{2, j\} \in \hat{E}(C_m)$ for all j, $4 \leq j \leq m$. By Lemma 10, it is sufficient to prove the existence of such a chord for $m = 4$. Indeed, if $e'_j \notin \hat{E}(C_m)$ for a given index j, then Lemma 10 applied to e_{j-1} (resp. e_j) and to the pair of adjacent edges $\{e_1, e_2\}$ implies that e'_{j-1} (resp. e'_{j+1}) exists in $\hat{E}(C_m)$. Then, $\{2, e'_{j-1}, j-1, e_{j-1}, j, e_j, j+1, e'_{j+1}, 2\}$ defines a cycle of length 4 as represented in Figure 15. By contradiction let us assume

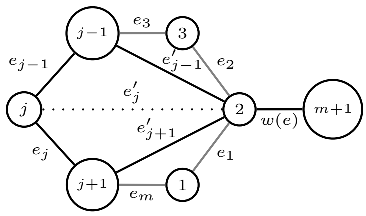


Figure 15: $w(e) < w_1 = w_2 < w_3 = \cdots = w_m = \hat{M}$

.

$e'_4 \notin \hat{E}(C_4)$. By the Pan condition, $\{1, 3\}$ is a maximum weight chord of C_4. Let us consider $i = 3$, $A_1 = \{2, 5\}$, $A_2 = \{4\}$, $A = A_1 \cup A_2$, and $B = A \cup \{1\}$ as represented in Figure 16. If $\{4, 5\} \in E$, then the Cycle condition ap-

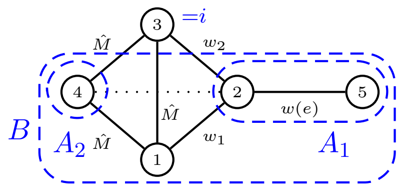


Figure 16: $w(e) < w_1 = w_2 < w_3 = w_4 = \hat{M}$.

plied to $\{1, e_1, 2, e, 5, \{5, 4\}, 4, e_4, 1\}$ implies $w(\{4, 5\}) = \hat{M} > w(e)$. Hence, we have either $\mathcal{P}_{\min}(A) = \{\{2\}, \{4, 5\}\}$ or $\mathcal{P}_{\min}(A) = \{\{2\}, \{4\}, \{5\}\}$. Moreover, $\mathcal{P}_{\min}(A \cup \{i\}) = \{\{2, 3, 4\}, \{5\}\}$ or $\{A \cup \{i\}\}$, and $\mathcal{P}_{\min}(B) = \{\{1, 2, 4\}, \{5\}\}$

or $\{B\}$. If $\mathcal{P}_{\min}(A \cup \{i\}) = \{\{2,3,4\},\{5\}\}$, then taking $A' = \{2,3,4\}$ we get $\mathcal{P}_{\min}(A)_{|A'} = \{\{2\},\{4\}\} \neq \{2,4\} = \mathcal{P}_{\min}(B)_{|A'}$ contradicting Theorem 3. Otherwise, taking $A' = A \cup \{i\}$, we get $\mathcal{P}_{\min}(A)_{|A'} = \{\{2\},\{4\},\{5\}\}$ or $\{\{2\},\{4,5\}\}$ and $\mathcal{P}_{\min}(B)_{|A'} = \{\{2,4\},\{5\}\}$ or $\{\{2,4,5\}\}$. Therefore, we always have $\mathcal{P}_{\min}(A)_{|A'} \neq \mathcal{P}_{\min}(B)_{|A'}$ and it contradicts Theorem 3.

Let us now prove $\{j,k\} \in \hat{E}(C_m)$ for all pairs of vertices j, k, with $3 \leq j \leq m-1$ and $k = 1$ or $j+2 \leq k \leq m$. We have $\{2,j\}$ and $\{2,k\}$ in $\hat{E}(C_m)$. Then, the Pan condition applied to $\tilde{C}_m = \{2, e'_j, j, e_j, j+1, \ldots, k, e'_k, 2\}$ and e

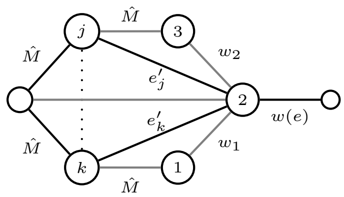


Figure 17: $\tilde{C}_m = \{2, e'_j, j, e_j, j+1, \ldots, k, e'_k, 2\}$ and $w'_j = w'_k = w_1 = w_2$.

as represented in Figure 17 implies that $\{j,k\}$ is a maximum weight chord of $\tilde{C}_m$. □

We finally establish necessary conditions on constant cycles and pans associated with constant cycles.

Proposition 19. *Let us assume that for all $\emptyset \neq S \subseteq N$, the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is convex and that the edge-weights have at most three different values $\sigma_1 < \sigma_2 \leq \sigma_3$. Then*

1. *If $|E_1| \geq 2$ (then $\sigma_2 = \sigma_3$), then every cycle with constant weight σ_2 is complete.*

2. *If $|E_1| = 1$ with $E_1 = \{e_1\}$ and if there exists a cycle C with constant weight σ_2, then there are only two different edge-weights ($\sigma_2 = \sigma_3$). Moreover, if C is not incident to e_1 and not linked to e_1 by an edge, then C is complete.*

3. *If $|E_1| = 1$ with $E_1 = \{e_1\}$ and $e_1 = \{1,2\}$, then for every cycle C with constant weight σ_2 or σ_3 and incident to 1 (resp. 2), $\{1,j\} \in E$ for all $j \in V(C) \setminus \{1\}$ (resp. $\{2,j\} \in E$ for all $j \in V(C) \setminus \{2\}$).*

4. *If $|E_1| = 1$ with $E_1 = \{e_1\}$ and $e_1 = \{1,2\}$, then for every cycle C with constant weight σ_2 or σ_3 and not adjacent to e_1 but linked to e_1 by an edge $e = \{2,k\}$ (of weight σ_2) with $k \in V(C)$, one of the following conditions is satisfied:*

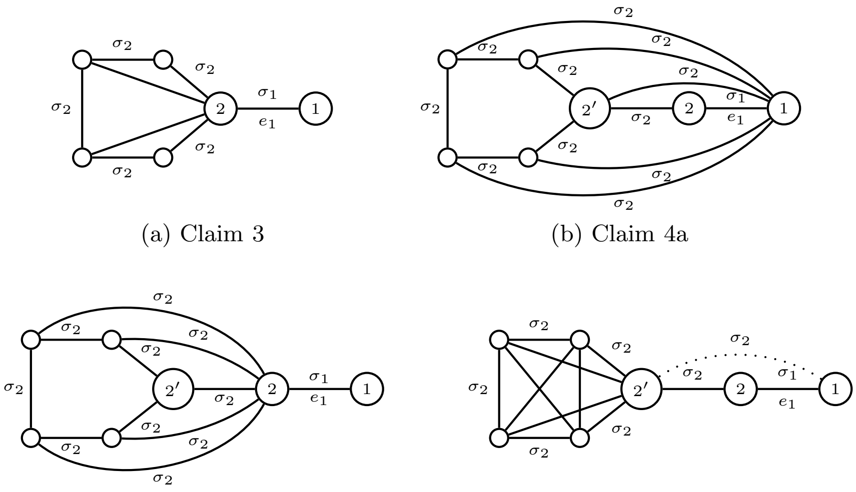
   (a) $\{1,j\} \in E$ for all $j \in V(C)$.

   (b) $\{2,j\} \in E$ for all $j \in V(C)$.

   (c) There is no edge $\{2,j\}$ in E with $j \in V(C) \setminus \{k\}$ and C is complete.

5. *Let us assume that the edge-weights have three different values $\sigma_1 < \sigma_2 < \sigma_3$. Let $e_1 = \{1, 2\}$ be the unique edge in E_1 and let us assume all edges in E_2 incident to 2. Then, every cycle C_m with $e_1 \notin E(C_m)$ is complete and $e_1 \notin \hat{E}(C_m)$.*

Situations corresponding to Claims 3 and 4 in Proposition 19 are represented in Figure 18.



(a) Claim 3

(b) Claim 4a

(c) Claim 4b

(d) Claim 4c

Figure 18: Situations of Claims 3 and 4 in Proposition 19.

Remark 2. If the edge-weights have three different values $\sigma_1 < \sigma_2 < \sigma_3$, then it follows from Claim 5 in Proposition 19 that the cycle C considered in Claim 3 is a triangle (of constant weight σ_3). As a triangle has no chord Claim 3 adds nothing in the particular case of three different edge-weights. But, *a priori*, we have to keep this case in Claim 3 to be able to prove Claim 5.

Proof. 1. By Proposition 13 the edge-weights have at most two different values $\sigma_1 < \sigma_2$. By Proposition 16, all edges in E_1 are incident to the same vertex j and no edge in E_2 is incident to j. Let us consider a cycle C with constant weight σ_2. By Lemma 10, every edge in E_2 is linked to j by an edge in E_1 as represented in Figure 19. Let us consider a given game (N, v), the $\mathcal{P}_{\min}$-

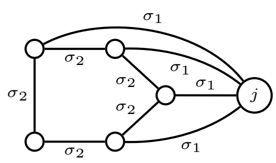


Figure 19: Every edge in E_2 is linked to j by an edge in E_1.

restricted game $(N, \overline{v})$, and the Myerson restricted game (N, v^M). We have $\overline{v}(A \cup \{j\}) = v^M(A) + v(j) = v^M(A)$ for all $A \subseteq N \setminus \{j\}$. Hence, for $i \in V(C)$

and $A \subseteq B \subseteq V(C) \setminus \{i, j\}$ the inequality

$$\overline{v}(B \cup \{j\} \cup \{i\}) - \overline{v}(B \cup \{j\}) \geq \overline{v}(A \cup \{j\} \cup \{i\}) - \overline{v}(A \cup \{j\}) \tag{12}$$

is equivalent to

$$v^M(B \cup \{i\}) - v^M(B) \geq v^M(A \cup \{i\}) - v^M(A). \tag{13}$$

As $(N, \overline{u_S})$ is convex, (12) and therefore (13) are satisfied with $v = u_S$. Then, u_S^M is convex if we restrict G to $V(C)$, and C has to be complete by Theorems 6 and 7.

