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A MATRIX-BASED APPROACH TO PROPERNESS AND INVERSION PROBLEMS FOR RATIONAL SURFACES

LAURENT BUSÉ AND CARLOS D'ANDREA

Abstract. We present a matrix-based approach for deciding if the parameterization of an algebraic space surface is invertible or not, and for computing the inverse of the parametrization if it exists.

Keywords: Rational Maps, Parameterizations, Inversion Matrices, Implicitization Matrices.

1. Introduction

Rational surfaces play an important role in the frame of practical applications, especially in Computer Aided Geometric Design (see [13, 17] and the references therein). These surfaces can be parameterized, i.e. can be seen as the image of a generically finite rational map

$$\begin{array}{rccc} \phi: & \mathbb{K}^2 & \dashrightarrow & \mathbb{K}^3 \\ & \mathbf{t}:=(t_1,t_2) & \mapsto & \left(\frac{p_1(\mathbf{t})}{q_1(\mathbf{t})},\frac{p_2(\mathbf{t})}{q_2(\mathbf{t})},\frac{p_3(\mathbf{t})}{q_3(\mathbf{t})}\right), \end{array} \tag{1}$$

where $\mathbb{K}$ denotes the ambient field, which we assume to be of characteristic zero. In the sequel we will address the following questions:

- *Properness problem*: decide if ϕ, co-restricted to its image, is invertible.
- *Inversion problem*: If ϕ, co-restricted to its image, is invertible, then compute its inverse.

Both questions have already been solved theoretically and algorithmically. For plane curves, the situation is very well-known: one can relate the properness and inversion problems to Lüroth's theorem, and there are different algorithmic procedures to solve them (see [13, 17, 21, 22]). For space surfaces, there exist some algorithmic approaches based on u-resultants [7] and on Gröbner bases in [19]. In [18] a complete algorithm is given to solve both problems by means of univariate resultants and GCD computations. Our starting point is the resultant matrix-based method presented in [13, Chapter 15] for inverting a parametrized algebraic surface, and in [5, §5] where it is used for computing the inverse image of a point of a parameterization.

In this paper, we introduce a general matrix-based approach for dealing with both problems. We will begin by reviewing the plane curves case; this will clarify our approach and help the reader to understand it. In section 3, we will introduce *inversion matrices* associated to parameterizations. We will show that their existence implies the properness of a given parameterization and also that they can be used in order to produce an inverse map in terms of quotient of determinants of some of its sub-matrices.

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The problem of deciding whether a given parameterization is proper or not is not an easy task, hence in general the construction of such matrices is a non trivial problem. In [4] some matrices called *Jacobian matrices* were introduced in order to deal with the properness problem. This is a first example of a family of inversion matrices; we will briefly recall this construction at the end of section 3.

In order to obtain other groups of inversion matrices, we introduce *implicitization matrices* in section 4. They are essentially square matrices whose determinant is a nontrivial multiple of the implicit equation of the given surface (although we do not need to compute the implicit equation in the process). We will prove that they characterize the properness of a given parameterization and moreover produce inversion matrices if the parameterization is proper. Then, we give a list of known algorithms to construct implicitization matrices. We end the paper with some illustrative examples.

We would like to emphasize that the scope of this paper is not to provide a complete algorithm to solve both properness and inversion problems, but to give general theorems showing that it is possible to test properness and to get inverse maps of a parameterization by using tools from elimination theory which have been developed in principle for other purposes: as soon as one has what we call an inversion matrix (or an implicitization matrix) it is possible to read almost immediately the inverse map from it.

A complete algorithm for computing the inverse map from the parameterization by using our approach requires only the construction of an implicitization matrix. So, the description of such an algorithm is only the description of the construction of an implicitization matrix. This is why we do not deal with algorithms in this paper, they can be found in the literature of matrices from elimination theory.

In section 4 we give a list of some general constructions yielding implicitization matrices and it turns out that this list already covers a lot of cases. For the general situation, no process is yet known (as far as we know), but one can also try to produce it case by case by computing particular syzygies.

It should be pointed out that our approach is very different from what has been done in [7, 18, 19] since we provide inversion formulas in terms of quotient of determinants of sub-matrices of a given matrix instead of producing expanded rational symbolic expressions. For instance, as a consequence of our results, we will show that, in the case where resultant matrices give implicitization matrices (this is always the case for plane curves for instance), the inversion map can be represented by sub-matrices of a resultant matrix which is built from the coefficients of the given parameterization; in other words, no symbolic computations are needed if we stay at this level of representation (note that this kind of representation in Computer Aided Geometric Design as already been widely used, for instance to deal with the surface/curve or surface/surface intersection problems). Moreover, it is interesting to know that certain big polynomials can be represented as determinants of certain matrices. In this paper we prove that inversion maps are given in terms of determinants (as soon as one has an implicitization map).

Finally, it should be mentioned that the two main theorems 3.2 and 4.2 proven in this paper are also valid for rational parameterizations of hypersurfaces (the proofs work verbatim), that is to say for rational parameterizations from $\mathbb{K}^{n-1}$ to $\mathbb{K}^n$, with $n \geq 2$, whose closed image is a hypersurface. We chose to stay in the context

of surfaces because of their important applications in Computer Aided Geometric Design.

