Adjunctions on the lattices of partitions and of partial partitions

Christian Ronse

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Abstract The complete lattice $\Pi(E)$ of partitions of a space E has been extended into $\Pi^*(E)$, the one of partial partitions of E (where the space covering axiom is removed). We recall the main properties of $\Pi^*(E)$, and exhibit two adjunctions (residuations) between $\Pi(E)$ and $\Pi^*(E)$. Given two spaces E_1 and E_2 (distinct or equal), we analyse adjunctions between $\Pi^*(E_1)$ and $\Pi^*(E_2)$, in particular those where the lower adjoint applies a set operator to each block of the partial partition; we also show how to build such adjunctions from adjunctions between $\mathcal{P}(E_1)$ and $\mathcal{P}(E_2)$ (the complete lattices of subsets of E_1 and E_2). They are then extended to adjunctions between $\Pi(E_1)$ and $\Pi(E_2)$. We obtain as particular case the adjunction on $\Pi(E)$ that was defined by Serra (for the upper adjoint) and Ronse (for the lower adjoint). We also study dilations from $\Pi^*(E_1)$ to an arbitrary complete lattice L; a particular case is given, for $L \subseteq [0, +\infty]$, by ultrametrics; then the adjoint erosion provides the corresponding hierarchy. We briefly discuss possible applications in image processing and in data clustering.

Keywords adjunction · dilation · erosion · partition · partial partition · one-block-preserving operator · blockwise extension of a set operator · ultrametric · hierarchy · clustering

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C. Ronse
LSIIT UMR 7005 CNRS-UdS,
Parc d'Innovation, Boulevard Sébastien Brant, BP 10413, 67412 ILLKIRCH CEDEX, FRANCE
E-mail: cronse@unistra.fr, URL: http://lsiit-miv.u-strasbg.fr/

1 Introduction

The notion of *adjunction* (also called *residuation*) plays an important role in lattice theory [5,8,13], and also in its applications to *mathematical morphology* [16,22,33,34], a branch of image processing. Let us recall it here.

Given two sets A and B, we will write $(\alpha,\beta) : A \rightleftharpoons B$, or say that (α,β) is $A \rightleftharpoons B$, if α is a map $A \to B$ and β is a map $B \to A$. Note that given a map ψ, we write $\psi(x)$ for the image of x by ψ (for example [14] writes $x\psi$). Following [16], the composition of a map ψ followed by a map ξ is written $\xi\psi$ rather than $\xi \circ \psi$ (because the symbol $\circ$ is used for the opening $X \circ B$ of X by B), thus we have $\xi\psi : x \mapsto \xi(\psi(x))$ (note that the composition must be read from right to left).

Let L and M be two posets (partially ordered sets), and consider $(\varepsilon,\delta) : M \rightleftharpoons L$. We say that (ε,δ) is an *adjunction* [13] (or a *residuation* [5]) if

$$\forall x \in L,\ \forall y \in M, \qquad \delta(x) \leq y \iff x \leq \varepsilon(y) \ . \tag{1}$$

Equivalently (ε,δ) is an adjunction if and only if δ and ε are isotone ($\forall x,x' \in L$, $x \leq x' \ \Rightarrow\ \delta(x) \leq \delta(x')$, $\forall y,y' \in M$, $y \leq y' \ \Rightarrow\ \varepsilon(y) \leq \varepsilon(y')$), $\delta\varepsilon$ is anti-extensive ($\forall y \in M$, $\delta\varepsilon(y) \leq y$) and $\varepsilon\delta$ is extensive ($\forall x \in L$, $\varepsilon\delta(x) \geq x$). Then δ is called the *lower adjoint* of ε and ε is called the *upper adjoint* of δ [13] ([5] says: δ is *residuated by* ε and ε is the *residual* of δ).

A close concept is that of a *Galois connection*, that is a pair $(\beta,\alpha) : M \rightleftharpoons L$ such that for $x \in L$ and $y \in M$, $y \leq \alpha(x) \ \Leftrightarrow\ x \leq \beta(y)$; equivalently, α and β are antitone ($\forall x,x' \in L$, $x \leq x' \ \Rightarrow\ \alpha(x) \geq \alpha(x')$, $\forall y,y' \in M$, $y \leq y' \ \Rightarrow\ \beta(y) \geq \beta(y')$), and $\alpha\beta$ and $\beta\alpha$ are extensive. It has been used widely in various theoretical and practical contexts [8,14].

Suppose now that L and M are complete lattices with universal bounds $\mathbf{0},\mathbf{1}$. In the evocative terminology of mathematical morphology [16,34], a map which commutes with the supremum operation (resp., with the infimum operation) is called a *dilation* (resp., an *erosion*). Thus a dilation $\delta : L \to M$ satisfies

$$\forall x_i \in L \ (i \in I), \qquad \delta\Big(\bigvee_{i\in I} x_i\Big) = \bigvee_{i\in I} \delta(x_i) \ , \tag{2}$$

in particular for $I = \emptyset$, $\delta(\mathbf{0}) = \mathbf{0}$; on the other hand an erosion $\varepsilon : M \to L$ satisfies

$$\forall y_i \in M \ (i \in I), \qquad \varepsilon\Big(\bigwedge_{i\in I} y_i\Big) = \bigwedge_{i\in I} \varepsilon(y_i) \ , \tag{3}$$

in particular for $I = \emptyset$, $\varepsilon(\mathbf{1}) = \mathbf{1}$. In the more precise, but less intuitive terminology of lattice theory, one says that δ is a *complete join-morphism* and ε is a *complete meet-morphism*. It is well-known [13,16] that in an adjunction (ε,δ), ε is an erosion and δ is a dilation, $\delta\varepsilon\delta = \delta$, $\varepsilon\delta\varepsilon = \varepsilon$. Conversely, given a dilation $\delta : L \to M$, there is a unique erosion $\varepsilon : M \to L$ such that (ε,δ) is an adjunction, and given an erosion $\varepsilon : M \to L$, there is a unique dilation $\delta : L \to M$ such that (ε,δ) is an adjunction.

In the Euclidean space $E = \mathbf{R}^n$ or its digital counterpart $E = \mathbf{Z}^n$, adjunctions can be built from the Minkowski operations. For every $p \in E$, the *translation* by p is the map $E \to E : x \mapsto x + p$; it transforms any subset X of E into its *translate by* p, $X_p = \{x + p \mid x \in X\}$. Then the *Minkowski addition* $\oplus$ [25] and *Minkowski subtraction* $\ominus$ [15] are defined as follows: for any $X, B \in \mathcal{P}(E)$ we set

$$\begin{aligned} X \oplus B &= \bigcup_{b \in B} X_b = \bigcup_{x \in X} B_x = \{x + b \mid x \in X, b \in B\} \ ; \\ X \ominus B &= \bigcap_{b \in B} X_{-b} = \{p \in E \mid B_p \subseteq X\} \ . \end{aligned} \tag{4}$$

We define then the *dilation by* B, $\delta_B : \mathcal{P}(E) \to \mathcal{P}(E) : X \mapsto X \oplus B$, and the *erosion by* B, $\varepsilon_B : \mathcal{P}(E) \to \mathcal{P}(E) : X \mapsto X \ominus B$ [16], see Figure 1. The set B is called the *structuring element* [22,33,34]. It is well-known that for the complete lattice $\mathcal{P}(E)$ (ordered by inclusion $\subseteq$), $(\varepsilon_B, \delta_B)$ is an adjunction, δ_B is a dilation (complete join-morphism) and ε_B is an erosion (complete meet-morphism).

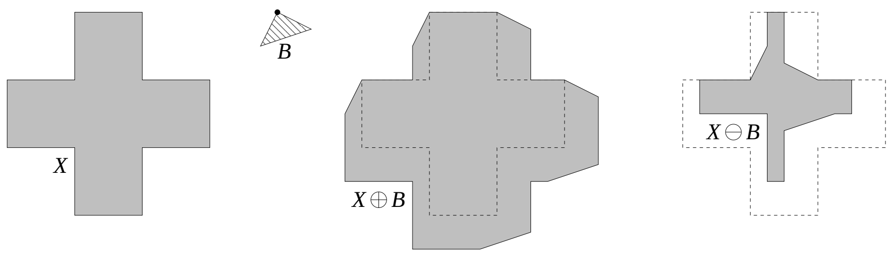

Fig. 1 From left to right: the set X, the structuring element B (the posision of the origin in E is indicated by a black dot), the dilation $X \oplus B$ of X by B, and the erosion $X \ominus B$ of X by B.

Note that this terminology follows the official standard introduced by Sternberg and confirmed by Heijmans and Ronse [4,16]; in the older books by Matheron [22] and Serra [33,34], the definitions given for the Minkowski operations, the dilation and erosion were slightly different, in that for some of them the structuring element has to be replaced by its transpose $\check{B} = \{-b \mid b \in B\}$; there are similar differences in the works of Soille [39]. See [17,38] for a discussion of these differences.

Adjunctions have been built for various types of spatial objects: in the complete lattice of numerical functions [16,33,34], for fuzzy sets and modal logic [4], etc.

Another complete lattice is the set $\Pi(E)$ of partitions of a set E, ordered by *refinement* ($\pi \leq \pi'$ if and only if every block of π is included in a block of π'). Since the seminal works of Dubreil [11] and Ore [28], many authors have studied it, notably analysing conditions for some identities to hold (such as the modular one), or investigating geometrical and combinatorial properties of $\Pi(E)$ when E is finite. However almost nothing has been done towards the analysis of basic lattice-theoretical operations on partitions (closures, openings, adjunctions). Let us note in this respect the study by Jordens and Sturm [19,20] of a relation between closure operators on partitions and closure operators on sets (in fact, a relation between their respective closure systems, that is, families of invariants). Recently, the topic has been revived by Serra and the author [29,31,37] in relation with image segmentation.

This paper is devoted to the construction and analysis of adjunctions on partitions. In [36], Serra introduced an erosion on partitions. Let ε be an erosion on $\mathcal{P}(E)$ such that $\varepsilon(\emptyset) = \emptyset$ (for example, when $E = \mathbf{R}^n$ or $\mathbf{Z}^n$, the erosion by a non-empty structuring element B). Then one derives from ε an erosion $\widehat{\varepsilon}$ on $\Pi(E)$; for a partition π, $\widehat{\varepsilon}(\pi)$ is obtained as follows:

1. Erode by ε all blocks of π, and discard all empty eroded blocks;
2. all points $p \in E$ which do not belong to an eroded block are constituted into singleton blocks $\{p\}$.

In other words,

$$\widehat{\varepsilon}(\pi) = \big\{\varepsilon(C) \bigm| C \in \pi,\ \varepsilon(C) \neq \emptyset\big\} \cup \Big\{\{p\} \Bigm| p \in E \setminus \Big(\bigcup_{C\in\pi} \varepsilon(C)\Big)\Big\} \ .$$

Serra [36] expressed $\widehat{\varepsilon}$ in terms of the *class* associated to a point (the unique block containing it); write $\mathsf{Cl}_\pi(p)$ for the class of point p in the partition π; then Serra stated that $\mathsf{Cl}_{\widehat{\varepsilon}(\pi)}(p) = \varepsilon(\mathsf{Cl}_\pi(p))$ if $p \in \varepsilon(\mathsf{Cl}_\pi(p))$, and $\mathsf{Cl}_{\widehat{\varepsilon}(\pi)}(p) = \{p\}$ if $p \notin \varepsilon(\mathsf{Cl}_\pi(p))$, but this formulation *is valid only if* ε *is anti-extensive* (i.e., $\varepsilon(X) \subseteq X$ for all $X \in \mathcal{P}(E)$). Indeed, if ε is not anti-extensive, we may have $p \notin \varepsilon(\mathsf{Cl}_\pi(p))$ but $p \in \varepsilon(\mathsf{Cl}_\pi(q))$ for some $\mathsf{Cl}_\pi(q) \neq \mathsf{Cl}_\pi(p)$, and in this case we have $\mathsf{Cl}_{\widehat{\varepsilon}(\pi)}(p) = \varepsilon(\mathsf{Cl}_\pi(q))$.

Then [30] described the lower adjoint. Let δ be the dilation on $\mathcal{P}(E)$ that is the lower adjoint of ε; the fact that $\varepsilon(\emptyset) = \emptyset$ is equivalent to $\forall X \in \mathcal{P}(E)$, $X \neq \emptyset \;\Rightarrow\; \delta(X) \neq \emptyset$. Then we can derive from δ a dilation $\widehat{\delta}$ on $\Pi(E)$; for a partition π, $\widehat{\delta}(\pi)$ is obtained as follows:

1. Remove all singleton blocks in π;
2. dilate by δ the remaining blocks;
3. recursively fuse all overlapping dilated blocks, until only disjoint blocks remain;
4. all points $p \in E$ which do not belong to a block are constituted into singleton blocks $\{p\}$.

In other words, $\widehat{\delta}(\pi)$ is the least partition π^* such that for every non-singleton block C of π, $\delta(C)$ is included in one block of π^*. Furthermore, given that (ε, δ) is an adjunction on $\mathcal{P}(E)$, [30] stated (without proof) that $(\widehat{\varepsilon}, \widehat{\delta})$ will be an adjunction on $\Pi(E)$.

We see that the operators $\widehat{\varepsilon}$ and $\widehat{\delta}$ involve respectively addition and removal of singleton blocks. The underlying reason will be understood later, let us say now that these operators combine several operators on partial partitions. A *partial partition* of E is a family π of non-empty and mutually disjoint subsets of E, called *blocks*; equivalently, it is a partition of a subset of E.

Let $\Pi^*(E)$ be the set of all partial partitions of E; in other words, $\Pi^*(E) = \bigcup_{A \in \mathcal{P}(E)} \Pi(A)$. Now $\Pi^*(E)$, with the same refinement order as $\Pi(E)$, is a complete lattice; in fact, the non-void infimum and supremum operations are the same in $\Pi^*(E)$ and $\Pi(E)$. The lattice $\Pi^*(E)$ was studied more than 30 years ago by Czekoslovak mathematicians: Draškovičová [9,10], and to a lesser extent Sturm [43].

In the context of image processing, the lattice $\Pi^*(E)$ and its main properties were "rediscovered" by the author [29]. Indeed, it was understood that the framework of partial partitions is more flexible than the one of partitions, many operations on partitions require the use of partial partitions. More fundamentally, it was noticed that several image segmentation algorithms produce a partial partition instead of a partition.

Therefore there is a practical interest for the study of the lattice-theoretical operators on partitions or partial partitions of a space E, constructed from similar operators on subsets of E. In this paper, we study adjunctions on partitions and partial partitions, in particular we describe the general form of dilations $\Pi^*(E_1) \to \Pi^*(E_2)$ and $\Pi(E_1) \to \Pi(E_2)$, in particular those that apply a set operator to each block, then we show how to build adjunctions $\Pi^*(E_2) \rightleftharpoons \Pi^*(E_1)$ and $\Pi(E_2) \rightleftharpoons \Pi(E_1)$ from adjunctions $\mathcal{P}(E_2) \rightleftharpoons \mathcal{P}(E_1)$. We also describe some possible applications of dilations and erosions on partial partitions to image segmentation and data clustering.

1.1 Paper organization

Subsection 1.2 summarizes our terminology and notation, mostly based on that of mathematical morphology [16,22,33,34], see Table 1. We follow it with Table 2 listing the notation introduced in this paper (in the order of first appearance). In Section 2 we recall the basic facts about partial partitions and the complete lattice that they make [9,10,29]; then we exhibit adjunctions $\Pi^*(E) \rightleftharpoons \Pi(E)$, $\Pi(E) \rightleftharpoons \Pi^*(E)$ and $\Pi^*(E) \rightleftharpoons \Pi^*(E)$ (Theorem 9). Next in Section 3, given two spaces E_1 and E_2 (equal or distinct), we relate adjunctions $\Pi(E_2) \rightleftharpoons \Pi(E_1)$ to those $\Pi^*(E_2) \rightleftharpoons \Pi^*(E_1)$ (Theorem 12), and we characterize dilations $\Pi^*(E_1) \to L$ and $\Pi(E_1) \to L$ for an arbitrary complete lattice L (Theorem 13); then we analyse in detail dilations $\Pi^*(E_1) \to \Pi^*(E_2)$ that apply a set operator to each block (Theorem 17), and finally we show how to construct adjunctions $\Pi^*(E_2) \rightleftharpoons \Pi^*(E_1)$ and $\Pi(E_2) \rightleftharpoons \Pi(E_1)$ from adjunctions $\mathcal{P}(E_2) \rightleftharpoons \mathcal{P}(E_1)$ (Theorem 23); for $E_1 = E_2 = E$, we obtain the adjunction $(\widehat{\varepsilon}, \widehat{\delta})$ from [30,36] described above. Section 4 characterizes dilations $\Pi^*(E_1) \to L$ and $\Pi(E_1) \to L$ in terms of *triangular maps*; a particular case will be given by ultrametric distances, with the corresponding hierarchy being given by the adjoint erosion $L \to \Pi(E_1)$. Possible applications in image processing and segmentation, or in clustering, are discussed in Section 5. The conclusion summarizes our results.

1.2 Terminology and notation

Mathematical morphology has developed an exhaustive terminology for the theory of operators on complete lattices [16,22,33,34]; we follow it generally, except when it leads to a confusion with the usual terminology in lattice theory, see Table 1 below. The two algebraic structures considered are the *poset* and the *complete lattice*; in other words, whenever we mention "the lattice X", this means implicitly "the complete lattice X".

Throughout this paper, we consider a "space" E, whose elements are called "points"; in fact E is an arbitrary set of size at least 2, although in practice E will be the Euclidean space $\mathbf{R}^n$, the digital space $\mathbf{Z}^n$, or a bounded interval in such spaces; sometimes we consider two spaces E_1 and E_2, that can be either distinct or equal (for example, $E_1 = \mathbf{R}^m$ and $E_2 = \mathbf{R}^n$, or $E_1 = \mathbf{R}^n$ and $E_2 = \mathbf{Z}^n$). Points of a space E (or of E_1, E_2) will be written $p, q, r, \ldots$, while subsets of E will be designated by $A, B, \ldots, Y, Z$ (except the empty set $\emptyset$). Partial partitions of E will be written $\pi, \pi', \pi_1, \pi^1, \ldots$.

An abstract complete lattice will be written $L, M, \ldots$, and its elements will be denoted with lower-case letters $a, b, \ldots, y, z$, except the least and greatest elements written $\mathbf{0}$ and $\mathbf{1}$ respectively; subsets of L will be designated by upper-case letters $A, B, \ldots, Y, Z$. Write $\leq$ for the order, and $\prec$ / $\succ$ for the predecessor / successor relation: $x \prec y$, or equivalently $y \succ x$, means that $x < y$ but there is no z with $x < z < y$; we say then that y *covers* x. Every complete lattice will be supposed to have at least 2 elements, in other words, $\mathbf{0} < \mathbf{1}$.

Recall that given two sets A and B, we will write $(\alpha, \beta) : A \rightleftharpoons B$, or say that (α, β) is $A \rightleftharpoons B$, if α is a map $A \to B$ and β is a map $B \to A$.

Morphological and standard terminology for lattice-theoretical concepts is summarized in Table 1; in italic we show our choice, it follows the morphological terminology, except when the term designates another concept in standard lattice-theoretical usage. For instance, morphology follows [3] in calling *atomic* a lattice where each non-zero element is a supremum of atoms; however [14] calls such a lattice *atomistic*, while in [5,13,14], the word "atomic" designates a lattice where each non-zero element majorates an atom, a weaker property; here we will abide by the traditional terminology, since we need both concepts: the lattice $\Pi^*(E)$ is atomic but not atomistic !

Note the many existing denominations for a subset of a complete lattice that is closed under arbitrary infima (in particular for the empty infimum, it contains $\mathbf{1}$), we choose *Moore family*; dually we call a *dual Moore family* a subset closed under arbitrary suprema (in particular, it contains $\mathbf{0}$) [4]. A Moore family or dual Moore family is itself a complete lattice for the order $\leq$. A subset of a complete lattice that is both a Moore family and a dual Moore family is a *complete sublattice* of that lattice.

Table 1 Morphological and traditional terminology for lattice-theoretical notions *(in italics, the terminology used in this paper)*

Morphological	*Traditional*	*Meaning*
sup-generating family	sup-basis	$S:\ \forall x,\ x = \bigvee\{s \in S \mid s \leq x\}$
inf-generating family	inf-basis	$S:\ \forall x,\ x = \bigwedge\{s \in S \mid s \geq x\}$
atom		$a:\ \mathbf{0} \prec a$
atomic [3]	*atomistic* [14]	L: $\forall x,\ x = \bigvee\{a \mid \mathbf{0} \prec a \leq x\}$
	atomic [5,13,14]	L: $\forall x\ \exists a,\ \mathbf{0} \prec a \leq x$
dual atom		$a:\ a \prec \mathbf{1}$
dually atomic	*dually atomistic*	L: $\forall x,\ x = \bigwedge\{a \mid x \leq a \prec \mathbf{1}\}$
	dually atomic	L: $\forall x\ \exists a,\ x \leq a \prec \mathbf{1}$
co-prime	*join-prime*	$x:\ x \leq y \vee z \Rightarrow x \leq y$ or $x \leq z$
strong co-prime	*complete join-prime*	$x:\ x \leq \bigvee_{i \in I} y_i \Rightarrow \exists i \in I,\ x \leq y_i$
inf-closed family, *Moore family* [4]	closure system [13], closure subset [5], closure range	$F:\ X \subseteq F \Rightarrow \bigwedge X \in F$
sup-closed family, *dual Moore family* [4]	kernel system, dual closure subset [5]	$F:\ X \subseteq F \Rightarrow \bigvee X \in F$
	directed subset [13]	$D:\ p, q \in D \Rightarrow \exists r \in D,\ p, q \leq r$
operator [16]	map [7], mapping [5], function [13]	$\psi : L \to M$
increasing [22,33]	*isotone* [5]	$\psi:\ x \leq y \Rightarrow \psi(x) \leq \psi(y)$
decreasing [16]	*antitone* [5]	$\psi:\ x \leq y \Rightarrow \psi(x) \geq \psi(y)$
dilation [34]	complete join-morphism	$\delta:\ \delta\Big(\bigvee_{i \in I} x_i\Big) = \bigvee_{i \in I} \delta(x_i)$
erosion [34]	complete meet-morphism	$\varepsilon:\ \varepsilon\Big(\bigwedge_{i \in I} x_i\Big) = \bigwedge_{i \in I} \varepsilon(x_i)$
	complete morphism	both dilation and erosion
adjunction [13]	residuation [5]	$(\varepsilon, \delta):\ \delta(x) \leq y \Leftrightarrow x \leq \varepsilon(y)$
lower adjoint [13]	residuated [5]	$\delta:\ (\varepsilon, \delta)$ adjunction
upper adjoint [13]	residual [5]	$\varepsilon:\ (\varepsilon, \delta)$ adjunction
extensive [3,22]	increasing [7]	$\psi:\ \psi(x) \geq x$
anti-extensive [22]	intensive, contracting, decreasing	$\psi:\ \psi(x) \leq x$
idempotent		$\psi:\ \psi(\psi(x)) = \psi(x)$
closing [22]	*closure* [5]	$\varphi:\ x \leq \varphi(y) \Leftrightarrow \varphi(x) \leq \varphi(y)$
opening [22]	dual closure [5], kernel operator [13]	$\gamma:\ x \geq \gamma(y) \Leftrightarrow \gamma(x) \geq \gamma(y)$
identity	*identity operator*	$\mathbf{id} : x \mapsto x$
monoid		$\mathsf{M}:\ \mathbf{id} \in \mathsf{M},\ \psi, \xi \in \mathsf{M} \Rightarrow \psi\xi \in \mathsf{M}$

Recall that for a poset P, a *directed subset* [13] of P is a non-empty $D \subseteq P$ such that every *finite* subset X of D has an upper bound (is majorated) in D.

Given two complete lattices L and M (equal or different), a map $L \to M$ is called an *operator*. Operators will be designated by lower-case Greek letters $\alpha, \ldots, \omega$ (except π, reserved for partial partitions). We write $\psi(x)$ for the image of x by ψ (for example [14] writes $x\psi$); thus the composition of operators is read from right to left: given $\psi : L \to M$ and $\xi : M \to N$, the *composition of* ψ *followed by* ξ is $\xi\psi : L \to N : x \mapsto \xi(\psi(x))$; when needed for typographical clarity, we will write $\xi \cdot \psi$ for $\xi\psi$ (we avoid the traditional notation $\xi \circ \psi$, since $\circ$ is used for the opening $X \circ B = (X \ominus B) \oplus B$). The set M^L of operators $L \to M$, with componentwise order: $\psi \leq \xi$ if and only if $\psi(x) \leq \xi(x)$ for all $x \in L$, is a complete lattice with componentwise supremum and infimum: $\left[\bigvee_{i \in I} \psi_i\right](x) = \left[\bigvee_{i \in I} \psi_i(x)\right]$, and similarly for $\bigwedge$.

We follow the algebraists' tradition of calling *isotone* (resp., *antitone*) an operator that preserves (resp., inverts) order; on the other hand morphology uses the analysts' denominations *increasing* (resp., *decreasing*), but this leads to confusion, as some textbooks (for instance [7]) use these terms to mean *extensive* and *anti-extensive*. Note that the isotone operators constitute a complete sublattice of the lattice of operators.

Recall from the introduction the notions of *adjunction* (residuation), *dilation* (complete join-morphism) and *erosion* (complete meet-morphism), see (1,2,3). Note that dilations form a dual Moore family while erosions form a Moore family, thus both families constitute complete lattices, and that the set of adjunctions constitutes a dual isomorphism between these two complete lattices. Recall also the composition rule for adjunctions [16]: given adjunctions $(\varepsilon_i, \delta_i) : L_i \rightleftharpoons L_{i-1}$ $(i = 1, \ldots, n)$, $(\varepsilon_1 \cdots \varepsilon_n, \delta_n \cdots \delta_1)$ is an adjunction $L_n \rightleftharpoons L_0$. In the sequel, we will use the following result, whose proof is left to the reader:

Lemma 1 *Let L be a complete lattice and let M be a complete sublattice of L. Define $\delta, \varepsilon : L \to L$ as follows:*

$$\forall\, x \in L, \quad \delta(x) = \bigwedge\{m \in M \mid m \geq x\} \text{ and } \varepsilon(x) = \bigvee\{m \in M \mid m \leq x\} \ .$$

Then δ is a closure, ε is an opening, and (ε, δ) is an adjunction on L.

A *complete morphism* is an operator that is both a dilation and an erosion, in other words, that is compatible with arbitrary suprema and infima, in particular with the zero and one.

When an operator is $L \to L$, we say that it is "on L". The *identity operator* on L is $\mathbf{id} : L \to L : x \mapsto x$. The set of operators on L, with the law of composition, is thus a *monoid* (i.e., composition is associative and admits the identity as neutral element), and the set of isotone operators is a sub-monoid of it. The power ψ^n of an operator ψ on L is defined by induction: $\psi^0 = \mathbf{id}$, $\psi^{n+1} = \psi\psi^n$.

An operator ψ on a complete lattice is *idempotent* if $\psi^2 = \psi$. A *closure* is an isotone, extensive and idempotent operator; equivalently, it is an operator

φ such that for all $x, y \in L$, $x \leq \varphi(y) \;\Leftrightarrow\; \varphi(x) \leq \varphi(y)$. An *opening* is an isotone, anti-extensive and idempotent operator; equivalently, it is an operator γ such that for all $x, y \in L$, $x \geq \gamma(y) \;\Leftrightarrow\; \gamma(x) \geq \gamma(y)$. For example, in an adjunction $(\varepsilon, \delta) : M \rightleftharpoons L$, $\delta\varepsilon$ is an opening on M and $\varepsilon\delta$ is a closure on L.

Finally, we introduce some new terminology:

Definition 2 Given two complete lattices L and M (equal or distinct) and an operator $\psi : L \to M$, we say that:

- ψ is *upper-regular* if $\psi(\mathbf{0}) = \mathbf{0}$;
- ψ is *lower-regular* if for all $x \in L$, $\psi(x) = \mathbf{0} \;\Rightarrow\; x = \mathbf{0}$;
- ψ is *connective*, if ψ is upper-regular ($\psi(\mathbf{0}) = \mathbf{0}$) and

$$\forall\, B \subseteq L, \quad \Big(B \neq \emptyset,\ \bigwedge B \neq \mathbf{0}\Big) \implies \psi\Big(\bigvee B\Big) = \bigvee_{b \in B} \psi(b) \ ; \tag{5}$$

- ψ *preserves separation* if ψ is upper-regular and

$$\forall\, x, y \in L \setminus \{\mathbf{0}\}, \quad x \wedge y = \mathbf{0} \implies \psi(x) \wedge \psi(y) = \mathbf{0} \ .$$

An adjunction $(\varepsilon, \delta) : M \rightleftharpoons L$ is called *regular* if ε is upper-regular, equivalently, if δ is lower-regular.