2. If there are three different edge-weights, then by Proposition 17 there is no cycle with constant weight σ_2. Let us consider a cycle C with constant weight σ_2 non-incident to $e_1 = \{1, 2\}$ and not linked by an edge to e_1. For any game (N, v) and for $A \subseteq N \setminus \{1, 2\}$ such that there is no edge linking A to $\{1, 2\}$, we have

$$\overline{v}(A \cup \{1, 2\}) = v^M(A) + v(\{1\}) + v(\{2\}) = v^M(A). \tag{14}$$

Hence, for $i \in V(C)$ and for $A \subseteq B \subseteq V(C) \setminus \{i\}$, the subsets A, B, $A \cup \{i\}$, and $B \cup \{i\}$ satisfy (14). Then, the inequality $\overline{v}((B \cup \{1, 2\}) \cup \{i\}) - \overline{v}(B \cup \{1, 2\}) \geq \overline{v}((A \cup \{1, 2\}) \cup \{i\}) - \overline{v}(A \cup \{1, 2\})$ is equivalent to: $v^M(B \cup \{i\}) - v^M(B) \geq v^M(A \cup \{i\}) - v^M(A)$. Therefore, taking $v = u_S$, we can conclude as in the previous case.

3. Let us consider $E_1 = \{e_1'\}$ with $e_1' = \{1', 2\}$ and a cycle $C = \{1, e_1, 2, e_2, \ldots, m, e_m, 1\}$ incident to 2. We can assume w.l.o.g. that C has constant weight σ_2. Note that, by the Cycle condition, e_1' cannot be a chord of C. By contradiction let us assume $\{2, 4\} \notin E$. Let us consider $i = 3$, $A_1 = \{4\}$, $A_2 = \{1', 2\}$, $A = A_1 \cup A_2$, and $B = (V(C) \setminus \{i\}) \cup \{1'\}$ as represented in Figure 20. Then,

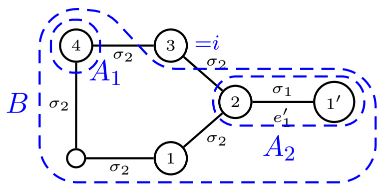


Figure 20: Cycle C incident to 2.

$\mathcal{P}_{\min}(A) = \{\{1'\}, \{2\}, \{4\}\}$ or $\{\{1', 4\}, \{2\}\}$, $\mathcal{P}_{\min}(A \cup \{i\}) = \{\{1'\}, \{2, 3, 4\}\}$ or $\{A \cup \{i\}\}$, and $\mathcal{P}_{\min}(B) = \{\{1'\}, V(C) \setminus \{3\}\}$ or $\{B\}$. Then, for any $A' \in \mathcal{P}_{\min}(A \cup \{i\})$ containing 2, we have $\{2\} \in \mathcal{P}_{\min}(A)_{|A'}$ but $\{2\} \notin \mathcal{P}_{\min}(B)_{|A'}$, contradicting Theorem 3. Hence, we have $e = \{2, 4\} \in E$ and the Star condition applied to $\{e, e_1', e_2\}$ implies $w(e) = \sigma_2$. Then, by the same reasoning on the cycle $\{1, e_1, 2, e, 4, e_4, \ldots, m, e_m, 1\}$, we have $\{2, 5\} \in E$. Iterating we get the result.

4. Let us consider $E_1 = \{e'_1\}$ with $e'_1 = \{1', 2'\}$ and a cycle $C = \{1, e_1, 2, e_2, \ldots, m, e_m, 1\}$ with constant weight σ_2 or σ_3 and not incident to e_1 but linked to e_1 by an edge $e = \{2, 2'\}$ of weight σ_2.

If $e' = \{2', j\} \in E$ for some $j \in V(C) \setminus \{2\}$, then Claim 3 applied to the cycles $C' = \{2', e, 2, e_2, 3, \ldots, e_{j-1}, j, e', 2'\}$ and $C'' = \{2', e', j, e_j, j+1 \ldots, e_m, 1, e_1, 2, e, 2'\}$ incident to e'_1 as represented in Figure 21 implies $\{2', k\} \in$

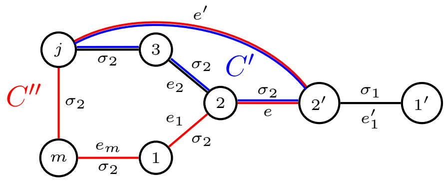


Figure 21: C' and C''.

E for all $k \in V(C)$. Hence, Claim 4b is satisfied.

Let us now assume $\{2', j\} \notin E$ for all $j \in V(C) \setminus \{2\}$. If $\{1', 2\} \in E$, then we can apply the same reasoning as before (interchanging the roles of $1'$ and $2'$). Then, either $\{1', k\} \in E$ for all $k \in V(C)$ and Claim 4a is satisfied or $\{1', k\} \notin E$ for all $k \in V(C) \setminus \{2\}$. Therefore, we assume henceforth that the following condition is satisfied:

$$\text{(15)} \qquad \begin{array}{l}\text{There is no edge } \{2', l\} \text{ with } l \in V(C) \setminus \{2\} \text{ and if } \{1', 2\} \in E \\ \text{there is also no edge } \{1', l\} \text{ with } l \in V(C) \setminus \{2\}.\end{array}$$

We now prove that C is complete, i.e., that Claim 4c is satisfied. By contradiction let us assume $\{2, 4\} \notin E$ (the proof is similar to the one of Claim 3). Let us consider $i = 3$, $A_1 = \{4\}$, $A_2 = \{1', 2', 2\}$, $A = A_1 \cup A_2$, and $B = (V(C) \setminus \{i\}) \cup A_2$ as represented in Figure 22. By (15), $\{2', 4\} \notin E$.

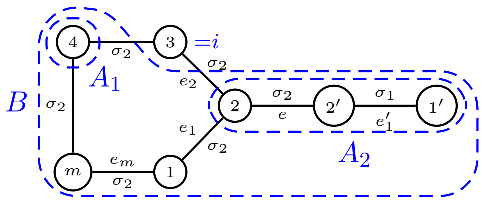


Figure 22: C linked to e'_1 by an edge.

We consider several cases:

1. If $\{1', 2\} \in E$, then $\{1', 4\} \notin E$ by (15) and therefore $\mathcal{P}_{\min}(A) = \{\{1', 2, 2'\}, \{4\}\}$.
2. If $\{1', 2\} \notin E$ and $\{1', 4\} \in E$, then $\mathcal{P}_{\min}(A) = \{\{1', 4\}, \{2, 2'\}\}$.
3. If $\{1', 2\} \notin E$ and $\{1', 4\} \notin E$, then $\mathcal{P}_{\min}(A) = \{\{1'\}, \{2, 2'\}, \{4\}\}$.

In every case $\mathcal{P}_{\min}(A \cup \{i\}) = \{A \cup \{i\}\}$ or $\{\{1'\}, A \cup \{i\} \setminus \{1'\}\}$ and $\mathcal{P}_{\min}(B) = \{B\}$ or $\{\{1'\}, B \setminus \{1'\}\}$. Therefore, taking $A' = A \cup \{i\}$ or $A \cup \{i\} \setminus \{1'\}$, we have either $\mathcal{P}_{\min}(B)_{|A'} = \{A\}$ or $\{\{1'\}, A \setminus \{1'\}\}$ or $\{A \setminus \{1'\}\}$. As neither A nor $A \setminus \{1'\}$ is in $\mathcal{P}_{\min}(A)$, we get $\mathcal{P}_{\min}(A)_{|A'} \neq \mathcal{P}_{\min}(B)_{|A'}$, contradicting Theorem 3. Hence, $\{2, 4\} \in E$. Then, iterating as in the proof of Claim 3 we get $\{2,j\} \in \hat{E}(C)$ for all $j \in V(C)$. Let us now assume $\{j,k\} \notin E$ for two vertices j and k with $3 \leq j \leq m-1$ and $k = 1$ or $j+2 \leq k \leq m$. Let us consider the edges $e'_j := \{2,j\}$ and $e'_k := \{2,k\}$ in $\hat{E}(C)$. If $\{1',j\} \in E$ and $\{1',k\} \in E$, then we obtain two adjacent chordless cycles $\tilde{C}_j = \{2, e'_j, j, \{j,1'\}, 1', e'_1, 2', e, 2\}$ and $\tilde{C}_k = \{2, e'_k, k, \{k,1'\}, 1', e'_1, 2', e, 2\}$ (by (15) $\{2',j\}$, $\{2',k\}$, $\{1',2\}$ are not in E) with a common edge e'_1 in E_1 contradicting the Adjacent cycles condition. Hence, we can assume that at most one of the edges $\{1',j\}$ or $\{1',k\}$ is in E. We now consider the cycle $\tilde{C} = \{2, e'_j, j, e_j, j+1, \ldots, k, e'_k, 2\}$ and $i = 2$, $A_1 = \{j\}$, $A_2 = \{k\}$, $A_3 = \{1',2'\}$, $A = A_1 \cup A_2 \cup A_3$, $B_1 = V(\tilde{C}) \setminus \{2\}$, $B_2 = A_3$, and $B = B_1 \cup B_2$ as represented in Figure 23. To obtain $\mathcal{P}_{\min}(A)$, $\mathcal{P}_{\min}(A \cup \{i\})$ or $\mathcal{P}_{\min}(B)$