2. Preliminary: inversion of parameterized plane algebraic curves

Before dealing with the case (1) of surfaces we first quickly describe our approach to properness and inversion problems in the case of plane algebraic curves for the sake of clarity and completeness.

Suppose given a rational plane algebraic curve $\mathcal{C}$ parameterized by

$$\phi : \mathbb{K}^1 \to \mathbb{K}^2 : t \mapsto \left(\frac{p_1(t)}{q_1(t)}, \frac{p_2(t)}{q_2(t)}\right)$$

where we can assume, without loss of generality, that $\gcd(p_1, q_1) = \gcd(p_2, q_2) = 1$. We moreover assume that $m := \max(\deg(p_1), \deg(q_1)) \geq 1$ and similarly that $n := \max(\deg(p_2), \deg(q_2)) \geq 1$; if it is not the case, then $\mathcal{C}$ is a line and the properness and inversion problems are easy.

It is well-known that

$$\mathrm{Res}_{m,n}(p_1(t) - xq_1(t), p_2(t) - yq_2(t)) = C(x, y)^{\deg(\phi)}$$

where $C(x, y)$ denotes an implicit equation of the curve $\mathcal{C}$ and $\mathrm{Res}_{m,n}(-,-)$ the classical Sylvester resultant. This resultant can be computed as the determinant of the so-called Sylvester matrix $S_{m,n}(p_1(t) - xq_1(t), p_2(t) - yq_2(t))$ which satisfies the equality

$${}^tS_{m,n}(p_1(t) - xq_1(t), p_2(t) - yq_2(t)) \begin{pmatrix} t^{m+n-1} \\ t^{m+n-2} \\ \vdots \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} t^{n-1}(p_1(t) - xq_1(t)) \\ \vdots \\ t(p_1(t) - xq_1(t)) \\ p_1(t) - xq_1(t) \\ t^{m-1}(p_2(t) - yq_2(t)) \\ \vdots \\ t(p_2(t) - yq_2(t)) \\ p_2(t) - yq_2(t) \end{pmatrix}.$$

We denote by $\mathbb{M}$ the sub-matrix of the above Sylvester matrix obtained by erasing its last column. For all $i = 1, \ldots, m+n$, we also denote by Δ_i the signed determinant of $\mathbb{M}$ obtained by erasing the i^{th} row. In this way, we have

$$\mathrm{Res}_{m,n}(p_1(t) - xq_1(t), p_2(t) - yq_2(t)) = \sum_{i=1}^{m+n} c_i \Delta_i, \tag{2}$$

where the c_i's are the entries of the last column of the Sylvester matrix (and are hence polynomials in $\mathbb{K}[y]$), i.e. $p_2(t) - yq_2(t) = \sum_{i=0}^{m+n-1} c_i t^{m+n-1-i}$.

Proposition 2.1 ([4, section 2])**.** *With the above notation we have*

$$\deg(\phi) = 1 \Leftrightarrow \gcd(\Delta_1, \ldots, \Delta_{m+n}) \in \mathbb{K} \setminus \{0\}.$$

Moreover, if $\deg(\phi) = 1$ *then for all* $i = 1, \ldots, m+n-1$ *the rational map*

$$\mathbb{K}^2 \to \mathbb{K}^1 : (x, y) \mapsto \frac{\Delta_i}{\Delta_{i+1}}$$

is an inversion of ϕ*.*

It is important to notice that this proposition gives *closed universal inversion formulas*, that is to say that the inversion maps can be pre-computed in terms of the coefficient of the polynomials p_1, q_1, p_2 and q_2. Moreover, if we stay at the level of matrices, no symbolic computations are needed: the inversion formula is just a quotient of two determinants of sub-matrices of the Sylvester matrix which is itself directly built from the coefficients of the polynomials p_1, q_1, p_2 and q_2. In this way, we can *represent* an inversion map by two sub-matrices of a Sylvester matrix; when an inverse image is required we just have to instantiate the corresponding point in these sub-matrices and then compute the quotient of two determinants of *numeric* matrices. This is very similar to the fact that the Sylvester matrix *represents* the implicit equation of $\mathcal{C}$.

Finally, note that we could use in the above process the Bezout matrix instead of the Sylvester matrix that we chosen for simplicity. For a definition and properties of the Bezout matrix, see [9].

Example 2.2. *Consider the following easy example of the unitary circle parameterized by*

$$\phi : \mathbb{K}^1 \to \mathbb{K}^2 : t \mapsto \left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2} \right).$$

The associated Sylvester matrix is

$$S_{2,2}(2t - x(1+t^2), 1 - t^2 - y(1+t^2)) = \begin{pmatrix} -x & 0 & -1-y & 0 \\ 2 & -x & 0 & -1-y \\ -x & 2 & 1-y & 0 \\ 0 & -x & 0 & 1-y \end{pmatrix}$$

from we extract the matrix

$$\mathbb{M} = \begin{pmatrix} -x & 0 & -1-y \\ 2 & -x & 0 \\ -x & 2 & 1-y \\ 0 & -x & 0 \end{pmatrix}.$$

The 3×3 *minors of the matrix* $\mathbb{M}$ *are then*

$$\Delta_1 = 2x(y-1),\ \Delta_2 = -2x^2,\ \Delta_3 = -2x(y+1),\ \Delta_4 = 2x^2 - 4(y+1).$$

