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Optimality of randomized strategies in a Markovian replacement model

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Abstract. We study a replacement system with discrete-time Markovian deterioration and finite state space $\{0, \dots, N\}$. State 0 stands for a new system, and the larger the state the worse the condition of the system with N as the failure state. We impose the condition that the long-term fraction of replacements in state N should not be larger than some fixed number. We prove that a generalized control-limit policy maximizes the expected time between two successive replacements and we explain explicitly how to derive this (randomized) optimal policy. Some numerical examples are given.

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1 Introduction

We consider a model for a system which deteriorates stochastically over time but may be replaced by a new system. The state of the system is an element of the set $I = \{0, 1, \dots, N\}$, 0 being the best and N being the worst state. We assume that the state of the system is detected by inspection at times $n = 0, 1, \dots$ and that a decision to replace the system can be taken immediately after inspection.

The most famous replacement models of this type are those of Derman [4] and Ross [11]. They assume operating cost/rewards which are higher/lower as the system gets worse, constant replacement cost and Markovian deterioration. Optimal replacement strategies for these models are *control-limit policies*, which are policies prescribing replacement if the system state exceeds a particular level. Also in many related models optimal policies are of the control-limit type, e.g. Stadje and Zuckerman [12], Parlar and Perry [9], Perry and

Posner [10] and Bruns [2]. A comprehensive review of replacement models is given in Valdes-Florez [13], Jensen [6] and in the book of Aven and Jensen [1].

The model we are interested in has Markovian deterioration (also known as *Derman's condition*) but there are no cost involved. Rather we want the fraction of replacements when the state of the system is N (the *failure state*) to be not larger than a fixed $\varepsilon_0 \in [0, 1]$. A system which is in this bad state has to be replaced. We will show that there is an optimal bang-bang strategy (a control-limit policy with a randomized threshold). Since this is a control-limit replacement policy with a randomized threshold, it may be interpreted as a generalized control-limit policy.

An example of a system whose status is an element of $\{0, \ldots, N\}$ is a parallel N-component system: the status reveals the number of failed components; the machine functions if at least one component is working. Therefore only status N identifies a failed system. Such a system with parallel components has been dealt with by Nakagawa [8], for example. With his model, which will be mentioned again later, the deterioration is caused by shocks. Since the components are identical and independent, every component fails with a constant probability p. Nakagawa used constant cost c_2 for a replacement and constant cost c_1 $(>c_2)$ for a replacement of a failed system and discovered that a control-limit policy minimizes the long-term average cost.

The remainder of this paper is organized as follows. In Section 2 we introduce some basic notation and terminology. In Section 3 we change the model to a cost model by introducing a cost c for every replacement. In this new model we do not consider the restriction regarding the fraction of replacements taking place in state N. The structure of some strategies minimizing the average cost of this new system will be obtained. Using this result we find two different kinds of strategies optimizing our original model in Section 4. In Section 5 we present formulas to compute the optimal strategies and in Section 6 we present some numerical examples.

2 Preliminaries

Our model deals with a system starting at time 0. X_0 stands for the initial state of the system. For all $n \in \mathbb{N}$ we call the time period $[n-1, n)$ the nth interval.

We let p_{ij} be the probability of deterioration from state i to state $j \geq i$ in one interval. We assume Markovian Deterioration (MD), that is, $\sum_{j=k}^{N} p_{ij}$ is non-decreasing in i for all $k \in I$.

We suppose that $p_{ii} < 1$ (to exclude trivialities) and $p_{0N} > 0$, which by MD implies that $p_{iN} > 0$ so that the failure state N is reached from anywhere in one step with positive probability.

At the end of each interval an inspection takes place after which the manager of the system can choose between two actions: to replace or not to replace. If the state of the system is N it has to be replaced. An admissible (randomized) strategy δ can be represented as a family of random variables $\{\delta^{(n)}(i), i \in I, n \in \mathbb{N}\}$ with $P(\delta^{(n)}(i) \in \{0, 1\}) = 1$ for all $i \in I, n \in \mathbb{N}$ and $\delta^{(n)}(N) = 1$ for every $n \in \mathbb{N}$. Decision $\delta^{(n)}(i) = 1$ stands for replacing a system in time n being in state i and $\delta^{(n)}(i) = 0$ stands for not replacing it. If $\delta^{(n)} = \delta^{(1)}$ holds for every $n \in \mathbb{N}$, we use $\delta(i)$ instead of $\delta^{(n)}(i)$. The space of all admissible strategies is denoted by Π.

We let X_n^- be the state before and X_n be the state after the n-th action.

Obviously, under any strategy δ the processes $(X_n^-)_{n\in\mathbb{N}}$ and $(X_n)_{n\in\mathbb{Z}^+}$ are Markov chains. We let $(q_{ij}^{\delta})_{i,j\in I}$ and $(\tilde{q}_{ij}^{\delta})_{i,j\in I}$ be the transition probabilities, and $(\pi_\delta(i))_{i\in I}$ and $(\tilde{\pi}_\delta(i))_{i\in I}$ be the stationary distributions of the stochastic processes $(X_n)_{n\in\mathbb{N}}$ and $(X_n^-)_{n\in\mathbb{N}}$ under strategy δ, respectively. There exists exactly one distribution $(\pi_\delta(i))_{i\in I}$ since $p_{ii} < 1$, $p_{ij} = 0$ for $j < i$ and $\delta^{(n)}(N) = N$ for every $n \in \mathbb{N}$ leads to the fact that 0 is positive recurrent in $(X_n)_{n\in\mathbb{N}}$ and I is irreducible. Because of the identity

$$\tilde{\pi}_\delta(j) = \sum_{i=0}^{j} \pi_\delta(i) p_{ij} \quad i, j \in I$$

there exists exactly one distribution $(\tilde{\pi}_\delta(i))_{i\in I}$, too. We observe that in a stationary setting

$$\begin{aligned}\pi_\delta(0) = P_\delta(X_n = 0) &= P_\delta(X_n = 0, X_n^- = 0) + P_\delta(X_n = 0, X_n^- \neq 0)\\ &= P_\delta(X_n^- = 0, X_{n-1} = 0) + P_\delta(X_n = 0, X_n^- \neq 0)\\ &= \pi_\delta(0) = \pi_\delta(0)p_{00} + P_\delta(X_n = 0, X_n^- \neq 0),\end{aligned}$$

so that

$$P_\delta(X_n = 0, X_n^- \neq 0) = \pi_\delta(0)(1 - p_{00}). \tag{1}$$

We will show later that both processes are ergodic. Hence, the expected cycle length, that is the time between two replacements, under δ is $\frac{1}{\pi_\delta(0)(1-p_{00})}$, since $\pi_\delta(0)(1 - p_{00})$ is the expected relative frequency of state zero occurring under strategy δ, excluding direct visits from state zero (in this case there was no replacement).

From $\{X_n^- = N\} \subset \{X_n = 0, X_n^- \neq 0\}$ and (1) we get

$$P_\delta(X_n^- = N \mid X_n = 0, X_n^- \neq 0) = \frac{P_\delta(X_n^- = N)}{P_\delta(X_n = 0, X_n^- \neq 0)} = \frac{\tilde{\pi}_\delta(N)}{\pi_\delta(0)(1 - p_{00})}. \tag{2}$$

As a consequence, we can phrase our problem in the following way. We look for a strategy $\delta \in \Pi$ which minimizes $\pi_\delta(0)(1 - p_{00})$ subject to the condition

$$\frac{\tilde{\pi}_\delta(N)}{\pi_\delta(0)(1 - p_{00})} \leq \varepsilon_0, \tag{3}$$

since from (2) we know that the latter fraction equals the probability that a replacement at time n (that is, the event $\{X_n = 0, X_n^- \neq 0\}$) was caused by a failure (that is, the event $\{X_n^- = N\}$).

