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Reparametrizing Swung Surfaces over the Reals

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Abstract

Let $\mathbb{K} \subseteq \mathbb{R}$ be a computable subfield of the real numbers (for instance, $\mathbb{Q}$). We present an algorithm to decide whether a given parametrization of a rational swung surface with coefficients in $\mathbb{K}(\mathtt{i})$, can be reparametrized over a real (i.e. embedded in $\mathbb{R}$) finite field extension of $\mathbb{K}$. Swung surfaces include, in particular, surfaces of revolution.

keywords: swung surfaces, revolution surfaces, real and complex surfaces, rational parametrization, ultraquadrics.

1 Introduction

A surface of revolution is a surface globally invariant by rotations around a certain line (the axis of revolution). The intersection of the surface with planes containing the revolution axis yields the so called *profile* curves. Revolution surfaces are well known since ancient times and very common objects in Differential Geometry and in Computer Aided Geometric Design. Still, they pose some interesting and challenging questions. One example is the recent work ([18]) devoted to computing the offset of revolution surfaces, provided the generatrix curve of the surface is implicitly given. Another recent paper deals with a new technique for implicitizing rational surfaces of revolution using μ-bases [20]. A basic question, such as efficiently determining, given the implicit equation of an algebraic surface, whether it is, or not, the equation of a surface of revolution, seems unsolved.

On the other hand, in the Geometric Modeling literature, revolution surfaces are often introduced under the assumption that they are generated by a profile plane curve (see e.g. [1], [7], [8]) subject to rotation around some axis. Since circles are rational curves, if the profile curve is rational, the revolution surface obtained by rotating it around a suitable axis will be rational, too. But the converse is not necessarily true (see Example 2.3).

In this paper we will work with *swung surfaces*, which are a natural extension of surfaces of revolution. More precisely, swung surfaces are produced by swinging around the z-axis a **profile curve** in the yz-plane along a **trajectory curve** in the xy-plane, see section 2 for more details. Assume that the profile curve is a plane rational curve parametrized by $(0, \phi_1(t), \phi_2(t))$ and the trajectory curve is also given by the parametrization $(\psi_1(s), \psi_2(s), 0)$. Then the corresponding swung surface is parametrized by

$$\mathcal{P}(s,t) = (\phi_1(t)\,\psi_1(s), \phi_1(t)\,\psi_2(s), \phi_2(t)) \tag{‡}$$

where we assume that the involved rational functions ϕ_i, ψ_j are defined over $\mathbb{K}(\mathtt{i})$, where $\mathbb{K}$ is a computable subfield of the reals. In fact, in the sequel, for the purpose of this paper, the equation (‡) above can be taken as the definition of **rational swung surface** $\mathcal{S}$.

Notice that when the trajectory curve is a circle, say, $(\psi_1(s) = (s^2-1)/(1+s^2), \psi_2(s) = 2s/(1+s^2), 0)$, the swung surfaces is the revolution surface obtained by rotating the profile curve around the z-axis. In particular, rational swung surfaces include all surfaces of revolution generated by rational profile curves, as well as many other surfaces, e.g. all quadrics. However we do not know whether every rational revolution surface is a swung surface, in the sense of having a parametrization of type (‡), cf. Example 2.3 below. Swung surfaces are thoroughly used in geometric aided design specially when the profile and trajectory curves are Bezier curves, and appear as part of the NURBS packages, see ([10]).

Let us describe the problem we will deal with in this framework. Assume we take as input a swung surface (‡) where the parametrization is given with coefficients over $\mathbb{K}(\mathtt{i})$, where $\mathbb{K}$ is a computable subfield of the reals (typically, the field $\mathbb{Q}$ of rational numbers, or an extension of $\mathbb{Q}$ such as $\mathbb{Q}(\sqrt[n]{\alpha})$, with $\alpha \in \mathbb{Q}^+$), and where $\mathtt{i}$ is the imaginary unit. That is, we suppose the proposed parametrization has coefficients of the kind $a+b\mathtt{i}$, with $a, b \in \mathbb{K}$. Yet, the swung surface might have a simpler parametrization, one involving real coefficients only. Then, our goal is to determine whether there is a change of parameters simplifying (in the sense of providing real coefficients) the given parametrization and, if so, to compute this parameter change. An obvious necessary condition for that is that the surface has "enough" real points. It turns out that in our case this is also a sufficient condition (see Theorem 4.2 and Corollary 4.5): the only requirement for the existence of a real reparametrization is that the surface should be "real", in the sense of having a two dimensional piece in $\mathbb{R}^3$ (see Section 3 for precisions on this concept).