Their gcd is a constant so we deduce that ϕ *is proper and we can check that the all the inversion formulas given in proposition 2 are equal, that is to say that*

$$\frac{\Delta_1}{\Delta_2} = \frac{\Delta_2}{\Delta_3} = \frac{\Delta_3}{\Delta_4} \in \mathrm{Frac}(\mathbb{K}[x,y]/I_{\mathcal{C}}),$$

where $I_{\mathcal{C}} = (x^2 + y^2 - 1)$ *is the defining ideal of* $\mathcal{C}$*; for instance:*

$$\frac{\Delta_1}{\Delta_2} - \frac{\Delta_2}{\Delta_3} = \frac{2x(y-1)}{-2x^2} - \frac{-2x^2}{-2x(y+1)} = -\frac{x^2+y^2-1}{x(y+1)} = 0 \in \mathrm{Frac}(\mathbb{K}[x,y]/I_{\mathcal{C}}).$$

We can also check we got inversion formulas, that is to say that

$$\frac{\psi(\Delta_1)}{\psi(\Delta_2)} = \frac{\psi(\Delta_2)}{\psi(\Delta_3)} = \frac{\psi(\Delta_3)}{\psi(\Delta_4)} = t \in \mathbb{K}(t)$$

where ψ *is the map* $\mathbb{K}[x,y] \to \mathbb{K}(t) : P(x,y) \mapsto P(\frac{p_1(t)}{q_1(t)}, \frac{p_2(t)}{q_2(t)})$*. For instance:*

$$\frac{\psi(\Delta_1)}{\psi(\Delta_2)} = \left(\frac{1+t^2}{2t} \right) \left(1 - \frac{1-t^2}{1+t^2} \right) = t.$$

In the following, we extend this approach to the more intricate case of algebraic space surfaces. To this task, we will introduce two distinct notions:
Inversion matrices: these matrices will play the role of the matrix $\mathbb{M}$ above. As soon as such a matrix exists we will prove that the given parameterization is birational and we will deduce inversion maps similarly to proposition 2.
Implicitization matrices: these matrices will play the role of the Sylvester matrix above. From such a matrix we can characterize the properness of the given parameterization ϕ and then produce an inversion matrix which is then used to get inversion maps.

3. Inversion by means of matrices

Suppose given a rational surface S parameterized by the map (1). In this section we develop a matrix-based approach to the inversion problem. To do this we introduce a certain class of matrices associated to parameterizations that we will call *inversion matrices*. We prove that if such a matrix exists for a given parameterization, then this parameterization is birational and we can derive an inversion map in terms of sub-matrices of this inversion matrix.

From now on, we will turn to projective geometry, so we will assume that our given rational surface S is embedded in $\mathbb{P}^3$ and parameterized by

$$\begin{array}{rccc} \phi: & \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^3 \\ & \mathbf{t} := (t_1 : t_2 : t_3) & \mapsto & (p_1(\mathbf{t}) : p_2(\mathbf{t}) : p_3(\mathbf{t}) : p_4(\mathbf{t})) \end{array} \tag{3}$$

where the $p_i(\mathbf{t})$'s are homogeneous polynomials of the same degree. Also, we will hereafter denote by $\mathbb{K}(S)$ (resp. $\mathbb{K}(\mathbb{P}^2)$) the field of rational functions over S (resp. over $\mathbb{P}^2$) (see e.g. [14, §I.3.]).

We recall that a *moving surface* of bi-degree $(m; n)$ is a bi-homogeneous polynomial in the sets of homogeneous variables $\mathbf{t} := (t_1, t_2, t_3)$ and $\underline{X} := (X_1, X_2, X_3, X_4)$ of degree $m \geq 1$ and $n \geq 1$ respectively. Thus a moving surface $M(\mathbf{t}; \underline{X})$ can be written as:

$$M(\mathbf{t}; \underline{X}) = \sum_{|\alpha|=n} A_\alpha(\mathbf{t}) \underline{X}^\alpha,$$

where $\alpha \in \mathbb{N}^4$ is a multi-index and $A_\alpha(\mathbf{t})$ is a homogeneous polynomial of degree m in $\mathbf{t}$. We will say that this moving surface *follows the parameterization* ϕ if M is identically zero after substituting X_i by $p_i(\mathbf{t})$ for $i = 1, 2, 3, 4$.

Definition 3.1. *A matrix* $\mathbb{M}$ *is called an* inversion matrix of ϕ *if it satisfies the two following conditions :*

(i) *There exists an integer* $m \geq 1$ *and a subset* $V = \{\mathbf{t}^{\alpha_1}, \ldots, \mathbf{t}^{\alpha_d}\}$ *of the monomial basis* $\{\mathbf{t}^\alpha, \alpha \in \mathbb{N}^3 \ |\alpha| = m\}$ *such that*

$${}^t\mathbb{M} \begin{pmatrix} \mathbf{t}^{\alpha_1} \\ \mathbf{t}^{\alpha_2} \\ \vdots \\ \mathbf{t}^{\alpha_d} \end{pmatrix} = \begin{pmatrix} M_1(\mathbf{t}; \underline{X}) \\ \vdots \\ M_{d-1}(\mathbf{t}; \underline{X}) \end{pmatrix},$$

where the polynomials $M_i(\mathbf{t}; \underline{X})$ *are moving surfaces following* ϕ *of bi-degree* $(m; d_i)$. *Moreover there exists* $(\mathbf{t}^{\beta_1}, \mathbf{t}^{\beta_2}, \mathbf{t}^{\beta_3}) \in V^3$ *satisfying* $t_1\mathbf{t}^{\beta_3} = t_3\mathbf{t}^{\beta_1}$ *and* $t_2\mathbf{t}^{\beta_3} = t_3\mathbf{t}^{\beta_2}$ *(and hence* $t_1\mathbf{t}^{\beta_2} = t_2\mathbf{t}^{\beta_1}$*).*