We let

$$\varepsilon_1 := \varepsilon_0(1 - p_{00}).$$

In the next Theorem we give a condition for a strategy satisfying (3) to exist.

Theorem 1. *There exists a strategy δ with $\frac{\tilde{\pi}_\delta(N)}{\pi_\delta(0)} \leq \varepsilon_1$ if and only if $p_{0N} \leq \varepsilon_1$.*

Proof: We note that

$$\tilde{\pi}_\delta(N) = \sum_{i=0}^{N} \pi_\delta(i) p_{iN} = \pi_\delta(0) p_{0N} + \sum_{i=1}^{N} \pi_\delta(i) p_{iN}. \tag{4}$$

The last summand is non-negative and vanishes if the strategy chosen is the control-limit policy with threshold one, defined as δ_1. Thus $p_{0N} = \frac{\tilde{\pi}_{\delta_1}(N)}{\pi_{\delta_1}(0)} = \min_{\delta \in \Pi} \frac{\tilde{\pi}_\delta(N)}{\pi_\delta(0)}$. ■

Next we define the strategies used in this paper:

Definition 1.

(i) *A control-limit policy with threshold i^*, denoted by δ_{i^*}, is a policy which prescribes replacement in state i if and only if $i \geq i^*$.*
(ii) *A pre-randomized bang-bang strategy with parameter $(i^*, p) \in I \times [0, 1]$, denoted by $(i^*, p)_{pre}$, is equal to the control-limit policy δ_{i^*} with probability $1 - p$ and to δ_{i^*+1} with probability p. Hence, $P(\delta^{(1)}(i^*) = i^*) = p$, $\delta^{(n)} = \delta^{(1)}$ for every $n \in \mathbb{N}$.*
(iii) *A post-randomized bang-bang strategy with parameter $(i^*, p) \in I \times [0, 1]$, denoted by $(i^*, p)_{post}$, is the strategy which prescribes for every machine seperately to choose the control-limit policy with threshold i^* or that with threshold $i^* + 1$ with probabilities $1 - p$ and p, respectively. Hence, $P(\delta^{(n)}(i^*) = i^*) = 1 - p$ for every $\in \mathbb{N}$ (independent on $(\delta^{(1)}, \ldots, \delta^{(n-1)})$.*

Using a pre-randomized strategy means making one decision before starting the process; using a post-randomized strategy means making a new decision for every machine. Obviously with the use of any bang-bang strategy δ (pre-, post-randomized) for the process $(X_n)_{n \in \mathbb{N}}$ only one stationary distribution exists because of the Markovian deterioration and the condition $p_{ii} < 1$ for every state i: if $\delta \in \Pi$ meaning the threshold i^* is less than or equal to N state 0 is reachable from every other state. In the opposite case $(i^* > N)$ this is valid for state N. Of course for the process $(X_n^-)_{n \in \mathbb{N}}$ there is only one stationary distribution, too. In the next Lemma some properties of the stationary probabilities are given. We write $\pi_{i^*}(i)$ instead of $\pi_{\delta_{i^*}}(i)$.

Lemma 1.

(i) $\pi_{i^*}(0)$ *is non-increasing in i^*.*
(ii) $\tilde{\pi}_{i^*}(N)$ *is non-decreasing in i^*.*
(iii) *If $\tilde{\pi}_{i^*}(N) = \tilde{\pi}_{i^*+1}(N)$ then $\pi_{i^*}(i) = \pi_{i^*+1}(i)$ or $p_{i,N} = p_{i+1,N} = \cdots = p_{i^*,N}$ holds for all $i \in \{0, \ldots, i^* - 1\}$.*
(iv) $\pi_{(i^*,p)_{pre}}(0)$ *and* $\pi_{(i^*,p)_{post}}(0)$ *are non-increasing in p on $[0, 1]$ for every threshold i^*.*
(v) $\pi_{(i^*,p)_{pre}}(i)$ *and* $\pi_{(i^*,p)_{post}}(i)$ *are continuous in p on $[0, 1]$ for every threshold i^* and for every state i.*
(vi) $\tilde{\pi}_{(i^*,p)_{pre}}(i)$ *and* $\tilde{\pi}_{(i^*,p)_{post}}(i)$ *are continuous in p on $[0, 1]$ for every threshold i^* and for every state i.*

This lemma will pe proven at the Appendix.

3 A cost model

In this section we consider the model of the previous section without the restriction on the percentage of replacements in the bad state N. We introduce the following cost function $d^{(c)}$ for $c \in \mathbb{R}^+$:

$$d^{(c)}(i,0) = 0, \quad i \in I, \quad d^{(c)}(i,1) = 1, \quad i \in \{1,\dots,N-1\} \quad \text{and}$$
$$d^{(c)}(N,1) = 1 + c. \tag{5}$$

The first component is the state before repair and the second one represents the action that is chosen at this replacement model. Hence we take the cost function used by Nakagawa [8] with $c_2 = 1$ and $c_1 = 1 + c$. His model was already described in the introduction. The value c may be interpreted as a penalty cost for being in the bad state N. Derman [4], p. 125, has shown that the unique strategy optimizing this average cost replacement model is a control-limit policy. Next we look at the average cost in this cost model:

With $\phi_\delta(c)$ being the average cost function of this cost replacement problem under strategy δ and cost function $d^{(c)}$, we have

$$\phi_{i^*}(c) = \pi_{i^*}(0)(1-p_{00}) + c\tilde{\pi}_{i^*}(N) = \pi_{i^*}(0)\left(1 - p_{00} + c\frac{\tilde{\pi}_{i^*}(N)}{\pi_{i^*}(0)}\right). \tag{6}$$

Hence, ϕ_{i^*} is obviously continuous on $\mathbb{R}^+$. (7)

The goal of this section is to find out how the strategy optimizing this cost model depends on the constant c. This result is given in Theorem 2 and helps us finding a strategy optimizing the original model in the next section. For the proof of Theorem 2 we need the next three lemmas:

Lemma 2. *Let* $\alpha := p_{0N} > 0$, *then*

$$\phi_\delta(c) = \alpha(d^{(c)}(N,1) + \tilde{V}^{(c)}_{\delta,1-\alpha}(0))), \tag{8}$$

where $\tilde{V}^{(c)}_{\delta,1-\alpha}$ *is the mean* $(1-\alpha)$ *discounted cost function of the cost model with functions*

$$\tilde{d}^{(c)} = d^{(c)} \cdot (1-\alpha), \quad \tilde{p}_{ij} = \frac{p_{ij} - \alpha 1_{\{N\}}(j)}{1-\alpha} \in [0,1).$$

Proof: A simple computation (e.g. Hernandez-Lerma, Lasserre [5], formula (4.2.15)) leads to

$$\begin{aligned}
\tilde{V}^{(c)}_{\delta,1-\alpha}(i) &= \sum_{j=i}^{N} \tilde{p}_{ij}(\tilde{d}^{(c)}(j,\delta(j)) + (1-\alpha)\tilde{V}^{(c)}_{\delta,1-\alpha}(j(1-\delta(j)))) \\
&= \sum_{j=i}^{N} \frac{p_{ij}}{1-\alpha}((1-\alpha)d^{(c)}(j,\delta(j)) + (1-\alpha)\tilde{V}^{(c)}_{\delta,1-\alpha}(j(1-\delta(j)))) \\
&\quad - \frac{\alpha}{1-\alpha}((1-\alpha)d^{(c)}(N,1) + (1-\alpha)\tilde{V}_{\delta,1-\alpha}(0)).
\end{aligned} \tag{9}$$

Thus

$$\tilde{V}^{(c)}_{\delta,1-\alpha}(i) + \alpha(d^{(c)}(N,N) + \tilde{V}^{(c)}_{\delta,1-\alpha}(0))$$
$$= \sum_{j=i}^{N} p_{ij}(d^{(c)}(j,\delta(j)) + \tilde{V}^{(c)}_{\delta,1-\alpha}(j(1-\delta(j)))\}. \tag{10}$$