When L is atomic, an isotone operator $\psi : L \to M$ is lower-regular if and only if for every atom a of L, $\psi(a) > \mathbf{0}$. Any upper-regular erosion preserves separation: for $x \wedge y = \mathbf{0}$, $\varepsilon(x) \wedge \varepsilon(y) = \varepsilon(x \wedge y) = \varepsilon(\mathbf{0}) = \mathbf{0}$. A composition of lower-regular (resp., upper-regular) operators is lower-regular (resp., upper-regular).

The denomination "connective" was suggested to us by Jean Serra; the meaning behind this wording will be explained in Section 5. Note that $\bigwedge \emptyset = \mathbf{1}$ while $\psi\big(\bigvee \emptyset\big) = \psi(\mathbf{0})$ and $\bigvee_{b \in \emptyset} \psi(b) = \mathbf{0}$, thus the extension of (5) to the case where $B = \emptyset$ is precisely the upper-regularity condition $\psi(\mathbf{0}) = \mathbf{0}$. Connective operators will play a central role in this paper.

Lemma 3 *Let L, M and N be three complete lattices (equal or distinct).*

1. *A connective operator $L \to M$ is isotone.*
2. *A dilation $L \to M$ is connective.*
3. *If $\psi : L \to M$ is connective, and $\delta : M \to N$ is a dilation, then $\delta\psi : L \to N$ is connective.*
4. *If $\psi : L \to M$ and $\xi : M \to N$ are connective, and ψ is lower-regular, then $\xi\psi : L \to N$ is connective.*
5. *The set of connective operators $L \to M$ is a dual Moore family of the lattice of all operators $L \to M$.*

Proof 1. Let $\psi : L \to M$ be connective, and let $a, b \in L$ such that $a \leq b$. Since ψ is upper-regular, for $a = \mathbf{0}$ we have $\psi(a) = \mathbf{0} \leq \psi(b)$. For $a > \mathbf{0}$ we have $a \wedge b = a \neq \mathbf{0}$ and $a \vee b = b$, so $\psi(b) = \psi(a \vee b) = \psi(a) \vee \psi(b)$, which means that $\psi(a) \leq \psi(b)$.

2. A dilation δ satisfies $\delta(\mathbf{0}) = \mathbf{0}$, and for $B \subseteq L$ with $B \neq \emptyset$, $\delta\big(\bigvee B\big) = \bigvee_{b \in B} \delta(b)$. Thus δ is connective.

3. We have $\psi(\mathbf{0}) = \mathbf{0}$ and $\delta(\mathbf{0}) = \mathbf{0}$, so $\delta\psi(\mathbf{0}) = \mathbf{0}$. Take $B \subseteq L$ with $B \neq \emptyset$ and $\bigwedge B \neq \mathbf{0}$; since ψ is connective, we have $\psi\big(\bigvee B\big) = \bigvee_{b \in B} \psi(b)$; since δ is a dilation, we get

$$\delta\psi\Big(\bigvee B\Big) = \delta\Big(\psi\Big(\bigvee B\Big)\Big) = \delta\Big(\bigvee_{b \in B} \psi(b)\Big) = \bigvee_{b \in B} \delta(\psi(b)) = \bigvee_{b \in B} \delta\psi(b) \enspace .$$

4. Since ψ and ξ are upper-regular, we have $\xi\psi(\mathbf{0}) = \xi(\psi(\mathbf{0})) = \xi(\mathbf{0}) = \mathbf{0}$. Let $B \subseteq L$ such that $\bigwedge B \neq \mathbf{0}$; since ψ is lower-regular, $\psi\big(\bigwedge B\big) \neq \mathbf{0}$, and as ψ is isotone, $\psi\big(\bigwedge B\big) \leq \bigwedge_{b \in B} \psi(b)$; thus $\bigwedge_{b \in B} \psi(b) \neq \mathbf{0}$, and as ξ is connective, $\xi\Big(\bigvee_{b \in B} \psi(b)\Big) = \bigvee_{b \in B} \xi(\psi(b))$; but ψ is connective, so $\psi\big(\bigvee B\big) = \bigvee_{b \in B} \psi(b)$, and we conclude that $\xi\big(\psi\big(\bigvee B\big)\big) = \bigvee_{b \in B} \xi\psi(b)$. Therefore $\xi\psi$ is connective.

5. Clearly the empty supremum of maps $L \to M$, namely the constant map $x \mapsto \mathbf{0}$ is a dilation, so it is connective. Given a non-void family of connective maps $\psi_i : L \to M$, $i \in I \neq \emptyset$, let $\psi = \bigvee_{i \in I} \psi_i$; we have $\psi(\mathbf{0}) = \bigvee_{i \in I} \psi_i(\mathbf{0}) = \mathbf{0}$; taking $B \subseteq L$ with $B \neq \emptyset$ and $\bigwedge B \neq \mathbf{0}$, for each $i \in I$ we have $\psi_i\big(\bigvee B\big) = \bigvee_{b \in B} \psi_i(b)$, so

$$\psi\Big(\bigvee B\Big) = \bigvee_{i \in I} \psi_i\Big(\bigvee B\Big) = \bigvee_{i \in I}\Big(\bigvee_{b \in B} \psi_i(b)\Big) = \bigvee_{b \in B}\Big(\bigvee_{i \in I} \psi_i(b)\Big) = \bigvee_{b \in B} \psi(b) \enspace ,$$

hence ψ is connective. $\square$

2 The lattice of partial partitions

We recall the essential facts about partial partitions and the complete lattice that they make. We adopt the terminology of [29]; some results proved there had previously been given in another form in [9,10]. Then we see that singleton blocks play a special role for the supremum operation on partial partitions, and show how they intervene in three adjunctions, respectively $\Pi^*(E) \rightleftharpoons \Pi(E)$, $\Pi(E) \rightleftharpoons \Pi^*(E)$ and on $\Pi^*(E)$

We consider an arbitrary space E having several elements called points. Every binary relation R on E can be identified with the set of ordered pairs $(x, y) \in E^2$ such that $x \; R \; y$; the *support* of R is the subset $\mathsf{supp}(R)$ of E comprising all $p \in E$ such that there is some $q \in E$ with $p \; R \; q$ or $q \; R \; p$. The support of a family $\mathcal{B}$ of subsets of E is the subset $\mathsf{supp}(\mathcal{B})$ of E comprising all points covered by at least one element of $\mathcal{B}$, in other words $\mathsf{supp}(\mathcal{B}) = \bigcup \mathcal{B}$.

A *partial equivalence* on E is a binary relation on E that is symmetric and transitive. Equivalently, it is a relation that forms an equivalence on its support. A partial equivalence is an equivalence relation iff it is reflexive, iff its support is E. A *partial partition* of E is a family π of subsets of E that are non-empty and mutually disjoint, in other words, such that every point of E belongs to at most one member of π. Equivalently, π is a partition of its

Table 2 Notation introduced in this paper

$(\alpha,\beta) : A \rightleftharpoons B$	$\alpha : A \to B$ and $\beta : B \to A$
$\mathbf{0}$ and $\mathbf{1}$	least and greatest elements of a complete lattice
$\mathsf{supp}(R)$	support of the binary relation R on E
$\mathsf{supp}(\mathcal{B})$	support of the family $\mathcal{B}$ of subsets of E
π	a partial partition
$\mathsf{PE}(\pi)$	partial equivalence corresponding to π
Cl_π	partial partition class map associated to π
cl	a partial partition class map
$\mathsf{PP}(\mathsf{cl})$	partial partition associated to cl
$\Pi(E)$	set of all partitions of E
$\Pi^*(E)$	set of all partial partitions of E
$\boldsymbol{\emptyset}$	empty partial partition
$\mathbf{0}_A$	identity partition of A into its singletons
$\mathbf{1}_A$	universal partition of A into a single block
$X \between Y$	$X \cap Y \neq \emptyset$
$\mathbf{1}_\bullet$	$A \mapsto \mathbf{1}_A$
$\mathbf{0}_\bullet$	$A \mapsto \mathbf{0}_A$
grind	block grinding $\pi \mapsto \mathbf{0}_{\mathsf{supp}(\pi)}$
blend	block blending $\pi \mapsto \mathbf{1}_{\mathsf{supp}(\pi)}$
IN	inclusion map $\Pi(E) \to \Pi^*(E) : \pi \mapsto \pi$
FS	$\Pi^*(E) \to \Pi(E) : \pi \mapsto \pi \cup \mathbf{0}_{E\setminus\mathsf{supp}(\pi)}$
RS	$\Pi^*(E) \to \Pi^*(E) : \pi \mapsto \pi \setminus \mathbf{0}_E$
$RSIN$	$RS \cdot IN : \Pi(E) \to \Pi^*(E) : \pi \mapsto RS(\pi)$
$INFS$	$IN \cdot FS : \Pi^*(E) \to \Pi^*(E) : \pi \mapsto FS(\pi)$
$\Pi^0(E)$	set of partial partitions of E without singleton blocks
$\mathsf{B}(\psi)$	$\pi \mapsto \bigvee_{B\in\pi} \mathbf{1}_{\psi(B)} \quad (\psi : \mathcal{P}(E_1) \to \mathcal{P}(E_2))$
$\mathsf{D}(\eta)$	$\pi \mapsto \bigcup_{B\in\pi} \eta(B) \quad (\eta : \mathcal{P}(E_2) \to \Pi^*(E_1)$ preserving separation)
$\pi_\bullet$	$E_2 \to \Pi^*(E_1) : x \mapsto \pi_x$
$\mathcal{P}_2(X)$	set of unordered pairs of distinct elements of X
θ	triangular map
$\theta^\sharp$	strongly triangular map
$X \stackrel{\delta}{\sim} Y$	$\delta(X) \cap \delta(Y) \neq \emptyset$

support $\mathsf{supp}(\pi)$. Every member of a partial partition is called a *block* [28]. A partial partition is a partition of E if and only if its support is E. There is a natural one-to-one correspondence between partial partitions of E and partial equivalences on E; write $\mathsf{PE}(\pi)$ for the partial equivalence on E corresponding to a partial partition π of E; then we have $\mathsf{supp}(\mathsf{PE}(\pi)) = \mathsf{supp}(\pi)$.

Partial equivalences have been used (cf. PER models, equilogical spaces) in programming semantics [1,26,32].

We now turn to the third formalism for a partial partition, in terms of the map associating to each point its class. Consider a map $\mathsf{cl} : E \to \mathcal{P}(E)$, and the following properties that it can satisfy:

(P1a) For any $p \in E$, $p \in \mathsf{cl}(p)$.
(P1b) For any $p \in E$, $\mathsf{cl}(p) = \emptyset$ or $p \in \mathsf{cl}(p)$.
(P2a) For any $p, q \in E$, $q \in \mathsf{cl}(p) \;\Rightarrow\; \mathsf{cl}(p) = \mathsf{cl}(q)$.

A map $\mathsf{cl} : E \to \mathcal{P}(E)$ is called

1. a *partial partition class map* on E if it satisfies (P1b) and (P2a);
2. a *partition class map* on E if it satisfies (P1a) and (P2a).

Note that in a partition class map, (P2a) can be replaced by the well-known:

(P2b) For any $p, q \in E$, $\mathsf{cl}(p) \cap \mathsf{cl}(q) \neq \emptyset \;\Rightarrow\; \mathsf{cl}(p) = \mathsf{cl}(q)$.

Indeed, (P1a) implies the equivalence between (P2a) and (P2b). However, *for a partial partition class map, we cannot replace* (P2a) *by* (P2b), since in general (P2b) *is weaker than* (P2a).

Proposition 4 *[29] There is a one-to-one correspondence between partial partitions on E and partial partition class maps on E, under which:*

– *To every partial partition π is associated the partial partition class map Cl_π given by*

$$\forall\, p \in E, \quad \mathsf{Cl}_\pi(p) = \begin{cases} \emptyset & \text{if } p \notin \mathsf{supp}(\pi) \ ; \\ C & \text{for } p \in C \in \pi, \text{ if } p \in \mathsf{supp}(\pi) \ ; \end{cases} \tag{6}$$

this C being unique.

– *To every partial partition class map cl is associated the partial partition*

$$\mathsf{PP}(\mathsf{cl}) = \{\mathsf{cl}(p) \mid p \in E,\ \mathsf{cl}(p) \neq \emptyset\} \ . \tag{7}$$

Furthermore, π is a partition if and only if Cl_π is a partition class map.

Then $\mathsf{Cl}_\pi(p)$ is called the *class of p in π*, it is either empty, or the unique block of π to which p belongs. Now the partial equivalence relation $\mathsf{PE}(\pi)$ corresponding to π satisfies:

$$\forall\, p, q \in E, \qquad p \;\mathsf{PE}(\pi)\; q \quad \Longleftrightarrow \quad q \in \mathsf{Cl}_\pi(p) \ . \tag{8}$$

Write $\Pi(E)$ for the set of all partitions of E, and $\Pi^*(E)$ for the set of all partial partitions of E. We have $\Pi^*(E) = \bigcup_{A \in \mathcal{P}(E)} \Pi(A)$. Now $\Pi(\emptyset) = \Pi^*(\emptyset)$ has a unique element, the empty partition having no block, we write it $\boldsymbol{\emptyset}$. Then $\boldsymbol{\emptyset} \in \Pi^*(E)$, and for every $p \in E$ we have $\mathsf{Cl}_{\boldsymbol{\emptyset}}(p) = \emptyset$. Formally, $\boldsymbol{\emptyset}$ is identical to the empty set $\emptyset$, but we use a slightly modified notation in order to distinguish the two roles of the empty set, as least element $\emptyset$ of the lattice $\mathcal{P}(E)$, and as least element $\boldsymbol{\emptyset}$ of the lattice $\Pi^*(E)$. For $A \in \mathcal{P}(E)$, let $\mathbf{0}_A$ be the partition of A into its singletons, and $\mathbf{1}_A$ the partition of A into a single block (or no block if $A = \emptyset$):

$$\mathbf{0}_A = \big\{\{p\} \mid p \in A\big\} \qquad \text{and} \qquad \mathbf{1}_A = \{A\} \setminus \{\emptyset\} = \begin{cases} \{A\} & \text{if } A \neq \emptyset \ , \\ \boldsymbol{\emptyset} & \text{if } A = \emptyset \ . \end{cases} \tag{9}$$

Following [28], we call $\mathbf{0}_A$ the *identity partition* of A and $\mathbf{1}_A$ the *universal partition* of A. Note that $\mathbf{0}_\emptyset = \mathbf{1}_\emptyset = \boldsymbol{\varnothing}$.

The well-known *refinement ordering* on partitions [28] extends to partial partitions. Given $\pi_1, \pi_2 \in \Pi^*(E)$, we say that π_1 is *finer* than π_2, or that π_2 is *coarser* than π_1, and write $\pi_1 \leq \pi_2$ (or $\pi_2 \geq \pi_1$), if and only if every block of π_1 is included in a block of π_2:

$$\pi_1 \leq \pi_2 \quad \iff \quad \forall\, C_1 \in \pi_1 \; \exists\, C_2 \in \pi_2,\; C_1 \subseteq C_2 \ .$$

The set of partial equivalences on E, ordered by inclusion, is a complete lattice where the infimum and supremum of a family of partial equivalences is given respectively by their intersection and the transitive closure of their union. Then partial partitions constitute, under refinement ordering, a complete lattice that is isomorphic to the one of partial equivalences:

Proposition 5 *[29] By the bijection between partial partitions and partial equivalences, the refinement relation on partial partitions corresponds to the inclusion order on partial equivalences:*

$$\forall\, \pi_1, \pi_2 \in \Pi^*(E), \quad \pi_1 \leq \pi_2 \iff \mathsf{PE}(\pi_1) \subseteq \mathsf{PE}(\pi_2) \ . \tag{10}$$

Therefore $\big(\Pi^*(E), \leq\big)$ *is a complete lattice, isomorphic to the lattice of partial equivalences. The refinement order corresponds to the inclusion of class maps:*

$$\forall\, \pi_1, \pi_2 \in \Pi^*(E), \qquad \pi_1 \leq \pi_2 \qquad \iff \qquad \forall\, p \in E,\ \mathsf{Cl}_{\pi_1}(p) \subseteq \mathsf{Cl}_{\pi_2}(p) \ . \tag{11}$$

Given a family $\{\pi_i \mid i \in I\}$ *of partial partitions, the class map of their infimum* $\bigwedge_{i \in I} \pi_i$ *is given by intersection of the respective class maps:*

$$\forall\, p \in E, \qquad \mathsf{Cl}_{\bigwedge_{i \in I} \pi_i}(p) = \bigcap_{i \in I} \mathsf{Cl}_{\pi_i}(p) \ . \tag{12}$$

The class map of their supremum $\bigvee_{i \in I} \pi_i$ *is given by* chaining *[28] class maps: for* $p, q \in E$, $q \in \mathsf{Cl}_{\bigvee_{i \in I} \pi_i}(p)$ *if and only if there is some integer* $n \geq 1$ *and a sequence* $x_0, \ldots, x_n$ *in* E *with* $x_0 = p$ *and* $x_n = q$*, such that for each* $t = 1, \ldots, n$ *there is some* $i(t) \in I$ *with* $x_t \in \mathsf{Cl}_{\pi_{i(t)}}(x_{t-1})$*. The least and greatest partial partitions are* $\boldsymbol{\varnothing}$ *and* $\mathbf{1}_E$*. Furthermore, the* support map $\mathsf{supp} : \Pi^*(E) \to \mathcal{P}(E) : \pi \mapsto \mathsf{supp}(\pi)$ *is a complete morphism.*

Note that (12) and the chaining construction for $\mathsf{Cl}_{\bigvee_{i \in I} \pi_i}(p)$ are also valid for I empty: the empty infimum gives as point class the empty intersection, that is, $\mathsf{Cl}_{\mathbf{1}_E}(p) = E$, while chaining in an empty family of partitions does not give any point, and we get $\mathsf{Cl}_{\boldsymbol{\varnothing}}(p) = \emptyset$.

Following [28], let us define the binary relation $\between$ on $\mathcal{P}(E)$ by $X \between Y \;\Leftrightarrow\; X \cap Y \neq \emptyset$. Given a family $\mathcal{B}$ of non-empty subsets of E, the *partial partition spanned by* $\mathcal{B}$ is $\bigvee_{B \in \mathcal{B}} \mathbf{1}_B$; for any two points $p, q \in E$, we say that p and q are *chained by* $\mathcal{B}$ if p and q belong both to one block of $\bigvee_{B \in \mathcal{B}} \mathbf{1}_B$, in other words if there are $B_1, \ldots, B_n \in \mathcal{B}$ ($n \geq 1$) such that $p \in B_1 \between \cdots \between B_n \ni q$. Then in

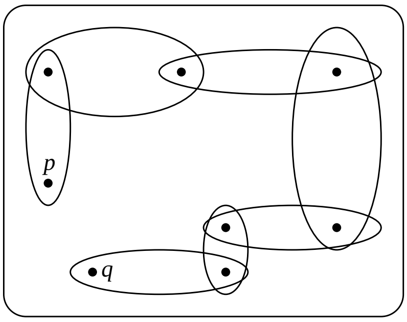

Fig. 2 A block (shown as a rounded rectangle) of the supremum of a family of partial partition is obtained by chaining blocks (shown as ellipses) of these partitions.

a supremum $\bigvee_{i \in I} \pi_i$ of partial partitions, two points $p, q \in E$ belong to the same block if and only if they are chained by $\bigcup_{i \in I} \pi_i$, see Figure 2.

Note that given partial partitions having pairwise disjoint supports ($i \neq j \Rightarrow \mathsf{supp}(\pi_i) \cap \mathsf{supp}(\pi_j) = \emptyset$), their supremum is their union: $\bigvee_{i \in I} \pi_i = \bigcup_{i \in I} \pi_i$. In particular, for any $\pi \in \Pi^*(E)$ we have $\pi = \bigcup_{C \in \pi} \mathbf{1}_C = \bigvee_{C \in \pi} \mathbf{1}_C$.

Given a family $\mathcal{B}$ of non-empty subsets of E, the least partial partition π such that every $B \in \mathcal{B}$ is included in one block of π, is $\bigvee_{B \in \mathcal{B}} \mathbf{1}_B$, it is a partition of $\mathsf{supp}(\mathcal{B})$; two points $p, q \in E$ belong to the same block of $\bigvee_{B \in \mathcal{B}} \mathbf{1}_B$ if and only if they are chained by $\mathcal{B}$.

For $A \in \mathcal{P}(E)$, the non-empty supremum and infimum operations in $\Pi^*(A)$ are inherited from $\Pi^*(E)$; in other words for a non-void $\{\pi_i \mid i \in I\} \subseteq \Pi^*(A)$, $\bigvee_{i \in I} \pi_i$ and $\bigwedge_{i \in I} \pi_i$ are the same in $\Pi^*(A)$ and in $\Pi^*(E)$.

A partial partition on E is a partition if and only if it majorates $\mathbf{0}_E$. Then $\big(\Pi(E), \leq\big)$ is a complete lattice whose non-empty supremum and infimum operations are inherited from $\Pi^*(E)$. For $A \in \mathcal{P}(E)$, the non-empty supremum and infimum operations in $\Pi(A)$ are inherited from $\Pi^*(A)$, hence from $\Pi^*(E)$; in other words a non-void supremum or infimum of partitions of A is the same in $\Pi(A)$, in $\Pi^*(A)$ or in $\Pi^*(E)$.

Let us now consider some properties of the lattice $\Pi^*(E)$. Many of them generalize known properties of $\Pi(E)$.

Proposition 6 *Let $\{\pi_i \mid i \in I\}$ be a non-empty family of partial partitions of E such that for every $p \in E$, the set $\{\mathsf{Cl}_{\pi_i}(p) \mid i \in I\}$ is directed. Then:*

1. *[29] The class map of the supremum of the π_i is the union of their respective class maps:*
$$\forall\, p \in E, \qquad \mathsf{Cl}_{\bigvee_{i \in I} \pi_i}(p) = \bigcup_{i \in I} \mathsf{Cl}_{\pi_i}(p) \ .$$
2. *For any $\pi \in \Pi^*(E)$,*
$$\pi \wedge \Big(\bigvee_{i \in I} \pi_i\Big) = \bigvee_{i \in I} (\pi \wedge \pi_i) \ .$$

These two results hold in particular in the following two situations:

A. The set $\{\pi_i \mid i \in I\}$ is directed.

B. For any two distinct $i, j \in I$, every non-singleton block of π_i is disjoint from every non-singleton block of π_j.

Proof Item 1 was shown in [29]. We prove item 2. Clearly, for every $p \in E$, the set $\{\mathsf{Cl}_{\pi\wedge\pi_i}(p) \mid i \in I\} = \{\mathsf{Cl}_\pi(p) \cap \mathsf{Cl}_{\pi_i}(p) \mid i \in I\}$ is directed. Combining this with item 1 and (12): for $p \in E$,

$$\mathsf{Cl}_{\pi\wedge\left(\bigvee_{i\in I}\pi_i\right)}(p) = \mathsf{Cl}_\pi(p) \cap \mathsf{Cl}_{\bigvee_{i\in I}\pi_i} = \mathsf{Cl}_\pi(p) \cap \left(\bigcup_{i\in I} \mathsf{Cl}_{\pi_i}(p)\right)$$
$$= \bigcup_{i\in I}\big(\mathsf{Cl}_\pi(p) \cap \mathsf{Cl}_{\pi_i}(p)\big) = \bigcup_{i\in I} \mathsf{Cl}_{\pi\wedge\pi_i}(p) = \mathsf{Cl}_{\bigvee_{i\in I}(\pi\wedge\pi_i)}(p) \ .$$

As explained in [29], if $\{\pi_i \mid i \in I\}$ is directed, then $\{\mathsf{Cl}_{\pi_i}(p) \mid i \in I\}$ is directed for each $p \in E$, so item A holds. Now in item B: for every $p \in E$, the set $\{\mathsf{Cl}_{\pi_i}(p) \mid i \in I\}$ contains at most one class of size > 1, all others being the singleton $\{p\}$ or the empty set; hence it is directed. $\square$

A particular instance of situation A is when $\{\pi_i \mid i \in I\}$ is a chain; for this case [43] gave the translation of result 1 in terms of partial equivalences, and result 2 was shown in [9]; but then the argument in [6] allows to extend such results from the particular case of a chain to the general situation of a directed set. Also result 2 for situation B was shown in [12] in the case of partitions. An example where result 1 is applied in situation B is that for $B \in \mathcal{P}(E)$ and $\pi \in \Pi^*(E)$, we have $\pi \vee \mathbf{0}_B = \pi \cup \mathbf{0}_{B\setminus\mathsf{supp}(\pi)}$. An example applying result 2 in situation B is that for $\pi, \pi' \in \Pi^*(E)$, we have $\pi \wedge \pi' = \bigvee_{B\in\pi'}(\pi \wedge \mathbf{1}_B)$.

Note that the dual of result 2 does not hold. For $E = \mathbf{R}^n$, take $p, q \in E$ with distance 2 between p and q, let A be the open ball of radius 1 centered about p, let B be the closed ball of radius 1 centered about q, and for $n \in \mathbf{N}^*$, let B_n be the open ball of radius $1 + 1/n$ centered about q; thus the B_n constitute a chain. We have $B = \bigcap_{n\in\mathbf{N}^*} B_n$, $A \cap B = \emptyset$, and for all $n \in \mathbf{N}^*$, $A \cap B_n \neq \emptyset$. Thus the partial partitions $\mathbf{1}_{B_n}$ form a decreasing sequence, but

$$\bigwedge_{n\in\mathbf{N}^*}(\mathbf{1}_A \vee \mathbf{1}_{B_n}) = \bigwedge_{n\in\mathbf{N}^*}(\mathbf{1}_{A\cup B_n}) = \mathbf{1}_{A\cup B}$$
$$> \{A, B\} = \mathbf{1}_A \vee \mathbf{1}_B = \mathbf{1}_A \vee \Big(\bigwedge_{n\in\mathbf{N}^*} \mathbf{1}_{B_n}\Big) \ .$$

Another counterexample for $E = \mathbf{Z}$ was given in [9].

Now we characterize the covering relation on $\Pi^*(E)$. The following result is easy to prove:

Proposition 7 *For $\pi, \pi' \in \Pi^*(E)$, we have $\pi \prec \pi'$ if and only if one of the following holds:*

1. *π' is obtained by merging two blocks of π:*

$$|\pi| \geq 2, \quad \exists C_1, C_2 \in \pi,\ C_1 \neq C_2, \quad \pi' = \big(\pi \setminus \{C_1, C_2\}\big) \cup \{C_1 \cup C_2\} \ .$$

2. π' is obtained by adding a singleton to π:

$$\mathsf{supp}(\pi) \subset E, \quad \exists\, p \in E \setminus \mathsf{supp}(\pi), \quad \pi' = \pi \cup \{\{p\}\} \ .$$

When $\pi, \pi' \in \Pi(E)$, only case 1 is possible.

Next, recall the definition of an *upper semi-modular* lattice: $x \succ x \wedge y \ \Rightarrow\ x \vee y \succ y$, or equivalently, $(x \succ z, y > z) \ \Rightarrow\ (x \leq y$ or $x \vee y \succ y)$ (other equivalent forms are given in [3,14]). It is known [3,14] that $\Pi(E)$ is upper semi-modular. Similarly, *the lattice $\Pi^*(E)$ is upper semi-modular* [9] (this is an easy consequence of Proposition 7). We have then a *height function* for partial partitions with finite support; we set

$$h(\pi) = 2|\mathsf{supp}(\pi)| - |\pi| \ . \tag{13}$$

Indeed, we check that $h(\mathbf{Ø}) = 0$ and that both cases of Proposition 7 give $h(\pi') = h(\pi)+1$. For $E = n$, the total height is $h(\mathbf{1}_E) = 2n-1$; for a partition π, $h(\pi) = 2n - |\pi|$ (but the height of π in the lattice $\Pi(E)$ is $n - |\pi|$).

It is known that the lattice $\Pi(E)$ is complemented [28]. It is easily seen that $\Pi^*(E)$ is not complemented, any $\pi \in \Pi^*(E)$ such that $\emptyset \neq \mathsf{supp}(\pi) \neq E$ has no complement; indeed $\pi \wedge \pi' = \mathbf{Ø}$ implies that $\mathsf{supp}(\pi) \cap \mathsf{supp}(\pi') = \emptyset$, from which we deduce that $\pi \vee \pi' \leq \mathbf{1}_{\mathsf{supp}(\pi)} \vee \mathbf{1}_{E \setminus \mathsf{supp}(\pi)} < \mathbf{1}_E$.