(ii) *The rank of* $\mathbb{M}$ *over the field* $\mathbb{K}(S)$ *is exactly* $d - 1$.

An inversion matrix $\mathbb{M}$ is thus a non-square matrix of size $d \times (d-1)$, where $d \geq 3$. Note that the condition (ii) they have to satisfy can actually be checked explicitly; one way to do this is described in [4, proposition 3.5 and remark 3.6].

Before stating the main result of this section we need to recall carefully what is the degree of the map ϕ, map that we now assume to be co-restricted to S. Roughly speaking the degree of ϕ is the finite number of preimages by ϕ of a sufficiently generic point on S. More precisely, the map ϕ induces an injective morphism $\phi^\sharp : \mathbb{K}(S) \to \mathbb{K}(\mathbb{P}^2) : f \to f \circ \phi$. Thus $\mathbb{K}(\mathbb{P}^2)$ is a finite extension field of $\mathbb{K}(S)$ and its degree is, by definition, the degree of ϕ; usually this is summarized by the formula $\deg(\phi) := [\mathbb{K}(\mathbb{P}^2) : \mathbb{K}(S)]$. It follows that ϕ is birational if and only if $\phi^\sharp$ is an isomorphism, i.e. ϕ has degree 1 (see e.g. [14, §I.4.]).

Theoreem 3.2. *With the above notation, if there exists an inversion matrix $\mathbb{M}$ of ϕ then ϕ is birational. Moreover, denoting by Δ_α the* signed *minor of $\mathbb{M}$ obtained by erasing the line indexed by the monomial $\mathbf{t}^\alpha$, an inversion of ϕ is given by the rational map*

$$S \subset \mathbb{P}^3 \to \mathbb{P}^2 : \underline{X} \mapsto (\Delta_{\beta_1}(\underline{X}) : \Delta_{\beta_2}(\underline{X}) : \Delta_{\beta_3}(\underline{X})),$$

where β_1, β_2 and β_3 are such that $t_1 t^{\beta_3} = t_3 t^{\beta_1}$ and $t_2 t^{\beta_3} = t_3 t^{\beta_2}$.

Proof. Since we assumed that the rank of $\mathbb{M}$ over $\mathbb{K}(S)$ is $d-1$ we deduce that the kernel of ${}^t\mathbb{M}$ is generated by the vector

$$\begin{pmatrix} \Delta_{\alpha_1} & \Delta_{\alpha_2} & \cdots & \Delta_{\alpha_d} \end{pmatrix}. \tag{4}$$

One has, for all $i = 1, \ldots, d$, $\phi^\sharp(\Delta_{\alpha_i}(\underline{X})) = \Delta_{\alpha_i}(\phi(\mathbf{t}))$ and hence it equals the *signed* $(d-1) \times (d-1)$ minor of $\phi^\sharp(\mathbb{M})$ obtained by erasing the line indexed by $\mathbf{t}^{\alpha_i}$. But by definition we have, in $\mathbb{K}[\mathbf{t}]$,

$${}^t\phi^\sharp(\mathbb{M}) \begin{pmatrix} \mathbf{t}^{\alpha_1} \\ \vdots \\ \mathbf{t}^{\alpha_d} \end{pmatrix} = \begin{pmatrix} M_1(\mathbf{t}; \phi(\mathbf{t})) \\ \vdots \\ M_{d-1}(\mathbf{t}; \phi(\mathbf{t})) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{5}$$

Consequently both vectors

$$\begin{pmatrix} \mathbf{t}^{\alpha_1} & \cdots & \mathbf{t}^{\alpha_d} \end{pmatrix} \text{ and } \begin{pmatrix} \phi^\sharp(\Delta_{\alpha_1}(\underline{X})) & \cdots & \phi^\sharp(\Delta_{\alpha_d}(\underline{X})) \end{pmatrix}$$

generate the kernel of ${}^t\phi^\sharp(\mathbb{M})$ over $\mathbb{K}(\mathbb{P}^2)$, and thus equal up to the multiplication by an invertible element of $\mathbb{K}(\mathbb{P}^2)$. In particular we have

$$\phi^\sharp\left(\frac{\Delta_{\beta_1}(\underline{X})}{\Delta_{\beta_2}(\underline{X})}\right) = \frac{t_1}{t_2},\ \phi^\sharp\left(\frac{\Delta_{\beta_1}(\underline{X})}{\Delta_{\beta_3}(\underline{X})}\right) = \frac{t_1}{t_3} \text{ and } \phi^\sharp\left(\frac{\Delta_{\beta_2}(\underline{X})}{\Delta_{\beta_3}(\underline{X})}\right) = \frac{t_2}{t_3},$$

which imply that $\phi^\sharp : \mathbb{K}(S) \to \mathbb{K}(\mathbb{P}^2)$ is an isomorphism.