Let us point out that it is not known, in general, whether a real surface, provided with a complex parametrization, has as well a real parametrization. We refer to the introduction of [17] for details on this problem. Therefore our result is a further step for settling down this general question. The fact that it is sufficient, in our context, to be real in order to have a real reparametrization is due, of course, to the close relation of the swung surfaces with a pair of curves and to the well known fact that, for curves, reality and complex rationality imply real parametrizability (see [12]). On the other hand, since the given parametrization of the swung surface does not univocally determine the associated pair of curves, but just the involved products $\phi_1(t)\psi_1(s)$, $\phi_1(t)\psi_2(s)$, some weaker conditions on these two curves have to be provided, as described in the statement of Theorem 4.2, item 1.

The algorithmic simplification of the coefficients of a parametrization (and more generally, that of simplifying a parametrization by regarding other features, such as its degree, etc.) is quite involved and has recently deserved quite a bit of attention. We refer the reader, for a detailed description of this general problem and references, to the Introduction of our recent paper [4]. There we have dealt with the case of parametric ruled surfaces, by using an *ad hoc* analysis that can not be easily generalized to include other types of surfaces. Yet, it can be said that the approach for the new case of rational swung surfaces shares with the previous one the need to adapt to the particular context the theory of *ultraquadrics* and *hypercircles* (cf. [3], [12], [11]), specifically created to handle over $\mathbb{R}$ the reparametrizing of a given complex parametrization.

We must briefly comment on an alternative approach to solve the proposed simplification problem. In fact, it is easy to observe that, given a parametrization (‡) over the complexes, the projection onto the z coordinate provides a rational map. Thus, for every value $z = z_0$ we obtain different (perhaps several) values t_0 of t, such that $z_0 = \phi_2(t_0)$ and, then, the fiber over z_0 is one (or more) rational curve $(\phi_1(t_0)\psi_1(s),\ \phi_1(t_0)\psi_2(s))$. Therefore, following [16] or [17], we are yield to discuss the existence of a real parametrization for this pencil of curves, by reducing it to the case of conics. Roughly speaking, this approach –if it could be carried out– relies on the theoretically well known birationality from rational curves and conics, while our approach, on the other hand, directly establishes such birational map from the family of curves to the –so called, see A– associated Weil variety.

One subtle point when dealing with reparametrizations is whether the input parametrization needs to be proper, that is, invertible. Although this is not a problem for curves, since it is well known (Lüroth's theorem, see, for instance [13]) that the existence of an improper rational mapping implies –and it is algorithmically easy to find– the existence of a birational parametrization [2], this is not the case, in general, for real surfaces (see Example 2.3). In Section 3, we address this issue, in order to allow improper parametrizations as potential inputs for our simplification goal.

Thus, we are able to state our main results on the existence and construction of real reparametrizations in the case of non-proper parametrizations of swung surfaces, by requiring, just, the birationality of the parametrizations for the two curves involved in the description of the surface. Starting from any (non-proper) parametrization of a swung surface, it is easy, computationally speaking, to obtain one of the same surface, but verifying the above requirement (through the algorithmic version of Lüroth's theorem, see [2]).

Section 4 contains the general statement for reparametrization of swung surfaces and its proof, relying on some technical aspects which are detailed in an Appendix. Moreover, we include in this Section a simpler reparametrizing statement in the particular case of classical surfaces of revolution. We conclude the paper (Section 5) with some detailed examples and the precise description and discussion of a pair of algorithms, based on our proposed method, as well as a table with running times for the performance of the implemented algorithms on a collection of surfaces. Computations have been obtained using the well known mathematical software Maple and Sage.

2 Swung Surfaces

As stated above, we will deal in this paper with the family of parametric or rational swung surfaces, that is, surfaces described parametrically in the form

$$(\phi_1(t)\,\psi_1(s), \phi_1(t)\,\psi_2(s), \phi_2(t))$$

where ϕ_i and ψ_j are rational functions over $\mathbb{K}(\mathtt{i})$, where $\mathbb{K}$ is a computable subfield of the reals.