The atoms of $\Pi^*(E)$ are the $\mathbf{1}_{\{p\}}$ ($p \in E$); they are complete join-primes, and when $|E| \geq 3$ no other element of $\Pi^*(E)$ is a join-prime. Given a partial partition π, the supremum of all atoms $\mathbf{1}_{\{p\}} \leq \pi$ is $\mathbf{0}_{\mathsf{supp}(\pi)}$. Thus $\Pi^*(E)$ is atomic, but not atomistic. A sup-generating family of $\Pi^*(E)$ is given by these atoms $\mathbf{1}_{\{p\}}$ ($p \in E$) and the join-irreducible non-atoms $\mathbf{1}_{\{p,q\}}$ ($p, q \in E$, $p \neq q$); in fact it is the least sup-generating family, any other one must contain it. The $\mathbf{1}_{\{p,q\}}$ are necessary to build the non-singleton blocks, while the $\mathbf{1}_{\{p\}}$ are used only to give the singleton blocks; indeed, in chaining, a singleton block is always redundant in the chain, except when the chain is reduced to that singleton. To be more precise, a partial partition π is the supremum of the $\mathbf{1}_{\{p\}}$ for all singleton blocks $\{p\} \in \pi$, and of the $\mathbf{1}_{\{p,q\}}$ for all pairs $\{p, q\} \subseteq B$ for all non-singleton blocks $B \in \pi$. Note that a partial partition π is *compact* (that is, $\pi \leq \bigvee_{i \in I} \pi_i$ implies $\pi \leq \bigvee_{i \in J} \pi_i$ for some *finite* $J \subseteq I$) if and only if it has finite support. Hence $\Pi^*(E)$ is compactly sup-generated (i.e., *algebraic* [14]).

On the other hand, it is well-known that $\Pi(E)$ is sup-generated by the compact atoms $\mathbf{1}_{\{p,q\}} \cup \mathbf{0}_{E \setminus \{p,q\}}$, and a partition is compact if and only if it is a supremum of a finite number of atoms; hence $\Pi(E)$ is a *geometric lattice* [3,14].

The dual atoms of $\Pi^*(E)$ are those of of $\Pi(E)$, namely the partitions $\mathbf{1}_A \cup \mathbf{1}_{E \setminus A}$ for $A \in \mathcal{P}(E) \setminus \{\emptyset, E\}$. Now $\Pi(E)$ is inf-generated by these dual atoms, so it is dually atomistic. On the other hand, for a partial partition π, the infimum of all dual atoms $\geq \pi$ is $\pi \cup \mathbf{0}_{E \setminus \mathsf{supp}(\pi)}$, the least partition $\geq \pi$ (see $FS(\pi)$ below). Thus $\Pi^*(E)$ is dually atomic, but not dually atomistic. The least inf-generating family of $\Pi^*(E)$ is given by these dual atoms $\mathbf{1}_A \cup \mathbf{1}_{E \setminus A}$, $A \in \mathcal{P}(E) \setminus \{\emptyset, E\}$, and the meet-irreducible dual non-atoms $\mathbf{1}_{E \setminus \{p\}}$ for $p \in E$.

Let us now consider two adjunctions between $\varPi^*(E)$ and $\mathcal{P}(E)$. Since the support map $\mathsf{supp} : \varPi^*(E) \to \mathcal{P}(E) : \pi \mapsto \mathsf{supp}(\pi)$ is a complete morphism, it has both an upper and a lower adjoint. Its upper adjoint is the erosion

$$\mathbf{1}_\bullet : \mathcal{P}(E) \to \varPi^*(E) : A \mapsto \mathbf{1}_A \ ,$$

while its lower adjoint is the dilation

$$\mathbf{0}_\bullet : \mathcal{P}(E) \to \varPi^*(E) : A \mapsto \mathbf{0}_A \ ,$$

in other words, for $\pi \in \varPi^*(E)$ and $A \in \mathcal{P}(E)$,

$$\mathsf{supp}(\pi) \subseteq A \iff \pi \leq \mathbf{1}_A \qquad \text{and} \qquad A \subseteq \mathsf{supp}(\pi) \iff \mathbf{0}_A \leq \pi \ .$$

Thus $\mathbf{1}_\bullet$ is an erosion and $\mathbf{0}_\bullet$ is a dilation:

$$\forall, \mathcal{B} \subseteq \mathcal{P}(E), \quad \mathbf{0}_{\bigcup \mathcal{B}} = \bigvee_{B \in \mathcal{B}} \mathbf{0}_B \quad \text{and} \quad \mathbf{1}_{\bigcap \mathcal{B}} = \bigwedge_{B \in \mathcal{B}} \mathbf{1}_B \ . \tag{14}$$

Furthermore, the map supp is surjective, while the maps $\mathbf{1}_\bullet$ and $\mathbf{0}_\bullet$ are injective. Thus $\mathbf{1}_\bullet$ and $\mathbf{0}_\bullet$ are *order-embeddings* of the poset $\mathcal{P}(E)$ into the poset $\varPi^*(E)$: for $A, B \in \mathcal{P}(E)$, $A \subseteq B \ \Leftrightarrow\ \mathbf{0}_A \leq \mathbf{0}_B \ \Leftrightarrow\ \mathbf{1}_A \leq \mathbf{1}_B$; in particular, $\mathbf{1}_A = \boldsymbol{\varnothing} \ \Leftrightarrow\ \mathbf{0}_A = \boldsymbol{\varnothing} \ \Leftrightarrow\ A = \emptyset$, i.e., $\mathbf{1}_\bullet$ and $\mathbf{0}_\bullet$ are are both upper- and lower-regular. Let us also note the following:

$$\forall\, \mathcal{B} \subseteq \mathcal{P}(E), \quad \Big(\mathcal{B} \neq \emptyset, \ \bigcap \mathcal{B} \neq \emptyset\Big) \implies \mathbf{1}_{\bigcup \mathcal{B}} = \bigvee_{B \in \mathcal{B}} \mathbf{1}_B \ . \tag{15}$$

Hence $\mathbf{1}_\bullet$ is connective and lower-regular.

From the two adjunctions $(\mathbf{1}_\bullet, \mathsf{supp})$ and $(\mathsf{supp}, \mathbf{0}_\bullet)$, we deduce two operators on $\varPi^*(E)$:

- the *block blending* closure **blend** : $\pi \mapsto \mathbf{1}_{\mathsf{supp}(\pi)}$, where all blocks of π are merged;
- the *block grinding* opening **grind** : $\pi \mapsto \mathbf{0}_{\mathsf{supp}(\pi)}$, where each block of π is pulverized into its singletons.

Furthermore, $(\mathbf{blend}, \mathbf{grind})$ is an adjunction on $\varPi^*(E)$, since for $\pi, \pi' \in \varPi^*(E)$,

$$\mathbf{0}_{\mathsf{supp}(\pi)} \leq \pi' \iff \mathsf{supp}(\pi) \subseteq \mathsf{supp}(\pi') \iff \pi \leq \mathbf{1}_{\mathsf{supp}(\pi')} \ .$$

We will now describe two adjunctions between $\varPi(E)$ and $\varPi^*(E)$, and a related adjunction on $\varPi^*(E)$. We define 3 operators:

- The inclusion map:

$$IN : \varPi(E) \to \varPi^*(E) : \pi \mapsto \pi \ .$$

- The filling of a partial partition by singleton blocks outside its support:

$$FS : \varPi^*(E) \to \varPi(E) : \pi \mapsto \pi \cup \mathbf{0}_{E \setminus \mathsf{supp}(\pi)} = \pi \vee \mathbf{0}_E \ .$$

– The removal of singleton blocks:

$$RS : \Pi^*(E) \to \Pi^*(E) : \pi \mapsto \{C \in \pi \mid |C| > 1\} = \pi \setminus \mathbf{0}_E \ .$$

Obviously, for $\pi \in \Pi^*(E)$,

$$FS(\pi) = \pi \iff \pi \vee \mathbf{0}_E = \pi \iff \mathbf{0}_E \leq \pi \iff \pi \in \Pi(E) \ .$$

We derive then by composition the two maps:

$$RSIN = RS \cdot IN : \Pi(E) \to \Pi^*(E) : \pi \mapsto RS(IN(\pi)) = RS(\pi) \ ;$$
$$INFS = IN \cdot FS : \Pi^*(E) \to \Pi^*(E) : \pi \mapsto IN(FS(\pi)) = FS(\pi) \ .$$

Lemma 8 *1. For any $\pi, \pi' \in \Pi^*(E)$, $RS(\pi) \leq \pi' \Leftrightarrow \pi \leq FS(\pi')$.*
2. For any $\pi \in \Pi^*(E)$ and $\pi' \in \Pi(E)$, $FS(\pi) \leq \pi' \Leftrightarrow \pi \leq \pi'$.

Proof 1. The two inequalities $RS(\pi) \leq \pi'$ and $\pi \leq FS(\pi')$ mean respectively:

(a) Every non-singleton block of π is included in a block of π'.
(b) Every block of π is included in a block of π' or in a singleton outside $\mathsf{supp}(\pi')$.

If (a) holds, then every block of π either is included in a block of π', or is a singleton; if it is a singleton, either it is in $\mathsf{supp}(\pi')$ hence included in a block of π', or it is outside $\mathsf{supp}(\pi')$ and included in itself; then (b) holds. If (b) holds, any non-singleton block of π may not be included in a singleton, hence it must be included in a block of π', so (a) holds.

2. We have $\pi \leq \pi \vee \mathbf{0}_E = FS(\pi)$, so $FS(\pi) \leq \pi' \Rightarrow \pi \leq \pi'$; as $\pi' \in \Pi(E)$, $\mathbf{0}_E \leq \pi'$, hence $\pi \leq \pi' \Rightarrow FS(\pi) = \pi \vee \mathbf{0}_E \leq \pi'$. □

Theorem 9 *1. $(FS, RSIN)$ is an adjunction $\Pi^*(E) \rightleftharpoons \Pi(E)$.*
2. (IN, FS) is an adjunction $\Pi(E) \rightleftharpoons \Pi^*(E)$.
3. FS is surjective, while IN and $RSIN$ are injective; $FS \cdot IN$ and $FS \cdot RSIN$ are the identity on $\Pi(E)$.
4. FS is a complete morphism.
5. $(INFS, RS)$ is an adjunction on $\Pi^*(E)$.
6. $INFS$ is a closure and RS is an opening.
7. For any $\pi \in \Pi^*(E)$, $RS(\pi) = RS(FS(\pi))$ and $FS(\pi) = FS(RS(\pi))$.

Proof 1. FS is $\Pi^*(E) \to \Pi(E)$ while $RSIN$ is $\Pi(E) \to \Pi^*(E)$; for $\pi \in \Pi(E)$ and $\pi' \in \Pi^*(E)$, $RSIN(\pi) = RS(\pi)$, so by item 1 of Lemma 8, $RSIN(\pi) \leq \pi' \Leftrightarrow RS(\pi) \leq \pi' \Leftrightarrow \pi \leq FS(\pi')$.

2. IN is $\Pi(E) \to \Pi^*(E)$ while FS is $\Pi^*(E) \to \Pi(E)$; for $\pi \in \Pi^*(E)$ and $\pi' \in \Pi(E)$, $IN(\pi') = \pi'$, so by item 2 of Lemma 8, $FS(\pi) \leq \pi' \Leftrightarrow \pi \leq \pi' \Leftrightarrow \pi \leq IN(\pi')$.

3. For $\pi \in \Pi(E)$, $\pi = FS(\pi)$, hence FS is surjective. It is known [8, 13] that in an adjunction, one adjoint is surjective if and only if the other adjoint is injective, and that the composition of the injective adjoint followed by the surjective one is the identity. From the two adjunctions $(FS, RSIN)$

and (IN, FS), IN and $RSIN$ are injective, while $FS \cdot IN$ and $FS \cdot RSIN$ are the identity on $\Pi(E)$.

4. Since FS has upper adjoint IN and lower adjoint $RSIN$, it is both a dilation and an erosion, hence a complete morphism.

5. $INFS$ is $\Pi^*(E) \to \Pi^*(E)$ while RS is $\Pi^*(E) \to \Pi^*(E)$; for $\pi, \pi' \in \Pi^*(E)$, $INFS(\pi') = FS(\pi')$, so by item 1 of Lemma 8, $RS(\pi) \leq \pi' \Leftrightarrow \pi \leq FS(\pi') \Leftrightarrow \pi \leq INFS(\pi')$.

6. From the adjunction (IN, FS), $INFS$ is a closure; as RS is the lower adjoint of that closure, it must be an opening [4,31].

7. Let $\pi \in \Pi^*(E)$; set $A = \mathsf{supp}(\pi) \setminus \mathsf{supp}(RS(\pi))$ and $B = E \setminus \mathsf{supp}(\pi)$. Then $\pi = RS(\pi) \cup \mathbf{0}_A$ and $FS(\pi) = RS(\pi) \cup \mathbf{0}_A \cup \mathbf{0}_B = RS(\pi) \cup \mathbf{0}_{A \cup B}$, where $RS(\pi)$ has no singleton block. The equalities $RS(\pi) = RS(FS(\pi))$ and $FS(\pi) = FS(RS(\pi))$ follow then. □

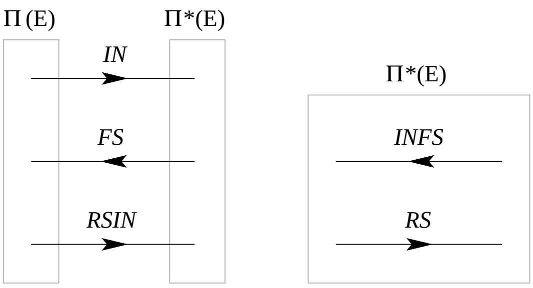

Fig. 3 The three adjunctions $(FS, RSIN) : \Pi^*(E) \rightleftharpoons \Pi(E)$, $(IN, FS) : \Pi(E) \rightleftharpoons \Pi^*(E)$ and $(INFS, RS) : \Pi^*(E) \rightleftharpoons \Pi^*(E)$ shown as pairs of arrows in opposite orientations, where the top arrow is the upper adjoint and the bottom arrow is the lower adjoint.

The three adjunctions $(FS, RSIN)$, (IN, FS) and $(INFS, RS)$ are illustrated by the diagrams of Figure 3. From the adjunction (IN, FS), it follows that for any $\pi \in \Pi^*(E)$, $FS(\pi)$ is the least $\pi' \in \Pi(E)$ such that $\pi \leq \pi'$. By item 7,

$$\forall \pi, \pi' \in \Pi^*(E), \qquad FS(\pi) = FS(\pi') \iff RS(\pi) = RS(\pi') \ . \tag{16}$$

Note that RS and **grind** are both at the same time dilations and openings on $\Pi^*(E)$, while their upper adjoints $INFS$ and **blend** are both at the same time erosions and closures on $\Pi^*(E)$; furthermore, for every $\pi \in \Pi^*(E)$,

$$\pi = RS(\pi) \vee \mathbf{grind}(\pi) = INFS(\pi) \wedge \mathbf{blend}(\pi) \ . \tag{17}$$

Corollary 10 *Let $\Pi^0(E) = \{RS(\pi) \mid \pi \in \Pi^*(E)\}$. Then $\Pi^0(E)$ is a dual Moore family of $\Pi^*(E)$. As a complete lattice, $\Pi^0(E)$ is isomorphic to $\Pi(E)$, the two maps $RSIN$ and FS provide the isomorphism $\Pi(E) \to \Pi^0(E)$ and the inverse isomorphism $\Pi^0(E) \to \Pi(E)$.*

Proof By item 1 of Theorem 9, $RSIN$ is a dilation, hence its image $\Pi^0(E)$ is closed under suprema. By item 7 of Theorem 9, $RSIN : \Pi(E) \to \Pi^0(E)$ and $FS : \Pi^0(E) \to \Pi(E)$ (in fact, the restrictions of $RSIN$ to range $\Pi^0(E)$, and of FS to domain $\Pi^0(E)$) are the inverses of each other. As $(FS, RSIN)$

is an adjunction (item 1 of Theorem 9), $RSIN$ and FS are isotone, thus they constitute an isomorphism $\mathit{\Pi}(E) \to \mathit{\Pi}^0(E)$ and its inverse. $\square$

Note that the least sup-generating family of $\mathit{\Pi}^0(E)$ is given by the $\mathbf{1}_{\{p,q\}}$ $(p, q \in E,\ p \neq q)$.

Corollary 10 can be used to match properties related to the supremum operations between $\mathit{\Pi}^*(E)$ and $\mathit{\Pi}(E)$. For example a partition π is compact in $\mathit{\Pi}(E)$ if and only if $RS(\pi)$ is compact in $\mathit{\Pi}^0(E)$; then we easily get that this is equivalent to $RS(\pi)$ being compact in $\mathit{\Pi}^*(E)$, in other words $\mathsf{supp}(RS(\pi))$ being finite.

Theorem 9 and Corollary 10 illuminate the role of singleton blocks in partial partitions, in particular the remark made above that when chaining blocks, a singleton block is always redundant in the chain, except when the chain is reduced to that singleton: this is because RS is a dilation. Thus a family $\mathcal{F} \subseteq \mathit{\Pi}(E)$ is sup-generated by a family $\mathcal{G} \subseteq \mathit{\Pi}(E)$ if and only if $\{RS(\pi) \mid \pi \in \mathcal{F}\}$ is sup-generated by $\{RS(\pi) \mid \pi \in \mathcal{G}\}$.

Recall that a *congruence* (resp., *complete congruence*) on a lattice (resp., complete lattice) is an equivalence relation compatible with binary joins and meets (resp., with arbitrary suprema and infima). It was shown in [28] that $\mathit{\Pi}(E)$ has only trivial congruences (the identity and the universal relation $E \times E$). This is no more the case in $\mathit{\Pi}^*(E)$:

Proposition 11 *[10] The relation on $\mathit{\Pi}^*(E)$ given by $FS(\pi) = FS(\pi')$, cf. (16), and the one given by $\mathsf{supp}(\pi) = \mathsf{supp}(\pi')$, are complete congruences. They are complements in the lattice of congruences on $\mathit{\Pi}^*(E)$. Furthermore, any congruence $\equiv$ on $\mathit{\Pi}^*(E)$ either is included in the first one,*

$$\pi \equiv \pi' \implies FS(\pi) = FS(\pi') \ ,$$

or contains the second one,

$$\mathsf{supp}(\pi) = \mathsf{supp}(\pi') \implies \pi \equiv \pi' \ .$$

Finally, the two adjunctions $(FS, RSIN)$ and (IN, FS) allows us to derive from operators on $\mathit{\Pi}^*(E)$ similar operators on $\mathit{\Pi}(E)$. For a closure φ on $\mathit{\Pi}^*(E)$, $FS \cdot \varphi \cdot RSIN$ will be a closure on $\mathit{\Pi}(E)$ (but the restriction of φ to $\mathit{\Pi}(E)$ will also be a closure), while for an opening γ on $\mathit{\Pi}^*(E)$, $FS \cdot \gamma \cdot IN$ will be an opening on $\mathit{\Pi}(E)$, see Figure 4. In the next section, we will see how adjunctions on $\mathit{\Pi}(E)$ can be obtained from adjunctions on $\mathit{\Pi}^*(E)$.

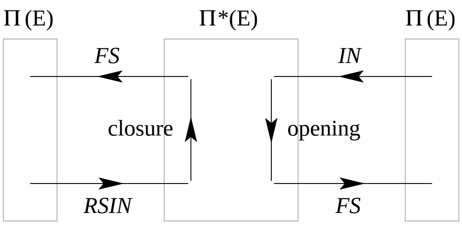

Fig. 4 Openings and closures on $\mathit{\Pi}^*(E)$ can be transformed into similar operators on $\mathit{\Pi}(E)$, by using the two adjunctions $(FS, RSIN)$ and (IN, FS) between $\mathit{\Pi}^*(E)$ and $\mathit{\Pi}(E)$.

3 Adjunctions on partial partitions

Throughout this section we consider two spaces E_1 and E_2 that may be distinct or equal; sometimes we even consider a third one, E_3. When $E_1 = E_2$, we will write E for E_1. We will analyse adjunctions $\Pi^*(E_2) \rightleftharpoons \Pi^*(E_1)$ and $\Pi(E_2) \rightleftharpoons \Pi(E_1)$. Dilations and erosions on (partial) partitions will be written $\overline{\delta}, \overline{\varepsilon}$, while δ, ε will denote dilations and erosions on sets.

In Subsection 3.1 we show how every adjunction $\Pi(E_2) \rightleftharpoons \Pi(E_1)$ arises by combining an adjunction $\Pi^*(E_2) \rightleftharpoons \Pi^*(E_1)$ with the operators FS, IN and $RSIN$; then we give a direct characterization of dilations $\Pi^*(E_1) \to L$ and $\Pi(E_1) \to L$ (where L is any complete lattice, for instance $\Pi^*(E_2)$ or $\Pi(E_2)$) in terms of connective maps $\mathcal{P}(E_1) \to L$. Subsection 3.2 analyses lower-regular dilations $\Pi^*(E_1) \to \Pi^*(E_2)$ that apply a set operator to each block. Subsection 3.3 shows how to build a regular adjunction $\Pi^*(E_2) \rightleftharpoons \Pi^*(E_1)$ from a regular adjunction $\mathcal{P}(E_2) \rightleftharpoons \mathcal{P}(E_1)$; in particular, we obtain the adjunction $(\widehat{\varepsilon}, \widehat{\delta})$ from [30,36] described in the Introduction. Finally, Subsection 3.4 characterizes connective maps $\mathcal{P}(E_1) \to \mathcal{P}(E_2)$ in terms of maps $E_2 \to \Pi^*(E_1)$. Section 4 will give an alternate characterization of dilations $\Pi^*(E_1) \to L$ and $\Pi(E_1) \to L$ (in particular, $\Pi^*(E_1) \to \Pi^*(E_2)$ and $\Pi(E_1) \to \Pi(E_2)$) in terms of *triangular maps*.

3.1 General results

We first show how the case of partitions reduces to that of partial partitions. For $i = 1, 2$, write $FS_i, IN_i, RSIN_i$ to mean that the operators $FS, IN, RSIN$ for $\Pi^*(E_i)$ and $\Pi(E_i)$.

Theorem 12 $(\overline{\varepsilon}, \overline{\delta}) : \Pi(E_2) \rightleftharpoons \Pi(E_1)$ *is an adjunction if and only if there is an adjunction* $(\overline{\varepsilon}^*, \overline{\delta}^*) : \Pi^*(E_2) \rightleftharpoons \Pi^*(E_1)$ *such that* $(\overline{\varepsilon}, \overline{\delta}) = (FS_1 \cdot \overline{\varepsilon}^* \cdot IN_2, FS_2 \cdot \overline{\delta}^* \cdot RSIN_1)$.

Proof For $i = 1, 2$: by Theorem 9, $(FS_i, RSIN_i)$ is an adjunction $\Pi^*(E_i) \rightleftharpoons \Pi(E_i)$, (IN_i, FS_i) is an adjunction $\Pi(E_i) \rightleftharpoons \Pi^*(E_i)$, while $FS_i \cdot IN_i$ and $FS_i \cdot RSIN_i$ are the identity on $\Pi(E_i)$.

Given an adjunction $(\overline{\varepsilon}^*, \overline{\delta}^*) : \Pi^*(E_2) \rightleftharpoons \Pi^*(E_1)$, by the composition rule for adjunctions, $(FS_1 \cdot \overline{\varepsilon}^* \cdot IN_2, FS_2 \cdot \overline{\delta}^* \cdot RSIN_1)$ will be an adjunction $\Pi(E_2) \rightleftharpoons \Pi(E_1)$. See Figure 5 (top).

Conversely, given an adjunction $(\overline{\varepsilon}, \overline{\delta}) : \Pi(E_2) \rightleftharpoons \Pi(E_1)$, set $(\overline{\varepsilon}^*, \overline{\delta}^*) = (IN_1 \cdot \overline{\varepsilon} \cdot FS_2, RSIN_2 \cdot \overline{\delta} \cdot FS_1)$; by the composition rule for adjunctions, it will be an adjunction $\Pi^*(E_2) \rightleftharpoons \Pi^*(E_1)$; then $(FS_1 \cdot \overline{\varepsilon}^* \cdot IN_2, FS_2 \cdot \overline{\delta}^* \cdot RSIN_1) = (FS_1 \cdot IN_1 \cdot \overline{\varepsilon} \cdot FS_2 \cdot IN_2, FS_2 \cdot RSIN_2 \cdot \overline{\delta} \cdot FS_1 \cdot RSIN_1) = (\overline{\varepsilon}, \overline{\delta})$. See Figure 5 (bottom). $\square$

Let us now give a general form for dilations $\Pi^*(E_1) \to L$ and $\Pi(E_1) \to L$ for an arbitrary complete lattice L; this will apply in particular to dilations $\Pi^*(E_1) \to \Pi^*(E_2)$ and $\Pi(E_1) \to \Pi(E_2)$.

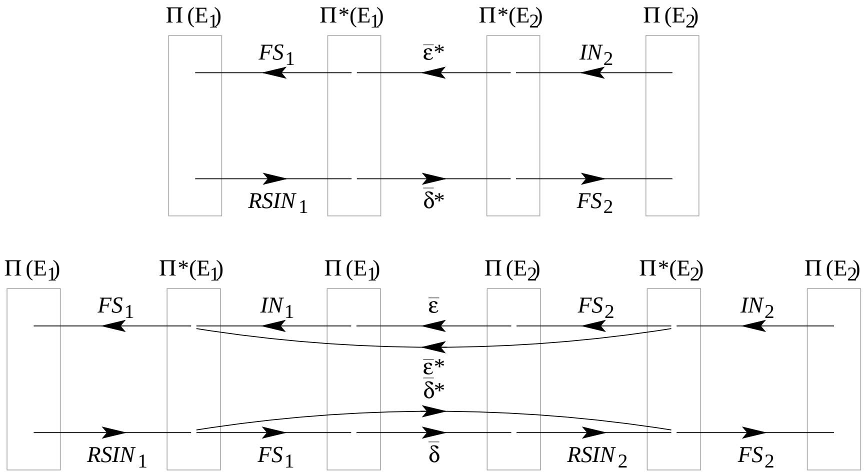

Fig. 5 Top: from an adjunction $(\overline{\varepsilon}^*, \overline{\delta}^*) : \Pi^*(E_2) \rightleftharpoons \Pi^*(E_1)$, we derive the adjunction $(FS_1 \cdot \overline{\varepsilon}^* \cdot IN_2, FS_2 \cdot \overline{\delta}^* \cdot RSIN_1) : \Pi(E_2) \rightleftharpoons \Pi(E_1)$. Bottom: from an adjunction $(\overline{\varepsilon}, \overline{\delta}) : \Pi(E_2) \rightleftharpoons \Pi(E_1)$, we derive the adjunction $(\overline{\varepsilon}^*, \overline{\delta}^*) = (IN_1 \cdot \overline{\varepsilon} \cdot FS_2, RSIN_2 \cdot \overline{\delta} \cdot FS_1) : \Pi^*(E_2) \rightleftharpoons \Pi^*(E_1)$, then $(FS_1 \cdot \overline{\varepsilon}^* \cdot IN_2, FS_2 \cdot \overline{\delta}^* \cdot RSIN_1) = (FS_1 \cdot IN_1 \cdot \overline{\varepsilon} \cdot FS_2 \cdot IN_2, FS_2 \cdot RSIN_2 \cdot \overline{\delta} \cdot FS_1 \cdot RSIN_1) = (\overline{\varepsilon}, \overline{\delta})$.

Theorem 13 *Let L be a complete lattice.*

1. *A map $\overline{\delta} : \Pi^*(E_1) \to L$ is a dilation if and only if there is a connective map $\xi : \mathcal{P}(E_1) \to L$ such that*

$$\forall \pi \in \Pi^*(E_1), \quad \overline{\delta}(\pi) = \bigvee_{B \in \pi} \xi(B) \ . \tag{18}$$

   Furthermore, $\overline{\delta}$ uniquely determines ξ by $\xi(\emptyset) = \mathbf{0}$ and

$$\forall A \in \mathcal{P}(E_1) \setminus \{\emptyset\}, \quad \xi(A) = \overline{\delta}(\mathbf{1}_A) \ .$$

2. *A map $\overline{\delta} : \Pi(E_1) \to L$ is a dilation if and only if there is a dilation $\overline{\delta}^* : \Pi^*(E_1) \to L$ such that $\overline{\delta} = \overline{\delta}^* \cdot RSIN_1$; this holds if and only if there is a connective map $\xi : \mathcal{P}(E_1) \to L$ such that*

$$\forall \pi \in \Pi(E_1), \quad \overline{\delta}(\pi) = \bigvee_{B \in RS_1(\pi)} \xi(B) \ . \tag{19}$$

   Furthermore, $\overline{\delta}$ uniquely determines the restriction of ξ to non-singletons, by $\xi(\emptyset) = \mathbf{0}$ and

$$\forall A \in \mathcal{P}(E_1),\ |A| \geq 2, \quad \xi(A) = \overline{\delta}(\mathbf{1}_A \cup \mathbf{0}_{E \setminus A}) \ .$$

Proof 1. Let $\overline{\delta} : \Pi^*(E_1) \to L$ be a dilation. Define $\xi : \mathcal{P}(E_1) \to L$ by $\xi(\emptyset) = \mathbf{0}$, and for $A \in \mathcal{P}(E_1) \setminus \{\emptyset\}$, $\xi(A) = \overline{\delta}(\mathbf{1}_A)$. Let $\mathcal{B} \subseteq \mathcal{P}(E_1)$ such that

$\mathcal{B} \neq \emptyset$ and $\bigcap \mathcal{B} \neq \emptyset$; by (15), $\mathbf{1}_{\bigcup \mathcal{B}} = \bigvee_{B \in \mathcal{B}} \mathbf{1}_B$. Applying the dilation $\overline{\delta}$, we get

$$\xi\Big(\bigcup \mathcal{B}\Big) = \overline{\delta}(\mathbf{1}_{\bigcup \mathcal{B}}) = \overline{\delta}\Big(\bigvee_{B \in \mathcal{B}} \mathbf{1}_B\Big) = \bigvee_{B \in \mathcal{B}} \overline{\delta}(\mathbf{1}_B) = \bigvee_{B \in \mathcal{B}} \xi(B) \ .$$

Thus ξ is connective. Now for $\pi \in \mathit{\Pi}^*(E_1)$, as $\pi = \bigvee_{B \in \pi} \mathbf{1}_B$ and $\overline{\delta}$ is a dilation, we get

$$\overline{\delta}(\pi) = \overline{\delta}\Big(\bigvee_{B \in \pi} \mathbf{1}_B\Big) = \bigvee_{B \in \pi} \overline{\delta}(\mathbf{1}_B) = \bigvee_{B \in \pi} \xi(B) \ ,$$

hence (18) holds.