Now let the rational map $\psi : S \subset \mathbb{P}^3 \to \mathbb{P}^2 : \underline{X} \mapsto (\psi_1(\underline{X}) : \psi_2(\underline{X}) : \psi_3(\underline{X}))$ be an inverse of ϕ and $\psi^\sharp : \mathbb{K}(\mathbb{P}^2) \to \mathbb{K}(S)$ its associated field embedding. By definition of $\mathbb{M}$ we have in $\mathbb{K}[S]$, and hence in $\mathbb{K}(S)$,

$${}^t\mathbb{M} \begin{pmatrix} \psi(\underline{X})^{\alpha_1} \\ \vdots \\ \psi(\underline{X})^{\alpha_d} \end{pmatrix} = \begin{pmatrix} M_1(\psi(\underline{X}); \underline{X}) \\ \vdots \\ M_{d-1}(\psi(\underline{X}); \underline{X}) \end{pmatrix}. \tag{6}$$

But for all i we have

$$\phi^\sharp(M_i(\psi(\underline{X}); \underline{X})) = \phi^\sharp(M_i(\psi^\sharp(\mathbf{t}); \underline{X}))) = M_i(\phi^\sharp \circ \psi^\sharp(\mathbf{t}), \phi^\sharp(\underline{X})) = M_i(\mathbf{t}, \phi(\mathbf{t})) = 0$$

in $\mathbb{K}(\mathbb{P}^2)$ and hence $M_i(\psi(\underline{X});\underline{X}) = 0$ in $\mathbb{K}(S)$. Therefore we deduce that the vector

$$\begin{pmatrix} \psi(\underline{X})^{\alpha_1} & \psi(\underline{X})^{\alpha_2} & \cdots & \psi(\underline{X})^{\alpha_d} \end{pmatrix} \tag{7}$$

generates the kernel of ${}^t\mathbb{M}^*$ over $\mathbb{K}(S)$. On the other hand we know that the vector (4) is also a generator of the kernel of ${}^t\mathbb{M}^*$ over $\mathbb{K}(S)$. It follows that vectors (4) and (7) must equal up to the multiplication by an invertible element of $\mathbb{K}(S)$, and the claimed result is proved. □

In general, the computation of an inversion matrix is not obvious. We will mainly obtain them from matrices coming from elimination theory that we will describe in section 4. There we will see that in most of the cases there exist algorithms for constructing inversion matrices. Moreover, we will also see that we can take advantage of the matrix formulation we have for producing closed and universal inversion formulas for some classes of surfaces by using resultant-based matrices.

We end by showing how we can deduce inversion matrices from Jacobian matrices; this was the main subject of [4].

Example 3.3 (Inversion matrices from jacobian matrices)**.** *In* [10]*, it is shown that one can construct a hybrid matrix whose determinant is a nonzero multiple of the resultant, having all rows except one of Sylvester style. In* [4]*, we show that the maximal minors of the Sylvester part of this matrix are subresultants and that we can solve the inversion problem by using them.*

This parameterization is extracted from [18, example P_1]*:*

$$p_1 = \frac{t_1}{t_1+t_2},\ p_2 = \frac{t_1^2 - t_1 + 1}{t_2+1},\ p_3 = t_1^2 + t_2.$$

By considering $F_1 := (t_1+t_2)X_1 - t_1$, $F_2 := (t_2+1)X_2 - (t_1^2 - t_1 + 1)$ *and* $F_3 := X_3 - t_1^2 + t_2$*, we get the following subresultant matrix:*

$$\mathbb{M} := \begin{pmatrix} X_2-1 & 1 & X_2 & -1 & 0 & 0 \\ X_3 & 0 & -1 & -1 & 0 & 0 \\ 0 & X_1-1 & X_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_1-1 & X_1 & 0 \\ 0 & 0 & 0 & 0 & X_1-1 & X_1 \end{pmatrix}.$$

All the maximal minors of this matrix are subresultants. Using theorem 3.2 we can solve the inverse problem. We obtain:

$$t_1 = -\frac{\Delta_{t_1}}{\Delta_1} = \frac{X_1(X_2-1-X_3)}{X_2X_1-1-X_2},\ \text{and } t_2 = \frac{\Delta_{t_2}}{\Delta_1} = \frac{X_2X_1 - X_2 - X_1 + 1 - X_1X_3 + X_3}{X_2X_1-1-X_2}.$$

4. Implicitization matrices

We keep the notation of section 3 where we developed a matrix-based approach to the properness and inversion problems. In this section we introduce a new kind of matrices, that we will call *implicitization matrices*. We will prove that, when it exists, such a matrix characterizes the properness of the parameterization ϕ. Moreover, if ϕ is proper we can extract from it an inversion matrix of ϕ, as defined in definition 3.1.

Hereafter we will denote by $F(X_1, X_2, X_3, X_4) \in \mathbb{K}[X_1, X_2, X_3, X_4]$ the implicit equation (which is actually defined up to multiplication by a non-zero constant in $\mathbb{K}$) of $S \subset \mathbb{P}^3$. Recall that it is the homogeneous polynomial of minimal degree

such that $F(p_1(t), p_2(t), p_3(t), p_4(t)) \equiv 0$ in $\mathbb{K}[t_1, t_2, t_3]$; its degree is the degree of the surface S that we denote by $\deg(S)$.