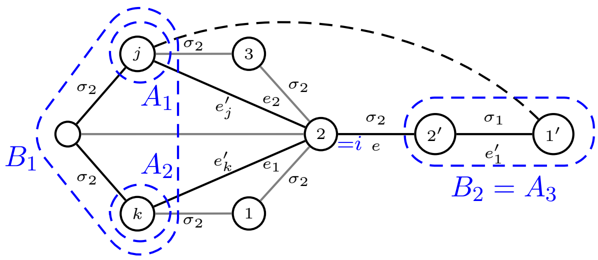


Figure 23: $\tilde{C} = \{2, e'_j, j, e_j, j+1, \ldots, k, e'_k, 2\}$ and $\{1',j\} \in E$.

we only have to delete the edge $e'_1 = \{1',2'\}$ of weight σ_1. As $\{1',j\} \notin E$ or $\{1',k\} \notin E$, $\{j,k\}$ cannot be a subset of any component of $\mathcal{P}_{\min}(A)$. Therefore, we can only have $\mathcal{P}_{\min}(A) = \{\{1'\},\{2'\},\{j\},\{k\}\}$ or $\{\{1',j\},\{2'\},\{k\}\}$ or $\{\{1',k\},\{2'\},\{j\}\}$. As $\{2,j\}$ and $\{2,k\}$ are in $\hat{E}(C)$ and as $i = 2$, j and k are connected in $G_{A \cup \{i\} \setminus \{1',2'\}}$. Hence, there exists $A' \in \mathcal{P}_{\min}(A \cup \{i\})$ with $\{j,k\} \subseteq A'$. As j and k are connected in $G_{B \setminus \{1',2'\}}$ there exists $B' \in \mathcal{P}_{\min}(B)$ with $\{j,k\} \subseteq B'$. Hence, $\{j,k\} \subseteq (B' \cap A') \in \mathcal{P}_{\min}(B)_{|A'}$. But $\{j,k\}$ cannot be a subset of any component of $\mathcal{P}_{\min}(A)$. Hence, $\mathcal{P}_{\min}(A)_{|A'} \neq \mathcal{P}_{\min}(B)_{|A'}$ contradicting Theorem 3. Therefore, $\{j,k\} \in \hat{E}(C)$.

5. By Proposition 17, there is a unique edge $e_1 = \{1,2\}$ in E_1, all edges in E_2 are incident to the same end-vertex 2 of e_1, and all edges in E_3 are linked to 2 by e_1 or by an edge in E_2. As $e_1 \notin E(C_m)$, an edge in $E(C_m)$ has weight σ_2 or σ_3.

Let us assume C_m non-constant. As edges in E_2 are incident to 2, $2 \in V(C_m)$ and e_1 is adjacent to C_m. By Proposition 18, $e_1 \notin \hat{E}(C_m)$ and C_m is complete.

Let us now assume C_m constant. As all edges in E_2 are incident to 2 they cannot form a cycle, therefore $E(C_m) \subseteq E_3$. Then, $2 \notin V(C_m)$ and $e_1 \notin \hat{E}(C_m)$. Let us assume $1 \notin V(C_m)$. Then, an edge e in $E(C_m)$ cannot be linked to vertex 2 by e_1, therefore e is linked to 2 by an edge in E_2. As $m \geq 3$ there exist at least two vertices i and j in $V(C_m)$ such that $\{2,i\}$ and $\{2,j\}$ are in E. C_m gives two obvious paths γ and γ' linking i and j. Let us consider the cycles $C'_m = \{2,i\} \cup \gamma \cup \{j,2\}$ and $C''_m = \{2,i\} \cup \gamma' \cup \{j,2\}$

as represented in Figure 24. By Case 1 (or Proposition 18) C'_m and C''_m are

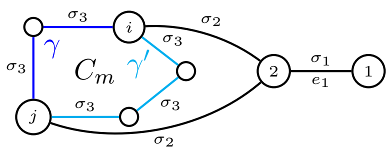


Figure 24: $C'_m = \{2, i\} \cup \gamma \cup \{j, 2\}$ and $C''_m = \{2, i\} \cup \gamma' \cup \{j, 2\}$.

complete. This implies $\{i, j\} \in E$ and $\{2, k\} \in E$ for all $k \in V(C_m)$. Hence, for any pair of vertices i, j in $V(C_m)$ the previous reasoning is valid and implies $\{i, j\} \in E$. Therefore, C_m is complete. Let us now assume $1 \in V(C_m)$. Claim 3 implies $\{1, i\} \in \hat{E}(C_m)$ for all $i \in V(C_m)$. As $m \geq 3$, there is at

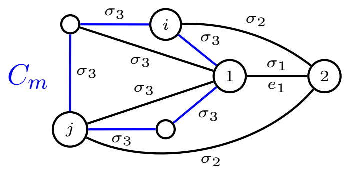
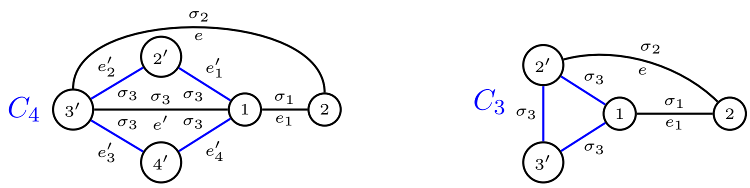


Figure 25: C_m with $m = 5$.

least one edge linking 2 to $V(C_m) \setminus \{1\}$. If there exist two edges $\{2, i\}$ and $\{2, j\}$ with i and j in $V(C_m) \setminus \{1\}$ as represented in Figure 25, then e_1 is a common edge for the triangles defined by $\{1, 2, i\}$ and $\{1, 2, j\}$, contradicting the Adjacent cycles condition. Hence, there is exactly one edge e linking 2 to $V(C_m) \setminus \{1\}$. If $m \geq 5$, then there is at least one edge in $E(C_m)$ neither



(a) (b)

Figure 26: C_4 and $\tilde{C}_3 = \{1, e_1, 2, e, 3', e', 1\}$. C_3 and $\tilde{C}_3 = \{1, e_1, 2, e, 2', e'_1, 1\}$.

incident to 1 nor linked to 2, a contradiction. If $m = 4$, then we necessarily have $e = \{2, 3'\}$ as represented in Figure 26a, otherwise we get the same contradiction. Let us denote by $e'_1 = \{1, 2'\}$, $e'_2 = \{2', 3'\}$, $e'_3 = \{3', 4'\}$, and $e'_4 = \{4', 1\}$ the edges in $E(C_4)$ and by $e' = \{1, 3'\}$ the chord of C_4 incident to 1. If $\{2', 4'\} \in E$, we are done. So, let us assume $\{2', 4'\} \notin E$. Let us consider $i = 1$, $A = \{2, 2', 4'\}$, and $B = A \cup \{3'\}$. Then, $\mathcal{P}_{\min}(A) = \{\{2\}, \{2'\}, \{4'\}\}$, $\mathcal{P}_{\min}(A \cup \{i\}) = \{\{2\}, \{1, 2', 4'\}\}$, and $\mathcal{P}_{\min}(B) = \{\{2\}, \{2', 3', 4'\}\}$. Taking $A' = \{1, 2', 4'\}$ we get $\mathcal{P}_{\min}(A)_{|A'} = \{\{2'\}, \{4'\}\} \neq \{2', 4'\} = \mathcal{P}_{\min}(B)_{A'}$ and it contradicts Theorem 3. Hence, $m = 3$ as represented in Figure 26b. $\square$

5 Graphs satisfying inheritance of convexity

We provide characterizations of weighted graphs satisfying inheritance of convexity with $\mathcal{P}_{\min}$. We start with connected weighted graphs.

5.1 Connected graphs with two edge-weights

Theorem 20. *Let $G = (N, E, w)$ be a connected weighted graph. Let us assume that the edge-weights have only two different values $\sigma_1 < \sigma_2$ and $|E_1| \geq 2$. Then, there is inheritance of convexity for $\mathcal{P}_{\min}$ if and only if*

1. *All edges in E_1 are incident to the same vertex 1 and all edges in E_2 are linked to 1 by an edge in E_1.*
2. *One of the following two equivalent conditions is satisfied:*
   - *(a) There is inheritance of convexity for $\mathcal{P}_M$ on the subgraph $G_1 = (N, E \setminus E_1)$.*
   - *(b) $G_1 = (N, E \setminus E_1)$ is cycle-complete.*

We give in Figure 27 an example of a graph satisfying conditions 1 and 2 of Theorem 20.

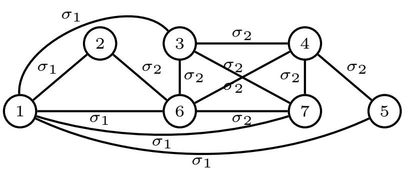


Figure 27: Every edge in E_2 is linked to 1 by an edge in E_1 and the cycle defined by $3, 4, 6, 7$ is complete.