Conversely, let $\xi : \mathcal{P}(E_1) \to L$ be connective, and define $\overline{\delta}$ by (18). By Lemma 3, ξ is isotone. Given $\pi, \pi' \in \mathit{\Pi}^*(E_1)$ such that $\pi \leq \pi'$, for every $B \in \pi$ there is $C \in \pi'$ with $B \subseteq C$, hence $\xi(B) \leq \xi(C)$; thus by (18) we get $\overline{\delta}(\pi) \leq \overline{\delta}(\pi')$. Hence $\overline{\delta}$ is isotone. By definition, $\overline{\delta}(\mathbf{\emptyset}) = \mathbf{0}$. Consider now a non-void family $\pi_i \in \mathit{\Pi}^*(E_1)$, $i \in I \neq \emptyset$, and let $\pi = \bigvee_{i \in I} \pi_i$ and $z = \bigvee_{i \in I} \overline{\delta}(\pi_i)$. For each $i \in I$ we have $\pi_i \leq \pi$; as $\overline{\delta}$ is isotone, $\overline{\delta}(\pi_i) \leq \overline{\delta}(\pi)$; hence $z = \bigvee_{i \in I} \overline{\delta}(\pi_i) \leq \overline{\delta}(\pi)$. Take any $C \in \pi$; as C is chained by blocks in $\bigcup_{i \in I} \pi_i$, for some $i \in I$ there is $B \in \pi_i$ such that $B \subseteq C$. By (18), $\xi(B) \leq \overline{\delta}(\pi_i)$, and by definition $\overline{\delta}(\pi_i) \leq z$, so $\xi(B) \leq z$. Let

$$\mathcal{B} = \big\{ X \mid B \subseteq X \subseteq C,\ \xi(X) \leq z \big\} \ .$$

Then $B \in \mathcal{B}$, so $\mathcal{B} \neq \emptyset$ and $\bigcap \mathcal{B} = B \neq \emptyset$. Let $Y = \bigcup \mathcal{B}$. Then $B \subseteq Y \subseteq C$, and since ξ is connective,

$$\xi(Y) = \bigvee \big\{ \xi(X) \mid B \subseteq X \subseteq C,\ \xi(X) \leq z \big\} \leq z \ .$$

Thus Y is the greatest element of $\mathcal{B}$. Suppose that $Y \subset C$. Now points of Y and $C \setminus Y$ must be chained by blocks in $\bigcup_{i \in I} \pi_i$; hence for some $j \in I$ there is $A \in \pi_j$ such that $A \subseteq C$, $A \between Y$ and $A \not\subseteq Y$. In the same way as $\xi(B)$, $\xi(A) \leq z$. Since ξ is connective and $A \cap Y \neq \emptyset$, $\xi(A \cup Y) = \xi(A) \cup \xi(Y) \leq z$, with $B \subseteq A \cup Y \subseteq C$, contradicting the fact that Y is the greatest element of $\mathcal{B}$. Therefore $Y = C$, so $\xi(C) \leq z$; as this holds for any $C \in \pi$, we get $\overline{\delta}(\pi) = \bigvee_{C \in \pi} \xi(C) \leq z$. From the double inequality follows the equality

$$\overline{\delta}\Big(\bigvee_{i \in I} \pi_i\Big) = \overline{\delta}(\pi) = z = \bigvee_{i \in I} \overline{\delta}(\pi_i) \ ,$$

thus $\overline{\delta}$ is a dilation. Since ξ is connective, it is upper-regular, so $\xi(\emptyset) = \mathbf{0}$; now for $A \in \mathcal{P}(E_1) \setminus \{\emptyset\}$, (18) gives $\overline{\delta}(\mathbf{1}_A) = \xi(A)$.

2. Given a dilation $\overline{\delta}^* : \mathit{\Pi}^*(E_1) \to L$, then $\overline{\delta} = \overline{\delta}^* \cdot RSIN_1$ is a dilation $\mathit{\Pi}(E_1) \to L$; conversely, given a dilation $\overline{\delta} : \mathit{\Pi}(E_1) \to L$, set $\overline{\delta}^* = \overline{\delta} \cdot FS_1$, then $\overline{\delta}^*$ is a dilation $\mathit{\Pi}^*(E_1) \to L$, and $\overline{\delta}^* \cdot RSIN_1 = \overline{\delta} \cdot FS_1 \cdot RSIN_1 = \overline{\delta}$. Now each such $\overline{\delta}^*$ corresponds to a connective map $\xi : \mathcal{P}(E_1) \to L$ for which (18) holds, and then $\overline{\delta} = \overline{\delta}^* \cdot RSIN_1$ will satisfy (19); conversely, from any

connective map ξ we can construct the dilation $\overline{\delta}^*$ by (18), and then $\overline{\delta}$ built by (19) will coincide with $\overline{\delta}^* \cdot RSIN_1$.

Here we have for $A \in \mathcal{P}(E_1)$ such that $|A| > 1$:

$$\xi(A) = \overline{\delta}^*(\mathbf{1}_A) = \overline{\delta}(FS_1(\mathbf{1}_A)) = \overline{\delta}(\mathbf{1}_A \cup \mathbf{0}_{E\setminus A}) \ ;$$

however for $p \in E_1$, the specific value of $\xi(\{p\})$ does not matter, i.e., ξ is undetermined on singletons. $\square$

From Lemma 3, it is easily seen that the correspondence $\xi \leftrightarrow \overline{\delta}$ between connective maps $\xi : \mathcal{P}(E_1) \to L$ and dilations $\overline{\delta} : \Pi^*(E_1) \to L$ satisfies the following properties:

- If $\xi_i \leftrightarrow \overline{\delta}_i$ for all $i \in I$, then $\bigvee_{i\in I} \xi_i \leftrightarrow \bigvee_{i\in I} \overline{\delta}_i$.
- Given a dilation $\delta : L \to M$ (where M is a complete lattice), if $\xi \leftrightarrow \overline{\delta}$, then $\delta\xi \leftrightarrow \delta\overline{\delta}$.

We see no simple characterization of connective maps $\mathcal{P}(E_1) \to \Pi^*(E_2)$. Note that for a dilation $\overline{\delta} : \Pi^*(E_1) \to \Pi^*(E_2)$ corresponding to a connective map $\xi : \mathcal{P}(E_1) \to \Pi^*(E_2)$, since RS_2 and FS_2 are dilations, the two dilations $RS_2 \cdot \overline{\delta}$ and $FS_2 \cdot \overline{\delta}$ will correspond to the connective maps $RS_2 \cdot \xi$ and $FS_2 \cdot \xi$ (cf. Lemma 3). Since supp is a dilation, the two maps $\mathsf{supp} \cdot \xi : \mathcal{P}(E_1) \to \mathcal{P}(E_2) : P \mapsto \mathsf{supp}(\xi(P))$ and $\mathsf{supp} \cdot RS_2 \cdot \xi : \mathcal{P}(E_1) \to \mathcal{P}(E_2) : P \mapsto \mathsf{supp}(RS_2(\psi(P)))$ will also be connective. In the next subsection, we will consider the particular case of dilations that can be characterized by connective maps $\mathcal{P}(E_1) \to \mathcal{P}(E_2)$.

3.2 Lower-regular one-block-preserving dilations

Recall from Definition 2 that a lattice operator ψ is *lower-regular* if and only if it satisfies $x > \mathbf{0} \;\Rightarrow\; \psi(x) > \mathbf{0}$. We will also require the following concept:

Definition 14 An operator $\overline{\xi} : \Pi^*(E_1) \to \Pi^*(E_2)$ is *one-block-preserving* if it transforms every one-block partial partition into a one-block partial partition (or $\boldsymbol{\emptyset}$): for every $A \in \mathcal{P}(E_1) \setminus \{\emptyset\}$ there is some $A' \in \mathcal{P}(E_2)$ with $\overline{\xi}(\mathbf{1}_A) = \mathbf{1}_{A'}$.

Now a lower-regular one-block-preserving dilation can be characterized in terms of the operator $A \mapsto A'$ for $\overline{\delta}(\mathbf{1}_A) = \mathbf{1}_{A'}$. We will thus consider operators on partial partitions applying a set operator to each block separately.

Definition 15 Given an operator $\psi : \mathcal{P}(E_1) \to \mathcal{P}(E_2)$, the *blockwise extension of* ψ is the operator

$$\mathsf{B}(\psi) : \Pi^*(E_1) \to \Pi^*(E_2) : \pi \mapsto \bigvee_{B\in\pi} \mathbf{1}_{\psi(B)} \ . \tag{20}$$

Note that ψ needs not to be defined on $\emptyset$, so one can consider that it is $\mathcal{P}(E_1) \setminus \{\emptyset\} \to \mathcal{P}(E_2)$; here we will rather assume $\psi(\emptyset) = \emptyset$, in other words consider upper-regular operators. Also,

$$\forall\, A \in \mathcal{P}(E_1) \setminus \{\emptyset\}, \quad \mathsf{B}(\psi)(\mathbf{1}_A) = \mathbf{1}_{\psi(A)} \tag{21}$$

and

$$\forall\, \pi \in \Pi^*(E_1), \quad \mathsf{supp}\big(\mathsf{B}(\psi)(\pi)\big) = \bigcup_{B\in\pi} \psi(B) \ . \tag{22}$$

The following will be useful in the rest of this section.

Lemma 16 *Consider an operator $\psi : \mathcal{P}(E_1) \to \mathcal{P}(E_2)$.*

1. *If ψ is isotone on $\mathcal{P}(E_1) \setminus \{\emptyset\}$, then ψ is lower-regular if and only if $\forall\, p \in E_1$, $\psi(\{p\}) \neq \emptyset$.*
2. *Assuming $E_1 = E_2 = E$: If ψ is isotone on $\mathcal{P}(E) \setminus \{\emptyset\}$, then ψ is extensive if and only if $\forall\, p \in E$, $p \in \psi(\{p\})$.*
3. *If ψ is isotone on $\mathcal{P}(E_1) \setminus \{\emptyset\}$, then $\mathsf{B}(\psi)$ is isotone.*
4. *If ψ is lower-regular, then $\mathsf{B}(\psi)$ is lower-regular.*
5. *$\mathsf{B}(\psi)$ is upper-regular.*
6. *Assuming $E_1 = E_2 = E$: If ψ is extensive, then $\mathsf{B}(\psi)$ is extensive.*

Proof 1. Obviously, if ψ is lower-regular, then $\forall\, p \in E_1$, $\psi(\{p\}) \neq \emptyset$. Conversely, suppose that $\forall\, p \in E_1$, $\psi(\{p\}) \neq \emptyset$. For any $X \in \mathcal{P}(E_1)$ such that $X \neq \emptyset$, taking $p \in X$ we have $X \supseteq \{p\}$; as ψ is isotone on $\mathcal{P}(E_1) \setminus \{\emptyset\}$, we get $\psi(X) \supseteq \psi(\{p\}) \neq \emptyset$, so ψ is lower-regular.

2. Obviously, if ψ is extensive, then $\forall\, p \in E_1$, $\{p\} \subseteq \psi(\{p\})$, that is, $p \in \psi(\{p\})$. Conversely, suppose that $\forall\, p \in E_1, p \in \psi(\{p\})$. For any $X \in \mathcal{P}(E_1)$ such that $X \neq \emptyset$, for all $p \in X$ we have $\{p\} \subseteq X$; as ψ is isotone on $\mathcal{P}(E_1) \setminus \{\emptyset\}$, we get $p \in \psi(\{p\}) \subseteq \psi(X)$; thus $X \subseteq \psi(X)$. Now obviously $\emptyset \subseteq \psi(\emptyset)$. Hence ψ is extensive.

3. Given $\pi, \pi' \in \Pi^*(E_1)$ such that $\pi \leq \pi'$, for every $B \in \pi$ there is $C \in \pi'$ with $B \subseteq C$, hence $\psi(B) \subseteq \psi(C)$ and so $\mathbf{1}_{\psi(B)} \leq \mathbf{1}_{\psi(C)}$; by (20) we deduce that $\mathsf{B}(\psi)(\pi) \leq \mathsf{B}(\psi)(\pi')$.

4. For $\pi \neq \boldsymbol{\emptyset}$, there is some $B \in \pi$, with $B \neq \emptyset$; since ψ is lower-regular, $\psi(B) \neq \emptyset$, so by (20), $\mathsf{B}(\psi)(\pi) \geq \mathbf{1}_{\psi(B)} > \boldsymbol{\emptyset}$.

5. By (20), $\mathsf{B}(\psi)(\boldsymbol{\emptyset}) = \boldsymbol{\emptyset}$.

6. This follows from (20) and the fact that $B \subseteq \psi(B)$, so $\mathbf{1}_B \leq \mathbf{1}_{\psi(B)}$. □

For sets, Definition 2 takes the following form: an operator $\psi : \mathcal{P}(E_1) \to \mathcal{P}(E_2)$ is *connective* if $\psi(\emptyset) = \emptyset$ and for any $\mathcal{B} \subseteq \mathcal{P}(E_1)$ such that $\mathcal{B} \neq \emptyset$ and $\bigcap \mathcal{B} \neq \emptyset$, we must have $\psi\big(\bigcup \mathcal{B}\big) = \bigcup_{B\in\mathcal{B}} \psi(B)$. Recall also Lemma 3. Thus, given a connective map ψ, Lemmas 3 and 16 give: (a) ψ and $\mathsf{B}(\psi)$ are isotone; (b) ψ is lower-regular if and only if $\forall\, p \in E_1$, $\psi(\{p\}) \neq \emptyset$, implying that $\mathsf{B}(\psi)$ is lower-regular; (c) (for $E_1 = E_2 = E$) ψ is extensive if and only if $\forall\, p \in E$, $p \in \psi(\{p\})$, implying that $\mathsf{B}(\psi)$ is extensive.

We now give our characterization of lower-regular one-block-preserving dilations.

Theorem 17 *A map $\overline{\delta} : \Pi^*(E_1) \to \Pi^*(E_2)$ is a lower-regular one-block-preserving dilation if and only if $\overline{\delta} = \mathsf{B}(\psi)$ for a lower-regular connective operator $\psi : \mathcal{P}(E_1) \to \mathcal{P}(E_2)$.*

Proof Let $\overline{\delta} : \Pi^*(E_1) \to \Pi^*(E_2)$ be a lower-regular one-block-preserving dilation. Thus for every $A \in \mathcal{P}(E_1) \setminus \{\emptyset\}$ there is some $A' \in \mathcal{P}(E_2) \setminus \{\emptyset\}$ with $\overline{\delta}(\mathbf{1}_A) = \mathbf{1}_{A'}$; we define then $\psi : \mathcal{P}(E_1) \to \mathcal{P}(E_2)$ by $\psi(A) = A'$, and we arbitrarily set $\psi(\emptyset) = \emptyset$. Clearly $A \neq \emptyset \;\Rightarrow\; \psi(A) \neq \emptyset$, i.e., ψ is lower-regular. Let $\xi : \mathcal{P}(E_1) \to \Pi^*(E_2)$ be given by $\xi(A) = \mathbf{1}_{\psi(A)}$ for all $A \in \mathcal{P}(E_1)$. Then for all $A \in \mathcal{P}(E_1)$, $\psi(A) = \mathsf{supp}\big(\mathbf{1}_{\psi(A)}\big) = \mathsf{supp}(\xi(A))$, that is, $\psi = \mathsf{supp} \cdot \xi$. Now $\xi(\emptyset) = \mathbf{1}_{\psi(\emptyset)} = \mathbf{1}_\emptyset = \boldsymbol{\emptyset}$, and for $A \in \mathcal{P}(E_1) \setminus \{\emptyset\}$, $\xi(A) = \overline{\delta}(\mathbf{1}_A)$, hence by Theorem 13, ξ is connective. But supp is a dilation; hence $\psi = \mathsf{supp} \cdot \xi$ is connective by Lemma 3. By Theorem 13, $\overline{\delta}$ satisfies (18). Replacing $\xi(B)$ by $\mathbf{1}_{\psi(B)}$ in (18), we get $\overline{\delta} = \mathsf{B}(\psi)$ according to (20).

Conversely, let $\psi : \mathcal{P}(E_1) \to \mathcal{P}(E_2)$ be lower-regular and connective. Let $\xi : \mathcal{P}(E_1) \to L$ be given by $\xi(A) = \mathbf{1}_{\psi(A)}$ for all $A \in \mathcal{P}(E_1)$, in other words $\xi = \mathbf{1}_\bullet \cdot \psi$. We saw in (15) that $\mathbf{1}_\bullet$ is connective; since ψ is lower-regular and connective, their composition $\xi = \mathbf{1}_\bullet \cdot \psi$ will be connective by Lemma 3. Now by (20), $\overline{\delta} = \mathsf{B}(\psi)$ satisfies (18), so by Theorem 13, $\mathsf{B}(\psi)$ is a dilation. Since ψ is lower-regular, $\mathsf{B}(\psi)$ is lower-regular by Lemma 16. □

It is easily seen that this bijection $\psi \leftrightarrow \mathsf{B}(\psi)$ is an isomorphism between the poset of lower-regular connective operators $\mathcal{P}(E_1) \to \mathcal{P}(E_2)$, and the one of lower-regular one-block-preserving dilations $\Pi^*(E_1) \to \Pi^*(E_2)$. Note that the poset of lower-regular connective operators $\mathcal{P}(E_1) \to \mathcal{P}(E_2)$ is closed under *non-void* joins, but that the poset of lower-regular one-block-preserving dilations $\Pi^*(E_1) \to \Pi^*(E_2)$ is *not* closed under non-void joins in the complete lattice of operators $\Pi^*(E_1) \to \Pi^*(E_2)$; indeed, the join of two or more one-block-preserving dilations is not necessarily one-block-preserving. More precisely, given a non-void family of lower-regular one-block-preserving dilations $\mathsf{B}(\psi_i)$, $i \in I \neq \emptyset$, the least one-block-preserving dilation above them is not $\bigvee_{i \in I} \mathsf{B}(\psi_i)$, but $\mathsf{B}\big(\bigvee_{i \in I} \psi_i\big)$. Now this bijection is also compatible with composition:

Corollary 18 *Let $\psi : \mathcal{P}(E_1) \to \mathcal{P}(E_2)$ and $\xi : \mathcal{P}(E_2) \to \mathcal{P}(E_3)$ be lower-regular and connective. Then $\mathsf{B}(\xi\psi) = \mathsf{B}(\xi)\mathsf{B}(\psi)$.*

Proof By Lemma 3, $\xi\psi$ is connective, and obviously it will be lower-regular. Hence $\mathsf{B}(\xi\psi)$ is a dilation. Now $\mathsf{B}(\xi)\mathsf{B}(\psi)$ is a composition of dilations, hence a dilation. For every $A \in \mathcal{P}(E_1) \setminus \{\emptyset\}$, (21) gives $\mathsf{B}(\xi\psi)(\mathbf{1}_A) = \mathbf{1}_{\xi\psi(A)} = \mathsf{B}(\xi)(\mathbf{1}_{\psi(A)}) = \mathsf{B}(\xi)\mathsf{B}(\psi)(\mathbf{1}_A)$. Now for any $\pi \in \Pi^*(E_1)$, $\pi = \bigvee_{B \in \pi} \mathbf{1}_B$, hence the two dilations $\mathsf{B}(\xi\psi)$ and $\mathsf{B}(\xi)\mathsf{B}(\psi)$ give

$$\mathsf{B}(\xi\psi)(\pi) = \bigvee_{B \in \pi} \mathsf{B}(\xi\psi)(\mathbf{1}_B) = \bigvee_{B \in \pi} \mathsf{B}(\xi)\mathsf{B}(\psi)(\mathbf{1}_B) = \mathsf{B}(\xi)\mathsf{B}(\psi)(\pi) \ .$$

□

Note that the assumption that ψ is lower-regular ($X \neq \emptyset \;\Rightarrow\; \psi(X) \neq \emptyset$) is crucial, as shows the following counterexample. Take $E_1 = E_2 = E$ and let $A \subset E$ such that $A \neq \emptyset$ and $|E \setminus A| \geq 2$. Define the operator $\zeta : \mathcal{P}(E) \to \mathcal{P}(E) : X \mapsto X \setminus A$. Then ζ is a dilation, and it is not lower-regular, as $\zeta(A) = \emptyset$. Then $\mathsf{B}(\zeta)$ is not a dilation on $\Pi^*(E)$. Indeed, for $Y \subseteq E \setminus A$ such that $Y \neq \emptyset$, (21) gives $\mathsf{B}(\zeta)(\mathbf{1}_{A\cup Y}) = \mathbf{1}_{\zeta(A\cup Y)} = \mathbf{1}_Y$, so for two distinct $p, q \in E \setminus A$ we have $\mathbf{1}_{A\cup\{p\}} \vee \mathbf{1}_{A\cup\{q\}} = \mathbf{1}_{A\cup\{p,q\}}$ and

$$\mathsf{B}(\zeta)(\mathbf{1}_{A\cup\{p,q\}}) = \mathbf{1}_{\{p,q\}} > \mathbf{1}_{\{p\}} \vee \mathbf{1}_{\{q\}} = \mathsf{B}(\zeta)(\mathbf{1}_{A\cup\{p\}}) \vee \mathsf{B}(\zeta)(\mathbf{1}_{A\cup\{q\}}) \enspace .$$

Consider now the map $\delta : \mathcal{P}(E) \to \mathcal{P}(E)$ defined by $\delta(\emptyset) = \emptyset$ and $\delta(X) = X \cup A$ for $X \neq \emptyset$. It is a lower-regular dilation; however $\mathsf{B}(\zeta\delta) \neq \mathsf{B}(\zeta)\mathsf{B}(\delta)$. Indeed, for two distinct $p, q \in E \setminus A$ we have

$$\mathsf{B}(\delta)(\mathbf{1}_{\{p\}} \vee \mathbf{1}_{\{q\}}) = \mathbf{1}_{\delta(\{p\})} \vee \mathbf{1}_{\delta(\{q\})} = \mathbf{1}_{A\cup\{p\}} \vee \mathbf{1}_{A\cup\{q\}} = \mathbf{1}_{A\cup\{p,q\}} \enspace ,$$

and from above we get $\mathsf{B}(\zeta)\mathsf{B}(\delta)(\mathbf{1}_{\{p\}} \vee \mathbf{1}_{\{q\}}) = \mathsf{B}(\zeta)(\mathbf{1}_{A\cup\{p,q\}}) = \mathbf{1}_{\{p,q\}}$; on the other hand $\zeta\delta(\{p\}) = \zeta(A \cup \{p\}) = \{p\}$ and similarly $\zeta\delta(\{q\}) = \{q\}$, so

$$\begin{aligned} \mathsf{B}(\zeta\delta)(\mathbf{1}_{\{p\}} \vee \mathbf{1}_{\{q\}}) &= \mathbf{1}_{\zeta\delta(\{p\})} \vee \mathbf{1}_{\zeta\delta(\{q\})} = \mathbf{1}_{\{p\}} \vee \mathbf{1}_{\{q\}} \\ &< \mathbf{1}_{\{p,q\}} = \mathsf{B}(\zeta)\mathsf{B}(\delta)(\mathbf{1}_{\{p\}} \vee \mathbf{1}_{\{q\}}) \enspace . \end{aligned}$$

The same happens for $\delta' : \mathcal{P}(E) \to \mathcal{P}(E)$ defined by $\delta'(X) = X$ if $X \subseteq A$, and $\delta'(X) = X \cup A$ if $X \not\subseteq A$: it is a lower-regular dilation, but $\mathsf{B}(\zeta\delta') \neq \mathsf{B}(\zeta)\mathsf{B}(\delta')$.

Let us now characterize lower-regular one-block-preserving dilations by their upper adjoints; this will lead in Subsection 3.4 to their representation by a map $E_2 \to \Pi^*(E_1)$.

Definition 19 Let the operator $\eta : \mathcal{P}(E_2) \to \Pi^*(E_1)$ preserve separation. The *disjoint application of* η is the operator

$$\mathsf{D}(\eta) : \Pi^*(E_2) \to \Pi^*(E_1) : \pi \mapsto \bigcup_{B\in\pi} \eta(B) \enspace . \tag{23}$$

The fact that η preserves separation guarantees that the $\eta(B)$ for $B \in \pi$ have mutually disjoint supports, hence that $\bigcup_{B\in\pi} \eta(B)$ is indeed a partial partition.

Proposition 20 *A map $\overline{\varepsilon} : \Pi^*(E_2) \to \Pi^*(E_1)$ is the upper adjoint of a lower-regular one-block-preserving dilation $\overline{\delta} : \Pi^*(E_1) \to \Pi^*(E_2)$ if and only if there is an upper-regular erosion $\eta : \mathcal{P}(E_2) \to \Pi^*(E_1)$ such that $\overline{\varepsilon} = \mathsf{D}(\eta)$.*

Proof Note that since η is an upper-regular erosion, it preserves separation, cf. the paragraph after Definition 2.

Suppose first that $(\overline{\varepsilon}, \overline{\delta})$ is an adjunction where $\overline{\delta} : \Pi^*(E_1) \to \Pi^*(E_2)$ is lower-regular and one-block-preserving. Define $\eta : \mathcal{P}(E_2) \to \Pi^*(E_1)$ by $\eta(X) = \overline{\varepsilon}(\mathbf{1}_X)$, in particular $\eta(\emptyset) = \overline{\varepsilon}(\boldsymbol{\emptyset})$. As $\overline{\varepsilon}$ is an erosion, $\eta(E_2) = \overline{\varepsilon}(\mathbf{1}_{E_2}) = \mathbf{1}_{E_1}$, and for a non-void family $X_i \in \mathcal{P}(E_2)$, $i \in I$, we have

$$\eta\Big(\bigcap_{i\in I} X_i\Big) = \overline{\varepsilon}\Big(\mathbf{1}_{\bigcap_{i\in I} X_i}\Big) = \overline{\varepsilon}\Big(\bigwedge_{i\in I} \mathbf{1}_{X_i}\Big) = \bigwedge_{i\in I} \overline{\varepsilon}(\mathbf{1}_{X_i}) = \bigwedge_{i\in I} \eta(X_i) \enspace ;$$

hence η is an erosion. Since $\overline{\delta}$ is lower-regular, $\overline{\varepsilon}$ is upper-regular, so $\eta(\emptyset) = \overline{\varepsilon}(\boldsymbol{\varnothing}) = \boldsymbol{\varnothing}$ and η is upper-regular. Take any $\pi \in \Pi^*(E_2)$ and $A \in \mathcal{P}(E_1) \setminus \{\emptyset\}$; then $\overline{\delta}(\mathbf{1}_A) = \mathbf{1}_{A'}$ for some $A' \in \mathcal{P}(E_2) \setminus \{\emptyset\}$. Now A is included in a block of $\overline{\varepsilon}(\pi)$ if and only if $\mathbf{1}_A \leq \overline{\varepsilon}(\pi)$, and by the adjunction $(\overline{\varepsilon}, \overline{\delta})$, this is equivalent to $\overline{\delta}(\mathbf{1}_A) \leq \pi$, that is, $\mathbf{1}_{A'} \leq \pi$; this means that A' is included in a block $B \in \pi$, that is, $\overline{\delta}(\mathbf{1}_A) = \mathbf{1}_{A'} \leq \mathbf{1}_B$; by the adjunction $(\overline{\varepsilon}, \overline{\delta})$ again, this is equivalent to $\mathbf{1}_A \leq \overline{\varepsilon}(\mathbf{1}_B) = \eta(B)$, that is, A is included in a block of $\eta(B)$. Hence A is included in a block of $\overline{\varepsilon}(\pi)$ if and only if it is included in a block of $\eta(B)$ for some $B \in \pi$, in other words in a block of $\bigcup_{B \in \pi} \eta(B)$ (indeed, since η preserves separation, $\bigcup_{B \in \pi} \eta(B)$ is a partial partition). By (23), this necessarily means that $\overline{\varepsilon}(\pi) = \mathsf{D}(\eta)(\pi)$.