Using $g^{(c)} := \alpha(d^{(c)}(N,N) + \tilde{V}^{(c)}_{\delta,1-\alpha}(0))$ and $h^{(c)} := \tilde{V}^{(c)}_{\delta,1-\alpha}$ (bounded because I is finite) we get $h^{(c)}(i) + g^{(c)} = \sum_{j=i}^{N} p_{ij}(d(j,\delta(j)) + h^{(c)}(j(1-\delta(j)))$. Hence $g^{(c)}$ equals $\phi_\delta(c)$ (e.g. Ross [11], p. 93). ■

Lemma 3. *Let* $0 \leq i^* < j^*$ *and* $\phi_{i^*}(c_0) = \phi_{j^*}(c_0)$ *for some* $c_0 \in \mathbb{R}^+$. *Then we have for* $c \in \mathbb{R}^+$*:*

$$\phi_{i^*}(c) \leq \phi_{j^*}(c) \Leftrightarrow c \geq c_0. \tag{11}$$

Proof: We have

$$\sum_{i=i^*}^{N} \tilde{\pi}_{i^*}(i) + c_0\tilde{\pi}_{i^*}(N) = \phi_{i^*}(c_0) = \phi_{j^*}(c_0) = \sum_{i=j^*}^{N} \tilde{\pi}_{j^*}(i) + c_0\tilde{\pi}_{j^*}(N).$$

Lemma 2 yields $\tilde{\pi}_{j^*}(N) \geq \tilde{\pi}_{i^*}(N)$. So if c_0 becomes larger (smaller), $\phi_{j^*}(c_0)$ will become smaller (larger) than or equal to $\phi_{i^*}(c_0)$. Hence this Lemma is also proven. ■

Lemma 4. *Let the cost c be fixed. If the control-limit policies with thresholds* i^* *and* j^* $(>i^*)$ *optimize the average cost, every control-limit policy with threshold* $i \in \mathbb{Z}^+$, $i^* < i < j^*$, *is optimal, too.*

This lemma will be proven at the Appendix.

Theorem 2. *There is a* j_0 *in* I *and there are positive real numbers* $c_{j_0} \geq c_{j_0+1} \geq \cdots \geq c_N$ *such that the control-limit policy* δ_{i^*} *with threshold*

$$i^* = \begin{cases} N & \text{if } 0 \leq c < c_N, \\ j & \text{if } c_{j+1} \leq c < c_j, \\ j_0 & \text{if } c \geq c_{j_0}, \end{cases}$$

is optimal. If $p_{0n} < p_{1N}$ *one can take* $j_0 = 1$.

Proof: First consider the case $c = 0$. Using the control-limit policy with threshold i^* the average cost are $\pi_{i^*}(0)(1-p_{00})$. These average cost decrease if i^* increases, so the control-limit policy with threshold N is optimal. In general, there are two possibilities:

Case a: The control-limit policy with threshold N, δ_N, is optimal for every $c \in \mathbb{R}^+$.

Case b: Suppose δ_N is not optimal for every $c > 0$, that is for some $c > 0$ and some state $i \neq N$, $\phi_N(c) > \phi_i(c)$. Then, let

$$\tilde{c} = \inf\{c > 0 \mid \phi_N(c) > \phi_i(c) \text{ for some } i \in I\}$$

and

$$i_0 = \min\{i \in I \mid \phi_N(\tilde{c}) > \phi_i(\tilde{c})\}.$$

$\phi_{i_0}(\tilde{c}) = \phi_N(\tilde{c})$ holds because of the continuity of the function $\phi_{i_0} - \phi_N$ (recall (6)) and $\phi_{i_0}(c) \geq \phi_N(c)$ for $c > \tilde{c}$. For all i with $i_0 < i \leq N$ let $c_i := \tilde{c}$. We remark that if $c = d_0$, every control-limit policy with a threshold $i \in \{i_0, \dots, N\}$ is optimal because of Lemma 4. For every c less than $\tilde{c}$ we have found an optimal strategy. For values of c which are larger then $\tilde{c}$ we use the following recursive procedure:

Case b(i): The control-limit policy with threshold i_0, δ_{i_0}, is optimal for every $c \in [\tilde{c}, \infty)$.

We choose $j_0 := \tilde{i}$ and $c_{\tilde{i}} = \tilde{c}$.

Case b(ii): Suppose δ_{i_0} is not optimal for every $c \in [\tilde{c}, \infty)$, that is, for some $c \in [\tilde{c}, \infty)$ and for some state $i \in \{0, \dots, i_0 - 1\}$, $\phi_i(c) < \phi_{i_0}(c)$. Then, let

$$\tilde{\tilde{c}} = \inf\{c \in [\tilde{c}, \infty) \mid \phi_{i_0}(c) > \phi_i(c) \text{ for some } i \in I\}$$

and

$$i_1 = \min\{i \in I \mid \phi_{i_0}(\tilde{\tilde{c}}) > \phi_i(\tilde{\tilde{c}})\}.$$

Now for all i with $i_1 < i \leq i_0$ let $c_i = \tilde{\tilde{c}}$. For every c less that $\tilde{\tilde{c}}$ we found the optimal strategy. Using i_1 and $\tilde{\tilde{c}}$ instead of i_0 and $\tilde{c}$ we repeat this procedure again. Recursively, we obtain a sequence $(c_i)_{i=j_0}^{N}$, where j_0 is state i^*, for which case a or b(ii) is valid for the first time. This will be the case at $j_0 = 1$ at the latest.

We prove the second part of this Theorem indirectly. Assume that $p_{0N} < p_{1N}$ and that $j_0 = 1$ can not be chosen to be 1. We recall the following identity which is valid for every threshold i^*:

$$\phi_{i^*}(c) = \pi_{i^*}(0)(1 - p_{00}) + \tilde{\pi}_{i^*}(N) \cdot c.$$

Case I: $\tilde{\pi}_{\delta_1}(N) < \tilde{\pi}_{\delta_{j_0}}(N)$. We get

$$\phi_{j_0}(c) - \phi_1(c) = (1 - p_{00})(\pi_{\delta_{j_0}}(0) - \pi_{\delta_1}(0)) + c(\tilde{\pi}_{\delta_{j_0}}(N) - \tilde{\pi}_{\delta_1}(N)). \tag{12}$$

The first summand is nonpositive. Since the factor of c is positive, there is a $c_0 \in \mathbb{R}$ so that the above term is positive for all values $c > c_0$. For these values c the control-limit strategy δ_1 is better than δ_{j_0}. This contradiction yields $j_0 = 1$.

Case II: $\tilde{\pi}_{\delta_1}(N) = \tilde{\pi}_{\delta_{j_0}}(N)$.

Lemma 1(ii) yields to $\tilde{\pi}_{\delta_1}(N) = \cdots = \tilde{\pi}_{\delta_{j_0}}(N)$ and Lemma 1(iii) to $\pi_{\delta_1}(0) = \cdots = \pi_{\delta_{j_0}}(0)$.