Proof. Conditions 2a and 2b are equivalent by Theorem 7 (van den Nouweland and Borm, 1991). By Proposition 16, Lemma 10 and Proposition 19 (Claim 1), Conditions 1 and 2 are necessary. We now prove their sufficiency. Let (N, v) be a convex game. We denote by $(N, \overline{v})$ (resp. (N, v^M)) the restricted game associated with $\mathcal{P}_{\min}$ (resp. $\mathcal{P}_M$) on G (resp. G_1). Let us consider $i \in N$ and subsets $A \subseteq B \subseteq N \setminus \{i\}$. We consider several cases to prove that the following inequality is satisfied:

$$\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq \overline{v}(A \cup \{i\}) - \overline{v}(A). \tag{16}$$

Let us first assume $E(B) \subseteq E_2$ (resp. $E(A \cup \{i\}) \subseteq E_2$). Then, $\mathcal{P}_{\min}(A)$ and $\mathcal{P}_{\min}(B)$ (resp. $\mathcal{P}_{\min}(A \cup \{i\})$) are singleton partitions and (16) is equivalent to $\overline{v}(B \cup \{i\}) \geq \overline{v}(A \cup \{i\})$ (resp. $\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq 0.$). This last inequality is satisfied as $(N, \overline{v})$ is superadditive (cf. Corollary 2).

Let us now assume $E(B) \cap E_1 \neq \emptyset$ and $E(A \cup \{i\}) \cap E_1 \neq \emptyset$. Then, we also have $E(B \cup \{i\}) \cap E_1 \neq \emptyset$. By Condition 1 any edge in E_1 is incident to 1, therefore we have $1 \in B$ and $i \neq 1$ as $B \subseteq N \setminus \{i\}$. Then, as $E(A \cup \{i\}) \cap E_1 \neq \emptyset$, we necessarily have $1 \in A$. If $E(A) \neq \emptyset$, Condition 1 implies $E(A) \cap E_1 \neq \emptyset$ and then $\overline{v}(A) = v^M(A)$. If $E(A) = \emptyset$, then we trivially have $\overline{v}(A) = v^M(A)$. Hence, (16) is equivalent to $v^M(B \cup \{i\}) - v^M(B) \geq v^M(A \cup \{i\}) - v^M(A)$, and by Condition 2 this last inequality is satisfied. $\square$

Theorem 21. *Let $G = (N, E, w)$ be a connected weighted graph. Let us assume that the edge-weights have only two different values $\sigma_1 < \sigma_2$ and $|E_1| = 1$. Let $e_1 = \{1, 2\}$ be the unique edge in E_1. Then, there is inheritance of convexity for $\mathcal{P}_{\min}$ if and only if*

1. *There exists at most one chordless cycle containing e_1.*
2. *For every cycle C with constant weight σ_2 either C is complete or all vertices of C are linked to the same end-vertex of e_1.*

We give in Figure 28 an example of a graph satisfying conditions 1 and 2 of Theorem 21.

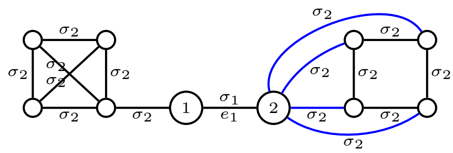


Figure 28: Either a constant cycle is complete or all its vertices are linked to one end-vertex of e_1.

Proof. Condition 1 is necessary by the Adjacent cycles condition. By Proposition 19 (Claims 2, 3, and 4) Condition 2 is also necessary. We now prove their sufficiency. Let (N, v) be a convex game and let us consider $i \in N$ and subsets $A \subseteq B \subseteq N \setminus \{i\}$. We consider several cases to prove that the following inequality is satisfied:

$$\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq \overline{v}(A \cup \{i\}) - \overline{v}(A). \tag{17}$$

Let us first assume $E(B) \subseteq E_2$ (resp. $E(A \cup \{i\}) \subseteq E_2$). Then, we can conclude as in Case 1 in the proof of Theorem 20.

Let us now assume $e_1 \in E(B)$ and $e_1 \in E(A \cup \{i\})$. Then, we also have $e_1 \in E(B \cup \{i\})$. If $e_1 \notin E(A)$, then $i = 1$ or 2 as $e_1 \in E(A \cup \{i\})$ but it contradicts $e_1 \in E(B)$ as $B \subseteq N \setminus \{i\}$. Therefore, we also have $e_1 \in E(A)$ and (17) is equivalent to $v^M(B \cup \{i\}) - v^M(B) \geq v^M(A \cup \{i\}) - v^M(A)$ where (N, v^M) is associated with $G_1 = (N, E \setminus E_1)$. Let $\mathcal{P}_M(A) = \{A_1, A_2, \ldots, A_p\}$ (resp. $\mathcal{P}_M(B) = \{B_1, B_2, \ldots, B_q\}$) be the partition of A (resp. B) into connected components in G_1. If there is no link between i and A, then $\mathcal{P}_M(A \cup \{i\}) = \{\mathcal{P}_M(A), \{i\}\}$ and $v^M(A \cup \{i\}) - v^M(A) = v(\{i\}) = 0$. Then, (17) is equivalent to $\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq 0$ and this last inequality is satisfied as $(N, \overline{v})$ is superadditive. Otherwise, we have $\mathcal{P}_M(A \cup \{i\}) = \{A_1 \cup \ldots \cup A_r \cup \{i\}, A_{r+1}, \ldots, A_p\}$ (resp. $\mathcal{P}_M(B \cup \{i\}) = \{B_1 \cup \ldots \cup B_s \cup \{i\}, B_{s+1}, \ldots, B_q\}$) with $1 \leq r \leq p$ (resp.

$1 \leq s \leq q$), after reordering if necessary. Then, setting $A' = A_1 \cup \ldots \cup A_r$ and $B' = B_1 \cup \ldots \cup B_s$, (17) is equivalent to

$$v(B' \cup \{i\}) - \sum_{j=1}^{s} v(B_j) \geq v(A' \cup \{i\}) - \sum_{j=1}^{r} v(A_j). \tag{18}$$

Let us observe that obviously $\mathcal{P}_M(A)$ is a refinement of $\mathcal{P}_M(B)_{|A}$. To complete the proof we need the following claim.

Claim A. *Conditions 1 and 2 imply $A_j \subseteq B_j$, for all j, $1 \leq j \leq r$, after renumbering if necessary.*

By contradiction, let us assume that two components A_1 and A_2 of $\mathcal{P}_M(A)_{|A'}$ are subsets of the same component $B_1 \in \mathcal{P}_M(B)$, after renumbering if necessary. Let $\tilde{e}_1 = \{i, k_1\}$ (resp. $\tilde{e}_2 = \{i, k_2\}$) be an edge linking i to A_1 (resp. A_2). As $i \notin \{1, 2\}$, $\tilde{e}_1$ and $\tilde{e}_2$ are in E_2. As B_1 is connected, there exists an elementary path γ in $\tilde{G}_{B_1}$ linking $k_1 \in A_1$ to $k_2 \in A_2$. We obtain a simple cycle $C = \{i, \tilde{e}_1, k_1\} \cup \gamma \cup \{k_2, \tilde{e}_2, i\}$ of constant weight σ_2. If C is complete, then $\{k_1, k_2\}$ is a chord of C. Condition 1 implies $\{k_1, k_2\} \neq e_1$. Then, $\{k_1, k_2\}$ links A_1 to A_2 in G_1, a contradiction. If C is not complete, then by Condition 2 all vertices of C are linked to the same end-vertex v of e_1. We can assume w.l.o.g. $v = 1$ as represented in Figure 29. As $e_1 \in E(A)$ and as k_1 and k_2 are in A, we have $\{1, k_1\}$ and $\{1, k_2\}$ in $E(A)$. We also have $\{1, k_1\} \neq e_1$ and $\{1, k_2\} \neq e_1$, otherwise e_1 would be a chord of a cycle contradicting Condition 1. Then, A_1 and A_2 are part of a connected component of A in G_1, a contradiction.

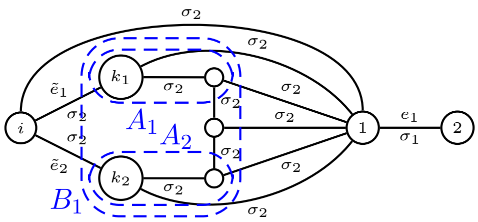


Figure 29: e_1 in $E(A)$ and k_1 in A_1, k_2 in A_2.

We now end the proof of Theorem 21. By Claim A, we have $\mathcal{P}_M(A') = \mathcal{P}_M(B')_{|A'}$. Then, Lemma 5 applied to $\mathcal{P}_M$ and the family $\mathcal{F}$ of connected subsets of N implies $v(B') - \sum_{j=1}^{s} v(B_j) \geq v(A') - \sum_{j=1}^{r} v(A_j)$. The convexity of (N, v) also implies $v(B' \cup \{i\}) - v(B') \geq v(A' \cup \{i\}) - v(A')$. Adding these last inequalities we obtain (18). □

Theorem 22. *Let $G = (N, E, w)$ be a connected weighted graph. Let us assume that the edge-weights have three different values $\sigma_1 < \sigma_2 < \sigma_3$. Then, there is inheritance of convexity for $\mathcal{P}_{\min}$ if and only if*

1. *There is only one edge $e_1 = \{1, 2\}$ in E_1.*
2. *Every edge in E_2 is incident to the same end-vertex 2 of e_1.*
3. *Every edge in E_3 is linked to 2 by e_1 or by an edge in E_2.*

4. *There exists at most one chordless cycle $\tilde{C}_m$ with $m = 3$ or 4 containing e_1.*

5. *$G_1 = (N, E \setminus E_1)$ is cycle-complete.*

Moreover, these conditions imply:

6. *If a cycle C_m does not contain e_1 and if $1 \in V(C_m)$, then $m = 3$ and such a cycle is unique, has constant weight σ_3, and is adjacent to a unique triangle $\tilde{C}_3$ containing e_1. Moreover, $\tilde{C}_3 = \{1, e_1, 2, e_2, 3, e_3, 1\}$ with $w_i = \sigma_i$ for $i \in \{1, 2, 3\}$, and $E(C_3) \cap E(\tilde{C}_3) = \{e_3\}$.*

Remark 3. The Star, Path, Cycle, Pan and Adjacent cycles conditions are straightforward consequences of Conditions 1 to 5 in Theorem 22.