Definition 4.1. *A square matrix $\mathbb{M}$ is an* implicitization matrix of the parameterization ϕ *if it satisfies the three following conditions :*

(1) *There exists an integer $m \geq 1$ and a subset $V = \{\mathbf{t}^{\alpha_1}, \ldots, \mathbf{t}^{\alpha_d}\}$ of the monomial basis $\{\mathbf{t}^{\alpha}, \alpha \in \mathbb{N}^3 \ |\alpha| = m\}$ such that*

$$ {}^t\mathbb{M} \begin{pmatrix} \mathbf{t}^{\alpha_1} \\ \mathbf{t}^{\alpha_2} \\ \vdots \\ \mathbf{t}^{\alpha_d} \end{pmatrix} = \begin{pmatrix} P(\mathbf{t}; \underline{X}) \\ M_1(\mathbf{t}; \underline{X}) \\ \vdots \\ M_{d-1}(\mathbf{t}; \underline{X}) \end{pmatrix}, $$

where polynomials $M_i(\mathbf{t}; \underline{X})$ are moving surfaces following ϕ of bi-degree $(m; d_i)$ and $P(\mathbf{t}; \underline{X})$ is an arbitrary bi-homogeneous polynomial with positive degree in variables $\underline{X}$. Moreover there exists $(\mathbf{t}^{\beta_1}, \mathbf{t}^{\beta_2}, \mathbf{t}^{\beta_3}) \in V^3$ satisfying $t_1\mathbf{t}^{\beta_3} = t_3\mathbf{t}^{\beta_1}$ and $t_2\mathbf{t}^{\beta_3} = t_3\mathbf{t}^{\beta_2}$ (and hence $t_1\mathbf{t}^{\beta_2} = t_2\mathbf{t}^{\beta_1}$).

(2) $\det(\mathbb{M}) = c.F(X_1, X_2, X_3, X_4)^{\delta}$ *where $\delta \in \mathbb{N} \setminus \{0\}$ and $c \in \mathbb{K} \setminus \{0\}$.*

(3) *If ϕ is birational, i.e. S is properly parameterized by ϕ, then $\delta = 1$.*

An implicitization matrix of ϕ is hence a square $d \times d$ matrix where $d \geq 3$. Of course the name *implicitization matrix* comes from the condition (2) in this definition. The following theorem shows that implicitization matrices characterize the properness of the map ϕ and moreover yield inversion matrices.

Theoreom 4.2. *Let $\mathbb{M}$ be an implicitization matrix of ϕ and denote by $\mathbb{M}^*$ the sub-matrix of $\mathbb{M}$ obtained by erasing its first column. Then the gcd of the maximal minors of $\mathbb{M}^*$ equals F^p with $p \in \mathbb{N}$.*

Moreover the following statements are equivalent:

(1) *ϕ is birational*

(2) *$p = 0$*

(3) *$\mathbb{M}^*$ is an inversion matrix of S.*

Proof. First the fact that the gcd of the maximal minors of $\mathbb{M}^*$ is a power of F follows immediately from the following equality in $\mathbb{K}[\underline{X}]$:

$$ \det(\mathbb{M}) = F^{\delta} = \sum_{|\alpha|=m} c_{\alpha}\Delta_{\alpha} \tag{8} $$

where the c_α are the coefficients of the erased column of $\mathbb{M}$, since F is an irreducible and homogeneous polynomial.

Now suppose that ϕ is birational (i.e. proper). Then we know that $\det(\mathbb{M}) = cF$. Looking at the formula (8) we deduce that $p = 0$ since the c_α's have positive degree (recall that P is supposed to have positive degree in variables $\underline{X}$). This implies that the rank of $\mathbb{M}^*$ over the field $\mathbb{K}(S)$ is $d - 1$ (recall that d is the size of the square matrix $\mathbb{M}$), that is to say that $\mathbb{M}^*$ is an inversion matrix.

Conversely, assume that $p = 0$, that is the gcd of the maximal minors of $\mathbb{M}^*$ is a constant. Then the rank of $\mathbb{M}^*$ over $\mathbb{K}(S)$ is $d - 1$ and $\mathbb{M}^*$ is an inversion matrix. By theorem 3.2, ϕ is then birational. □

Corollary 4.3. *If $\mathbb{M}$ is an implicitization matrix and ϕ is not birational, then $\det(\mathbb{M}) = F^{\delta}$, with $\delta > 1$.*

Proof. If $\delta = 1$ then $p = 0$ in theorem 4.2 and we would have that ϕ is birational. □

We now give a (non-exhaustive) list of known implicitization matrices; they can be divided into two distinct groups: the moving surfaces matrices and the resultant matrices.

Moving surfaces matrices. All matrices coming from the moving surfaces method introduced by Sederberg [20] can be used; they are by definition implicitization matrices. A lot of recent works have extended the foundational work of Sederberg. At this time, algorithms to construct an implicitization matrix of a given parameterization ϕ are available if:

- ϕ has no base points over $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ (see [8]),
- ϕ has l.c.i. base points (plus some other technical conditions) over $\mathbb{P}^2$ (see [3]) or $\mathbb{P}^1 \times \mathbb{P}^1$ (see [1]),
- ϕ has no base points on a certain projective toric variety (see [11]).