Thus we have $\phi_{j_0}(c) = \phi_1(c)$ for every $c \in \mathbb{R}^+$. So besides the control-limit policy with threshold j_0 the control-limit policy with threshold 1 is optimal, too. Again we obtain a contradiction. ■

If $p_{0N} \neq p_{1N}$, then $\pi_{\delta_1}(0) = \cdots = \pi_{\delta_{j_0}}(0)$ by Lemma 1(iii). If on the other hand the identity $p_{0N} = p_{1N} = \cdots = p_{j_0-1,N}$ holds, we have for every $j \in \{1, \dots, j_0\}$

$$\tilde{\pi}_{\delta_j}(N) = \sum_{i=0}^{j-1} \pi_{\delta_j}(i) p_{iN} = p_{0N} \sum_{i=0}^{j-1} \pi_{\delta_j}(i) = p_{0N}, \quad \text{so } \tilde{\pi}_{\delta_1}(N) = \cdots = \tilde{\pi}_{\delta_{j_0}}(N).$$

Now we show that the condition $p_{0n} < p_{1n}$ is necessary to get $j_0 = 1$: under the conditions $p_{0N} = p_{1N}$ and $p_{01} > 0$ the strategy δ_2 is better than strategy δ_1 for every penalty cost $c \in \mathbb{R}^+$, as we now compute:

$$\pi_{\delta_2}(0) = 1 - \pi_{\delta_2}(1) \le 1 - \pi_{\delta_2}(0) p_{01}, \quad \text{so } \pi_{\delta_2}(0) \le \frac{1}{1 + p_{01}} < 1 = \pi_{\delta_1}(0),$$

$$\begin{aligned} \tilde{\pi}_{\delta_2}(N) &= \pi_{\delta_2}(0) p_{0N} + \pi_{\delta_2}(1) p_{1N} \\ &= p_{0N}(\pi_{\delta_2}(0) + \pi_{\delta_2}(1)) = p_{0N} = \pi_{\delta_1}(0) p_{0N} = \tilde{\pi}_{\delta_1}(N), \end{aligned}$$

since $p_{0N} = p_{1N}$. Hence,

$$\phi_{\delta_2}(c) = \pi_{\delta_2}(0) + c\tilde{\pi}_{\delta_2}(N) < \pi_{\delta_1}(0) + c\tilde{\pi}_{\delta_2}(N) = \phi_{\delta_1}(c).$$

Under these conditions the probability of visiting state N is the same for the initial states 0 and 1, so the probability that a machine will reach the worst state N is the same under both strategies. But the number of replacements in every time interval $[0, T], T \in \mathbb{N}$ will be smaller under δ_2 than under δ_1. In the sequel we need the second part of Theorem 2, so that we assume

Probability condition: $0 > p_{0N} \neq p_{1N}$ and $p_{ii} > 0$ for every state $i \in I$.

which holds for the subsequent results. The second part of that condition was just repeated.

4 Optimal strategies in the original model

Now let us return to the original model. Using Theorem 2 of Section 3 we first prove that the search for an optimal strategy may be restricted to the class of pre-randomized strategies or to the class of post-randomized strategies. The existence and the construction of the optimal strategy will be studied in the next section.

Theorem 3. *For every strategy δ satisfying $\frac{\tilde{\pi}_\delta(N)}{\pi_\delta(0)} \le \varepsilon_1$, there are numbers $j \in I$ and $\lambda \in [0, 1]$ with*

$$\frac{\tilde{\pi}_{(j,\lambda)_{pre}}(N)}{\pi_{(j,\lambda)_{pre}}(0)} = \frac{\lambda\tilde{\pi}_{\delta_j}(N) + (1-\lambda)\tilde{\pi}_{\delta_{j+1}}(N)}{\lambda\pi_{\delta_j}(0) + (1-\lambda)\pi_{\delta_{j+1}}(0)} \le \varepsilon_1 \tag{13}$$

and

$$\pi_{(j,\lambda)_{pre}}(0) = \lambda\pi_{\delta_j}(0) + (1-\lambda)\pi_{\delta_{j+1}}(0) = \pi_\delta(0).$$

Proof: Lemma 1(i) yields to

$$\pi_{\delta_N}(0) \le \cdots \le \pi_{\delta_1}(0) = 1. \tag{14}$$

Since a system in state N has to be replaced under every strategy, δ_N is the strategy which replaces most rarely. Thus using δ_N the process $(X_n)_{n\in\mathbb{N}}$ visits state N most rarely so this strategy minimizes $\pi_\delta(0)$. So $\pi_\delta(0) \le \pi_{\delta_N}(0)$ for every strategy δ. Lemma 1 yields the same result if we look at the subset of control-limit policies only.

For every strategy δ we have

$$1 = \pi_{\delta_1}(0) \ge \pi_\delta(0) \ge \pi_{\delta_N}(0). \tag{15}$$

(14) yields that for every δ there is a $j \in I$ with

$$\pi_{\delta_{j+1}}(0) \le \pi_\delta(0) \le \pi_{\delta_j}(0). \tag{16}$$

Take $\lambda \in [0,1)$ such that $\pi_\delta(0) = \lambda\pi_{\delta_j}(0) + (1-\lambda)\pi_{\delta_{j+1}}(0)$. (17)

The cost model with $c := c_{j+1}$ defined in Theorem 2 will be optimized by both control-limit policies δ_j and δ_{j+1}. Therefore, every strategy δ satisfies

$$\phi_j(c) = \phi_{j+1}(c) \le \phi_\delta(c). \tag{18}$$

Hence

$$\begin{aligned}\pi_j(0)(1-p_{00}) + c\tilde{\pi}_j(N) &= \pi_{j+1}(0)(1-p_{00}) + c\tilde{\pi}_{j+1}(N)\\ &\le \pi_\delta(0)(1-p_{00}) + c\tilde{\pi}_\delta(N).\end{aligned} \tag{19}$$

Now we find that

$$\pi_{(j,\lambda)_{pre}}(i) = \lambda\pi_j(i) + (1-\lambda)\pi_{j+1}(i) \quad \text{for } i \in I, \tag{20}$$

$$\tilde{\pi}_{(j,\lambda)_{pre}}(i) = \lambda\tilde{\pi}_j(i) + (1-\lambda)\tilde{\pi}_{j+1}(i) \quad \text{for } i \in I. \tag{21}$$

The main idea of this proof stands behind the subsequent inequality.

$$\begin{aligned}&\pi_{(j,\lambda)_{pre}}(0)(1-p_{00}) + c\tilde{\pi}_{(j,\lambda)_{pre}}(N)\\ &\quad= (\lambda\pi_{\delta_j}(0) + (1-\lambda)\pi_{\delta_{j+1}}(0))(1-p_{00}) + c(\lambda\tilde{\pi}_{\delta_j}(N) + (1-\lambda)\tilde{\pi}_{\delta_{j+1}}(N))\\ &\quad= \lambda(\pi_{\delta_j}(0)(1-p_{00}) + c\tilde{\pi}_{\delta_j}(N)) + (1-\lambda)(\pi_{\delta_{j+1}}(0)(1-p_{00}) + c\tilde{\pi}_{\delta_{j+1}}(N))\\ &\quad\le \lambda(\pi_\delta(0)(1-p_{00}) + c\tilde{\pi}_\delta(N)) + (1-\lambda)(\pi_\delta(0)(1-p_{00}) + c\tilde{\pi}_\delta(N))\\ &\quad= \pi_\delta(0)(1-p_{00}) + c\tilde{\pi}_\delta(N).\end{aligned} \tag{22}$$

This, together with

$$\pi_{(j,\lambda)_{pre}}(0) = \lambda\pi_{\delta_j}(0) + (1-\lambda)\pi_{\delta_{j+1}}(0) = \pi_\delta(0)$$

gives us

$$\tilde{\pi}_{(j,\lambda)_{pre}}(N) \leq \tilde{\pi}_\delta(N).$$

Finally

$$\pi_{(j,\lambda)_{pre}}(0) = \pi_\delta(0)$$

yields

$$\frac{\tilde{\pi}_{(j,\lambda)_{pre}}(N)}{\pi_{(j,\lambda)_{pre}}(0)} \leq \frac{\tilde{\pi}_\delta(N)}{\pi_\delta(0)} \leq \varepsilon_1. \quad \blacksquare$$

For the proof of an analogous result regarding to the post-randomized strategies which is formulated in the next Theorem we need an auxiliary Lemma:

Lemma 5. (i) *For every strategy* $(i,p)_{pre}$, $i \in \{1,\dots,N-1\}$, $p \in [0,1]$ *we can choose* $q \in [0,1]$, *such that*

$$\phi_{(i,p)_{pre}}(c) = \phi_{(i,q)_{post}}(c) \quad \textit{for all } c \in \mathbb{R}^+. \tag{23}$$