Conversely, let $\overline{\varepsilon} = \mathsf{D}(\eta)$ for an upper-regular erosion $\eta : \mathcal{P}(E_2) \to \Pi^*(E_1)$. Let $\zeta : \Pi^*(E_1) \to \mathcal{P}(E_2)$ be the lower adjoint of η, and define $\psi : \mathcal{P}(E_1) \to \mathcal{P}(E_2)$ by $\psi(A) = \zeta(\mathbf{1}_A)$. For any $\pi \in \Pi^*(E_1)$ and $\pi' \in \Pi^*(E_2)$, we have the following equivalences:

$$
\begin{aligned}
\mathsf{B}(\psi)(\pi) \leq \pi' \iff \bigvee_{B \in \pi} \mathbf{1}_{\psi(B)} \leq \pi' \iff \forall\, B \in \pi,\ \mathbf{1}_{\psi(B)} \leq \pi' \\
\iff \forall\, B \in \pi,\ \exists\, C \in \pi',\ \psi(B) \subseteq C \iff \forall\, B \in \pi,\ \exists\, C \in \pi',\ \zeta(\mathbf{1}_B) \subseteq C \\
\iff \forall\, B \in \pi,\ \exists\, C \in \pi',\ \mathbf{1}_B \leq \eta(C) \\
\iff \forall\, B \in \pi,\ \exists\, C \in \pi',\ \exists\, D \in \eta(C),\ B \subseteq D \\
\iff \forall\, B \in \pi,\ \exists\, D \in \bigcup_{C \in \pi'} \eta(C),\ B \subseteq D \iff \pi \leq \bigcup_{C \in \pi'} \eta(C) = \overline{\varepsilon}(\pi') \ .
\end{aligned}
$$

Here we used successively (20), the meaning (9) of $\mathbf{1}_{\psi(B)}$, the adjunction (η, ζ), and (23). Thus $(\overline{\varepsilon}, \mathsf{B}(\psi))$ is an adjunction, and the dilation $\mathsf{B}(\psi)$ will be one-block-preserving by (21). By (23), $\overline{\varepsilon}(\boldsymbol{\varnothing}) = \boldsymbol{\varnothing}$, so $\overline{\varepsilon}$ is upper-regular, hence its lower adjoint $\mathsf{B}(\psi)$ is lower-regular. $\square$

Example 21 Let $E_1 = E_2 = E$. For an integer $n \geq 0$, consider a strict chain $\pi_0 < \cdots < \pi_n = \mathbf{1}_E$ in $\Pi^*(E)$. Let $\Omega = \Pi^*(E) \cap \mathcal{P}\big(\bigcup_{i=0}^n \pi_i\big)$, the set of all partial partitions of E having each block belonging to some π_i, $i = 0, \ldots, n$; thus for $\pi \in \Pi^*(E)$, we have

$$
\pi \in \Omega \iff \forall\, p \in E,\ \mathsf{Cl}_\pi(p) \in \{\emptyset\} \cup \{\mathsf{Cl}_{\pi_i}(p) \mid i = 0, \ldots, n\} \ .
$$

Given $\Upsilon \subseteq \Omega$, for any $p \in E$, $\{\mathsf{Cl}_\pi(p) \mid \pi \in \Upsilon\} \subseteq \{\emptyset\} \cup \{\mathsf{Cl}_{\pi_i}(p) \mid i = 0, \ldots, n\}$, hence it is a chain; by (12) and Proposition 6, $\mathsf{Cl}_{\bigwedge \Upsilon}(p)$ and $\mathsf{Cl}_{\bigvee \Upsilon}(p)$ are the least and greatest among all $\mathsf{Cl}_\pi(p)$, $\pi \in \Upsilon$, so $\bigwedge \Upsilon, \bigvee \Upsilon \in \Omega$: Ω is a complete sublattice of $\Pi^*(E)$.

Define $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ by $\psi(\emptyset) = \emptyset$, and for $X \in \mathcal{P}(E) \setminus \{\emptyset\}$, $\psi(X)$ is the least $B \in \bigcup_{i=0}^n \pi_i$ such that $X \subseteq B$. Alternately, let t be the least $i = 0, \ldots, n$ such that X is included in a block of π_i, and then let $\psi(X)$ be that block of π_t containing X:

$$
\begin{aligned}
&\forall\, X \in \mathcal{P}(E) \setminus \{\emptyset\}, \quad \psi(X) = \mathsf{Cl}_{\pi_t}(p) \text{ for any } p \in X \ , \\
&\text{where } t = \min\big\{i = 0, \ldots, n \mid X \subseteq \mathsf{Cl}_{\pi_i}(p)\big\} \ .
\end{aligned}
$$

For any $\pi \in \Pi^*(E)$, $\mathsf{B}(\psi)(\pi)$ is the least $\pi' \in \Omega$ such that $\pi' \geq \pi$. By Lemma 1, $\mathsf{B}(\psi)$ is both a closure and a dilation, and its upper adjoint $\overline{\varepsilon}$ is the opening given by: for any $\pi \in \Pi^*(E)$, $\overline{\varepsilon}(\pi)$ is the greatest $\pi' \in \Omega$ such that $\pi' \leq \pi$. Here $\overline{\varepsilon} = \mathsf{D}(\eta)$ for the erosion $\eta : \mathcal{P}(E_2) \to \Pi^*(E)$ given by $\eta(\emptyset) = \boldsymbol{\emptyset}$, and for $Y \in \mathcal{P}(E) \setminus \{\emptyset\}$, $\eta(Y)$ is the greatest $\pi' \in \Omega$ such that $\mathsf{supp}(\pi') \subseteq Y$. Alternately,

$$\forall Y \in \mathcal{P}(E) \setminus \{\emptyset\},\ \forall p \in E,\ S_p = \big\{i = 0, \ldots, n \mid \mathsf{Cl}_{\pi_i}(p) \subseteq Y\big\} \ ,$$
$$\mathsf{Cl}_{\eta(Y)}(p) = \mathsf{Cl}_{\pi_t}(p) \text{ for } t = \max S_p \text{ if } S_p \neq \emptyset,\ \mathsf{Cl}_{\eta(Y)}(p) = \emptyset \text{ otherwise} \ .$$

For instance, let $E = \{0, \ldots, 2^n - 1\}^2$, a digital square of size $2^n \times 2^n$, and for $i = 0, \ldots, n$, let π_i be the partition of E into the $2^i \times 2^i$-squares

$$\{(a2^i + x, b2^i + y) \mid 0 \leq x, y \leq 2^i - 1\}, \quad 0 \leq a, b \leq 2^{n-i} - 1 \ . \tag{24}$$

Here for $X \in \mathcal{P}(E) \setminus \{\emptyset\}$, $\psi(X)$ is the least square of the form (24) enclosing X, while $\eta(X)$ will be the quad-tree partition of X. See Figure 6 for $n = 3$.

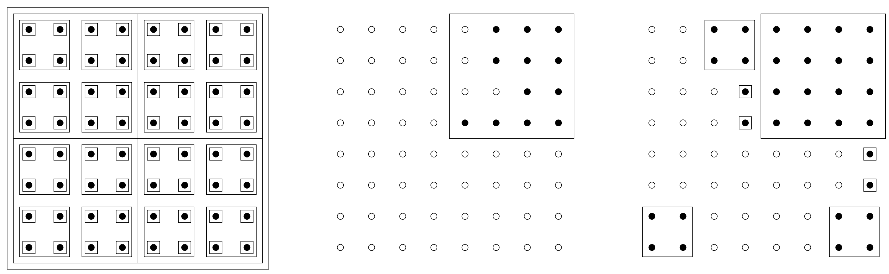

Fig. 6 Left: let $E_1 = E_2 = E$ be the 8×8 square $\{0, \ldots, 7\}^2$ (points of E are shown as filled dots); for $i = 0, \ldots, 3$, π_i is the partition of E into $2^i \times 2^i$-squares, cf. (24). Middle: the set X (filled dots) and the least enclosing square $\psi(X)$. Right: the set Y (filled dots) and its quad-tree partition $\eta(Y)$ into maximal squares.

3.3 Regular adjunctions derived from set adjunctions

We will now study the construction, from set adjunctions (ε, δ), of adjunctions $\big(\mathsf{B}(\varepsilon), \mathsf{B}(\delta)\big)$ on partial partitions.

Recall from Definition 2 that an operator $\psi : \mathcal{P}(E_1) \to \mathcal{P}(E_2)$ *preserves separation* if $\psi(\emptyset) = \emptyset$ and for any $X, Y \in \mathcal{P}(E_1) \setminus \{\emptyset\}$ such that $X \cap Y = \emptyset$, we must have $\psi(X) \cap \psi(Y) = \emptyset$. For example, when $E_1 = E_2$, every anti-extensive operator preserves separation.

Proposition 22 *1. Let $\psi : \mathcal{P}(E_1) \to \mathcal{P}(E_2)$ preserve separation. Then for any $\pi \in \Pi^*(E_1)$ we have*

$$\mathsf{B}(\psi)(\pi) = \{\psi(B) \mid B \in \pi,\ \psi(B) \neq \emptyset\} \ . \tag{25}$$

2. Let $\psi : \mathcal{P}(E_1) \to \mathcal{P}(E_2)$ *and* $\xi : \mathcal{P}(E_2) \to \mathcal{P}(E_3)$ *preserve separation. Then* $\xi\psi : \mathcal{P}(E_1) \to \mathcal{P}(E_3)$ *preserves separation and* $\mathsf{B}(\xi)\mathsf{B}(\psi) = \mathsf{B}(\xi\psi)$*.*

Proof 1. By defininition, $\mathbf{1}_{\psi(B)} = \{\psi(B)\}$ if $\psi(B) \neq \emptyset$, and is void otherwise (9). Since ψ preserves separation, the non-void $\psi(B)$, $B \in \pi$, will be disjoint. Hence

$$\bigvee_{B\in\pi} \mathbf{1}_{\psi(B)} = \bigvee_{\substack{B\in\pi \\ \psi(B)\neq\emptyset}} \{\psi(B)\} = \{\psi(B) \mid B \in \pi,\ \psi(B) \neq \emptyset\} \ .$$

2. We have $\xi\psi(\emptyset) = \xi(\psi(\emptyset)) = \xi(\emptyset) = \emptyset$. Let $X, Y \in \mathcal{P}(E_1) \setminus \{\emptyset\}$. If $\psi(X) = \emptyset$, as $\xi(\emptyset) = \emptyset$, we get $\xi\psi(X) = \emptyset$, so $\xi\psi(X) \cap \xi\psi(Y) = \emptyset$; similarly $\psi(Y) = \emptyset$ gives $\xi\psi(X) \cap \xi\psi(Y) = \emptyset$. If $\psi(X) \neq \emptyset \neq \psi(Y)$, as ψ preserves separation, $\psi(X) \cap \psi(Y) = \emptyset$, and as ξ preserves separation, $\xi\psi(X) \cap \xi\psi(Y) = \emptyset$. Therefore $\xi\psi$ preserves separation.

For any $\pi \in \Pi^*(E)$, applying (25) successively for ψ and for ξ, we get (cf. Figure 7):

$$\begin{aligned} \mathsf{B}(\xi)\mathsf{B}(\psi)(\pi) &= \mathsf{B}(\xi)\big(\{\psi(B) \mid B \in \pi,\ \psi(B) \neq \emptyset\}\big) \\ &= \{\xi\psi(B) \mid B \in \pi,\ \psi(B) \neq \emptyset,\ \xi\psi(B) \neq \emptyset\} \\ &= \{\xi\psi(B) \mid B \in \pi,\ \xi\psi(B) \neq \emptyset\} = \mathsf{B}(\xi\psi)(\pi) \ . \end{aligned}$$

Here we used the fact that, since $\xi(\emptyset) = \emptyset$, $\xi\psi(B)) \neq \emptyset \ \Rightarrow\ \psi(B) \neq \emptyset$. $\square$

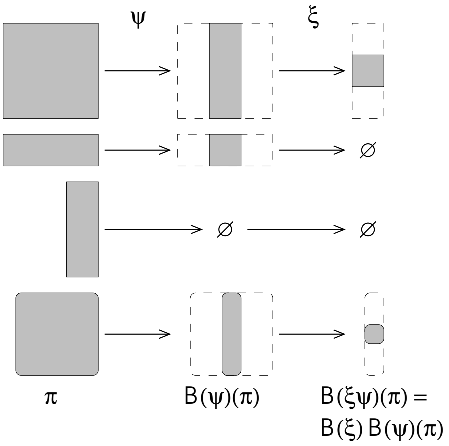

Fig. 7 Let $E_1 = E_2 = \mathbf{Z}^2$. Here $\psi, \xi : \mathcal{P}(\mathbf{Z}^2) \to \mathcal{P}(\mathbf{Z}^2)$ are erosions respectively by a horizontal segment and a vertical segment. We have $\mathsf{B}(\xi\psi)(\pi) = \mathsf{B}(\xi)\mathsf{B}(\psi)(\pi)$.

Recall from Definition 2 that an adjunction $(\varepsilon, \delta) : \mathcal{P}(E_2) \rightleftharpoons \mathcal{P}(E_1)$ is *regular* if and only if ε is *upper-regular* (that is, $\varepsilon(\emptyset) = \emptyset$), if and only if δ is *lower-regular* (that is, $\forall\, X \in \mathcal{P}(E_1),\ X \neq \emptyset \ \Rightarrow\ \delta(X) \neq \emptyset$); a necessary and sufficient condition is that $\forall\, p \in E_1,\ \delta(\{p\}) \neq \emptyset$. Given a regular adjunction (ε, δ), every adjunction (ε', δ') such that $\varepsilon' \leq \varepsilon$ (equivalently, $\delta' \geq \delta$) will be regular. For instance, an anti-extensive erosion is upper-regular and an extensive dilation is lower-regular. Now every upper-regular erosion preserves separation:

$$X \cap Y = \emptyset \ \Rightarrow \ \varepsilon(X) \cap \varepsilon(Y) = \varepsilon(X \cap Y) = \varepsilon(\emptyset) = \emptyset.$$

The following result is a specialization of Theorem 17 and Corollary 18; indeed a dilation δ is connective (Lemma 3).

Theorem 23 *1. For any regular adjunction $(\varepsilon, \delta) : \mathcal{P}(E_2) \rightleftharpoons \mathcal{P}(E_1)$, $\mathsf{B}(\varepsilon)$ satisfies (25) and $\big(\mathsf{B}(\varepsilon), \mathsf{B}(\delta)\big)$ is an adjunction $\Pi^*(E_2) \rightleftharpoons \Pi^*(E_1)$.*

2. Given two regular adjunctions $(\varepsilon_1, \delta_1) : \mathcal{P}(E_2) \rightleftharpoons \mathcal{P}(E_1)$ and $(\varepsilon_2, \delta_2) : \mathcal{P}(E_3) \rightleftharpoons \mathcal{P}(E_2)$, the adjunction $(\varepsilon_1\varepsilon_2, \delta_2\delta_1) : \mathcal{P}(E_3) \rightleftharpoons \mathcal{P}(E_1)$ is regular, $\mathsf{B}(\varepsilon_1\varepsilon_2) = \mathsf{B}(\varepsilon_1)\mathsf{B}(\varepsilon_2)$ and $\mathsf{B}(\delta_2\delta_1) = \mathsf{B}(\delta_2)\mathsf{B}(\delta_1)$.

Proof 1. Since ε is upper-regular, it preserves separation and (25) follows from Proposition 22. Let $\pi \in \Pi^*(E_1)$ and $\pi' \in \Pi^*(E_2)$. We have:

$$\begin{aligned}
\mathsf{B}(\delta)(\pi) \leq \pi' \iff \bigvee_{B \in \pi} \mathbf{1}_{\delta(B)} \leq \pi' \iff \forall B \in \pi,\ \mathbf{1}_{\delta(B)} \leq \pi' \\
\iff \forall B \in \pi,\ \exists C \in \pi',\ \delta(B) \subseteq C \iff \forall B \in \pi,\ \exists C \in \pi',\ B \subseteq \varepsilon(C) \\
\iff \forall B \in \pi,\ \exists C \in \pi',\ B \subseteq \varepsilon(C),\ \varepsilon(C) \neq \emptyset \\
\iff \pi \leq \{\varepsilon(C) \mid C \in \pi',\ \varepsilon(C) \neq \emptyset\} \iff \pi \leq \mathsf{B}(\varepsilon)(\pi') \ .
\end{aligned}$$

Here we used successively (20), the meaning (9) of $\mathbf{1}_{\delta(B)}$, the adjunction (ε, δ), the fact that $B \neq \emptyset$, and (25).

2. By the composition rule for adjunctions, $(\varepsilon_1\varepsilon_2, \delta_2\delta_1)$ is an adjunction $\mathcal{P}(E_3) \rightleftharpoons \mathcal{P}(E_1)$. Clearly $\varepsilon_1\varepsilon_2(\emptyset) = \varepsilon_1(\emptyset) = \emptyset$, thus $(\varepsilon_1\varepsilon_2, \delta_2\delta_1)$ is regular. Since ε_1 and ε_2 satisfy the requirements of item 2 of Proposition 22, $\mathsf{B}(\varepsilon_1\varepsilon_2) = \mathsf{B}(\varepsilon_1)\mathsf{B}(\varepsilon_2)$. Now $\mathsf{B}(\delta_2\delta_1) = \mathsf{B}(\delta_2)\mathsf{B}(\delta_1)$ by Corollary 18. Alternately, $\big(\mathsf{B}(\varepsilon_1\varepsilon_2), \mathsf{B}(\delta_2\delta_1)\big)$ and $\big(\mathsf{B}(\varepsilon_1)\mathsf{B}(\varepsilon_2), \mathsf{B}(\delta_2)\mathsf{B}(\delta_1)\big)$ are two adjunctions $\Pi^*(E_3) \rightleftharpoons \Pi^*(E_1)$, then the upper adjoints are equal, $\mathsf{B}(\varepsilon_1\varepsilon_2) = \mathsf{B}(\varepsilon_1)\mathsf{B}(\varepsilon_2)$, if and only if the lower adjoints are equal, $\mathsf{B}(\delta_2\delta_1) = \mathsf{B}(\delta_2)\mathsf{B}(\delta_1)$. □

Note that the regularity assumption is crucial, as shows the following counterexample. Take again $E_1 = E_2 = E$ and $A \subset E$ such that $A \neq \emptyset$ and $|E \setminus A| \geq 2$. We saw after the proof of Theorem 17 that the dilation $\zeta : \mathcal{P}(E) \to \mathcal{P}(E) : X \mapsto X \setminus A$ is not lower-regular, and that $\mathsf{B}(\zeta)$ is not a dilation on $\Pi^*(E)$. The upper adjoint of ζ is the erosion $\eta : \mathcal{P}(E) \to \mathcal{P}(E) : X \mapsto X \cup A$, it is not upper-regular. Not only $\mathsf{B}(\eta)$ does not satisfy (25), but it is not an erosion. Indeed, for $Y \subseteq E \setminus A$ such that $Y \neq \emptyset$, (21) gives $\mathsf{B}(\eta)(\mathbf{1}_Y) = \mathbf{1}_{\eta(Y)} = \mathbf{1}_{A \cup Y}$, so for two distinct $p, q \in E \setminus A$ we have

$$\begin{aligned}
\mathsf{B}(\eta)(\mathbf{1}_{\{p\}}) \wedge \mathsf{B}(\eta)(\mathbf{1}_{\{q\}}) = \mathbf{1}_{A \cup \{p\}} \wedge \ \mathbf{1}_{A \cup \{q\}} = \mathbf{1}_A \ , \\
\text{but} \quad \mathsf{B}(\eta)(\mathbf{1}_{\{p\}} \wedge \mathbf{1}_{\{q\}}) = \mathsf{B}(\eta)(\boldsymbol{\emptyset}) = \boldsymbol{\emptyset} \ .
\end{aligned}$$

Combining Theorems 12 and 23, from a regular adjunction $(\varepsilon, \delta) : \mathcal{P}(E_2) \rightleftharpoons \mathcal{P}(E_1)$ we derive the adjunction $\big(FS_1 {\cdot} \mathsf{B}(\varepsilon) {\cdot} IN_2, FS_2 {\cdot} \mathsf{B}(\delta) {\cdot} RSIN_1\big) : \Pi(E_2) \rightleftharpoons$

$\Pi(E_1)$. For $E_1 = E_2 = E$, this is indeed the adjunction $(\widehat{\varepsilon}, \widehat{\delta})$ from [30,36] that we described in the Introduction.

Similarly, dilations (resp., erosions) on partial partitions can be obtained by composition or join (resp., meet) of dilations $\mathsf{B}(\delta)$, RS and **grind** (resp., erosions $\mathsf{B}(\varepsilon)$, $INFS$ and **blend**) for a regular adjunction (ε, δ). For example, given two regular adjunctions $(\varepsilon_1, \delta_1)$ and $(\varepsilon_2, \delta_2)$, we obtain the adjunction

$$\big(INFS\,\mathsf{B}(\varepsilon_2) \wedge \mathbf{blend}\,\mathsf{B}(\varepsilon_1), \mathsf{B}(\delta_2)\,RS \vee \mathsf{B}(\delta_1)\,\mathbf{grind}\big) \enspace .$$

If $\delta_1 = \delta_2$ (equivalently, $\varepsilon_1 = \varepsilon_2$), then by (17) this adjunction reduces to $\big(\mathsf{B}(\varepsilon_1), \mathsf{B}(\delta_1)\big)$. Suppose that $\delta_1 < \delta_2$ (equivalently, $\varepsilon_1 > \varepsilon_2$); by (17) the adjunction becomes:

$$\big(INFS\,\mathsf{B}(\varepsilon_2) \wedge \mathsf{B}(\varepsilon_1), \mathsf{B}(\delta_2)\,RS \vee \mathsf{B}(\delta_1)\big) \enspace .$$

Now $\mathsf{B}(\delta_2)\,RS \vee \mathsf{B}(\delta_1)\,\mathbf{grind} = \mathsf{B}(\delta_2)\,RS \vee \mathsf{B}(\delta_1)$ is one-block-preserving. Here according to Theorem 17 we have $\psi(\{p\}) = \delta_1(\{p\})$ and $\psi(X) = \delta_2(X)$ for $|X| \geq 2$. We illustrate such a dilation in Figure 8.

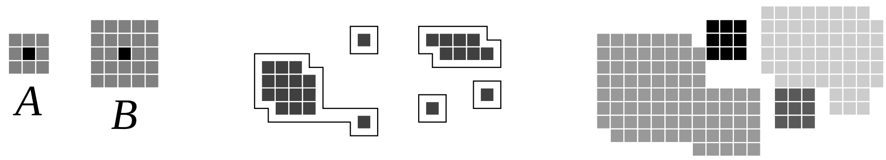

Fig. 8 Let $E_1 = E_2 = \mathbf{Z}^2$. Pixels are shown as small squares. Left: the two structuring elements A and B (centered about the origin, shown in black). Middle: a partition π with 3 singleton blocks and 2 non-singleton blocks (each block is surrounded by a closed line). We apply on singleton blocks the dilation δ_1 by A, and on non-singleton blocks the dilation δ_2 by B, and merge overlapping dilated blocks (the two rightmost ones). Right: the resulting partition $\big(\mathsf{B}(\delta_2)\,RS \vee \mathsf{B}(\delta_1)\big)(\pi)$ has 4 blocks (each one is shown with a distinctive grey-level).

3.4 Dual characterization

We will characterize lower-regular connective operators $\psi : \mathcal{P}(E_1) \to \mathcal{P}(E_2)$ by maps $E_2 \to \Pi^*(E_1)$. We will also specialize this characterization to the case where ψ is a dilation. Application-minded readers can skip this subsection and go directly to Section 5.

Lemma 24 *There is a one-to-one correspondence between erosions $\eta : \mathcal{P}(E_2) \to \Pi^*(E_1)$ and maps $\pi_\bullet : E_2 \to \Pi^*(E_1) : x \mapsto \pi_x$, given by:*

- *for every $Y \in \mathcal{P}(E_2)$, $\eta(Y) = \bigwedge_{x \in E_2 \setminus Y} \pi_x$;*
- *for every $x \in E_2$, $\pi_x = \eta(E_2 \setminus \{x\})$.*

Furthermore, η is upper-regular if and only if $\bigwedge_{x \in E_2} \pi_x = \mathbf{\emptyset}$, if and only if $\forall\, p \in E_1,\ \exists\, x \in E_2,\ p \notin \mathsf{supp}(\pi_x)$.

Proof If η is an erosion, for every $x \in E_2$ set $\pi_x = \eta(E_2 \setminus \{x\})$. Then for any $Y \in \mathcal{P}(E_2)$ we have $Y = \bigcap_{x \in E_2 \setminus Y} (E_2 \setminus \{x\})$, so

$$\eta(Y) = \eta\Big(\bigcap_{x \in E_2 \setminus Y} (E_2 \setminus \{x\}) \Big) = \bigwedge_{x \in E_2 \setminus Y} \eta(E_2 \setminus \{x\}) = \bigwedge_{x \in E_2 \setminus Y} \pi_x \enspace .$$

Conversely, given a map $\pi_\bullet : E_2 \to \Pi^*(E_1) : x \mapsto \pi_x$, the operator $\eta : \mathcal{P}(E_2) \to \Pi^*(E_1) : Y \mapsto \bigwedge_{x \in E_2 \setminus Y} \pi_x$ is an erosion. Indeed, $\eta(E_2)$ is the empty infimum in $\Pi^*(E_1)$, that is $\eta(E_2) = \mathbf{1}_{E_1}$, while for a non-void family $Y_i \in \mathcal{P}(E_2)$, $i \in I \neq \emptyset$, we have

$$\begin{aligned} \eta\Big(\bigcap_{i \in I} Y_i \Big) &= \bigwedge_{x \in E_2 \setminus \bigcap_{i \in I} Y_i} \pi_x = \bigwedge_{x \in \bigcup_{i \in I} (E_2 \setminus Y_i)} \pi_x \\ &= \bigwedge_{i \in I} \Big(\bigwedge_{x \in E_2 \setminus Y_i} \pi_x \Big) = \bigwedge_{i \in I} \eta(Y_i) \enspace . \end{aligned}$$

Then for $Y = E_2 \setminus \{x\}$, x is the only element of $E_2 \setminus Y$, thus $\eta(E_2 \setminus \{x\}) = \pi_x$.

Now η is upper-regular if and only if $\eta(\emptyset) = \mathbf{\emptyset}$; but $\eta(\emptyset) = \bigwedge_{x \in E_2 \setminus \emptyset} \pi_x$, so the condition is $\bigwedge_{x \in E_2} \pi_x = \mathbf{\emptyset}$. Since $\mathbf{\emptyset}$ is the unique partition with empty support, this is equivalent to $\emptyset = \mathsf{supp}\big(\bigwedge_{x \in E_2} \pi_x \big) = \bigcap_{x \in E_2} \mathsf{supp}(\pi_x)$, in other words every $p \in E_1$ satisfies $p \notin \bigcap_{x \in E_2} \mathsf{supp}(\pi_x)$, that is, for some $x \in E_2$, we have $p \notin \mathsf{supp}(\pi_x)$ □

Theorem 25 *There is a dual isomorphism between the poset of lower-regular connective operators $\psi : \mathcal{P}(E_1) \to \mathcal{P}(E_2)$ and the poset of maps $\pi_\bullet : E_2 \to \Pi^*(E_1) : x \mapsto \pi_x$ such that $\bigwedge_{x \in E_2} \pi_x = \mathbf{\emptyset}$, where $\pi_\bullet$ corresponds to ψ in two ways:*

– *The upper adjoint of $\mathsf{B}(\psi)$ is the erosion $\overline{\varepsilon} : \Pi^*(E_2) \to \Pi^*(E_1)$ given by*

$$\forall \pi \in \Pi^*(E_2), \quad \overline{\varepsilon}(\pi) = \bigcup_{B \in \pi} \bigwedge_{x \in E_2 \setminus B} \pi_x \enspace . \tag{26}$$

–

$$\forall X \in \mathcal{P}(E_1),\ \forall x \in E_2, \quad x \notin \psi(X) \iff \mathbf{1}_X \leq \pi_x \enspace , \tag{27}$$

in other words if and only if X is included in a block of π_x.

Given a non-void family ψ_i, $i \in I \neq \emptyset$ of lower-regular connective operators, corresponding each to $\pi^i_\bullet$, $\bigvee_{i \in I} \psi_i : X \mapsto \bigcup_{i \in I} \psi_i(X)$ will correspond to $\bigwedge_{i \in I} \pi^i_\bullet : x \mapsto \bigwedge_{i \in I} \pi^i_x$.