This method of moving surfaces is still under development (see for instance [6, 23]) and the list above will probably be extended in a near future.

Resultant matrices. The implicitization problem can be solved using some resultant computations, as it is illustrated in [5, 13]. The computation of a resultant often involves the construction of a matrix which sometimes is an implicitization matrix. At their most typical, resultant matrix give universal formulas for a particular class of parameterizations. They are thus very interesting since they allow the design of a pre-computed inversion formula of a specified class of surfaces. Given a parameterization ϕ, resultant matrices which are implicitization matrices of ϕ are known if

- if ϕ parameterizes a Steiner surface (see [15], and also [2])
- if ϕ has no base point on $\mathbb{P}^1 \times \mathbb{P}^1$ (see [10] and [12][1])
- if ϕ has no base point in a certain toric variety (see [16])

In the list above we only mentioned general constructions and it should be pointed out that a lot of constructions adapted to particular cases exist.

5. Some illustrative examples

Below we exhibit three examples in order to illustrate our matrix-based approach to the inversion and properness problem.

5.1. This example is taken from [20]. Consider the following parameterization of a cubic surface with 6 base points:

$$\begin{aligned}
p_1 &= t_1^2t_2 + 2t_2^3 + t_1^2t_3 + 4t_1t_2t_3 + 4t_2^2t_3 + 3t_1t_3^2 + 2t_2t_3^2 + 2t_3^3,\\
p_2 &= -t_1^3 - 2t_1t_2^2 - 2t_1^2t_3 - t_1t_2t_3 + t_1t_3^2 - 2t_2t_3^2 + 2t_3^3,\\
p_3 &= -t_1^3 - 2t_1^2t_2 - 3t_1t_2^2 - 3t_1^2t_3 - 3t_1t_2t_3 + 2t_2^2t_3 - 2t_1t_3^2 - 2t_2t_3^2,\\
p_4 &= t_1^3 + t_1^2t_2 + t_2^3 + t_1^2t_3 + t_2^2t_3 - t_1t_3^2 - t_2t_3^2 - t_3^3.
\end{aligned}$$

If we pick the following moving planes:

$$\begin{aligned}
M_1 &:= t_1X_1 + t_2X_2 + t_3X_3,\\
M_2 &:= t_1(X_2 + X_4) + t_2(2X_2 - X_3) + t_3(X_2 + 2X_4),\\
M_3 &:= t_1(X_3 - X_2) + t_2(-X_1 + 2X_4) + t_3(X_1 - X_2),
\end{aligned}$$

[1] see also [8] where it is explained that the rows of Dixon and Bezoutian matrices are moving planes, and hence that any hybrid combination of them give an implicitization matrix

we can construct the following matrix indexed by the monomials t_1, t_2, t_3, whose determinant is the implicit equation of the surface:

$$\begin{pmatrix} X_3 - X_2 & -X_1 + 2X_4 & X_1 - X_2 \\ -X_2 - X_4 & X_3 - 2X_2 & -X_2 - 2X_4 \\ X_1 & X_2 & X_3 \end{pmatrix}.$$

If we erase the first row of this matrix, Theorem 4.2 tells us that the inverse of this parameterization equals $(\Delta_{t_1} : \Delta_{t_2} : \Delta_{t_3})$, which itself equals

$$\left(\begin{vmatrix} X_3 - 2X_2 & -X_2 - 2X_4 \\ X_2 & X_3 \end{vmatrix} : \begin{vmatrix} X_2 + X_4 & -X_2 - 2X_4 \\ -X_1 & X_3 \end{vmatrix} : \begin{vmatrix} -X_2 - X_4 & X_3 - 2X_2 \\ X_1 & X_2 \end{vmatrix} \right).$$

5.2. The following example appears in [11]. Consider the following parameterization:

$$\begin{aligned} p_1 &= t_3^3 + t_1 t_3^2 - t_2 t_3^2 + t_1 t_2 t_3 - t_1^2 t_2 - t_1 t_2^2, \\ p_2 &= t_3^3 + t_1 t_3^2 - t_2 t_3^2 - t_1 t_2 t_3 + t_1^2 t_2 - t_1 t_2^2, \\ p_3 &= t_3^3 - t_1 t_3^2 + t_2 t_3^2 - t_1 t_2 t_3 - t_1^2 t_2 + t_1 t_2^2, \\ p_4 &= t_3^3 - t_1 t_3^2 - t_2 t_3^2 + t_1 t_2 t_3 - t_1^2 t_2 + t_1 t_2^2. \end{aligned}$$

There are two moving planes and one moving quadric of degree one that follow the surface:

$$\begin{aligned} &M_1 := t_1(X_4 - X_3) + t_3(X_1 - X_2), \\ &M_2 := t_1(X_2 - X_3 + 2X_4) + t_2(X_2 + X_3) + t_3(-X_2 - X_3 + 2X_4), \\ &M_3 := t_1(X_1X_2 + X_1X_3) + t_2(X_1X_3 - X_1X_4 + X_2^2 + X_2X_4) + \\ &+ t_3(-2X_1^2 + X_2^2 + X_2X_4 - X_3X_4 + X_4^2). \end{aligned}$$