(ii) *For every strategy* $(i,q)_{post}$, $i \in \{1,\dots,N-1\}$, $q \in [0,1]$ *we can choose*

$$p := \frac{\frac{q}{\pi_{\delta_{i^*}}(0)}}{\frac{q}{\pi_{\delta_{i^*}}(0)} + \frac{1-q}{\pi_{\delta_{i^*+1}}(0)}}$$

such that

$$\phi_{(i,q)_{post}}(c) = \phi_{(i,p)_{pre}}(c) \quad \textit{for all } c \in \mathbb{R}^+ \tag{24}$$

Proof: If $p = 0$ then $\phi_{(i,0)_{pre}}(c) = \phi_{\delta_i}(c) = \phi_{(i,0)_{post}}(c) = \phi_{(i,0)_{post}}(c)$ and if $p = 1$ then $\phi_{(i,1)_{pre}}(c) = \phi_{(i+1,0)_{pre}}(c) = \phi_{(i+1,0)_{post}}(c) = \phi_{(i,1)_{post}}(c) = \phi_{(i,1)_{post}}$, so that 23 holds true. For $p \in (0,1)$ we look at the two assertions separately:

1. $\phi_{(i,p)_{pre}}(c) = p\phi_{i+1}(c) + (1-p)\phi_i(c)$.

Thus for every i_0 and c the graph of the function $g^c_{i_0} : p \to \phi_{(i_0,p)_{pre}}(c)$ is a straight line joining $(0, \phi_{i^*}(c))$ and $(1, \phi_{i^*+1}(c))$.

According to Lemma 1, $\phi_{(i_0,p)_{post}}$ is continuous in p. Since $\phi_{(i,0)_{pre}} \equiv \phi_{(i,0)_{post}}$ and $\phi_{(i,1)_{pre}} \equiv \phi_{(i,1)_{post}}$ hold and g^c_i is linear, there is a $q \in [0,1]$ for which $\phi_{(i_0,p)_{pre}}(c) = \phi_{(i_0,q)_{post}}(c)$.

2. For every $n \in \mathbb{N}$ we define the random variable τ_n as the time at which the n-th visit to state 0 takes place, not counting visits from zero. Let $\tilde{C}^c_\delta(J)$ be the cost during the interval $J \subset \mathbb{Z}^+$ using any strategy δ such that zero is positive recurrent for the process (X_n) and using 'penalty cost' c. Since state zero is positive recurrent, we have

$$\phi_\delta(c) = \lim_{n\to\infty} \frac{E(\tilde{C}^c_\delta([\tau_{n-1},\tau_n]))}{E(\tau_n - \tau_{n-1})} = \frac{E(\tilde{C}^c_\delta([\tau_1,\tau_2]))}{E(\tau_2 - \tau_1)}. \tag{25}$$

Hence,

$$\phi_{(i^*,q)_{post}}(c)$$

$$= \frac{E(\tilde{C}^c_\delta([\tau_1,\tau_2])\,|\,\delta=\delta_{i^*})\cdot P(\delta=\delta_{i^*}) + E(\tilde{C}^c_\delta([\tau_1,\tau_2])\,|\,\delta=\delta_{i^*+1}))\cdot P(\delta=\delta_{i^*+1})}{P(\delta=\delta_{i^*})\cdot E(\tau_2-\tau_1\,|\,\delta=\delta_{i^*}) + P(\delta=\delta_{i^*+1})\cdot E(\tau_2-\tau_1\,|\,\delta=\delta_{i^*+1})}$$

$$= \frac{E(\tilde{C}^c_{\delta_{i^*}}([\tau_1,\tau_2]))\cdot(1-q) + E(\tilde{C}^c_{\delta_{i^*+1}}([\tau_1,\tau_2]))\cdot q}{\frac{1-q}{\pi_{\delta_{i^*}}(0)} + \frac{q}{\pi_{\delta_{i^*+1}}(0)}}$$

$$= E(\tilde{C}^c_{\delta_{i^*}}([\tau_1,\tau_2]))\cdot\pi_{\delta_{i^*}}(0)\cdot\frac{\frac{1-q}{\pi_{\delta_{i^*}}(0)}}{\frac{1-q}{\pi_{\delta_{i^*}}(0)} + \frac{q}{\pi_{\delta_{i^*+1}}(0)}}$$

$$+ E(\tilde{C}^c_{\delta_{i^*+1}}([\tau_1,\tau_2]))\cdot\pi_{\delta_{i^*+1}}(0)\cdot\frac{\frac{q}{\pi_{\delta_{i^*+1}}(0)}}{\frac{1-q}{\pi_{\delta_{i^*}}(0)} + \frac{q}{\pi_{\delta_{i^*+1}}(0)}}$$

$$= \phi_{i^*}(c)\cdot(1-p) + \phi_{i^*+1}(c)\cdot p = \phi_{(i^*,p)_{pre}}. \quad \blacksquare$$

Theorem 4. *For every strategy δ satisfying the inequality $\frac{\tilde{\pi}_\delta(N)}{\pi_\delta(0)} \le \varepsilon_1$ there are numbers $j \in \mathbb{Z}^+$ and $\mu \in [0,1]$ such that:*

$$\frac{\tilde{\pi}_{(j,\mu)_{post}}(N)}{\pi_{(j,\mu)_{post}}(0)} \le \varepsilon_1 \quad \textit{and} \quad \pi_{(j,\mu)_{post}}(0) = \pi_\delta(0).$$

Proof: Due to (16) there exists a number $j \in \mathbb{Z}^+$ with $\pi_{\delta_{j+1}}(0) \le \pi_\delta(0) \le \pi_{\delta_j}(0)$. Because of $\delta_j = \delta_{(j,0)_{post}}$, $\delta_{j+1} = \delta_{(j,1)_{post}}$ and the fact that $\pi_{(j,\mu)_{post}}(0)$ is continuous and non-increasing on the unit interval, there exists a $\mu \in [0,1]$ with $\pi_\delta(0) = \pi_{(j,\mu)_{post}}(0)$. Because of the last Lemma there exists a $\lambda \in [0,1]$, such that we have for $c := c_{j+1}$:

$$\begin{aligned}\pi_{(j,\mu_2)_{post}}(0)(1-p_{00}) + c\tilde{\pi}_{(j,\mu_2)_{post}}(N) &= \phi_{(j,\mu_2)_{post}}(c) \overset{(24)}{=} \phi_{(j,\lambda)_{pre}}(c)\\ &= \pi_{(j,\lambda)_{pre}}(0)(1-p_{00}) + c\tilde{\pi}_{(j,\lambda)_{pre}}(N)\\ &\le \pi_\delta(0)(1-p_{00}) + c\tilde{\pi}_\delta(N). \qquad (26)\end{aligned}$$

To see (26), note that for every $\lambda \in [0,1]$ and for all $i \in I$ we have:

$$\pi_{(j,\lambda)_{pre}}(i) \overset{(20)}{=} \lambda\pi_{\delta_j}(i) + (1-\lambda)\pi_{\delta_{j+1}}(i) \quad \text{and}$$

$$\tilde{\pi}_{(j,\lambda)_{pre}}(i) \overset{(21)}{=} \lambda\tilde{\pi}_{\delta_j}(i) + (1-\lambda)\tilde{\pi}_{\delta_{j+1}}(i).$$

$$\pi_{(j,\mu)_{post}}(0) = \pi_\delta(0) \quad \text{yields} \quad \tilde{\pi}_{(j,\mu)_{post}}(N) \le \tilde{\pi}_\delta(N),$$

$$\text{so } \frac{\tilde{\pi}_{(j,\mu)_{post}}(N)}{\pi_{(j,\mu)_{post}}(0)} \le \frac{\tilde{\pi}_\delta(N)}{\pi_\delta(0)} \le \varepsilon_1. \quad \blacksquare$$

Summarizing, for every strategy in the class $\left\{\delta : \frac{\tilde{\pi}_\delta(N)}{\pi_\delta(0)} \leq \varepsilon_1\right\}$ there exists a pre-randomized bang-bang strategy yielding the same value $\frac{1}{\pi_\delta(0)(1-p_{00})}$ which is to be minimized. Moreover, there exists a post-randomized bang-bang strategy having the same value. Hence the search for an optimal strategy can be restricted to the class of pre-randomized strategies or to the class of post-randomized strategies.