Proof of theorem 22. By Proposition 17 and the Adjacent cycles condition, Conditions 1 to 4 are necessary. By Proposition 19 (Claim 5), Condition 5 is necessary. We now prove their sufficiency. Let us consider a convex game (N, v), $i \in N$ and subsets $A \subseteq B \subseteq N \setminus \{i\}$. We have to prove that the following inequality is satisfied:

$$\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq \overline{v}(A \cup \{i\}) - \overline{v}(A). \tag{19}$$

Let us note that if i is not linked to A, then (19) is trivially satisfied as $(N, \overline{v})$ is superadditive (cf. Corollary 2). If $i = 2$, Conditions 1 and 2 imply $E(A) \subseteq E(B) \subseteq E_3$. Then, $\mathcal{P}_{\min}(A)$ and $\mathcal{P}_{\min}(B)$ are singletons partitions and (19) is equivalent to $\overline{v}(B \cup \{i\}) \geq \overline{v}(A \cup \{i\})$. This last inequality is satisfied as $(N, \overline{v})$ is superadditive.

We thereafter assume i linked to A by at least one edge and $i \neq 2$, and consider several cases.

Case 1 Let us assume $2 \notin A$. Conditions 1 and 2 imply $E(A) \subseteq E(A \cup \{i\}) \subseteq E_3$. Then, $\mathcal{P}_{\min}(A)$ and $\mathcal{P}_{\min}(A \cup \{i\})$ are singleton partitions and $\overline{v}(A \cup \{i\}) - \overline{v}(A) = v(\{i\}) = 0$. As $(N, \overline{v})$ is superadditive (cf. Corollary 2), (19) is satisfied.

Case 2 Let us assume $2 \in A$ and $1 \in A$. Then, e_1 belongs to $E(A)$, $E(A \cup \{i\})$, $E(B)$, $E(B \cup \{i\})$, and (19) is equivalent to $v^M(B \cup \{i\}) - v^M(B) \geq v^M(A \cup \{i\}) - v^M(A)$ where (N, v^M) is the Myerson restricted game associated with $G_1 = (N, E \setminus \{e_1\})$. By Condition 5, G_1 is cycle-complete. Then, (N, v^M) is convex by Theorem 7 and therefore (19) is satisfied.

Case 3 Let us assume $2 \in A$ and $1 \in B \setminus A$. Then, $e_1 \in E(B) \setminus E(A)$, $i \notin \{1, 2\}$, $e_1 \notin E(A \cup \{i\})$, and $e_1 \in E(B \cup \{i\})$. As in Case 2, $\overline{v}(B \cup \{i\}) - \overline{v}(B) = v^M(B \cup \{i\}) - v^M(B)$ where (N, v^M) is associated with $G_1 = (N, E \setminus \{e_1\})$. As i is linked to A in G, and as $i \notin \{1, 2\}$, i is also linked to A in G_1. Let $\hat{A} = \{A_1, \ldots, A_p\}$ with $p \geq 1$ be the set of connected components of A in G_1 linked to i. By Conditions 2 and 3, any edge in G_1 is either incident to 1 or 2 or linked to 2 by an edge in E_2. Therefore, $\hat{A}$ is only made up of one component containing 2, and possibly singleton components (the component containing 2 may be reduced to a singleton). Note that if $\hat{A}$ contains a singleton different from 2, then the edge $\{i, 2\}$ exists in G_1. Hence, there necessarily is an element in $\hat{A}$ containing 2. Let us assume $2 \in A_1$ after renumbering if necessary. Then, as $A_2, \dots, A_p$ are singletons, we get $v^M(A \cup \{i\}) - v^M(A) = v(\bigcup_{j=1}^{p} A_j \cup \{i\}) - v(A_1)$. By Condition 5, G_1 is cycle-complete. Then, (N, v^M) is convex by Theorem 7 and this implies

$$\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq v\left(\bigcup_{j=1}^{p} A_j \cup \{i\}\right) - v(A_1). \tag{20}$$

As $e_1 \notin E(A)$, if $E(A) \neq \emptyset$, then by Condition 3 any edge in $E(A) \cap E_3$ is linked to 2 by an edge in $E(A) \cap E_2$. Hence, $E(A) \cap E_2 \neq \emptyset$ and $\overline{v}(A) = v^M(A)$ where (N, v^M) is associated with $\tilde{G}_3 := (N, E_3)$. If $E(A) = \emptyset$, we trivially have $\overline{v}(A) = v^M(A)$. As $e_1 \notin E(A \cup \{i\})$, we have by the same reasoning $\overline{v}(A \cup \{i\}) = v^M(A \cup \{i\})$ where (N, v^M) is associated with $\tilde{G}_3$. Let $\tilde{A}$ be the set of connected components of A in $\tilde{G}_3$ linked to i. Note that, by Conditions 2 and 3, $\{2\}$ is a singleton component in $\tilde{G}_3$ and cannot belong to $\tilde{A}$. If i is linked to $A_1 \setminus \{2\}$, then $\tilde{A} = \{\tilde{A}_{1,1}, \tilde{A}_{1,2}, \dots, \tilde{A}_{1,r}, A_2, \dots, A_p\}$ with $r \geq 1$ and $\emptyset \neq \tilde{A}_{1,j} \subset A_1$ for all j, $1 \leq j \leq r$. Let us assume $r \geq 2$. There exists $k_1 \in \tilde{A}_{1,1}$ (resp. $k_2 \in \tilde{A}_{1,2}$) such that $\{i, k_1\} \in E$ (resp. $\{i, k_2\} \in E$). As $\tilde{A}_{1,1} \subseteq A_1$ and $\tilde{A}_{1,2} \subseteq A_1$, there is a path γ in A_1 linking k_1 to k_2. Then, $\{i, k_1\} \cup \gamma \cup \{k_2, i\}$ induces a cycle C in G_1 as represented in Figure 30. By Condition 5, C is

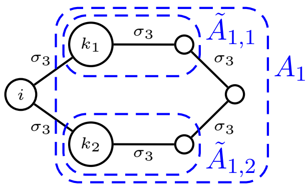


Figure 30: Cycle C.

complete in G. As $1 \notin A$ and $2 \notin \tilde{A}_{1,1}$ and $2 \notin \tilde{A}_{1,2}$, $\{k_1, k_2\} \in E_3$ and links $\tilde{A}_{1,1}$ to $\tilde{A}_{1,2}$ in $\tilde{G}_3$, a contradiction. Hence, $r = 1$ and we have

$$\overline{v}(A \cup \{i\}) - \overline{v}(A) = v\left(\tilde{A}_{1,1} \cup \bigcup_{j=2}^{p} A_j \cup \{i\}\right) - v(\tilde{A}_{1,1}). \tag{21}$$

If i is not linked to $A_1 \setminus \{2\}$, then (21) is still satisfied setting $\tilde{A}_{1,1} = \emptyset$. As $\tilde{A}_{1,1} \subseteq A_1$, we have $(\tilde{A}_{1,1} \cup \bigcup_{j=2}^{p} A_j \cup \{i\}) \cap A_1 = \tilde{A}_{1,1}$ and $(\tilde{A}_{1,1} \cup \bigcup_{j=2}^{p} A_j \cup \{i\}) \cup A_1 = \bigcup_{j=1}^{p} A_j \cup \{i\}$. Therefore, the convexity of (N, v) implies

$$v\left(\bigcup_{j=1}^{p} A_j \cup \{i\}\right) - v(A_1) \geq v\left(\tilde{A}_{1,1} \cup \bigcup_{j=2}^{p} A_j \cup \{i\}\right) - v(\tilde{A}_{1,1}). \tag{22}$$

Finally, (20), (22), and (21) imply (19).

Case 4 Let us assume $2 \in A$, $1 \notin B$, and $i \neq 1$. Then, $e_1 \notin E(A)$, $e_1 \notin E(A \cup \{i\})$, $e_1 \notin E(B)$, and $e_1 \notin E(B \cup \{i\})$. By the same reasoning as in Case 2, we have $\overline{v}(A) = v^M(A)$, $\overline{v}(A \cup \{i\}) = v^M(A \cup \{i\})$, $\overline{v}(B) = v^M(B)$, and $\overline{v}(B \cup \{i\}) = v^M(B \cup \{i\})$ where (N, v^M) is associated with $\tilde{G}_3 := (N, E_3)$. Then, (19) is equivalent to $v^M(B \cup \{i\}) - v^M(B) \geq v^M(A \cup \{i\}) - v^M(A)$. Let C_m be a cycle in $\tilde{G}_3$. Then, $E(C_m) \subseteq E_3$ and by Condition 5, C_m is a complete cycle in G. By Conditions 2 and 3, an edge is in E_2 if and only if it is incident to 2. Therefore, $2 \notin V(C_m)$ and any chord of C_m is in E_3. Then, C_m is also a complete cycle in $\tilde{G}_3$. Hence, $\tilde{G}_3$ is cycle-complete. Then, (N, v^M) is convex by Theorem 7 and (19) is satisfied.