In this case, the transpose of the following matrix is an implicitization matrix

$$\begin{pmatrix} X_4 - X_3 & 0 & X_1 - X_2 \\ X_2 - X_3 + 2X_4 & X_2 + X_3 & -X_2 - X_3 + 2X_4 \\ X_1X_2 + X_1X_3 & X_1X_3 - X_1X_4 + X_2^2 + X_2X_4 & -2X_1^2 + X_2^2 + X_2X_4 - X_3X_4 + X_4^2 \end{pmatrix}.$$

By erasing the last row, we obtain an inversion $(\Delta_{t_1} : \Delta_{t_2} : \Delta_{t_3})$ where

$$\Delta_{t_1} = \begin{vmatrix} 0 & X_1 - X_2 \\ X_2 + X_3 & -X_2 - X_3 + 2X_4 \end{vmatrix},$$

$$\Delta_{t_2} = \begin{vmatrix} X_3 - X_4 & X_1 - X_2 \\ -X_2 + X_3 - 2X_4 & -X_2 - X_3 + 2X_4 \end{vmatrix},$$

$$\Delta_{t_3} = \begin{vmatrix} X_4 - X_3 & 0 \\ X_2 - X_3 + 2X_4 & X_2 + X_3 \end{vmatrix}.$$

5.3. We will compute the inverse of the surface parameterized by

$$X_1 = \tfrac{t_1^2}{t_1^2+t_2^2+1} \qquad X_2 = \tfrac{2}{t_1^2+t_2^2+1} \qquad X_3 = \tfrac{t_1+t_2}{t_1^2+t_2^2+1}.$$

Note that here we are working with "affine" variables, i.e. we set $X_4 = t_3 = 1$. We will use the Dixon formulation for the resultant. In order to do so, consider the polynomials

$$F_1 = (t_1^2+t_2^2+1)X_1 - 2t_1,\ F_2 = (t_1^2+t_2^2+1)X_2 - 2t_2,\ F_3 = (t_1^2+t_2^2+1)X_3 - (t_1^2+t_2^2-1).$$

The Dixon matrix of the resultant of F_1, F_2, F_3 is the 6×6 matrix

$$\mathbb{D} = \begin{pmatrix} X_1 & 0 & 0 & X_1-1 & 0 & X_1 \\ X_2-2 & 0 & 0 & X_2 & 0 & X_2 \\ X_3 & -1 & -1 & X_3 & 0 & X_3 \\ 0 & -X_2-2X_1+2 & 2X_1 & 0 & -2X_3 & 0 \\ -2X_1+2-X_2 & 0 & -2X_3 & 0 & X_2 & 0 \\ 2X_1 & -2X_3 & 0 & 0 & X_2 & 0 \end{pmatrix}$$

whose columns are indexed by the monomials $1, t_1, t_2, t_1^2, t_1t_2, t_2^2$ (in that order). So, it turns out that the transpose of $\mathbb{D}$ is an implicitization matrix, and we can recover the inverse by deleting any row and considering the ratios $(\Delta_1 : \Delta_{t_1} : \Delta_{t_2})$. For instance, by deleting the last row we get

$$t_1 = \frac{\det\begin{pmatrix} X_1 & 0 & X_1-1 & 0 & X_1 \\ X_2-2 & 0 & X_2 & 0 & X_2 \\ X_3 & -1 & X_3 & 0 & X_3 \\ 0 & 2X_1 & 0 & -2X_3 & 0 \\ -X_2-2X_1+2 & -2X_3 & 0 & X_2 & 0 \end{pmatrix}}{\det\begin{pmatrix} 0 & 0 & X_1-1 & 0 & X_1 \\ 0 & 0 & X_2 & 0 & X_2 \\ -1 & -1 & X_3 & 0 & X_3 \\ -X_2-2X_1+2 & 2X_1 & 0 & -2X_3 & 0 \\ 0 & -2X_3 & 0 & X_2 & 0 \end{pmatrix}},$$

$$t_2 = \frac{\det\begin{pmatrix} X_1 & 0 & X_1-1 & 0 & X_1 \\ X_2-2 & 0 & X_2 & 0 & X_2 \\ X_3 & -1 & X_3 & 0 & X_3 \\ 0 & -X_2-2X_1+2 & 0 & -2X_3 & 0 \\ -X_2-2X_1+2 & 0 & 0 & X_2 & 0 \end{pmatrix}}{\det\begin{pmatrix} 0 & 0 & X_1-1 & 0 & X_1 \\ 0 & 0 & X_2 & 0 & X_2 \\ -1 & -1 & X_3 & 0 & X_3 \\ -X_2-2X_1+2 & 2X_1 & 0 & -2X_3 & 0 \\ 0 & -2X_3 & 0 & X_2 & 0 \end{pmatrix}},$$

that is to say

$$t_1 = \frac{-2X_3(4X_3^2+X_2^2-2X_2)}{-X_2(X_2^2+4X_2X_1-2X_2-4X_3^2)} \quad \text{and} \quad t_2 = \frac{4X_3(X_2+2X_1-2)}{X_2^2+4X_2X_1-2X_2-4X_3^2}.$$

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