5 Construction of optimal strategies

First we show how to find for an optimal pre-randomized strategy:

From the monotonicity of $\pi_{i^*}(0)$ and $\tilde{\pi}_{i^*}(N)$ in i^*, as described in Lemma 1, we conclude the monotonocity of $\pi_{(i^*,p)_{pre}}(0)$ and $\tilde{\pi}_{(i^*,p)_{pre}}(N)$ in i^* and p from the equalities

$$\pi_{(i^*,p)_{pre}}(i) = (1-p)\pi_{i^*}(i) + p\pi_{i^*+1}(i) \quad \text{and}$$

$$\tilde{\pi}_{(i^*,p)_{pre}}(i) = (1-p)\tilde{\pi}_{i^*}(i) + p\tilde{\pi}_{i^*+1}(i).$$

Thus $\frac{\tilde{\pi}_{(i^*,p)_{pre}}(N)}{\pi_{(i^*,p)_{pre}}(0)}$ and $\frac{1}{\pi_{(i^*,p)_{pre}}(0)}$ are non-decreasing in i^* and in p. Since the first term may be not larger than ε_1 and the second has to be maximized, we look for parameters i^* and p with

$$\frac{\tilde{\pi}_{(i^*,p)_{pre}}(N)}{\pi_{(i^*,p)_{pre}}(0)} = \varepsilon_1.$$

Thus we get

$$i^* = \max\left\{ i \,\middle|\, \frac{\tilde{\pi}_i(N)}{\pi_i(0)} \leq \varepsilon_1 \right\} \tag{27}$$

and the value p is obtained as solution of the equation

$$\frac{(1-p)\tilde{\pi}_{i^*}(N) + p\tilde{\pi}_{i^*+1}(N)}{(1-p)\pi_{i^*}(0) + p\pi_{i^*+1}(0)} = \varepsilon_1.$$

Thus,

$$p = \frac{\varepsilon_1\pi_{i^*+1}(0) - \tilde{\pi}_{i^*+1}(N)}{(\tilde{\pi}_{i^*}(N) - \tilde{\pi}_{i^*+1}(N)) - \varepsilon_1(\pi_{i^*}(0) - \pi_{i^*+1}(0))}. \tag{28}$$

Hence we have obtained an optimal strategy in the class of pre-randomized bang-bang strategies. Then the strategy optimizes our replacement system in the whole class Π.

Now we are also able to compute an optimal post-randomized bang-bang strategy. We simply have to compute the second parameter, because the first is identical to that of the optimal pre-randomized bang-bang strategy. We obtain the second parameter q using Lemma 5. If $p = 0$ we choose $q = 0$ and if $p \neq 0$ we know that

$$p = \frac{\frac{q}{\pi_{i^*+1}(0)}}{\frac{1-q}{\pi_{i^*}(0)} + \frac{q}{\pi_{i^*+1}(0)}},$$

so that

$$q = \frac{\pi_{i^*+1}(0)}{\pi_{i^*+1}(0) - \left(1 - \frac{1}{p}\right)\pi_{i^*}(0)}. \tag{29}$$

6 Numerical examples

We conclude this paper with some numerical results generated by a C-program, which solves (27), (28) and (29) after computing the stationary probabilities $(\pi_{i^*}(i))$ and $(\tilde{\pi}_{i^*}(i))$. The following transition probabilities have the Markovian deterioration property:

$$\tilde{p}_{ij} = \begin{cases} \left(\frac{i+1}{j+1}\right)^{\beta} - \left(\frac{i+1}{j+2}\right)^{\beta} & N > j \geq i, \\ \left(\frac{i+1}{N+1}\right)^{\beta} & j = N, \\ 0 & \text{otherwise.} \end{cases}$$

In Table 1 optimal strategies are given for various values of N, β and ε_0. The identity

$$p = 0 \Leftrightarrow q = 0 \quad \text{else} \quad p = \frac{\frac{q}{\pi_{i^*+1}(0)}}{\frac{1-q}{\pi_{i^*}(0)} + \frac{q}{\pi_{i^*+1}(0)}},$$

where q is the parameter of the post-randomized strategy and p is the parameter of the pre-randomized strategy yields to

$$q = 0 \Rightarrow p = q \tag{30}$$

$$q > 0 \Rightarrow p = \frac{\frac{q}{\pi_{i^*+1}(0)}}{\frac{1-q}{\pi_{i^*}(0)} + \frac{q}{\pi_{i^*+1}(0)}} \geq \frac{\frac{q}{\pi_{i^*+1}(0)}}{\frac{1-q}{\pi_{i^*+1}(0)} + \frac{q}{\pi_{i^*+1}(0)}} = p. \tag{31}$$

Hence,

$$p \geq q.$$

Comparisons of both optimal strategies would be an interesting topic for further research.

7 Appendix

Proof of Lemma 1:

(i) For any subsets $A, B \subset I$ and a family of random variables $(Z_n)_{n \in \mathbb{N}}$ with values in I and $P(Z_{n+1} = j \,|\, Z_n = i) = p_{ij}$, $i, j \in I$, $n \in \mathbb{N}$, let

$$A \overset{\geq n}{\to} B := \{Z_0 \in A, Z_1 \notin B, \ldots, Z_{n-1} \notin B, \exists m \geq n : Z_m \in B\} \tag{32}$$

Table 1. Some examples for optimal strategies

N	β	ε_0	p_{0N}	i^*	$p : (i^*, p)_{pre}$	$p : (i^*, p)_{post}$
2	0.75	0.75	0.738	1	0.056	0.047
2	0.75	0.9	0.738	1	0.663	0.619
2	0.75	0.99	0.738	1	0.968	0.962
10	0.5	0.5	0.426	1	0.793	0.768
10	0.5	0.9	0.426	7	0.915	0.912
10	0.5	0.99	0.426	9	0.789	0.785
100	0.75	0.1	0.053	3	0.710	0.694
100	0.75	0.25	0.053	14	0.908	0.907
100	0.75	0.5	0.053	39	0.082	0.082
100	0.75	0.75	0.053	67	0.824	0.824
100	0.75	0.99	0.053	98	0.656	0.656
100	0.9	0.1	0.029	6	0.827	0.821
100	0.9	0.25	0.029	20	0.648	0.646
100	0.9	0.5	0.029	45	0.758	0.758
100	0.9	0.9	0.029	88	0.842	0.842
100	0.9	0.99	0.029	98	0.879	0.878
100	1.0	0.05	0.02	4	0.053	0.050
100	1.0	0.5	0.02	49	0.501	0.500
100	1.0	0.9	0.02	89	0.900	0.900
100	1.0	0.99	0.02	98	0.990	0.990
100	2.0	0.001	0.0004	2	0.192	0.172
100	2.0	0.01	0.0004	9	0.098	0.096
100	2.0	0.1	0.0004	30	0.938	0.938
100	2.0	0.9	0.0004	94	0.816	0.816
100	2.0	0.99	0.0004	99	0.493	0.492

and

$$P(T(i) = m) := P(Z_1 \neq i, \dots, Z_{m-1} \neq i, Z_m = i \mid Z_0 = i) \tag{33}$$