Case 5 Let us assume $2 \in A$ and $i = 1$. Then, $e_1 \notin E(A)$, $e_1 \notin E(B)$ but $e_1 \in E(A \cup \{i\})$ and $e_1 \in E(B \cup \{i\})$. Let $\hat{A} = \{A_1, \ldots, A_p\}$ (resp. $\hat{B} = \{B_1, \ldots, B_q\}$) be the set of connected components of A (resp. B) in $G_1 = (N, E \setminus \{e_1\})$. By Conditions 2 and 3, any edge in G_1 is either incident to 1 or 2 or linked to 2 by an edge in E_2. Therefore, $\hat{A}$ (resp. $\hat{B}$) is only made up of one component containing 2, and possibly singleton components (the component containing 2 may be reduced to a singleton). Let A_1 (resp. B_1) be the component containing 2. Then, we have $A_1 \subseteq B_1$ and $A_2, \ldots, A_p$ (resp. $B_2, \ldots, B_q$) are singletons. Let $\tilde{A}$ (resp. $\tilde{B}$) be the set of connected components of A (resp. B) in $\tilde{G}_3 := (N, E_3)$. By Conditions 2 and 3, $\{2\}$ is a singleton component in $\tilde{G}_3$. Therefore, we have $\tilde{A} = \{\tilde{A}_{1,1}, \tilde{A}_{1,2}, \ldots, \tilde{A}_{1,r}, A_2, \ldots, A_p\}$ (resp. $\tilde{B} = \{\tilde{B}_{1,1}, \tilde{B}_{1,2}, \ldots, \tilde{B}_{1,s}, B_2, \ldots, B_q\}$) where $\{\tilde{A}_{1,1}, \tilde{A}_{1,2}, \ldots, \tilde{A}_{1,r}\}$ (resp. $\{\tilde{B}_{1,1}, \tilde{B}_{1,2}, \ldots, \tilde{B}_{1,s}\}$) with $r \geq 1$ (resp. $s \geq 1$) is the partition of A_1 (resp. B_1) in $\tilde{G}_3$ and $\tilde{A}_{1,1} = \tilde{B}_{1,1} = \{2\}$. Note that $\tilde{A}_{1,j}$ (resp. $\tilde{B}_{1,j}$) is linked to 2 in G (and G_1) i.e., there exists $k_j \in \tilde{A}_{1,j}$ (resp. $l_j \in \tilde{B}_{1,j}$) such that $\{2, k_j\} \in E$ (resp. $\{2, l_j\} \in E$) for all j, $2 \leq j \leq r$ (resp. $2 \leq j \leq s$).

Claim B. *We can assume $\tilde{A}_{1,j} \subseteq \tilde{B}_{1,j}$ for all j, $1 \leq j \leq r$, after renumbering if necessary.*

Proof of Claim B. We have $\tilde{A}_{1,1} = \tilde{B}_{1,1} = \{2\}$. By contradiction, let us assume $\tilde{A}_{1,2} \subseteq \tilde{B}_{1,2}$ and $\tilde{A}_{1,3} \subseteq \tilde{B}_{1,2}$, after renumbering if necessary. Let γ be a simple path in $\tilde{B}_{1,2}$ linking $k_2 \in \tilde{A}_{1,2}$ to $k_3 \in \tilde{A}_{1,3}$. Then, $\{2, k_2\} \cup \gamma \cup \{k_3, 2\}$ induces a cycle C in G_1 as represented in Figure 31. By Condition 5, C is

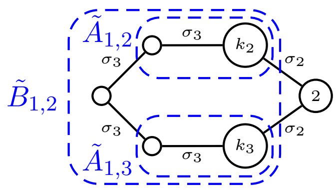


Figure 31: Cycle C.

complete in G. Then, $\{k_2, k_3\} \in E(A) \cap E_3$ and links $\tilde{A}_{1,2}$ to $\tilde{A}_{1,3}$ in $\tilde{G}_3$, a contradiction. $\square$

The partition of A_1 (resp. B_1) in $\tilde{G}_3$ is $\mathcal{P}_M(A_1) = \{\{2\}, \tilde{A}_{1,2}, \ldots, \tilde{A}_{1,r}\}$ (resp. $\mathcal{P}_M(B_1) = \{\{2\}, \tilde{B}_{1,2}, \ldots, \tilde{B}_{1,s}\}$). Claim B implies $\mathcal{P}_M(B_1)_{|A_1} = \mathcal{P}_M(A_1)$. Then, as (N, v) is convex, Lemma 5 implies

$$v(B_1) - \sum_{j=1}^{s} v(\tilde{B}_{1,j}) \geq v(A_1) - \sum_{j=1}^{r} v(\tilde{A}_{1,j}). \tag{23}$$

As $e_1 \notin E(A)$ (resp. $e_1 \notin E(B)$) we have $\overline{v}(A) = v^M(A)$ (resp. $\overline{v}(B) = v^M(B)$) where (N, v^M) is associated with $\tilde{G}_3$. We get $\overline{v}(A) = \sum_{j=1}^{r} v(\tilde{A}_{1,j})$ (resp. $\overline{v}(B) = \sum_{j=1}^{s} v(\tilde{B}_{1,j})$). As $e_1 \in E(A \cup \{i\})$ (resp. $e_1 \in E(B \cup \{i\})$), we have $\overline{v}(A \cup \{i\}) = v^M(A \cup \{i\})$ (resp. $\overline{v}(B \cup \{i\}) = v^M(B \cup \{i\})$) where (N, v^M) is now associated with G_1.

Case 5.1 Let us first assume i is not linked to A in G_1. Then, $\overline{v}(A \cup \{i\}) = v(\{i\}) + \sum_{j=1}^{p} v(A_j) = v(A_1)$. If i is not linked to B in G_1, then we also have $\overline{v}(B \cup \{i\}) = v(B_1)$ and (19) is satisfied as it is equivalent to (23). If i is linked to B in G_1, let $\hat{B}'$ be the set of connected components of B linked to i in G_1. We have either $\hat{B}' = \{B_1, \ldots, B_{q'}\}$ with $1 \leq q' \leq q$ or $\hat{B}' = \{B_2, \ldots, B_{q'}\}$ with $2 \leq q' \leq q$, after renumbering if necessary and therefore, either $\overline{v}(B \cup \{i\}) - \overline{v}(B) = v(\bigcup_{j=1}^{q'} B_j \cup \{i\}) - \sum_{j=1}^{s} v(\tilde{B}_{1,j})$ or $\overline{v}(B \cup \{i\}) - \overline{v}(B) = v(B_1) + v(\bigcup_{j=2}^{q'} B_j \cup \{i\}) - \sum_{j=1}^{s} v(\tilde{B}_{1,j})$. As (N, v) is superadditive, we have in any case

$$\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq v(B_1) + v\left(\bigcup_{j=2}^{q'} B_j \cup \{i\}\right) - \sum_{j=1}^{s} v(\tilde{B}_{1,j}). \tag{24}$$

As (N, v) is superadditive and zero-normalized, we have $v\left(\bigcup_{j=2}^{q'} B_j \cup \{i\}\right) \geq 0$. Then, (24) and (23) imply (19).

Case 5.2 Let us now assume i is linked to A in G_1. Let $\hat{A}'$ be the set of connected components of A linked to i in G_1. We have either $\hat{A}' = \{A_1, \ldots, A_{p'}\}$ with $1 \leq p' \leq p$ or $\hat{A}' = \{A_2, \ldots, A_{p'}\}$ with $2 \leq p' \leq p$, after renumbering if necessary.

Claim C. *There is at most one element of $\hat{A}'$ included in B_1.*

Proof of Claim C. Let us assume w.l.o.g. $A_2 \subseteq B_1$ and $A_3 \subseteq B_1$. Let $\tilde{e}_2 = \{i, k_2\}$ (resp. $\tilde{e}_3 = \{i, k_3\}$) be an edge linking i to A_2 (resp. A_3) in G_1. As B_1 is connected there is a path P connecting k_2 to k_3. Then, $\tilde{e}_2$, $\tilde{e}_3$, and P induce a cycle C in G_1. By Condition 5, C is complete in G. Then, $\{k_2, k_3\}$ links A_2 and A_3 in G_1, a contradiction. $\square$

Let us first assume $A_1 \in \hat{A}'$. Then, $B_1 \in \hat{B}'$, and therefore (19) is equivalent to

$$v\left(\bigcup_{j=1}^{q'} B_j \cup \{i\}\right) - \sum_{j=1}^{s} v(\tilde{B}_{1,j}) \geq v\left(\bigcup_{j=1}^{p'} A_j \cup \{i\}\right) - \sum_{j=1}^{r} v(\tilde{A}_{1,j}). \tag{25}$$

The partition of $\bigcup_{j=1}^{p'} A_j$ (resp. $\bigcup_{j=1}^{q'} B_j$) in $\tilde{G}_3$ is $\mathcal{P}_M(\bigcup_{j=1}^{p'} A_j) = \{\{2\}, \tilde{A}_{1,2}, \ldots, \tilde{A}_{1,r}, A_2, \ldots, A_{p'}\}$ (resp. $\mathcal{P}_M(\bigcup_{j=1}^{q'} B_j) = \{\{2\}, \tilde{B}_{1,2}, \ldots, \tilde{B}_{1,s}, B_2, \ldots, B_{q'}\}$). By Claim B, we have $\tilde{A}_{1,j} \subseteq \tilde{B}_{1,j}$ for all j, $1 \leq j \leq r$. As $A_1 \subseteq B_1$, Claim C implies $A_j \not\subseteq \tilde{B}_{1,k}$ for all j, $2 \leq j \leq p'$, and all k, $2 \leq k \leq s$. Therefore, we have $\mathcal{P}_M(\bigcup_{j=1}^{q'} B_j)_{|\bigcup_{j=1}^{p'} A_j} = \mathcal{P}_M(\bigcup_{j=1}^{p'} A_j)$, and as (N, v) is convex Lemma 5 implies

$$v\left(\bigcup_{j=1}^{q'} B_j\right) - \sum_{j=1}^{s} v(\tilde{B}_{1,j}) \geq v\left(\bigcup_{j=1}^{p'} A_j\right) - \sum_{j=1}^{r} v(\tilde{A}_{1,j}). \tag{26}$$