Proof By Theorem 17, we have a bijection $\psi \leftrightarrow \mathsf{B}(\psi)$ between lower-regular connective operators $\mathcal{P}(E_1) \to \mathcal{P}(E_2)$, and lower-regular one-block-preserving dilations $\Pi^*(E_1) \to \Pi^*(E_2)$. By Proposition 20, the latter are in a bijection with their upper adjoints $\overline{\varepsilon} : \Pi^*(E_2) \to \Pi^*(E_1)$ that take the form $\mathsf{D}(\eta)$, cf. (23), for an upper-regular erosion $\eta : \mathcal{P}(E_2) \to \Pi^*(E_1)$; the relation $\overline{\varepsilon} \leftrightarrow \eta$ is obviously a bijection. By Lemma 24, these η are in bijection with maps

$\pi_\bullet : E_2 \to \Pi^*(E_1)$ such that $\bigwedge_{x \in E_2} \pi_x = \mathbf{Ø}$. Then (23) combined with the first item in Lemma 24 gives for every $\pi \in \Pi^*(E_2)$:

$$\overline{\varepsilon}(\pi) = \bigcup_{B \in \pi} \eta(B) = \bigcup_{B \in \pi} \bigwedge_{x \in E_2 \setminus B} \pi_x \ ,$$

that is (26). Note that for since η is upper-regular, in other words $\bigwedge_{x \in E_2} \pi_x = \mathbf{Ø}$, the $\eta(B) = \bigwedge_{x \in E_2 \setminus B} \pi_x$ for $B \in \pi$ have mutually disjoint supports, so that $\bigcup_{B \in \pi} \eta(B)$ is indeed a partial partition.

Let $X \in \mathcal{P}(E_1)$ and $x \in E_2$. Then by (21), the adjunction $(\overline{\varepsilon}, \mathsf{B}(\psi))$ and (26), we obtain the following equivalences:

$$\begin{aligned} x \notin \psi(X) &\iff \psi(X) \subseteq E_2 \setminus \{x\} \iff \mathbf{1}_{\psi(X)} \leq \mathbf{1}_{E_2 \setminus \{x\}} \\ \iff \mathsf{B}(\psi)(\mathbf{1}_X) \leq \mathbf{1}_{E_2 \setminus \{x\}} &\iff \mathbf{1}_X \leq \overline{\varepsilon}(\mathbf{1}_{E_2 \setminus \{x\}}) \iff \mathbf{1}_X \leq \pi_x \ , \end{aligned}$$

that is (27).

Let ψ_i, $i \in I \neq \emptyset$ be lower-regular connective operators, corresponding each to $\pi^i_\bullet$. Clearly $\bigvee_{i \in I} \psi_i : X \mapsto \bigcup_{i \in I} \psi_i(X)$ is lower-regular and connective, while $\bigwedge_{i \in I} \pi^i_\bullet : x \mapsto \bigwedge_{i \in I} \pi^i_x$ satisfies

$$\bigwedge_{x \in E_2} \bigwedge_{i \in I} \pi_x = \bigwedge_{i \in I} \bigwedge_{x \in E_2} \pi_x = \bigwedge_{i \in I} \mathbf{Ø} = \mathbf{Ø} \ .$$

By (27), for $X \in \mathcal{P}(E_1)$ and $x \in E_2$ we have

$$\begin{aligned} x \notin \Big(\bigvee_{i \in I} \psi_i\Big)(X) = \bigcup_{i \in I} \psi_i(X) &\iff \forall\, i \in I,\ x \notin \psi_i(X) \\ \iff \forall\, i \in I,\ \mathbf{1}_X \leq \pi^i_x &\iff \mathbf{1}_X \leq \bigwedge_{i \in I} \pi^i_x = \Big(\bigwedge_{i \in I} \pi^i_\bullet\Big)(x) \ , \end{aligned}$$

so by (27) $\bigvee_{i \in I} \psi_i$ will correspond to $\bigwedge_{i \in I} \pi^i_\bullet$. Thus the bijection $\psi \leftrightarrow \pi_\bullet$ reverses the order, it is a dual isomorphism between the two posets. ☐

If we do not assume that ψ is lower-regular, applying Theorem 13 to the case where $L = \mathcal{P}(E_2)$, we obtain a bijection between connective maps $\psi : \mathcal{P}(E_1) \to \mathcal{P}(E_2)$ and dilations $\zeta : \Pi^*(E_1) \to \mathcal{P}(E_2)$, given by $\zeta(\pi) = \bigcup_{B \in \pi} \psi(B) = \mathsf{supp}\big(\mathsf{B}(\psi)(\pi)\big)$ for all $\pi \in \Pi^*(E_1)$. Then (27) gives a bijection between connective maps ψ and maps $\pi_\bullet : E_2 \to \Pi^*(E_1) : x \mapsto \pi_x$, without the assumption and that $\bigwedge_{x \in E_2} \pi_x = \mathbf{Ø}$. The erosion $\eta : \mathcal{P}(E_2) \to \Pi^*(E_1) : B \mapsto \bigwedge_{x \in E_2 \setminus B} \pi_x$ will be the upper adjoint of the dilation ζ. The correspondence $\psi \leftrightarrow \zeta$ is an isomorphism of dual Moore families, the one $\eta \leftrightarrow \pi_\bullet$ is an isomorphism of Moore families, and $\psi \leftrightarrow \pi_\bullet$ is a dual isomorphism.

Proposition 26 *Suppose that $E_1 = E_2 = E$. Let $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ be a lower-regular connective operator, corresponding to $\pi_\bullet : E \to \Pi^*(E) : x \mapsto \pi_x$. Then ψ is extensive if and only if $\forall\, p \in E$, $p \notin \mathsf{supp}(\pi_p)$, and this implies that $\mathsf{B}(\psi)$ is extensive.*

Proof By Lemma 3, ψ is isotone, and by Lemma 16, ψ is extensive if and only if $\forall p \in E,\ p \in \psi(\{p\})$. By (27), this is equivalent to $\forall p \in E_2,\ \mathbf{1}_{\{p\}} \not\leq \pi_p$, that is, $p \notin \mathsf{supp}(\pi_p)$. By Lemma 16 again, the extensivity of ψ implies that of $\mathsf{B}(\psi)$. $\square$

Example 27 Let $E = \mathbf{R}$ or $\mathbf{Z}$. For $p, q \in E$ such that $p \leq q$, let $[p, q] = \{x \in E \mid p \leq x \leq q\}$; a subset X of E is *convex* if and only if for all $p, q \in X$ such that $p \leq q$, $[p, q] \subseteq X$. Otherwise, the *convex hull* of X is the least convex subset of E containing X. Note that X is convex if and only if it is *connected* (in $\mathbf{R}$: for the usual topological connectivity; in $\mathbf{Z}$: in the graph-theoretical sense, if we put an edge between z and $z + 1$ for all $z \in \mathbf{Z}$). Let $\Pi^*_{conv}(E)$ be the set of all partial partitions of E whose blocks are all convex. It can be checked that chaining or intersecting convex blocks leads to convex blocks, so $\Pi^*_{conv}(E)$ is a complete sublattice of $\Pi^*(E)$.

For every $x \in E$, let

$$\pi_x = \big\{ \{y \in E \mid y < x\} \, , \, \{y \in E \mid y > x\} \big\} \ .$$

Then we obtain by (27): for any $X \in \mathcal{P}(E)$, $\psi(X) = \{x \in E \mid \exists p, q \in X,\ p \leq x \leq q\}$; thus for $p \in E$, $\psi(\{p\}) = \{p\}$, while for $|X| \geq 2$, $\psi(X) = \bigcup\{[p, q] \mid p, q \in X,\ p < q\}$. In other words, ψ is the convex hull operator. We note that for all $x \in E$, $x \notin \mathsf{supp}(\pi_x)$, and indeed ψ is extensive. The extensive dilation $\mathsf{B}(\psi)$ transforms $\pi \in \Pi^*(E)$ into the least $\pi' \in \Pi^*_{conv}(E)$ with $\pi' \geq \pi$. By Lemma 1, $\mathsf{B}(\psi)$ is a closure, and its upper adjoint $\overline{\varepsilon}$ is the opening such that for $\pi \in \Pi^*(E)$, $\overline{\varepsilon}(\pi)$ is the greatest $\pi' \in \Pi^*_{conv}(E)$ with $\pi' \leq \pi$. The erosion $\eta : \mathcal{P}(E_2) \to \Pi^*(E)$ of (23) will split a set into the partial partition of its maximal convex subsets, in other words, of its connected components.

Let us return to the example of Figure 8, namely the one-block-preserving dilation $\mathsf{B}(\delta_2)\, RS \vee \mathsf{B}(\delta_1)\, \mathbf{grind} = \mathsf{B}(\delta_2)\, RS \vee \mathsf{B}(\delta_1)$ for $\delta_1 < \delta_2$. For every $x \in E_2$, define $B_x, C_x \in \mathcal{P}(E_1)$ by $p \in B_x \ \Leftrightarrow\ x \in \delta_1(\{p\})$ and $p \in C_x \ \Leftrightarrow\ x \in \delta_2(\{p\})$; since $\delta_1 < \delta_2$, we have $B_x \subseteq C_x$; now the blocks of π_x are $E_1 \setminus C_x$ and the singletons in $C_x \setminus B_x$.

Proposition 28 *Let $\psi : \mathcal{P}(E_1) \to \mathcal{P}(E_2)$ be a lower-regular connective operator, corresponding to $\pi_\bullet : E_2 \to \Pi^*(E_1) : x \mapsto \pi_x$. Then the following are equivalent:*

1. *ψ is a dilation.*
2. *For any two distinct $p, q \in E_1$, $\psi(\{p, q\}) = \psi(\{p\}) \cup \psi(\{q\})$.*
3. *$\forall x \in E_2$, π_x has at most one block.*
4. *The upper adjoint of $\mathsf{B}(\psi)$ is one-block-preserving.*

Proof Obviously $1 \Rightarrow 2$.

$2 \Rightarrow 3$. By 2 we have for any two distinct $p, q \in E_1$ and for any $x \in E_2$: $x \notin \psi(\{p, q\}) \ \Leftrightarrow\ x \notin \psi(\{p\}) \cup \psi(\{q\}) \ \Leftrightarrow\ x \notin \psi(\{p\})$ and $x \notin \psi(\{q\})$; by (27), this means that $\mathbf{1}_{\{p,q\}} \leq \pi_x \ \Leftrightarrow\ \mathbf{1}_{\{p\}} \leq \pi_x$ and $\mathbf{1}_{\{q\}} \leq \pi_x$, in other words, $\mathbf{1}_{\{p,q\}} \leq \pi_x \ \Leftrightarrow\ \{p, q\} \subseteq \mathsf{supp}(\pi_x)$. This is equivalent to π_x having at most one block.

$3 \Rightarrow 4$. By 3, for every $x \in E_2$ there is some $B(x) \in \mathcal{P}(E_1)$ such that $\pi_x = \mathbf{1}_{B(x)}$. Let $\overline{\varepsilon}$ be the upper adjoint of $\mathsf{B}(\psi)$. By (26) and (14), for every $A \in \mathcal{P}(E_2) \setminus \{\emptyset\}$ we have

$$\overline{\varepsilon}(\mathbf{1}_A) = \bigwedge_{x \in E_2 \setminus A} \pi_x = \bigwedge_{x \in E_2 \setminus A} \mathbf{1}_{B(x)} = \mathbf{1}_{\bigcap_{x \in E_2 \setminus A} B(x)} \ ,$$

so $\overline{\varepsilon}$ is one-block-preserving.

$4 \Rightarrow 1$. Let $\overline{\varepsilon}$ be the upper adjoint of $\mathsf{B}(\psi)$. By 4, there is $\xi : \mathcal{P}(E_2) \to \mathcal{P}(E_1)$ such that for every $B \in \mathcal{P}(E_2) \setminus \{\emptyset\}$, $\overline{\varepsilon}(\mathbf{1}_B) = \mathbf{1}_{\xi(B)}$; we set also $\xi(\emptyset) = \emptyset$; as $\overline{\varepsilon}$ is upper-regular, we have $\overline{\varepsilon}(\mathbf{1}_\emptyset) = \overline{\varepsilon}(\boldsymbol{\emptyset}) = \boldsymbol{\emptyset} = \mathbf{1}_\emptyset = \mathbf{1}_{\xi(\emptyset)}$. For any $A \in \mathcal{P}(E_1) \setminus \{\emptyset\}$, (21) gives $\mathsf{B}(\psi)(\mathbf{1}_A) = \mathbf{1}_{\psi(A)}$; now $\psi(\emptyset) = \emptyset$, so $\mathsf{B}(\psi)(\mathbf{1}_\emptyset) = \mathsf{B}(\psi)(\boldsymbol{\emptyset}) = \boldsymbol{\emptyset} = \mathbf{1}_\emptyset = \mathbf{1}_{\psi(\emptyset)}$. Thus for $A \in \mathcal{P}(E_1)$ and $B \in \mathcal{P}(E_2)$, $\mathsf{B}(\psi)(\mathbf{1}_A) = \mathbf{1}_{\psi(A)}$ and $\overline{\varepsilon}(\mathbf{1}_B) = \mathbf{1}_{\xi(B)}$. Then by the adjunction $(\overline{\varepsilon}, \mathsf{B}(\psi))$ and the fact that $X \mapsto \mathbf{1}_X$ is an order-embedding, we get

$$\begin{aligned} &\psi(A) \subseteq B \iff \mathbf{1}_{\psi(A)} \leq \mathbf{1}_B \iff \mathsf{B}(\psi)(\mathbf{1}_A) \leq \mathbf{1}_B \\ &\iff \mathbf{1}_A \leq \overline{\varepsilon}(\mathbf{1}_B) \iff \mathbf{1}_A \leq \mathbf{1}_{\xi(B)} \iff A \subseteq \xi(B) \ . \end{aligned}$$

Hence (ξ, ψ) is an adjunction, so ψ is a dilation. □

In Example 27 (here $E_1 = E_2 = E$), ψ is not a dilation, and we see indeed that π_x has two blocks.

Example 29 Let us give some instances of regular adjunctions, and characterize the map $\pi_\bullet$ corresponding to each lower adjoint.

1. In the Euclidean space $E = \mathbf{R}^n$ or the digital space $E = \mathbf{Z}^n$, the adjunction $(\varepsilon_B, \delta_B)$, with $\delta_B : X \mapsto X \oplus B$ and $\varepsilon_B : X \mapsto X \ominus B$, is regular if and only if $B \neq \emptyset$. For $x \in E$, the unique block of π_x is $E \setminus \check{B}_x$, where $\check{B}_x = \{x - b \mid b \in B\}$.
2. If $E_1 \subseteq E_2$, the restriction and inclusion pair $(res, inc) : \mathcal{P}(E_2) \rightleftharpoons \mathcal{P}(E_1)$, given by $inc(X) = X$ $(X \in \mathcal{P}(E_1))$ and $res(Y) = Y \cap E_1$ $(Y \in \mathcal{P}(E_2))$, is a regular adjunction. Then the adjunction $\big(\mathsf{B}(res), \mathsf{B}(inc)\big) : \Pi^*(E_2) \rightleftharpoons \Pi^*(E_1)$ is given by $\mathsf{B}(inc)(\pi) = \pi$ for $\pi \in \Pi^*(E_1)$, and

$$\mathsf{B}(res)(\pi) = \pi \wedge \mathbf{1}_{E_1} = \{B \cap E_1 \mid B \in \pi,\ B \cap E_1 \neq \emptyset\}$$

   for $\pi \in \Pi^*(E_2)$. For $x \in E_2$, the unique block of π_x is $E_1 \setminus \{x\}$ if $x \in E_1$, and E_1 if $x \in E_2 \setminus E_1$.
3. For any two spaces E_1 and E_2, the retro-projection and projection pair $(retr, proj) : \mathcal{P}(E_1) \rightleftharpoons \mathcal{P}(E_1 \times E_2)$, given by $retr(Y) = Y \times E_2$ for $Y \in \mathcal{P}(E_1)$ and $proj(X) = \{p \in E_1 \mid \exists\, q \in E_2,\ (p, q) \in X\}$ for $X \in \mathcal{P}(E_1 \times E_2)$, is a regular adjunction. For $x \in E_1$, the unique block of π_x is $(E_1 \setminus \{x\}) \times E_2$.

We see that each of the 3 examples, for every x, π_x will have at most one block.

Given a lower-regular connective operator $\psi : \mathcal{P}(E_1) \to \mathcal{P}(E_2)$, it is possible to design other ones by modifying the associated π_x. Let ζ be an operator on $\Pi^*(E_1)$ that *reduces the support*: for any $\pi \in \Pi^*(E_1)$, $\mathsf{supp}(\zeta(\pi)) \subseteq \mathsf{supp}(\pi)$, equivalently, $\zeta(\pi) \leq \mathbf{1}_{\mathsf{supp}(\pi)}$, in other words: $\zeta \leq \mathbf{blend}$. Then for any $\pi_\bullet : E_2 \to \Pi^*(E_1)$ such that $\bigwedge_{x \in E_2} \pi_x = \mathbf{\emptyset}$ we get

$$\bigwedge_{x \in E_2} \zeta(\pi_x) \leq \bigwedge_{x \in E_2} \mathbf{blend}(\pi_x) = \mathbf{blend}\big(\bigwedge_{x \in E_2} \pi_x\big) = \mathbf{blend}(\mathbf{\emptyset}) = \mathbf{\emptyset} \enspace ,$$

hence $\bigwedge_{x \in E_2} \zeta(\pi_x) = \mathbf{\emptyset}$. Thus by Theorem 25: if $\pi_\bullet$ corresponds to the lower-regular connective operator $\psi : \mathcal{P}(E_1) \to \mathcal{P}(E_2)$, then $\pi^\zeta_\bullet : E_2 \to \Pi^*(E_1) : x \mapsto \pi^\zeta(x) = \zeta(\pi_x)$ will correspond to a lower-regular connective operator $\psi^\zeta : \mathcal{P}(E_1) \to \mathcal{P}(E_2)$; it is in general possible to derive the expression of ψ^ζ from that of ψ, with the help of (27). Note that if ζ does not increase the number of blocks of a partial partition, then by Proposition 28, for any lower-regular dilation $\delta : \mathcal{P}(E_1) \to \mathcal{P}(E_2)$, δ^ζ will be a dilation: indeed, δ corresponds to $\pi_\bullet$, where each π_x has at most one block, so each $\pi^\zeta(x) = \zeta(\pi_x)$ will have at most one block.

Example 30 Let the lower-regular connective operator $\psi : \mathcal{P}(E_1) \to \mathcal{P}(E_2)$ correspond to $\pi_\bullet : E_2 \to \Pi^*(E_1)$. We give three simple instances of $\zeta \leq \mathbf{blend}$, with the corresponding expression for ψ^ζ.

- $\zeta = \mathbf{blend}$. Here $\pi_x^{\mathbf{blend}} = \mathbf{blend}(\pi_x)$ has at most one block, which is the union of all blocks of π_x. By Proposition 28, $\psi^{\mathbf{blend}}$ is a lower-regular dilation; in fact, for all $X \in \mathcal{P}(E_1)$, $\psi^{\mathbf{blend}}(X) = \bigcup_{p \in X} \psi(\{p\})$. Alternately, we see that for every $x \in E_2$, $\pi_x^{\mathbf{blend}}$ is the least partial partition $\geq \pi_x$ having at most one block, so by the dual isomorphism $\psi \leftrightarrow \pi_\bullet$, $\psi^{\mathbf{blend}}$ is the greatest dilation $\leq \psi$.
- $\zeta = RS$. Then ψ^{RS} satisfies: for $p \in E_1$, $\psi^{RS}(\{p\}) = \bigcap_{q \in E_1 \setminus \{p\}} \psi(\{p, q\})$ (thus $\psi^{RS}(\{p\}) \supseteq \psi(\{p\})$), and $\psi^{RS}(X) = \psi(X)$ for $|X| \geq 2$. As RS does not increase the number of blocks of a partial partition, for a lower-regular dilation $\delta : \mathcal{P}(E_1) \to \mathcal{P}(E_2)$, δ^{RS} will be a dilation.
- Take $\pi_0 \in \Pi^*(E_1)$, and set $\zeta : \pi \mapsto \pi \wedge \pi_0$. For any $X \in \mathcal{P}(E_1)$, $\psi^\zeta(X) = \psi(X)$ if $\mathbf{1}_X \leq \pi_0$ (i.e., X is included in a block of π_0), and $\psi^\zeta(X) = E_2$ otherwise.

4 Triangular maps and hierarchies

In Theorem 13, we characterized a dilation $\overline{\delta} : \Pi^*(E_1) \to L$ by its behaviour on blocks, which is given by a connective map $\mathcal{P}(E_1) \to L$. As we said after its proof, we see no simple characterization of connective maps $\mathcal{P}(E_1) \to \Pi^*(E_2)$. In the case where $\overline{\delta}$ is $\Pi^*(E_1) \to \Pi^*(E_2)$, lower-regular and one-block-preserving, the corresponding connective map simplifies to one $\mathcal{P}(E_1) \to \mathcal{P}(E_2)$, that is characterized by a map $E_2 \to \Pi^*(E_1)$.

Recall that we write X^2 for the set of ordered pairs (x, y) of elements of a set X. In this section, we will characterize connective maps $\mathcal{P}(E_1) \to L$ by maps $E_1^2 \to L$ satisfying some conditions. This is somewhat analogous to the characterization of (partial) partitions by (partial) equivalence relations, which are defined as subsets of E_1^2. When L is a complete chain of non-negative real numbers, a particular case of this map $E_1^2 \to L$ is given by ultrametric distances [21], and then the upper adjoint erosion of the dilation $\Pi^*(E_1) \to L$ gives the corresponding hierarchy [2,18].

Our starting point is the possibility to characterize a dilation by its behaviour on a sup-generating family of the lattice. Now the least sup-generating family of $\Pi^*(E_1)$ is given by all $\mathbf{1}_{\{p\}}$ ($p \in E_1$) and $\mathbf{1}_{\{p,q\}}$ ($p, q \in E_1$, $p \neq q$); they are linked by the following relations: for any two distinct $p, q \in E_1$, $\mathbf{1}_{\{p\}} \vee \mathbf{1}_{\{q\}} \leq \mathbf{1}_{\{p,q\}}$, and for any three mutually distinct $p, q, r \in E_1$, $\mathbf{1}_{\{p,r\}} \leq \mathbf{1}_{\{p,q\}} \vee \mathbf{1}_{\{q,r\}}$; thus any dilation $\overline{\delta} : \Pi^*(E_1) \to \Pi^*(E_2)$ will satisfy

$$\overline{\delta}(\mathbf{1}_{\{p\}}) \vee \overline{\delta}(\mathbf{1}_{\{q\}}) \leq \overline{\delta}(\mathbf{1}_{\{p,q\}}) \quad \text{and} \quad \overline{\delta}(\mathbf{1}_{\{p,r\}}) \leq \overline{\delta}(\mathbf{1}_{\{p,q\}}) \vee \overline{\delta}(\mathbf{1}_{\{q,r\}}) \enspace .$$

These inequalities suggest the following definition:

Definition 31 Let X be a set, L be a complete lattice, and let $\theta : X^2 \to L$. We say that θ is *triangular* if θ is symmetric,

$$\forall\, x, y \in X, \quad \theta(x, y) = \theta(y, x) \enspace ,$$

and satisfies

$$\forall\, x, y, z \in X,\ x \neq y \neq z \neq x \quad \Longrightarrow \quad \theta(x, z) \leq \theta(x, y) \vee \theta(y, z) \enspace . \tag{28}$$

We say that θ is *strongly triangular* if θ is triangular and satisfies

$$\forall\, x, y \in X,\ x \neq y \quad \Longrightarrow \quad \theta(x, x) \leq \theta(x, y) \enspace . \tag{29}$$

For any set X, let $\mathcal{P}_2(X)$ be the set of unordered pairs $\{x, y\}$ of distinct elements of X.

Lemma 32 *Let X be a set, L be a complete lattice, and let $\theta : X^2 \to L$.*

1. *θ is* strongly triangular *if and only if θ is symmetric and satisfies*

$$\forall\, x, y, z \in X, \quad \theta(x, z) \leq \theta(x, y) \vee \theta(y, z) \enspace , \tag{30}$$

whether x, y, z are distinct or not.

2. *If θ is triangular, then the least strongly triangular map $\geq \theta$ is $\theta^\sharp$ given by*

$$\forall\, x, y \in X, \quad \theta^\sharp(x, y) = \theta(x, x) \vee \theta(x, y) \vee \theta(y, y) \enspace , \tag{31}$$

in particular $\theta^\sharp(x, x) = \theta(x, x)$ for all $x \in X$. Furthermore, for any $B \subseteq X$,

$$\bigvee_{(x,y) \in B^2} \theta^\sharp(x, y) = \bigvee_{(x,y) \in B^2} \theta(x, y) \enspace . \tag{32}$$

3. If θ is strongly triangular, then for any $B \subseteq X$,

$$|B| \geq 2 \quad \Longrightarrow \quad \bigvee_{(x,y)\in B^2} \theta(x,y) = \bigvee_{\{x,y\}\in \mathcal{P}_2(B)} \theta(x,y) \ . \tag{33}$$

Proof 1. Let θ be symmetric. For any $x, y, z \in X$ (distinct or not), the inequality $\theta(x,z) \leq \theta(x,y) \vee \theta(y,z)$ has three cases: (a) for x, y, z pairwise distinct, it corresponds to (28); (b) for $x = z \neq y$, it means $\theta(x,x) \leq \theta(x,y) \vee \theta(y,x) = \theta(x,y)$, that is (29); (c) for $x = y$ or $y = z$ we get the trivial inequalities $\theta(x,z) \leq \theta(x,x) \vee \theta(x,z)$ and $\theta(x,y) \leq \theta(x,y) \vee \theta(y,y)$ respectively. Now (28) and (29) together mean that θ is strongly triangular.

2. Clearly $\theta^\sharp$ given by (31) is $\geq \theta$ and satisfies $\theta^\sharp(x,x) = \theta(x,x) \vee \theta(x,x) \vee \theta(x,x) = \theta(x,x)$. By (31) again, $\theta^\sharp(x,x) = \theta(x,x) \leq \theta^\sharp(x,y)$. Now

$$\begin{aligned}
&\theta^\sharp(x,z) = \theta(x,x) \vee \theta(x,z) \vee \theta(z,z) \qquad \text{and} \\
&\theta^\sharp(x,y) \vee \theta^\sharp(y,z) = \theta(x,x) \vee \theta(x,y) \vee \theta(y,y) \vee \theta(y,z) \vee \theta(y,z) \ ;
\end{aligned}$$

since θ is triangular, $\theta(x,z) \leq \theta(x,y) \vee \theta(y,z)$, so $\theta^\sharp(x,z) \leq \theta^\sharp(x,y) \vee \theta^\sharp(y,z)$. Hence $\theta^\sharp$ is strongly triangular. Now given a strongly triangular $\theta' : X^2 \to L$ such that $\theta' \geq \theta$, we have $\theta(x,x) \leq \theta'(x,x) \leq \theta'(x,y)$, $\theta(y,y) \leq \theta'(y,y) \leq \theta'(y,x) = \theta'(x,y)$ and $\theta(x,y) \leq \theta'(x,y)$, so by (31), $\theta^\sharp(x,y) \leq \theta'(x,y)$. Therefore $\theta^\sharp$ is the least strongly triangular map $\geq \theta$. Finally (31) gives

$$\bigvee_{(x,y)\in B^2} \theta^\sharp(x,y) = \Big(\bigvee_{x\in B} \theta(x,x)\Big) \vee \Big(\bigvee_{(x,y)\in B^2} \theta(x,y)\Big) \vee \Big(\bigvee_{y\in B} \theta(y,y)\Big) \ ,$$

and since $\bigvee_{x\in B} \theta(x,x) = \bigvee_{y\in B} \theta(y,y) \leq \bigvee_{(x,y)\in B^2} \theta(x,y)$ (the supremum of a smaller set is smaller), this gives $\bigvee_{(x,y)\in B^2} \theta(x,y)$. Therefore (32) holds.

3. Let $|B| \geq 2$. The ordered pairs in B^2 are the (x,x) for $x \in B$, and the (x,y) and (y,x) for $\{x,y\} \in \mathcal{P}_2(B)$. Thus

$$\bigvee_{(x,y)\in B^2} \theta(x,y) = \Big(\bigvee_{x\in B} \theta(x,x)\Big) \vee \Big(\bigvee_{\{x,y\}\in \mathcal{P}_2(B)} \big[\theta(x,y) \vee \theta(y,x)\big]\Big) \ .$$

Since θ is strongly triangular, $\theta(x,x) \leq \theta(x,y) = \theta(y,x)$, so we get (33). $\square$

We easily see by induction that for all $n \geq 2$, for every mutually distinct $p_0, \ldots, p_n \in X$, a triangular map θ satisfies

$$\theta(p_0,p_n) \leq \theta(p_0,p_1) \vee \cdots \vee \theta(p_{n-1},p_n) = \bigvee_{i=1}^{n} \theta(p_{i-1},p_i) \ ;$$

when θ is strongly triangular, the inequality holds also if $p_0, \ldots, p_n$ are not necessarily mutually distinct.