We define $q_{ij}^n(i^*)$ as the probability that the process $(X_n)_{n \in \mathbb{N}}$, using the bang-bang strategy with threshold i^* and starting in state i, will reach state j after at least n steps. If $i \neq j$ the process $(X_n)_{n \in \mathbb{Z}^+}$ has to leave state i before reaching it again. Furthermore, let $(p_{ij}^n) = P^n$. The condition $p_{ii} \neq 0$ for every $i \in I$ yields that the process $(Z_n)_{n \in \mathbb{N}}$ will visit a state of $\{i^* + 1, \dots, N\}$ after a visit in i^* in a finite number of steps. If $i = j$ the process has to leave state i before reaching it again. Thus we have for $n \in \mathbb{N}$:

$$\{\{i\} \overset{\geq n}{\to} \{i^*, i^* + 1, \dots, N\}\} \subset \{\{i\} \overset{\geq n}{\to} \{i^* + 1, i^* + 2, \dots, N\}\},$$

so

$$P(\{i\} \overset{\geq n}{\to} \{i^*, i^* + 1, \dots, N\}) \leq P(\{i\} \overset{\geq n}{\to} \{i^* + 1, i^* + 2, \dots, N\}),$$

that is $q_{i0}^n(i^*) \leq q_{i0}^n(i^* + 1)$ for all $n \in \mathbb{N}$. Hence

$$\begin{aligned} q_{ii}^n(i^*) &= \sum_{m=1}^{n-1} q_{i0}^m(i^*) P(Z_n = i, Z_{n-1} \neq i, \dots, Z_{m+1} \neq i \mid Z_m = 0) \\ &\quad + \sum_{m=n}^{\infty} P(Z_m = i, Z_{m-1} \neq i, \dots, Z_1 \neq i \mid Z_0 = 0) \qquad (34) \\ &\leq \sum_{m=1}^{n-1} q_{i0}^m(i^* + 1) P(Z_n = i, Z_{n-1} \neq i, \dots, Z_{m+1} \neq i \mid Z_m = 0) \\ &\quad + \sum_{m=n}^{\infty} P(Z_m = i, Z_{m-1} \neq i, \dots, Z_1 \neq i \mid Z_0 = 0) = q_{ii}^n(i^* + 1). \qquad (35) \end{aligned}$$

To see (34) recall that $q_{ii}^n(i^*)$ is the probability that X_k conditioned to start at i and to use δ_{i^*} first visits state i after visiting a state which is different from i after at least n steps. Thus during the time between these two visits there is a visit to state 0. Now we define the random variable X as the number of steps until the first visit at state zero and the random variable Y as the number of steps to the subsequent visit at state i. Then we get:

$$\begin{aligned} P(X + Y \geq n) &= \sum_{m=1}^{n-1} P(X + Y \geq n, Y = m) + P(Y \geq n) \\ &= \sum_{k=1}^{n-1} P(X \geq n - m, Y = m) + P(Y \geq n). \qquad (36) \end{aligned}$$

Hence equation (34) holds.

Furthermore we have for all $i < i^*$:

$$\frac{1}{\pi_{i^*}(i)} = E_{\delta_{i^*}}(T(i)) = \sum_{n\in\mathbb{N}} P_{\delta_{i^*}}(T(i) \geq n) = \sum_{n\in\mathbb{N}} q_{ii}^n(i^*)$$

$$\leq \sum_{n\in\mathbb{N}} q_{ii}^n(i^*+1) = E_{\delta_{i^*+1}}(T(i)) = \frac{1}{\pi_{i^*+1}(i)}. \tag{37}$$

Therefore $\pi_{i^*}(i)$ is non-increasing in $i^* \in \{i+1, \ldots, N\}$ and the first part of the Lemma is proven.

(ii) Part (i) yields the existence of a constant $a_{i^*}(i) \geq 0$ for every state $i < i^*$ such that

$$\pi_{i^*}(i) = \pi_{i^*+1}(i) + a_{i^*}(i). \tag{38}$$

We have

$$\sum_{i=0}^{i^*-1} a_{i^*}(i) = \sum_{i=0}^{i^*-1} \pi_{i^*}(i) - \sum_{i=0}^{i^*-1} \pi_{i^*+1}(i) = \pi_{i^*+1}(i^*)$$

and

$$\tilde{\pi}_{i^*}(N) = \sum_{i=0}^{i^*-1} \pi_{i^*}(i) p_{iN} = \sum_{i=0}^{i^*-1} \pi_{i^*+1}(i) p_{iN} + \sum_{i=0}^{i^*-1} a_{i^*}(i) p_{iN} \tag{39}$$

$$\leq \sum_{i=0}^{i^*-1} \pi_{i^*+1}(i) p_{iN} + p_{i^*N} \sum_{i=0}^{i^*-1} a_{i^*}(i) \quad \text{since } p_{iN} = \sum_{k=N}^{N} p_{ik} \uparrow (i)$$

$$\overset{(7)}{=} \sum_{i=0}^{i^*-1} \pi_{i^*+1}(i) p_{iN} + \pi_{i^*+1}(i^*) p_{i^*N} = \sum_{i=0}^{i^*} \pi_{i^*+1}(i) p_{iN}$$

$$= \tilde{\pi}_{i^*+1}(N). \tag{40}$$

which proves the second part of the Lemma.

(iii) $\tilde{\pi}_{i^*}(N) = \tilde{\pi}_{i^*+1}(N)$ yields $\sum_{i=0}^{i^*-1} a_{i^*}(i) p_{i^*N} = \sum_{i=0}^{i^*-1} a_{i^*}(i) p_{i,N}$ and because of $p_{i,N} \leq p_{i^*,N}$ for every $i \in \{0, \ldots, i^*-1\}$ the subsequent statement holds:

If $a_{i^*}(i) = 0$ then $\pi_{i^*}(i) = \pi_{i^*+1}(i)$ or $p_{iN} = p_{i+1,N} = \cdots = p_{i^*N}$.

Hence part (iii) is also proven.

(iv) For all $j \in \{1, \ldots, i^*-1\}$, $i \in \{0, \ldots, i^*\}$ the following identities hold: $q_{ij}^{(i^*,p)_{post}} = p_{ij}$,

$$q_{i,i^*}^{(i^*,p)_{post}} = p \cdot p_{i,i^*} \quad \text{and}$$

$$q_{i0}^{(i^*,p)_{post}} = p_{i0} + (1-p) p_{ii^*} + \sum_{k=i^*+1}^{N} p_{ik} \quad \text{so } q_{i0}^{(i^*,p)_{post}}.$$

Thus $q_{i,i^*}^{(i^*,p)_{post}}$ is non-decreasing and $q_{i0}^{(i^*,p)_{post}}$ is non-increasing in p. This means $\pi_{(i^*,p)_{post}}(0)$ is non-increasing in p. The reason is that this probability is equal to the reciprocal of the mean number of steps the process $(X_n)_{n=1}^{\infty}$ uses from starting at state zero, leaving it and returning the first time: if we increase the value p, probability $q_{i,0}^{(i^*,p)}$ will decrease or not change and $q_{i,i^*}^{(i^*,p)}$ will increase or not change for every $i \in \{0,\dots,i^*-1\}$; hence this mean number either increases or stays the same. Part (iv) of the Lemma is therefore also proven.