Moreover, the convexity of (N, v) implies

$$v\left(\bigcup_{j=1}^{q'} B_j \cup \{i\}\right) - v\left(\bigcup_{j=1}^{q'} B_j\right) \geq v\left(\bigcup_{j=1}^{p'} A_j \cup \{i\}\right) - v\left(\bigcup_{j=1}^{p'} A_j\right). \tag{27}$$

(26) and (27) imply (25). Let us now assume $A_1 \notin \hat{A}'$. If $B_1 \notin \hat{B}'$, then (19) is equivalent to

$$v(B_1)+v\left(\bigcup_{j=2}^{q'} B_j \cup \{i\}\right)-\sum_{j=1}^{s} v(\tilde{B}_{1,j}) \geq v(A_1)+v\left(\bigcup_{j=2}^{p'} A_j \cup \{i\}\right)-\sum_{j=1}^{r} v(\tilde{A}_{1,j}). \tag{28}$$

We have $A_j \not\subseteq B_1$ for all j, $2 \leq j \leq p'$ (otherwise B_1 is linked to i, a contradiction). Hence, $\bigcup_{j=2}^{p'} A_j \subseteq \bigcup_{j=2}^{q'} B_j$ and the superadditivity of (N, v) implies $v(\bigcup_{j=2}^{q'} B_j \cup \{i\}) \geq v(\bigcup_{j=2}^{p'} A_j \cup \{i\})$. This last inequality and (23) imply (28). Finally, if $B_1 \in \hat{B}'$, then (19) is equivalent to

$$v\left(\bigcup_{j=1}^{q'} B_j \cup \{i\}\right)-\sum_{j=1}^{s} v(\tilde{B}_{1,j}) \geq v(A_1)+v\left(\bigcup_{j=2}^{p'} A_j \cup \{i\}\right)-\sum_{j=1}^{r} v(\tilde{A}_{1,j}). \tag{29}$$

If $\bigcup_{j=2}^{p'} A_j \subseteq \bigcup_{j=2}^{q'} B_j$, then (28) is satisfied and it implies (29) as (N, v) is superadditive. Otherwise, we can assume w.l.o.g. $A_2 \subseteq B_1$. Claim C implies $\bigcup_{j=3}^{p'} A_j \subseteq \bigcup_{j=2}^{q'} B_j$. Then, we have $(\bigcup_{j=2}^{p'} A_j \cup \{i\}) \cap B_1 = A_2$ and the convexity of (N, v) implies

$$v\left(\bigcup_{j=2}^{p'} A_j \cup \{i\} \cup B_1\right) + v(A_2) \geq v\left(\bigcup_{j=2}^{p'} A_j \cup \{i\}\right) + v(B_1). \tag{30}$$

Moreover, we have $\bigcup_{j=2}^{p'} A_j \cup B_1 \subseteq \bigcup_{j=1}^{q'} B_j$ and A_2 is a singleton. As (N, v) is superadditive and zero-normalized, (30) implies $v(\bigcup_{j=1}^{q'} B_j \cup \{i\}) \geq v(\bigcup_{j=2}^{p'} A_j \cup \{i\}) + v(B_1)$, and therefore

$$(31)\quad v\left(\bigcup_{j=1}^{q'} B_j \cup \{i\}\right) - \sum_{j=1}^{s} v(\tilde{B}_{1,j}) \geq v\left(\bigcup_{j=2}^{p'} A_j \cup \{i\}\right) + v(B_1) - \sum_{j=1}^{s} v(\tilde{B}_{1,j}).$$

Finally, (31) and (23) imply (29).

We now prove that Conditions 1 to 5 imply Condition 6. By Conditions 1 to 3, e_1 is the unique edge in E_1, all edges in E_2 are incident to 2, and all edges in E_3 are linked to 2 by e_1 or by an edge in E_2. By Condition 5, C_m is complete in G_1, so that e_1 cannot be a chord of C_m. Moreover, as $1 \in V(C_m)$ and $e_1 \notin \hat{E}(C_m)$, we have $2 \notin V(C_m)$. Then, C_m has constant weight σ_3 and any chord of C_m has weight σ_3. If there exist two edges $\{2, i\}$ and $\{2, j\}$ with i and j in $V(C_m) \setminus \{1\}$ as represented in Figure 32a with $m = 5$, then e_1 is a common edge for the triangles defined by $\{1, 2, i\}$ and $\{1, 2, j\}$, contradicting Condition 4. Hence, there is at most one edge linking 2 to $V(C_m) \setminus \{1\}$. This implies $m = 3$ and there is exactly one edge linking 2 to $V(C_m) \setminus \{1\}$ as represented in Figure 32b (otherwise there is at least one edge in $\hat{E}(C_m)$ neither incident to 1 nor linked to 2 by an edge in E_2). Finally, let us assume

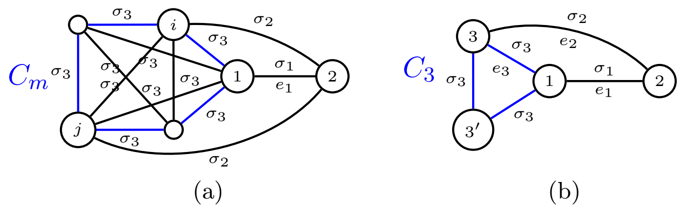


Figure 32: C_5 and C_3 linked to 2 by an edge in E_2.

that there is a second triangle C'_3 with $1 \in V(C'_3)$ and $e_1 \notin E(C'_3)$. If C'_3 is adjacent to C_3, then C_3 and C'_3 induce a cycle of size 4 incident to 1 and not containing e_1, a contradiction. Otherwise, there exists $e \in E_2$ (resp. $e' \in E_2$) linking 2 to $V(C_3) \setminus \{1\}$ (resp. $V(C'_3) \setminus \{1\}$). Then, there are two triangles containing e_1, contradicting Condition 4. $\square$

Remark 4. Let us note that Conditions 1 and 2 of Theorem 20, Conditions 1 and 2 of Theorem 21, and Conditions 1 to 5 of Theorem 22 are consequences of the necessary conditions established in Section 4. To obtain these last conditions we only needed to assume inheritance of convexity with $\mathcal{P}_{\min}$ for the family of unanimity games. Therefore, Theorems 20, 21, and 22 imply that for the correspondence $\mathcal{P}_{\min}$ there is inheritance of convexity if and only if there is inheritance of convexity for the family of unanimity games. This result was already observed in (Skoda, 2017b).

5.2 Disconnected graphs

We now consider the case of disconnected weighted graphs. A connected component is said to be constant if all its edges have the same weight. We prove that if there is inheritance of convexity for $\mathcal{P}_{\min}$, then the underlying graph G has to be connected or has only one component with non-constant weight.

Proposition 23. *Let $G = (N, E, w)$ be a weighted graph. Let us assume that for all $\emptyset \neq S \subseteq N$ the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is convex.*

1. *If G has a connected component with three different weights $\sigma_1 < \sigma_2 < \sigma_3$, then all other connected components of G are singletons.*

2. *Suppose that G has a connected component with two different weights $\sigma_1 < \sigma_2$ and that the component has two distinct edges e_1 and e_2 with $w_1 = w_2 = \sigma_1$. Then all other connected components of G are singletons.*

3. *If G has a connected component with two different weights $\sigma_1 < \sigma_2$ and if $|E_1| = 1$, then all other connected components of G have constant weight σ_2 or are singletons.*

To prove Proposition 23 we need the following lemma.

Lemma 24. *Let us assume that for all $\emptyset \neq S \subseteq N$ the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is convex. Let $e_1 = \{1, 2\}$ and $e_2 = \{2, 3\}$ be two adjacent edges with $w_1 < w_2$ and e be an edge non-incident to 1. Then, we have $w(e) \geq w_2$ and if moreover e is not linked to 2 by an edge, then $w(e) = w_2$.*

Proof. We set $e = \{j, k\}$. If e is incident to 2, Star condition implies $w(e) = w_2$. If e is incident to 3, Path condition implies $w(e) \geq w_2$. Hence, we can assume e non-incident to the vertices 1, 2, and 3. By contradiction, let us assume $w(e) < w_2$. Let us consider $i = 1$ and the subsets $A = \{2, 3\}$ and $B = A \cup \{j, k\}$, as represented in Figure 33. As $w(e) < w_2$, there is a block

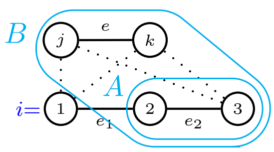


Figure 33: $w_1 < w_2$ and $w(e) < w_2$.

B' of $\mathcal{P}_{\min}(B)$ such that $A \subseteq B'$. As $w_1 < w_2$, we have $\sigma(A \cup \{i\}) < w_2$ and $\mathcal{P}_{\min}(A \cup \{i\}) = \{A \cup \{i\}\}$ or $\{A, \{i\}\}$. Then, taking $A' = A$ we get $\mathcal{P}_{\min}(B)_{|A'} = \{A\} \neq \{\{2\}, \{3\}\} = \mathcal{P}_{\min}(A)_{|A'}$ and it contradicts Theorem 3 applied with $\mathcal{F} = 2^N \setminus \{\emptyset\}$. If e is not linked to 2 by an edge, Lemma 10 implies $w(e) \leq \max(w_1, w_2) = w_2$ and therefore $w(e) = w_2$. $\square$

We can now prove Proposition 23.