Clearly, the two sets of triangular maps and of strongly triangular maps $X^2 \to L$ are dual Moore families of the complete lattice $L^{(X^2)}$ of all maps $X^2 \to L$. We have the following link between connective and triangular maps.

Proposition 33 *Let L be a complete lattice. For any map $\xi : \mathcal{P}(E_1) \to L$, the following three statements are equivalent:*

1. *ξ is connective.*
2. *ξ satisfies the following three conditions:*
   (a) $\xi(\emptyset) = \mathbf{0}$;
   (b) the map $\theta : E_1^2 \to L$ defined by $\theta(p,p) = \xi(\{p\})$ for $p \in E_1$, and $\theta(p,q) = \xi(\{p,q\})$ for two distinct $p,q \in E_1$, is strongly triangular;
   (c) for any $X \subseteq E_1$ such that $|X| \geq 2$, $\xi(X) = \bigvee_{P \in \mathcal{P}_2(X)} \xi(P)$.
3. *There is a triangular map $\theta : E_1^2 \to \Pi^*(E_2)$ such that for any $X \subseteq E_1$, $\xi(X) = \bigvee_{(p,q)\in X^2} \theta(p,q)$.*

Moreover, the map defined in item 2(b) is the unique strongly triangular map satisfying the condition of item 3.

Proof 1 $\Rightarrow$ 2. By definition, ξ is upper-regular, thus (a) holds. For any distinct $p,q \in E_1$, $\{p\} \subseteq \{p,q\}$, and as ξ is isotone (Lemma 3), we have $\xi(\{p\}) \leq \xi(\{p,q\})$, that is $\theta(p,p) \leq \theta(p,q)$, cf. (29). For any three mutually distinct $p,q,r \in E_1$, $\{p,q\} \cap \{q,r\} = \{p\} \neq \emptyset$ and $\{p,q\} \cup \{q,r\} = \{p,q,r\}$; as ξ is connective, we get $\xi(\{p,q,r\}) = \xi(\{p,q\}) \vee \xi(\{q,r\})$, and as ξ is isotone, $\xi(\{p,r\}) \leq \xi(\{p,q,r\})$; hence $\xi(\{p,r\}) \leq \xi(\{p,q\}) \vee \xi(\{q,r\})$, that is $\theta(p,r) \leq \theta(p,q) \vee \theta(q,r)$, cf. (28). Thus (b) holds. Now take $X \subseteq E_1$ such that $|X| \geq 2$; for $p \in X$ we have $X = \bigcup_{q \in X\setminus\{p\}} \{p,q\}$ with $\bigcap_{q\in X\setminus\{p\}} \{p,q\} = \{p\} \neq \emptyset$, and as ξ is connective, we get $\xi(X) = \bigvee_{q\in X\setminus\{p\}} \xi(\{p,q\})$; as ξ is isotone, for any $P \in \mathcal{P}_2(X)$, $\xi(P) \leq \xi(X)$; thus

$$\xi(X) = \bigvee_{q \in X\setminus\{p\}} \xi(\{p,q\}) \leq \bigvee_{P\in\mathcal{P}_2(X)} \xi(P) \leq \xi(X) \ ,$$

which gives (c).

2 $\Rightarrow$ 3. Take θ as in 2(b); thus θ is strongly triangular, hence triangular. Let $X \subseteq E_1$. If $|X| \geq 2$, by 2(b,c) and (33) we have

$$\xi(X) = \bigvee_{P\in\mathcal{P}_2(X)} \xi(P) = \bigvee_{\{p,q\}\in\mathcal{P}_2(X)} \theta(p,q) = \bigvee_{(p,q)\in X^2} \theta(p,q) \ .$$

For $x \in E_1$, $\xi(\{x\}) = \theta(x,x) = \bigvee_{(p,q)\in\{x\}^2} \theta(p,q)$. By 2(a), $\xi(\emptyset) = \mathbf{0}$, while $\bigvee_{(p,q)\in\emptyset^2} \theta(p,q) = \bigvee \emptyset = \mathbf{0}$. Therefore $\xi(X) = \bigvee_{(p,q)\in X^2} \theta(p,q)$ whatever the size (0, 1 or ≥ 2) of X.

3 $\Rightarrow$ 1. Here $\xi(\emptyset) = \bigvee \emptyset = \mathbf{0}$. Let $\mathcal{B} \subseteq \mathcal{P}(E_1)$ such that $\mathcal{B} \neq \emptyset$ and $\bigcap \mathcal{B} \neq \emptyset$. Take $p \in \bigcap \mathcal{B}$. Let $(q,r) \in \left(\bigcup \mathcal{B}\right)^2$. Now $r \in B$ for some $B \in \mathcal{B}$, so if $q = p$ or $q = r$ we get $q \in B$ too, thus $(q,r) \in B^2$, that is, $(q,r) \in \bigcup_{B\in\mathcal{B}} B^2$. The same conclusion is reached if $r = p$. In these cases, $\theta(q,r)$ is one of the $\theta(x,y)$ for $(x,y) \in \bigcup_{B\in\mathcal{B}} B^2$. There remains the case where $p \neq q \neq r \neq p$; as θ is triangular, $\theta(q,r) \leq \theta(q,p) \vee \theta(p,r)$, with $(q,p),(p,r) \in \bigcup_{B\in\mathcal{B}} B^2$. Hence in any case we get $\theta(q,r) \leq \bigvee\{\theta(x,y) \mid (x,y) \in \bigcup_{B\in\mathcal{B}} B^2\}$, so

$$\bigvee_{(q,r)\in\left(\bigcup\mathcal{B}\right)^2} \theta(q,r) \leq \bigvee\Big\{\theta(x,y) \mid (x,y) \in \bigcup_{B\in\mathcal{B}} B^2\Big\} = \bigvee_{B\in\mathcal{B}} \bigvee_{(x,y)\in B^2} \theta(x,y) \ .$$

Since $\bigcup_{B\in\mathcal{B}} B^2 \subseteq \left(\bigcup\mathcal{B}\right)^2$, we have

$$\bigvee\Big\{\theta(x,y) \mid (x,y)\in \bigcup_{B\in\mathcal{B}} B^2\Big\} \leq \bigvee_{(q,r)\in\left(\bigcup\mathcal{B}\right)^2} \theta(q,r) \enspace .$$

The two inequalities give then the equality

$$\bigvee_{(q,r)\in\left(\bigcup\mathcal{B}\right)^2} \theta(q,r) = \bigvee_{B\in\mathcal{B}} \bigvee_{(x,y)\in B^2} \theta(x,y) \enspace .$$

By definition of ξ, we have then $\xi\big(\bigcup\mathcal{B}\big) = \bigvee_{B\in\mathcal{B}} \xi(B)$. Therefore ξ is connective.

We showed above that the strongly triangular map θ defined in 2(b) satisfies the condition $\xi(X) = \bigvee_{(p,q)\in X^2} \theta(p,q)$ of item 3. Conversely, given a strongly triangular map θ satisfying it, we get $\xi(\{x\}) = \bigvee_{(p,q)\in\{x\}^2} \theta(p,q) = \theta(x,x)$ and for $x \neq y$,

$$\xi(\{x,y\}) = \bigvee_{(p,q)\in\{x,y\}^2} \theta(p,q) = \theta(x,x)\vee\theta(y,y)\vee\theta(x,y)\vee\theta(y,x) \enspace ,$$

and since θ is strongly triangular, this simplifies to $\xi(\{x,y\}) = \theta(x,y)$. Hence θ is as in 2(b). $\square$

Combining Theorems 13 with Lemma 32 and Proposition 33, we easily get:

Proposition 34 *Let L be a complete lattice.*

1. *A map $\overline{\delta} : \Pi^*(E_1) \to L$ is a dilation if and only if there is a triangular map $\theta : E_1^2 \to L$ such that*

$$\forall\, \pi\in\Pi^*(E_1), \quad \overline{\delta}(\pi) = \bigvee_{B\in\pi} \bigvee_{(p,q)\in B^2} \theta(p,q) \enspace . \tag{34}$$

   Furthermore, the map $\theta^\sharp : E_1^2 \to L$ given by $\theta^\sharp(p,p) = \overline{\delta}(\mathbf{1}_{\{p\}})$ for $p\in E_1$, and $\theta^\sharp(p,q) = \overline{\delta}(\mathbf{1}_{\{p,q\}})$ for two distinct $p,q\in E_1$, is the unique strongly triangular map satisfying (34).

2. *A map $\overline{\delta} : \Pi(E_1) \to L$ is a dilation if and only if there is a triangular map $\zeta : E_1^2 \to L$ such that for any $\pi\in\Pi(E_1)$,*

$$\overline{\delta}(\pi) = \bigvee_{B\in RS_1(\pi)} \bigvee_{\{p,q\}\in\mathcal{P}_2(B)} \zeta(p,q) \enspace .$$

   Furthermore, we can take ζ such that for any two distinct $p,q\in E_1$, $\zeta(p,q) = \overline{\delta}(\mathbf{1}_{\{p,q\}} \cup \mathbf{0}_{E\setminus\{p,q\}})$.

Obviously, for any triangular map θ satisfying (34), the strongly triangular map $\theta^\sharp$ given after (34) is the one of item 2 of Lemma 32.

Corollary 35 *Let $(\overline{\varepsilon}, \overline{\delta})$ be an adjunction $L \rightleftharpoons \mathit{\Pi}^*(E_1)$, and let $\theta^\sharp$ be the strongly triangular map satisfying (34). Then for any $t \in L$, $\overline{\varepsilon}(t)$ is the partial partition corresponding to the partial equivalence relation linking p and q whenever $\theta^\sharp(p,q) \leq t$:*

$$\forall\, p, q \in E_1, \quad p \; \mathsf{PE}(\overline{\varepsilon}(t)) \; q \iff \theta^\sharp(p,q) \leq t \ . \tag{35}$$

Proof Let $P = \{p, q\}$ for $p \neq q$ and $P = \{p\}$ for $p = q$. Then $p \; \mathsf{PE}(\overline{\varepsilon}(t)) \; q$ means that p and q are members of a same block of $\overline{\varepsilon}(t)$, in other words, $\mathbf{1}_P \leq \overline{\varepsilon}(t)$; equivalently $\overline{\delta}(\mathbf{1}_P) \leq t$ by the adjunction $(\overline{\varepsilon}, \overline{\delta})$; now $\overline{\delta}(\mathbf{1}_P) = \theta^\sharp(p,q)$ in both cases $p \neq q$ and $p = q$, so (35) follows. □

4.1 Ultrametrics and hierarchies

Let us now consider the particular case where L is the real interval $[0, \top]$ for $0 < \top < +\infty$, or a discrete interval $\{t_0, \ldots, t_n\}$ for $0 = t_0 < \cdots < t_n = \top$; thus L is a complete chain with 0 and $\top$ as least and greatest elements. Set $E_1 = E$. Consider an adjunction $(\overline{\varepsilon}, \overline{\delta}) : L \rightleftharpoons \mathit{\Pi}^*(E)$ and the corresponding strongly triangular map $\theta^\sharp$. For all $x, y, z \in E$, it satisfies the inequality $\theta^\sharp(x,z) \leq \theta^\sharp(x,y) \vee \theta^\sharp(y,z)$, cf. (30), which generalizes the triangular inequality $d(x,z) \leq d(x,y) + d(y,z)$ satisfied by a metric d. Metric spaces where the triangular inequality is replaced by (30) were introduced by [21], they are called *ultrametric spaces*. Hence this inequality (30) has been called the *ultrametric triangular inequality* or the *ultratriangular inequality*. We can thus extend to $\theta^\sharp$ some concepts defined in metric spaces, for example:

- The *ball of radius t centered about* a point $p \in E$: $B(t,p) = \{q \in E \mid \theta^\sharp(p,q) \leq t\}$.
- The *diameter* of a set $X \subseteq E$: $diam(X) = \sup\{\theta^\sharp(p,q) \mid p, q \in X\}$; it is also the minimum radius of a ball containing X.

In fact the diameter is the connective map corresponding to $\theta^\sharp$ according to Proposition 33. Then for $\pi \in \mathit{\Pi}^*(E)$, $\overline{\delta}(\pi)$ is the supremum of diameters of all blocks of π, while for $t \in L$, $\overline{\varepsilon}(t)$ is the partial partition of E into all balls of radius t. Since $\overline{\varepsilon}$ is an erosion and L is a chain, the partial partitions $\overline{\varepsilon}(t)$ form a chain whose greatest element is $\overline{\varepsilon}(\top) = \mathbf{1}_E$.

Now assume that $\theta^\sharp(x,x) = 0$ for all $x \in E$. Here (29) is always satisfied, thus if $\theta^\sharp$ is triangular, it will be strongly triangular. This identity $\theta^\sharp(x,x) = 0$ means that a ball is never empty, it must contain its center. Then $(E, \theta^\sharp)$ is a *pseudometric space*. Now for every $t \in L$, $\overline{\varepsilon}(t)$ will be a partition. Since $\overline{\varepsilon}$ is an erosion $L \to \mathit{\Pi}^*(E)$ and a map $L \to \mathit{\Pi}(E)$, and since $\mathit{\Pi}(E)$ is a Moore family in $\mathit{\Pi}^*(E)$, we get that $\overline{\varepsilon}$ will be an erosion $L \to \mathit{\Pi}(E)$.

Finally, on top of $\theta^\sharp(x,x) = 0$ for all $x \in E$, we can assume $\theta^\sharp(x,y) > 0$ for $x \neq y$; in other words, for all $x, y \in E$, $\theta^\sharp(x,y) = 0 \;\Leftrightarrow\; x = y$. This means that every ball of radius 0 is the singleton made of its center. Then $(E, \theta^\sharp)$ is a metric space; since it satisfies the ultratriangular inequality, it is called a *ultrametric space* and $\theta^\sharp$ is called an *ultrametric distance* [21]. Here

$\overline{\varepsilon}(0) = \mathbf{0}_E$. The partitions $\overline{\varepsilon}(t)$, $t \in L$, form a chain with least element $\overline{\varepsilon}(0) = \mathbf{0}_E$ and greatest element $\overline{\varepsilon}(\top) = \mathbf{1}_E$, in other words a *hierarchy* [2,18]; in fact this hierarchy is *indexed*, each $t \in L$ is the index of the partition $\overline{\varepsilon}(t)$, with $t < t' \Rightarrow \overline{\varepsilon}(t) \leq \overline{\varepsilon}(t')$. This hierarchy can also be expressed in terms of the family $\mathcal{H} = \bigcup_{t \in L} \overline{\varepsilon}(t)$ of all blocks of all partitions $\overline{\varepsilon}(t)$ [27]. We have

1. $E \in \mathcal{H}$.
2. $\forall\, p \in E$, $\{p\} \in \mathcal{H}$.
3. $\forall\, X, Y \in \mathcal{H}$, $X \subseteq Y$ or $Y \subseteq X$ or $X \cap Y = \emptyset$.

Then the elements of $\mathcal{H}$ can also be indexed: to each $B \in \mathcal{H}$ we associate the index $\mu(B)$ which is the least $t \in L$ such that $B \in \overline{\varepsilon}(t)$. Note that this least t always exists, because $\overline{\varepsilon}$ is an erosion, and by (12) an infimum of partitions having each B as a block, will also have B as a block. In fact, $\mu(B) = diam(B) = \overline{\delta}(\mathbf{1}_B)$.

The equivalence between ultrametrics and hierarchies is due to [2,18], although this theory has always been expressed in the case where L is discrete. Let us thus assume that $L = \{t_0, \ldots, t_n\}$ for $0 = t_0 < \cdots < t_n = \top$. One can represent the indexed hierarchy of partitions by a *dendrogram* as in Figure 9. We already encountered such a discrete hierarchy in the quad-tree decomposition of sets, see Figure 6.

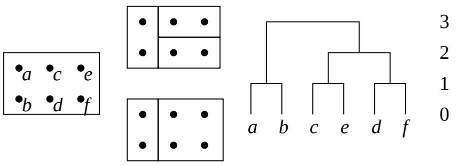

Fig. 9 Here $L = \{0, 1, 2, 3\}$. Left: the space $E = \{a, b, c, d, e, f\}$. Middle: the partitions $\overline{\varepsilon}(1)$ (top) and $\overline{\varepsilon}(2)$ (bottom); we have $\overline{\varepsilon}(0) = \mathbf{0}_E$ and $\overline{\varepsilon}(3) = \mathbf{1}_E$. Right: the corresponding dendrogram; the horizontal lines at height t represent the merging of blocks of $\overline{\varepsilon}(t-1)$ into a block of $\overline{\varepsilon}(t)$.

Here $\overline{\varepsilon}$ is a complete morphism $L \to \Pi(E)$ (both a dilation and an erosion). Its lower adjoint is $\overline{\delta}$, while its upper adjoint is the erosion $\overline{\zeta} : \Pi(E) \to L$ associating to a partition π the greatest t such that the distance between two points belonging to distinct blocks of π is always $> t$. In other words, $\overline{\zeta}(\pi)$ is the greatest radius t such that all balls of radius t are included in blocks of π, while $\overline{\delta}(\pi)$ is the least radius t such that all blocks of π are included in balls of radius t:

$$\overline{\zeta}(\pi) = \max\{t \in L \mid \overline{\varepsilon}(t) \leq \pi\} \;, \qquad \overline{\delta}(\pi) = \min\{t \in L \mid \pi \leq \overline{\varepsilon}(t)\} \;.$$

Note that $(\overline{\varepsilon}\overline{\zeta}, \overline{\varepsilon}\overline{\delta})$ is an adjunction, $\overline{\varepsilon}\overline{\zeta}$ is an opening, $\overline{\varepsilon}\overline{\delta}$ is a closure, and for every $\pi \in \Pi(E)$, $\overline{\varepsilon}\overline{\zeta}(\pi)$ is the greatest $\overline{\varepsilon}(t)$, $t \in L$, such that $\overline{\varepsilon}(t) \leq \pi$, while $\overline{\varepsilon}\overline{\delta}(\pi)$ is the least $\overline{\varepsilon}(t)$, $t \in L$, such that $\pi \leq \overline{\varepsilon}(t)$. This is exactly the situation described in Lemma 1.

Everything that we said in Example 21 remains true. In particular, the map $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ such that $\psi(\emptyset) = \emptyset$, and associating to each non-empty set the least member of $\mathcal{H}$ containing it, in other words $X \mapsto B(diam(X), p)$ for any $p \in X$, is connective. Note that these two properties, of $\overline{\varepsilon}$ being a dilation and ψ being connective, are valid because L is a finite chain, they will generally fail in other cases.

Ultrametrics and hierarchies have been used in various domains of sciences, in particular for image segmentation [23,24,27].

4.2 Set-valued triangular maps

Let us now return to dilations $\Pi^*(E_1) \to \Pi^*(E_2)$. We see no simple characterization of triangular maps $E_1^2 \to \Pi^*(E_2)$. Note that for a (strongly) triangular map $\psi : E_1^2 \to \Pi^*(E_2)$, the maps $E_1^2 \to \Pi^*(E_2) : P \mapsto RS(\psi(P))$ and $E_1^2 \to \Pi(E_2) : P \mapsto FS(\psi(P))$ are (strongly) triangular; we get also the two (strongly) triangular maps $E_1^2 \to \mathcal{P}(E_2)$ given by $P \mapsto \mathsf{supp}(\psi(P))$ and $P \mapsto \mathsf{supp}(RS(\psi(P)))$.

By Lemmas 3 and 16, a connective map $\psi : \mathcal{P}(E_1) \to \mathcal{P}(E_2)$ will be lower-regular if and only if $\forall\, p \in E_1,\ \psi(\{p\}) \neq \emptyset$. We obtain thus the following analogue of Theorem 17:

Proposition 36 *A map $\overline{\delta} : \Pi^*(E_1) \to \Pi^*(E_2)$ is a lower-regular one-block-preserving dilation if and only if there is a triangular map $\theta : E_1^2 \to \mathcal{P}(E_2)$ satisfying $\theta(p,p) \neq \emptyset$ for every $p \in E_1$, such that*

$$\forall\, \pi \in \Pi^*(E_1), \quad \overline{\delta}(\pi) = \bigvee_{B \in \pi} \bigvee_{(p,q) \in B^2} \mathbf{1}_{\theta(p,q)} \ . \tag{36}$$

There is a unique strongly triangular map $\theta^\sharp : E_1^2 \to \mathcal{P}(E_2)$ satisfying (36), and $\theta^\sharp(p,p) \neq \emptyset$ for every $p \in E_1$.

Consider the connective operator ψ of Theorem 17; then $\theta^\sharp(p,p) = \psi(\{p\})$ for $p \in E_1$, and $\theta^\sharp(p,q) = \psi(\{p,q\})$ for two distinct $p, q \in E_1$. The condition $\theta(p,p) \neq \emptyset$ ensures the fact that ψ is lower-regular.

Proposition 36 allows us to build dilations on partial partitions by specifying triangular maps on sets. For instance, using the identity $A \setminus C \subseteq (A \setminus B) \cup (B \setminus C)$, it follows that for any map $\omega : E_1 \to \mathcal{P}(E_2)$, the map

$$\theta^\sharp : E_1^2 \to \mathcal{P}(E_2) : (p,q) \mapsto \big(\omega(p) \setminus \omega(q)\big) \cup \big(\omega(q) \setminus \omega(p)\big)$$

will be strongly triangular.

Example 37 Let $E_1 = E_2 = E$, and to every $x \in E$ we associate a subset B_x of E containing x. For every two $p, q \in E$, set

$$\theta^\sharp(p,q) = \{p,q\} \cup \big(B_p \setminus B_q\big) \cup \big(B_q \setminus B_p\big) \ ,$$

in particular $\theta^\sharp(p,p) = \{p\}$ for every $p \in E$. Then $\theta^\sharp$ is strongly triangular and $\theta^\sharp(p,p) \neq \emptyset$, thus $\theta^\sharp$ satisfies the requirements of Proposition 36.

This map is illustrated in Figure 10 for $E = \mathbf{R}^2$, the Euclidean plane provided with the Euclidean metric d, where for every $x \in \mathbf{R}^2$ we set $B_x = B_r(x) = \{y \in \mathbf{R}^2 \mid d(x,y) \leq r\}$, the closed ball of radius r centered about x.

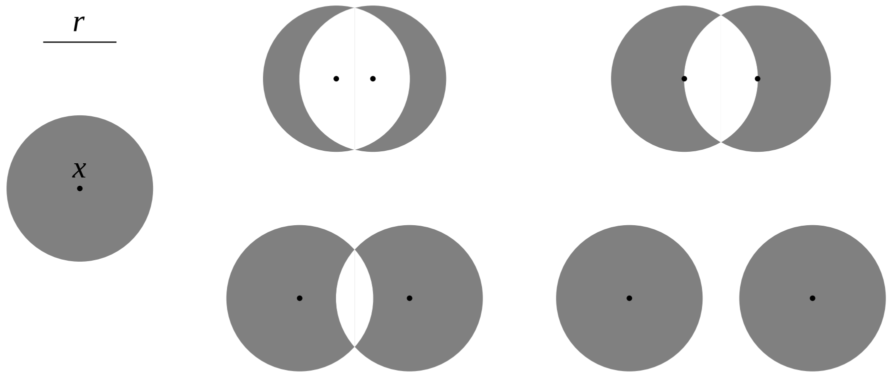

Fig. 10 Top left: the radius r. Bottom left: for $x \in \mathbf{R}^2$ (the black dot), $B_r(x)$ (the grey disk). Middle and right: we show four configurations of two distinct points p, q (as black dots), and $\theta^\sharp(p,q)$ consists of the two points p and q plus the two grey crescents (which become disks when $d(p,q) > 2r$).

Here the corresponding connective map ψ satisfies: for $p \in E$ we have $\psi(\{p\}) = \{p\}$, while for $X \in \mathcal{P}(E)$ such that $|X| \geq 2$,

$$\psi(X) = X \cup \bigcup_{(p,q) \in X^2} \left(B_p \setminus B_q\right) \enspace .$$

For a lower-regular dilation δ, $\mathsf{B}(\delta)$ satisfies the conditions of Proposition 36 with $\theta^\sharp(p,p) = \delta(\{p\})$ and $\theta^\sharp(p,q) = \delta(\{p,q\})$ for $p \neq q$; in particular $\theta^\sharp$ is strongly triangular (see also Lemma 3 and Proposition 28).

Following Theorem 17, Theorem 25 characterized a lower-regular one-block-preserving dilation expressed as $\mathsf{B}(\psi)$ for a lower-regular operator ψ, in terms of partial partitions π_x associated to each $x \in E_2$. We can instead follow Proposition 36, and in (27) replace the subsets of E_1 by ordered pairs.

For every $x \in E_2$, let $\stackrel{x}{\sim} = \mathsf{PE}(\pi_x)$, the partial equivalence relation on E_1 that corresponds to the partial partition π_x. Now (27) is expressed by

$$\forall\, x \in E_2,\ \forall\, p, q \in E_1, \quad x \notin \theta^\sharp(p,q) \iff p \stackrel{x}{\sim} q \enspace . \tag{37}$$

Independently of Theorem 25, we can notice that (37) gives a bijection between maps $\theta^\sharp : E_1^2 \to \mathcal{P}(E_2)$, and map associating to each $x \in E_2$ a relation $\stackrel{x}{\sim}$ on E_1. Now $\theta^\sharp$ is symmetric if and only if each $\stackrel{x}{\sim}$ is symmetric; in this case, using (30) in Lemma 32, $\theta^\sharp$ is strongly triangular if and only if for any $p, q, r \in E_1$ and $x \in E_2$, $[x \notin \theta^\sharp(p,q), x \notin \theta^\sharp(q,r)] \;\Rightarrow\; x \notin \theta^\sharp(p,r)$, in other words the symmetric relation $\stackrel{x}{\sim}$ is also transitive, that is, a partial equivalence relation. Now the condition $\theta^\sharp(p,p) \neq \emptyset$ means that $\forall\, p \in E_1$, $\exists\, x \in E_2$, $x \in \theta^\sharp(p,p)$, i.e., $p \stackrel{x}{\not\sim} p$; this is equivalent to the fact that the intersection of the partial equivalences $\stackrel{x}{\sim}$, for all $x \in E_2$, is empty (cf. the last sentence of Lemma 24 with π_x in place of $\stackrel{x}{\sim}$). Therefore we obtain directly the bijection between the strongly triangular maps $\theta^\sharp$ with $\theta^\sharp(p,p) \neq \emptyset$ of Proposition 36, and the maps associating to each $x \in E_2$ a partial equivalence $\overset{x}{\sim}$ on E_1 with $\bigcap_{x \in E_2} \overset{x}{\sim} = \emptyset$, corresponding to the maps $\pi_\bullet : E_2 \to \Pi^*(E_1) : x \mapsto \pi_x$ with $\bigwedge_{x \in E_2} \pi_x = \mathbf{Ø}$ of Theorem 25. Note that if for each $i \in I$, $\theta_i^\sharp$ corresponds to the family of $\underset{i}{\overset{x}{\sim}}$ for $x \in E_2$, then $\bigvee_{i \in I} \theta_i^\sharp$ will correspond to $\bigcap_{i \in I} \underset{i}{\overset{x}{\sim}}$, cf. the last sentence of Theorem 25.

For instance, in Example 37 (where $E_1 = E_2 = E$), for every $x \in E$, π_x has the two blocks $\check{B}_x \setminus \{x\}$ and $E \setminus \check{B}_x$, where $\check{B}_x = \{y \in E \mid x \in B_y\}$ (since $x \in B_x$, we have $x \in \check{B}_x$). In the particular case of Figure 10, where $E = \mathbf{R}^2$ and $B_x = B_r(x)$, the closed ball of radius r centered about x, we have $\check{B}_x = B_r(x)$. Here $x \notin \mathsf{supp}(\pi_x)$, so by Proposition 26, ψ and $\mathsf{B}(\psi)$ are extensive. Since π_x has two blocks, by Proposition 28, ψ is not a dilation.

5 Applications in image processing and in data clustering

We will briefly describe some existing or potential applications of dilations and erosions on partial partitions; they center around the grouping of points or objects on the basis of spatial proximity or similarity of attributes.

The first type of application is clustering (Subsection 5.1): given a collection of mutually disjoint subsets of space, regroup them into clusters. Clustering can be applied in unsupervised classification, but also in morphological image processing: given a set of mutually disjoint connected markers that serve as seeds for growing regions, regroup them (on the basis of proximity) into clusters that will serve as new markers, leading thus to a smaller number of regions.

The second type of application in image processing is image *segmentation* (Subsection 5.2), that is, extracting from a grey-level or colour image a partition of the underlying space into relatively homogeneous zones corresponding to the distinct objects seen in the image. In some way segmentation can be seen as a clustering of image points on the basis of their proximity, both in space and in the set of grey-level or colour values.