(v) The continuity of $\pi_{(i^*,p)_{pre}}$ in p follows from the identity

$$\pi_{(i^*,p)_{pre}} = (1-p)\pi_{i^*} + p\pi_{i^*+1}.$$

Now we look at $\pi_{(i^*,p)_{post}}$. Define $Q^{(i^*,p)_{post}}$ as the square matrix of size i^*+1 with elements $(q_{ji}^{(i^*,p)_{post}})_{\{(i,j)\in I^2\}}$. $(x_0,\dots,x_{i^*}) = (\pi_{(i^*,p)_{post}}(0),\dots,\pi_{(i^*,p)_{post}}(N))$ is the unique solution of the system of equations

$$\begin{pmatrix} & Q^{(i^*,p)_{post}} - I_{i^*+1} & \\ 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ \vdots \\ x_{i^*} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Thus the rank of the $(i^*+2)\times(i^*+1)$ matrix

$$\begin{pmatrix} & Q^{(i^*,p)_{post}} - E_{i^*+1} & \\ 1 & \dots & 1 \end{pmatrix}$$

is i^*+1. Eliminate one row which is linear dependent on the others and call the resulting non-singular matrix $A(p) = (a_{ij}(p))_{\{0\le i,j\le i^*\}}$, where $(A(p))^{-1} = (b_{ij}(p))_{\{0\le i,j\le i^*\}}$. Then

$$\pi_{(i^*,p)_{post}} = (b_{ij}(p))e_{i^*+1} = \frac{(a_{ji}^*(p))}{|(a_{ij})(p)|}e_{i^*+1}. \tag{41}$$

Since the entries of $(a_{ji}^*(p))$ are determinants and thus polynomials in p, $\pi_{(i^*,p)_{post}}(i)$ is continuous in p. Thus part (iv) is proven completely and it remains to prove the last part.

(vi) To prove the continuity of the probabilities $\tilde{\pi}_{(i^*,p)_{post}}(i)$, we are faced with the problem that the process (X_n^-) using strategy $\delta_{(i^*,p)_{post}}$ does not form a Markov process, since the probability $P(X_n^- = j \mid X_{n-1} = i^*, X_{n-2} = i^*)$ is in general not equal to the probability $P(X_n^- = j \mid X_{n-1}^- = i^*, X_{n-2} < i^*)$. We avoid this problem by splitting the state $X_n^- = i^*$ into states $X_n^- = (i^*, i^*)$ if $X_{n-1} = X_n^- = i^*$ and $X_n^- = (i^*, <i^*)$, if $X_{n-1} < X_n^- = i^*$. Then the process $(X_n^-)_{n\in\mathbb{N}}$ forms a Markov chain with the following transition probabilities: for $i \in \{0,\dots,i^*-1, i^*+1,\dots,N)$ we have

$$\tilde{q}_{ij}^{(i^*,p)_{post}} = p_{ij}1_{\{i<i^*\}} + p_{0j}1_{\{i>i^*\}}, \quad \tilde{q}_{(i^*,i^*),(i^*,i^*)}^{(i^*,p)_{post}} = p_{i^*i^*},$$

$$\tilde{q}_{(i^*,<i^*),(i^*,i^*)}^{(i^*,p)_{post}} = (1-p)p_{i^*i^*},$$

$$\tilde{q}^{(i^*,p)_{post}}_{(i^*,i^*),(i^*,<i^*)} = 0, \quad \tilde{q}^{(i^*,p)_{post}}_{(i^*,<i^*),(i^*,<i^*)} = p \cdot p_{0i^*}, \quad \tilde{q}^{(i^*,p)_{post}}_{(i^*,i^*),j} = p_{i^*,j},$$

$$\tilde{q}^{(i^*,p)_{post}}_{(i^*,<i^*),j} = (1-p) \cdot p_{0j} + p p_{i^*j}, \quad \tilde{q}^{(i^*,p)_{post}}_{j,(i^*,i^*)} = 0,$$

$$\tilde{q}^{(i^*,p)_{post}}_{j,(i^*,<i^*)} = p_{ji^*} 1_{\{j<i^*\}} + p_{0i^*} 1_{\{j>i^*\}}.$$

Defining the corresponding matrix $\tilde{Q}^{(i^*,p)_{post}}$ for these probabilities ($\dim \tilde{Q}^{(i^*,p)_{post}} = \dim Q^{(i^*,p)_{post}} + 1$), the continuity of the probabilities $\tilde{\pi}_{(i^*,p)_{post}}$ can be proven similiarly to the continuity of the probabilities $\pi_{(i^*,p)_{post}}$. ■

Proof of Lemma 4: From Lemma 2 we recall (8):

$$\phi_\delta(c) = (1-\alpha)(d^{(c)}(N,1) + \tilde{V}^{(c)}_{\delta,\alpha}(0)) \quad \text{with } \alpha := 1 - p_{0N} < 1.$$

We have $\phi_{\delta_{i^*}} \leq \phi_\delta$ and $\phi_{\delta_{j^*}} \leq \phi_\delta$ for every strategy $\delta \in \Pi$.

Hence $\tilde{V}^{(c)}_{\delta_{i^*},\alpha}(0) \leq \tilde{V}^{(c)}_{\delta,\alpha}(0)$ and $\tilde{V}^{(c)}_{\delta_{j^*},\alpha}(0) \leq \tilde{V}^{(c)}_{\delta,\alpha}(0)$ for every $\delta \in \Pi$.

A control-limit strategy with threshold i^* optimizes $V^{(c)}_{\delta,\alpha}(0)$ in δ, if and only if:

$$\left(\alpha \sum_{j=0}^{N} \tilde{p}_{ij} \tilde{V}^{(c)}_{\alpha}(j) \geq (1-\alpha) + \alpha \sum_{j=0}^{N} \tilde{p}_{0j} \tilde{V}^{(c)}_{\alpha}(j) \text{ or } \sum_{n=0}^{\infty} \tilde{p}^{(n)}_{0i} = 0\right)$$

$$\text{for } i \in \{i^*, i^*+1, \ldots, N-1\}$$

and

$$\left(\alpha \sum_{j=0}^{N} \tilde{p}_{ij} \tilde{V}^{(c)}_{\alpha}(j) \leq (1-\alpha) + \alpha \sum_{j=0}^{N} \tilde{p}_{0j} \tilde{V}^{(c)}_{\alpha}(j) \text{ or } \sum_{n=0}^{\infty} \tilde{p}^{(n)}_{0i} = 0\right)$$

$$\text{for } i \in \{1, 2, \ldots, i^*-1\}.$$

We prove this indirectly. First we consider states i satisfying $\sum_{n=0}^{\infty} p^{(n)}_{0i} > 0$. If the inequality regarding this state i is not fulfilled, we introduce the strategy δ which is equal to δ_{i^*} except at state i, where it takes the other action. Then we have $\tilde{V}^{(c)}_{\delta,\alpha} < \tilde{V}^{(c)}_{\delta_{i^*},\alpha}$, i.e. δ_{i^*} is not optimal. Now let i satisfy $\sum_{n=0}^{\infty} \tilde{p}^{(n)}_{0i} = 0$. Starting at state 0 it is almost sure that state i will never be visited. Thus for every strategy δ the value $\tilde{V}^{(c)}_{\delta,\alpha}(0)$ is independent of the behaviour of δ at i. Using the function $\tilde{V}^{(c,n)}_{\alpha}$ to denote the discounted cost up to the nth interval it is standard to prove that $\tilde{V}^{(c)}_{\alpha}$ is non-decreasing, since $\tilde{d}$ is non-decreasing and the Markovian deterioration of the (p_{ij}) yields the Markovian deterioration of the $(\tilde{p}_{ij})$ (e.g. Ross, [11], pp. 37). If the control-limit policies with thresholds i^* and j^* are both optimal, we get the following:

$$f(i) := \sum_{j=0}^{N} \tilde{p}_{ij} \tilde{V}^{(c)}_{\alpha}(j) \text{ is constant in } \{i^*, \ldots, j^*\} \Big\backslash \left\{ i \,\middle|\, \sum_{n=0}^{\infty} p^{(n)}_{0i} = 0 \right\}.$$

If the state is an element of $\{i \mid \sum_{n=0}^{\infty} p^{(n)}_{0i} = 0\}$ both actions are optimal, so all control-limit policies with thresholds $\{i^*, \dots, j^*\}$ minimize $\tilde{V}_{\delta,\alpha}(0)$ and hence also ϕ_δ in δ. ■

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