Proof of Proposition 23. 1. Let us consider edges e_1, e_2, e_3 in the same connected component of weights $\sigma_1 < \sigma_2 < \sigma_3$. By Theorem 22 we can assume $e_1 = \{1,2\}$, $e_2 = \{2,3\}$, and e_3 incident to 1 or 3. Let e be an edge that is not connected to e_1. Applying Lemma 24 to the pair of edges $\{e_1, e_2\}$ we get $w(e) = \sigma_2$. If e_3 is incident to 1 (resp. 3), then Lemma 24 applied to $\{e_1, e_3\}$ (resp. $\{e_2, e_3\}$) implies $w(e) = \sigma_3$, a contradiction.

2. Let us now consider edges e_1, e_2, e_3 in the same connected component of weights $w_1 = w_2 = \sigma_1$ and $w_3 = \sigma_2$. By Theorem 20 we can assume $e_1 = \{1,2\}$, $e_2 = \{1,3\}$, and e_3 incident to 3. By contradiction, let e be an edge that is not connected to e_1. Lemma 24 applied to $\{e_2, e_3\}$ implies $w(e) = w_3 = \sigma_2$. Then, by Lemma 10 applied to $\{e_1, e_2\}$ (we have $w(e) = \sigma_2 > \sigma_1 = w_1 = w_2$), e has to be linked to 1, a contradiction.

3. Let $e_1 = \{1,2\}$ be the unique edge with weight σ_1 and let us consider an edge $e_2 = \{2,3\}$ of weight $w_2 = \sigma_2 > \sigma_1$ adjacent to e_1. Let e be an edge that is not connected to e_1. Applying Lemma 24 to $\{e_1, e_2\}$, we get $w(e) = \sigma_2$. □

5.3 Complexity analysis

Using the characterizations previously obtained, we finally investigate the complexity of the following decision problem: "Given a weighted graph $G = (N, E, w)$, is there inheritance of convexity for $\mathcal{P}_{\min}$?". Throughout this section we assume that G is represented by its adjacency matrix $A = (a_{ij})$ defined by

$$a_{ij} = \begin{cases} w_{ij} & \text{if } \{i,j\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

We first show that cycle-completeness of $G_1 = (N, E \setminus E_1)$ can be verified in polynomial time. Although G_1 corresponds to an unweighted subgraph of G, it can be represented by the matrix obtained from A by assigning value 0 to all entries associated with edges in E_1. We recall that a connected graph is *biconnected* if it remains connected after the removal of any vertex and its incident edges. A *biconnected component* of a graph is a maximal biconnected subgraph. We say that a biconnected component is complete if it corresponds to a complete subgraph. Noting that a graph is cycle-complete if and only if all its biconnected components are complete, we can easily check the cycle-completeness of a given graph. Tarjan (1972) proposed a polynomial algorithm based on a depth-first search procedure for finding all the biconnected components of an undirected graph. With the adjacency matrix representation, Tarjan's algorithm would compute all biconnected components of G_1 in $O(n^2)$ time. Then, verifying completeness of a given component only requires to check the entries of the corresponding submatrix. As two biconnected components cannot have any edge in common, we can check if all biconnected components are complete in $O(n^2)$ time. This implies the following result.

Lemma 25. *Cycle-completeness of $G_1 = (N, E \setminus E_1)$ can be verified in $O(n^2)$ time.*

We now consider the remaining conditions on cycles required in Theorems 21 and 22.

Lemma 26. *Let $G = (N, E, w)$ be a connected weighted graph. Let us assume $|E_1| = 1$ and let $e_1 = \{1, 2\}$ be the unique edge in E_1. The existence and uniqueness of a chordless cycle containing e_1 can be verified in $O(n^2)$ time.*

Proof. We can check the existence of a path linking 1 and 2 in $G_1 = (N, E \setminus E_1)$ with a Breadth First Search (BFS) algorithm in $O(n^2)$ time. If it exists, then the BFS algorithm returns a shortest path P and $P \cup \{e_1\}$ corresponds to a chordless cycle. Then, it remains to check that there is no path $P' \neq P$ linking 1 and 2 in G_1 such that $P' \cup \{e_1\}$ corresponds to a chordless cycle. Let us note that if a path $P' \neq P$ linking 1 and 2 in G_1 contains all vertices of P, then $P' \cup \{e_1\}$ cannot correspond to a chordless cycle as at least one edge of P is necessarily a chord of $P' \cup \{e_1\}$. Moreover, if there is a path $P' \neq P$ linking 1 and 2 in G_1 which does not contain all vertices of P, then at least one vertex in $V(P) \setminus \{1, 2\}$ is not an articulation point[2] in G_1. Tarjan's algorithm returns the articulation points (and the biconnected components) of G_1 in $O(n^2)$ time. Then, it is sufficient to check that each vertex in $V(P) \setminus \{1, 2\}$ belongs to the set of articulation points of G_1. □

Lemma 27. *Let $G = (N, E, w)$ be a connected weighted graph. Let us assume $|E_1| = 1$. Let $e_1 = \{1, 2\}$ be the unique edge in E_1, and let $G_1 = (N, E \setminus E_1)$. Let us assume that the following condition is satisfied:*

1. *There exists at most one chordless cycle containing e_1.*

Then the following conditions are equivalent:

2. *For every cycle C in G_1 either C is complete or all vertices of C are linked to the same end-vertex of e_1.*
3. *For every biconnected component $\tilde{C}$ of G_1 with at least three vertices either $\tilde{C}$ is complete or all vertices of $\tilde{C}$ are linked to the same end-vertex of e_1.*

Moreover, these conditions can be verified in $O(n^2)$ time.

Proof. As any cycle belongs to a biconnected component, Condition 3 obviously implies Condition 2. Let us assume Condition 2 satisfied and let $\tilde{C}$ be a non-complete biconnected component in G_1. Then there exist i and j in $V(\tilde{C})$ with $\{i, j\} \notin E \setminus E_1$. Let k be a vertex in $V(\tilde{C}) \setminus \{i, j\}$. As $\tilde{C}$ is a biconnected component, there exists a simple cycle C containing i, j, and k. As $\{i, j\} \notin E \setminus E_1$, Condition 2 implies that i, j, and k are linked to the same

[2] A vertex in a graph is an articulation point if its removal disconnects the graph.

end-vertex of e_1. We can repeat this reasoning for any k in $V(\tilde{C}) \setminus \{i, j\}$. Finally, all vertices in $V(\tilde{C})$ are either all linked to 1 or all linked to 2, otherwise we get a contradiction to Condition 1. Let us now investigate the complexity. By Lemma 26, Condition 1 can be verified in $O(n^2)$ time. Then, Condition 3 (which is equivalent to Condition 2) can be checked in $O(n^2)$ time as follows. We can obtain all biconnected components of G_1 with Tarjan's algorithm in $O(n^2)$ time. Then, for any biconnected component $\tilde{C}$, we check if $a_{1i} \neq 0$ for all i in $V(\tilde{C})$ or $a_{2i} \neq 0$ for all i in $V(\tilde{C})$. If these conditions are not satisfied then we check the completeness of $\tilde{C}$. □

Proposition 28. *Inheritance of convexity for $\mathcal{P}_{\min}$ can be decided in $O(n^2)$ time.*

Proof. Let $\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$, with $k \leq |E|$, be the set of edge-weights in G with $\sigma_1 < \sigma_2 < \ldots < \sigma_k$. We apply the following procedure. We first count in A the number k of edge-weights and the number n_i of occurences of σ_i, i.e., $n_i = |E_i|$, for all i, $1 \leq i \leq k$. If $k > 3$ or if $k = 3$ and $n_1 > 1$, then we stop as there is no inheritance of convexity for $\mathcal{P}_{\min}$ by Proposition 13. Otherwise, we use the characterizations given in Theorems 20, 21, and 22 to solve the decision problem in the remaining cases described below. If a contradiction is found for a given case, then we stop the associated procedure as it implies there is no inheritance of convexity.

Let us assume $k = 2$ and $n_1 \geq 2$. We have to check Conditions 1 and 2b of Theorem 20. Let i^* be the smallest index $i \in \{1, \ldots, n\}$ such that row i of A contains at least two values σ_1. If there are indices $i \neq i^*$ and $j \neq i^*$ with $a_{ij} = \sigma_1$, then we stop as it contradicts Condition 1. If row i^* contains a value σ_2, we stop. If there are indices $i \neq i^*$ and $j \neq i^*$ such that $a_{ij} = \sigma_2$, $a_{i^*j} \neq \sigma_1$ and $a_{ii^*} \neq \sigma_1$, we stop as it still contradicts Condition 1. Otherwise, it only remains to check Condition 2b, i.e., whether G_1 is cycle-complete. By Lemma 25, it can be done in $O(n^2)$ time.

Let us now assume $k = 2$ and $n_1 = 1$. By Lemmas 26 and 27, Conditions 1 and 2 of Theorem 21 can be verified in $O(n^2)$ time.

Let us finally assume $k = 3$. We have to verify Conditions 1 to 5 of Theorem 22. As $n_1 = 1$, Condition 1 is satisfied. Let us assume w.l.o.g. $E_1 = \{e_1\}$ with $e_1 = \{1, 2\}$. Let i^* be the smallest index $i \in \{1, \ldots, n\}$ such that row i of A contains the value σ_2. If $i^* \notin \{1, 2\}$, we stop as it contradicts Condition 2. Otherwise, if there are indices $i > i^*$ and $j > i$ such that $a_{ij} = \sigma_2$, then we stop as it still contradicts Condition 2. If row i^* contains σ_3 or if there are indices $i \in \{3, \ldots, n\}$ and $j \in \{3, \ldots, n\}$ such that $a_{ij} = \sigma_3$, $a_{ii^*} \neq \sigma_2$ and $a_{i^*j} \neq \sigma_2$, we stop as it contradicts Condition 3. Otherwise, it only remains to check Conditions 4 and 5. By Lemmas 25 and 26, these last conditions can be verified in $O(n^2)$ time. □

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