Both clustering and segmentation rely on some notion of connectivity, so we will start by recalling the essential facts of the morphological theory of partial connections, then give some links between it and the connective operators studied in this paper.

A *partial connection* [29,31] on $\mathcal{P}(E)$ is a family $\mathcal{C}$ of subsets of E, comprising the empty set, such that for $\mathcal{B} \subseteq \mathcal{C}$, $\bigcap \mathcal{B} \neq \emptyset \Rightarrow \bigcup \mathcal{B} \in \mathcal{C}$. When $\mathcal{C}$ comprises also all singletons, one says that it is a *connection* [35] (or *connectivity class*). Elements of $\mathcal{C}$ are called *connected*. This notion generalizes the usual connectivities (topological, arc-based, etc.), which constitute indeed connections, and permits the notion of a *connected component* of a set X [29, 31,35]: it is a non-void connected subset Y of X that is maximal for the inclusion: $Y \in \mathcal{P}(X) \cap \mathcal{C}$, $Y \neq \emptyset$, $\forall Z \in \mathcal{P}(X)$, $Y \subset Z \Rightarrow Z \notin \mathcal{C}$; then the family of connected components of X constitute a partial partition of X, and when $\mathcal{C}$ is a connection, they form a partition.

A dual Moore family of $\mathcal{P}(E)$ is a partial connection, in the same way as a dilation is a connective operator. One sees that the relation between connective operators and dilations is analogous to that between partial connections and dual Moore families: a property about $\bigcup \mathcal{B}$ is restricted to the case where $\bigcap \mathcal{B} \neq \emptyset$. Our use of the word “connective” to describe such a map comes from this analogy. Also we have:

Proposition 38 *Given a partial connection $\mathcal{C}$ on $\mathcal{P}(E_2)$ and a lower-regular connective operator $\psi : \mathcal{P}(E_1) \to \mathcal{P}(E_2)$, then $\psi^{-1}(\mathcal{C}) = \{X \in \mathcal{P}(E_1) \mid \psi(X) \in \mathcal{C}\}$ is a partial connection on $\mathcal{P}(E_1)$.*

This result was given in [29] in the restricted case where ψ is a dilation, but the proof works also for ψ connective (and it is similar to the proof of item 4 of Lemma 3).

5.1 Clustering

Clustering amounts to merging some blocks of the partial partition made by the initial sets. In practice, given a partial partition π, we build a coarser one π', and we take $\pi' \wedge \mathbf{blend}(\pi)$. Then $\mathsf{supp}(\pi' \wedge \mathbf{blend}(\pi)) = \mathsf{supp}(\pi') \cap \mathsf{supp}(\mathbf{blend}(\pi))) = \mathsf{supp}(\pi') \cap \mathsf{supp}(\pi) = \mathsf{supp}(\pi)$, and for $p \in \mathsf{supp}(\pi)$, we have $\mathsf{Cl}_{\pi' \wedge \mathbf{blend}(\pi)}(p) = \mathsf{Cl}_{\pi'}(p) \cap \mathsf{Cl}_{\mathbf{blend}(\pi)}(p) = \mathsf{Cl}_{\pi'}(p) \cap \mathsf{supp}(\pi)$; as $\pi \leq \pi'$, $\mathsf{Cl}_{\pi'}(p) \cap \mathsf{supp}(\pi)$ includes all blocks of π included in $\mathsf{Cl}_{\pi'}(p)$, and excludes all other blocks of π, disjoint from $\mathsf{Cl}_{\pi'}(p)$. Hence two distinct blocks $B, C \in \pi$ will be merged in $\pi' \wedge \mathbf{blend}(\pi)$ if and only if they are included in the same block of π', see Figure 11.

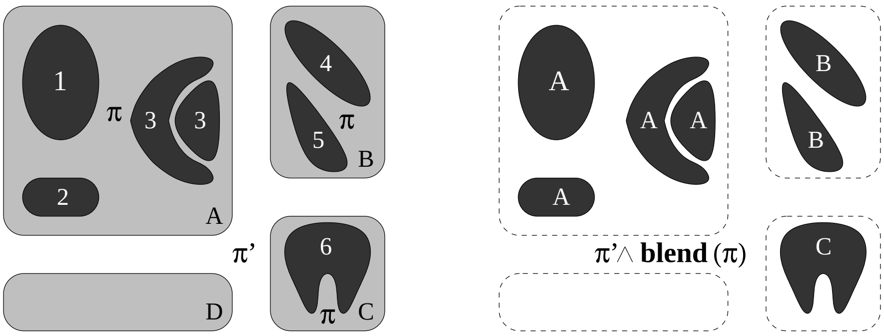

Fig. 11 Left: the partial partitions π and π', with $\pi \leq \pi'$; the 6 blocks of π are shown in dark grey and labelled 1, ..., 6 (note that block 3 is not connected), while the 4 blocks of π' are shown in light grey and labelled A, ..., D. Right: in $\pi' \wedge \mathbf{blend}(\pi)$, we fuse together blocks of π that are included in a same block of π', we get thus 3 blocks labelled A, B, C according to the corresponding blocks of π'.

Thus clustering can be seen as a support-preserving extensive operator on $\Pi^*(E_1)$, taking the form $\overline{\psi} \wedge \mathbf{blend}$ for an extensive operator $\overline{\psi}$ on $\Pi^*(E)$. We will consider in particular the case where $\overline{\psi}$ is the closure $\mathsf{B}(\varepsilon)\mathsf{B}(\delta)$ for a regular adjunction $(\varepsilon, \delta) : \mathcal{P}(E_2) \rightleftharpoons \mathcal{P}(E_1)$; since **blend** is also a closure, $\overline{\psi} \wedge \mathbf{blend}$ will be a closure. Let us define the binary relation $\overset{\delta}{\sim}$ on $\mathcal{P}(E_1)$ by $X \overset{\delta}{\sim} Y$ if and only if $\delta(X) \between \delta(Y)$, in other words if and only if there is some $p \in X$ and $q \in Y$ such that $\delta(\{p\}) \cap \delta(\{q\}) \neq \emptyset$. We have then the following:

Proposition 39 *Let (ε, δ) be a regular adjunction $\mathcal{P}(E_2) \rightleftharpoons \mathcal{P}(E_1)$.*

1. *For any $\pi \in \Pi^*(E_1)$, two distinct blocks $B, C \in \pi$ will be merged in $\mathsf{B}(\varepsilon)\mathsf{B}(\delta)(\pi) \wedge \mathbf{blend}(\pi)$ if and only if $\delta(B)$ and $\delta(C)$ are included in the same block of $\mathsf{B}(\delta)(\pi)$, if and only if there is a sequence $B_0, \ldots, B_n$ in π $(n \geq 1)$ such that $B = B_0 \overset{\delta}{\sim} \cdots \overset{\delta}{\sim} B_n = C$.*
2. *Suppose that $E_1 \subseteq E_2$ and δ is extensive (equivalently, ε is anti-extensive). Then for any $\pi \in \Pi^*(E_1)$ and $\pi' \in \Pi^*(E_2)$ such that $\mathsf{B}(\delta)(\pi) \leq \pi'$, we have $\pi' \wedge \mathbf{blend}(\pi) = \mathsf{B}(\varepsilon)(\pi') \wedge \mathbf{blend}(\pi)$.*

Proof Let $\pi \in \Pi^*(E_1)$. For any $D \in \mathsf{B}(\delta)(\pi)$, by (20) D is obtained through chaining some $\delta(A)$ with $A \in \pi$, hence every such A gives $\varepsilon(D) \supseteq A \neq \emptyset$. By (25) we get then

$$\mathsf{B}(\varepsilon)\mathsf{B}(\delta)(\pi) = \{\varepsilon(D) \mid D \in \mathsf{B}(\delta)(\pi),\ \varepsilon(D) \neq \emptyset\} = \{\varepsilon(D) \mid D \in \mathsf{B}(\delta)(\pi)\}.$$

By the above discussion, any two distinct blocks $B, C \in \pi$ will be merged in $\mathsf{B}(\varepsilon)\mathsf{B}(\delta)(\pi) \wedge \mathbf{blend}(\pi)$ if and only if they are included in a same block of $\mathsf{B}(\varepsilon)\mathsf{B}(\delta)(\pi)$, that is, $B, C \subseteq \varepsilon(D)$ for some $D \in \mathsf{B}(\delta)(\pi)$, in other words $\delta(B), \delta(C) \subseteq D$. Since D is obtained by chaining some $\delta(A)$ with $A \in \pi$, this means that $\delta(B)$ and $\delta(C)$ must also be chained in this way: we have $B_0, \ldots, B_n \in \pi$ with $B = B_0$ and $B_n = C$, and $\delta(B_0) \between \cdots \between \delta(B_n)$, that is, $B_0 \overset{\delta}{\sim} \cdots \overset{\delta}{\sim} B_n$.

Now let $E_1 \subseteq E_2$, with δ extensive and ε anti-extensive. Then $\mathsf{B}(\delta)$ is extensive by Lemma 16, so $\pi \leq \mathsf{B}(\delta)(\pi) \leq \pi'$. Let $B \in \pi$. As $\mathsf{B}(\delta)(\pi) \leq \pi'$, by (20) $\delta(B)$ must be included in a block of $\mathsf{B}(\delta)(\pi)$, itself included in a block of π': for some $A \in \pi'$, $\delta(B) \subseteq A$, and as $B \subseteq \delta(B)$, we get $B \subseteq A$. For any $D \in \pi'$, if $B \subseteq D$, we must have $D = A$, and as $\delta(B) \subseteq A$, we get $B \subseteq \varepsilon(A) = \varepsilon(D)$; conversely if $B \subseteq \varepsilon(D)$, as $\varepsilon(D) \subseteq D$, we get $B \subseteq D$. Hence $B \subseteq D \iff B \subseteq \varepsilon(D)$. By (25), any block of $\mathsf{B}(\varepsilon)(\pi')$ takes the form $\varepsilon(D)$ for some $D \in \pi'$. Therefore any two distinct blocks $B, C \in \pi$ will be merged in $\pi' \wedge \mathbf{blend}(\pi)$ if and only if $B, C \subseteq D$ for some $D \in \pi'$, equivalently $B, C \subseteq \varepsilon(D) \in \mathsf{B}(\varepsilon)(\pi')$, if and only if B and C will be merged in $\mathsf{B}(\varepsilon)(\pi') \wedge \mathbf{blend}(\pi)$. Thus $\pi' \wedge \mathbf{blend}(\pi) = \mathsf{B}(\varepsilon)(\pi') \wedge \mathbf{blend}(\pi)$. □

Our first example deals with digital connected components. Let $E_1 = \mathbf{Z}^n$ and $E_2 = (\frac{1}{2}\mathbf{Z})^n$. We consider a Euclidean norm N with integer values on E_1, and the adjacency relation $\sim$ on E_1 corresponding to N by $p \sim q \Leftrightarrow N(p-q) = 1$ (this is for example the case for the 4- and 8-adjacencies on $\mathbf{Z}^2$, corresponding to the L_1 and L_∞ norms). Define the adjacency relation $\approx$ on E_2 by $p \approx q \Leftrightarrow N(p-q) = 1/2$. Then

$$\forall\, p, q \in E_1, \quad \Big(p = q \text{ or } p \sim q\Big) \iff \Big(\exists\, r \in E_2,\ p \approx r \approx q\Big) \ . \tag{38}$$

Let $\delta^{\approx} : \mathcal{P}(E_1) \to \mathcal{P}(E_2)$ be the dilation adding to a set its neighbours according to $\approx$:

$$\forall\, X \in \mathcal{P}(E_1), \delta^{\approx}(X) = X \cup \{q \in E_2 \mid \exists\, p \in X,\ p \approx q\} \ ; \tag{39}$$

let $\varepsilon^{\approx} : \mathcal{P}(E_2) \to \mathcal{P}(E_1)$ be the upper adjoint erosion. Given two disjoint $X, Y \in \mathcal{P}(E_1)$, we have $\delta^{\approx}(X) \between \delta^{\approx}(Y)$ if and only if there are $p \in X$ and $q \in Y$ such that $\delta^{\approx}(\{p\}) \between \delta^{\approx}(\{q\})$, and by (38) this means that $p \sim q$. Thus by item 1 of Proposition 39, $\mathsf{B}(\varepsilon^{\approx})\mathsf{B}(\delta^{\approx}) \wedge \mathbf{blend}$ acts on a partial partition π by merging adjacent blocks (according to $\sim$), in other words $\mathsf{B}(\varepsilon^{\approx})\mathsf{B}(\delta^{\approx})(\pi) \wedge \mathbf{blend}(\pi)$ will be the join of π and of the partial partition of all connected components of $\mathsf{supp}(\pi)$. See Figure 12. In particular, for a subset X of E_1,

$$\mathsf{B}(\varepsilon^{\approx})\mathsf{B}(\delta^{\approx})(\mathbf{0}_X) \wedge \mathbf{blend}(\mathbf{0}_X) = \mathsf{B}(\varepsilon^{\approx})\mathsf{B}(\delta^{\approx})(\mathbf{0}_X) \wedge \mathbf{1}_X$$

will be the partial partition of all connected components of X. Since $\mathbf{Z}^n \subset (\frac{1}{2}\mathbf{Z})^n$, we can apply item 2 of Proposition 39 to the case where $\pi' = \mathsf{B}(\delta^{\approx})(\pi)$, thus we get that $\mathsf{B}(\varepsilon^{\approx})\mathsf{B}(\delta^{\approx}) \wedge \mathbf{blend} = \mathsf{B}(\delta^{\approx}) \wedge \mathbf{blend}$.

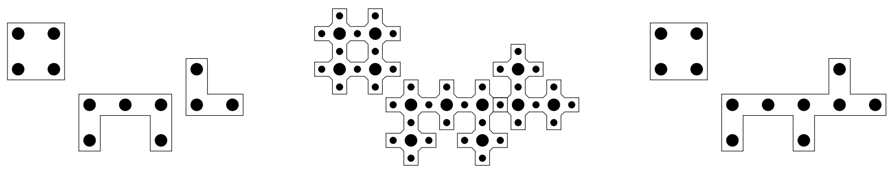

Fig. 12 Points of $E_1 = \mathbf{Z}^2$ and of $E_2 \setminus E_1 = (\frac{1}{2}\mathbf{Z})^2 \setminus \mathbf{Z}^2$ are shown as big dots and small dots respectively. We consider the 4-adjacency $\sim$ on E_1. Left: a partial partition π of E_1 with 3 blocks, each one is shown surrounded by a closed line. Middle: apply $\delta^{\approx}$ to each block of π; the two blocks of π that are 4-adjacent have their dilates overlapping, so $\mathsf{B}(\delta^{\approx})(\pi)$ has two blocks. Right: the resulting clustering $\mathsf{B}(\varepsilon^{\approx})\mathsf{B}(\delta^{\approx})(\pi) \wedge \mathbf{blend}(\pi)$; the two 4-adjacent blocks of π, whose dilates overlapped, are merged.

This type of clustering can be generalized to an arbitrary set E provided with an irreflexive and symmetric adjacency relation $\sim$. Set $E_1 = E$ and $E_2 = E^2$, and define the adjacency $\approx$ by setting

$$\forall\, p, q \in E, \quad \begin{cases} p \approx (p,q) \ , \\ q \approx (p,q) \ \Leftrightarrow \ p \sim q \ . \end{cases}$$

Then (38) holds, and for $\delta^{\approx} : \mathcal{P}(E_1) \to \mathcal{P}(E_2)$ given by (39) and its upper adjoint $\varepsilon^{\approx} : \mathcal{P}(E_2) \to \mathcal{P}(E_1)$, $\mathsf{B}(\varepsilon^{\approx})\mathsf{B}(\delta^{\approx}) \wedge \mathbf{blend}$ will, in a partial partition, merge blocks that are adjacent according to $\sim$.

Now assume that $E_1 = E_2 = E$. Let δ be an extensive dilation on $\mathcal{P}(E)$ (thus δ is regular), and let ε be the upper adjoint of δ. By item 2 of Proposition 39, $\mathsf{B}(\varepsilon)\mathsf{B}(\delta) \wedge \mathbf{blend} = \mathsf{B}(\delta) \wedge \mathbf{blend}$. This operator is a support-preserving closure on $\mathit{\Pi}^*(E)$. By item 1 of Proposition 39, for $\pi \in \mathit{\Pi}^*(E)$, two distinct blocks $B, C \in \pi$ will be merged in the same block of $\mathsf{B}(\delta)(\pi) \wedge \mathbf{blend}(\pi)$ if and only if there is a sequence $B_0, \ldots, B_n$ in π ($n \geq 1$) such that $B_0 = B$,

$B_n = C$, and $B_{t-1} \stackrel{\delta}{\sim} B_t$ for $t = 1, \ldots, n$. For example if $\delta(\{p\})$ is the ball of radius r centered about p, $\mathsf{B}(\delta) \wedge \mathbf{blend}$ will fuse any two blocks such that the minimum distance between their respective points is $\leq 2r$, see Figure 13.

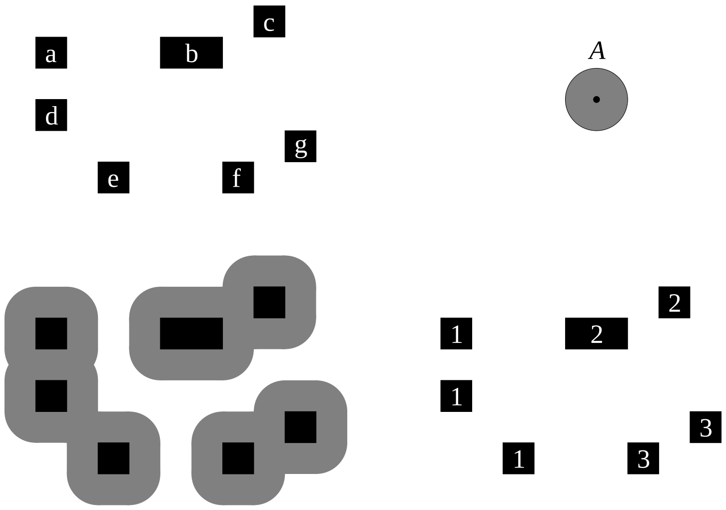

Fig. 13 Let $E = \mathbf{Z}^2$. Top left: a partial partition π with 7 blocks labelled a, ..., g. Top right: take the dilation $\delta : X \mapsto X \oplus A$ by the disk A of radius r centered about the origin (shown as a black dot); δ is extensive. Bottom left: the dilates $\delta(B)$ (in grey, with B in black) of all $B \in \pi$; now for $B, C \in \pi$, $\delta(B)$ and $\delta(C)$ overlap iff the minimum distance between points of B and those of C is $\leq 2r$; then $\mathsf{B}(\delta)(\pi)$ has 3 blocks. Bottom right: restrict these 3 blocks to $\mathsf{supp}(\pi)$, we obtain $\mathsf{B}(\delta)(\pi) \wedge \mathbf{blend}(\pi)$, whose 3 blocks are labelled 1, 2 and 3.

This example resembles the construction of a "second generation connectivity" defined in [35]. Given a connection $\mathcal{C}$ on $\mathcal{P}(E)$ and an extensive dilation δ on $\mathcal{P}(E)$ such that for any $p \in E$, $\delta(\{p\}) \in \mathcal{C}$, then $\mathcal{C}^\delta = \{X \in \mathcal{P}(E) \mid \delta(X) \in \mathcal{C}\}$ is a connection containing $\mathcal{C}$; it is a particular case of Proposition 38 and it has been called the "clustering connectivity". For any subset X of E, each connected component of X for the connection $\mathcal{C}^\delta$ is of the form $X \cap C$, where C is a connected component of $\delta(X)$ for the connection $\mathcal{C}$. Equivalently, two connected components B, C of X (for $\mathcal{C}$) are clustered (for $\mathcal{C}^\delta$) whenever their dilates $\delta(B), \delta(C)$ belong to the same connected component of $\delta(X)$.

The difference between $\mathsf{B}(\delta) \wedge \mathbf{blend}$ and $\mathcal{C}^\delta$ is that in the former, two blocks B, C are merged if $\delta(B)$ and $\delta(C)$ overlap, while in the latter they are also merged if they are adjacent. If we return to the above example of $E_1 = \mathbf{Z}^n$ with an adjacency relation $\sim$ on E_1 corresponding to a Euclidean norm N, and take $\mathcal{C}$ to be the connection consisting of all connected subsets of E_1 for that adjacency, then the clustering operated by the connection $\mathcal{C}^\delta$ will be constructed as follows. As above we take $E_2 = (\frac{1}{2}\mathbf{Z})^n$ with the adjacency relation $\approx$ satisfying (38), define the dilation $\delta^{\approx} : \mathcal{P}(E_1) \to \mathcal{P}(E_2)$ by (39), and its upper adjoint $\varepsilon^{\approx} : \mathcal{P}(E_2) \to \mathcal{P}(E_1)$; then for an extensive dilation δ

on $\mathcal{P}(E_1)$ such that $\delta(\{p\}) \in \mathcal{C}$ for all $p \in E$, the clustering according to $\mathcal{C}^\delta$ is given by the operator

$$\mathsf{B}(\varepsilon)\mathsf{B}(\varepsilon^{\approx})\mathsf{B}(\delta^{\approx})\mathsf{B}(\delta) \wedge \mathbf{blend} = \mathsf{B}(\varepsilon\varepsilon^{\approx})\mathsf{B}(\delta^{\approx}\delta) \wedge \mathbf{blend} \ .$$

Indeed, for $\pi \in \Pi^*(E_1)$, two distinct blocks $B, C \in \pi$ will be merged if $\delta^{\approx}\delta(B) \between \delta^{\approx}\delta(C)$, in other words $\delta(B)$ is adjacent (for $\sim$) to $\delta(C)$. Applying item 2 of Proposition 39 with $\pi' = \mathsf{B}(\delta^{\approx})\mathsf{B}(\delta)(\pi) = \mathsf{B}(\delta^{\approx}\delta)(\pi)$ for the dilation $\delta^{\approx}\delta : \mathcal{P}(E_1) \to \mathcal{P}(E_2)$, then with $\pi' = \mathsf{B}(\varepsilon^{\approx})\mathsf{B}(\delta^{\approx})\mathsf{B}(\delta)(\pi)$ for the dilation $\delta : \mathcal{P}(E_1) \to \mathcal{P}(E_1)$ (noting that $\mathsf{B}(\varepsilon^{\approx})\mathsf{B}(\delta^{\approx})$ is a closure on $\Pi^*(E)$), we obtain:

$$\mathsf{B}(\varepsilon)\mathsf{B}(\varepsilon^{\approx})\mathsf{B}(\delta^{\approx})\mathsf{B}(\delta) \wedge \mathbf{blend} =$$
$$\mathsf{B}(\varepsilon^{\approx})\mathsf{B}(\delta^{\approx})\mathsf{B}(\delta) \wedge \mathbf{blend} = \mathsf{B}(\delta^{\approx})\mathsf{B}(\delta) \wedge \mathbf{blend} \ .$$

5.2 Image segmentation

The interest in lattice-theoretical aspects of partitions has recently been revived by studies applying mathematical morphology to the problem of image *segmentation*. In particular the theory of *connective segmentation* [29,31,37] associates to any grey-level or colour image F defined on a space E a (partial) connection $\mathcal{C}^F$, and then the segmentation of F on any subset A of E will be the (partial) partition of connected components of A according to $\mathcal{C}^F$. For example if $E = \mathbf{Z}^n$ and $\mathcal{C}^F$ is the family of all digitally connected subsets of E on which F is constant, then the segmentation of F on $A \in \mathcal{P}(E)$ will be its partition into flat zones. An associated problem is the design of so-called *connected filters* for image simplification; such filters coarsen the image segmentation partition [31,37].

Connective segmentation involves an opening on (partial) partitions [29], and many segmentation algorithms operate by growing several regions in parallel, in other words through extensive operators on partial partitions. However, to our knowledge, dilations and erosions on (partial) partitions have not been used in relation to image segmentation, except for one instance using an erosion on image partitions: the work by Serra [36] on multivariate images, where to each point p of space is asssociated a vector of n heterogeneous values $(v_1(p), \ldots, v_n(p))$. Most segmentation algorithms given in the literature are described in the case of univalued images (associating to each point p a single value $v(p)$). Hence in order to segment a multivariate image, the following sequence of operations is performed:

1. Segment separately the n univalued images $v_1, \ldots, v_n$, resulting in n partitions $\pi_1, \ldots, \pi_n$.
2. Apply to each partition π_i ($i = 1, \ldots, n$) the erosion $FS \cdot \mathsf{B}(\varepsilon) \cdot IN$ for an anti-extensive set erosion ε, resulting in n partitions $\pi'_1, \ldots, \pi'_n$.
3. For $i = 1, \ldots, n$, with the partition π'_i one associates the numerical function f_i defined by $f_i(p) = -d(p, B_i(p))$, where d is a metric and $B_i(p)$ is the boundary of the class $\mathsf{Cl}_{\pi'_i}(p)$.

4. Segment the univalued function $f_1 + \cdots + f_n$.

Soille [40–42] has devised an image segmentation algorithm combining several segmentations with a criterion. Take a chain $\pi_0 < \cdots < \pi_n$ of prior segmentation partitions of a grey-level image I (they are obtained by varying a numerical parameter in the segmentation algorithm). Given a "constraining" homogeneity criterion cons (always satisfied by π_0), build the partition π such that for every point p, $\mathsf{Cl}_\pi(p) = \mathsf{Cl}_{\pi_t}(p)$, where t is the greatest $i = 0, \ldots, n$ such that on the set $\mathsf{Cl}_{\pi_i}(p)$, the image I satisfies the criterion cons. This construction has some resemblance with that of Example 21; a possible application of the latter could be: given a chain $\pi_0 < \cdots < \pi_n$ of prior segmentation partitions of an image,

- from a "seed" partial partition π, we construct the final partition $\mathsf{B}(\psi)(\pi)$ (the least partial partition $\geq \pi$ with blocks from $\pi_0, \ldots, \pi_n$);
- from an "enclosing" partition π, we construct the final partition $\overline{\varepsilon}(\pi)$ (the greatest partial partition $\leq \pi$ with blocks from $\pi_0, \ldots, \pi_n$).

Other examples of hierarchical segmentations are studied in [27].

Note also from Example 21 that the quad-tree decomposition of a set is an erosion $\eta : \mathcal{P}(E) \to \Pi^*(E)$, cf. Figure 6.

6 Conclusion

We have made an exhaustive study of adjunctions on the lattice of partitions, on the one of partial partitions, or between both. First Theorem 9 describes three fundamental adjunctions between $\Pi^*(E)$ and $\Pi(E)$. Then Theorem 12 shows how any adjunction $\Pi(E_2) \rightleftharpoons \Pi(E_1)$ can be derived from one $\Pi^*(E_2) \rightleftharpoons \Pi^*(E_1)$, combined with those of Theorem 9. Theorem 13 characterizes dilations on partial partitions in terms of connective maps. This characterization is further developed in Theorem 17 in the case of lower-regular one-block-preserving dilations, with the upper adjoint erosion characterized in Proposition 20. Finally Theorem 23 shows how any regular adjunction on sets leads to a regular adjunction on partial partitions. It is from them that the adjunction on partitions given by [30,36] is built.

Furthermore, in Subsection 3.4 we have characterized lower-regular connective operators $\mathcal{P}(E_1) \to \mathcal{P}(E_2)$ in terms of a map associating to each point of E_2 a partial partition of E_1, see Theorem 25. In Section 4 we expressed (for an arbitrary complete lattice L) connective operators $\mathcal{P}(E_1) \to L$, hence dilations $\Pi^*(E_1) \to L$, in terms of strongly triangular maps. A particular case of our theory is given by ultrametrics [21] and hierarchies [2,18].

Besides the use of in [36] of an erosion on partitions for the segmentation of multivariate images, we have given several examples of image processing or clustering operators, some of them based on connectivity, that can be expressed in our framework.

Through this work we see that the framework of partial partitions is more flexible than that of partitions. In fact, as explained in [29,31], many image processing operators produce a partial partition instead of a partition; in order to impose the framework of partitions, the usual practice is to fill the space with singleton blocks (i.e., to apply the operator FS), but then we lose the distinction between meaningful singleton blocks from the original partial partition and dummy singleton blocks added only to fill space.

In [30] it was shown how dilations on partitions (of the form $FS \cdot \mathsf{B}(\delta) \cdot RSIN$ for an extensive set dilation δ) can be used to produce so-called geodesic operators. This can of courses be extended to partial partitions (using the dilation $\mathsf{B}(\delta)$). This suggests a new research topic on the theory of connectivity and geodesic operators on partial connections.

Another promising research track is the analysis of idempotent operators in $\Pi^*(E)$ that are involved in image segmentation:

- on the one hand anti-extensive operators that act by splitting blocks, they model segmentation approaches that split space into homogeneous image regions, cf. [29,31,37,40–42];
- on the other hand extensive operators that model region growing approaches, in particular block clustering (cf. above) or block closing as in [19,20].

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