Thus, the intersection of the resulting surface with the planes $z = k$, i.e. perpendicular to the z axis, produces copies of the of the trajectory (ψ) curve dilated with the y values $\phi_2(t_0)$ of the profile curve as augmentation factor. Notice that we obtain as many curves as points of intersection of the given plane with the profile curve, i.e., as solutions t_0 of the equation $\phi_2(t) = k$. Alternatively, consider the plane $y = \lambda x$ that contains the z-axis and take any s_0 such that this plane intersects the trajectory curve at $u_0 = (\psi_1(s_0), \psi_2(s_0), 0)$. Then, referred to the canonical basis of $y = \lambda x$ given by $u_0/||u_0||$ and $e_3 = (0, 0, 1)$, the intersection of the surface with the plane is the curve $||u_0||\phi_1(t)u_0 + \phi_2(t)e_3$ which is the profile curve distorted horizontally by the scalar $||u_0|| = \sqrt{(\psi_1(s_0))^2 + (\psi_2(s_0))^2}$. Thus, if we imagine the profile curve as being joined to the z-axis with a horizontal elastic arm, the surface can be produced mechanically as the contour obtained by stretching ϕ horizontally with factor $\sqrt{(\psi_1(s))^2 + (\psi_2(s))^2}$ as the yz plane rotates or swings around the z-axis.

Since these surfaces are initially described with, perhaps, complex coefficients, we will consider the geometric object defined by the parametrization in $\mathbb{C}^3$ and, thus, we will denote the surface as $\mathcal{S}_{\mathbb{C}}$. It is important to remark here that the relation between the complex and real parts of this surface will play an important role in what follows. Yet, we want to discard, for the rest of this paper, the case of parametrizations $(\phi_1(t)\,\psi_1(s), \phi_1(t)\,\psi_2(s), \phi_2(t))$ that do not produce a true surface in $\mathbb{C}^3$, i.e. such that the Jacobian of the parametrization has, generically, rank smaller than 2. This excludes precisely the following cases:

- when both ϕ_1, ϕ_2 are constant (since, then, ϕ does not describe a true curve)
- when both ψ_1, ψ_2 are constant (since, then, ψ does not describe a true curve)
- when ϕ_1 is identically zero (since, then, $\mathcal{S}_{\mathbb{C}}$ is just a line, the z-axis).
- when ϕ_2 is constant and ψ_2, ψ_1 are proportional (since then $\mathcal{S}_{\mathbb{C}}$ is just a line $\{c_1x = y, z = c_2\}$ or $\{c_1y = x, z = c_2\}$, with c_1, c_2 some constants)

Leaving apart these degenerate cases, this family of surfaces includes, in particular, surfaces of revolution – when the trajectory curve, is the unit circle–

with rational profile curve, but it extends also to other surfaces that are not of revolution, as all quadrics (after a suitably parametrization) as well as other kinds of surfaces, as shown in the following examples.

Example 2.1. Consider a cone with apex at (x_0, y_0, z_0) and a directrix curve parametrized by $(\phi_1(t), \phi_2(t), \phi_3(t))$, so that the cone is the union of straight lines passing through the apex and a point at the directrix. After a suitable translation we may assume that the apex is the origin of coordinates. Then the cone is parametrized as

$$s\,(\phi_1(t), \phi_2(t), \phi_3(t))$$

Now, considering as new parameter $T = s\,\phi_3(t)$, we can reparametrize the cone as

$$\left(T\frac{\phi_1(t)}{\phi_3(t)}, T\frac{\phi_2(t)}{\phi_3(t)}, T\right)$$

yielding a parametrization of the kind $(\phi_1(t)\,\psi_1(s), \phi_1(t)\,\psi_2(s), \phi_2(t))$. Cones are, then, swung surfaces and our contribution in this paper applies to these surfaces, too, after performing a translation of the apex to the origin.

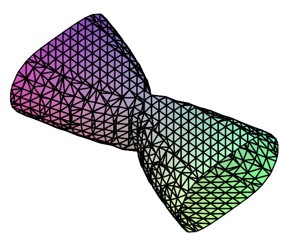

Figure 1: A swung surface: $-y^2 + x^4 + z^2y^2 = 0$

Example 2.2. Let

$$P := \big(2ts/(t^2+1), 2ts^2/(t^2+1), (t^2-1)/(t^2+1)\big)$$

According to our definition this is a parametric swung surface, with profile curve the circle $(0, 2t/(t^2+1), (t^2-1)/(t^2+1))$ and trajectory curve the parabola (s, s^2). Its implicit equation is $-y^2 + x^4 + z^2y^2 = 0$. See Figure 1.

For another example of this kind, take

$$Q := (2ts/(t^2+1), 2ts^2/(t^2+1), t^3)$$

Again, this is a rational swung surface with profile curve the cubic $(2t/(t^2+1), t^3)$, swinging along the parabola (s, s^2) as trajectory curve. See Figure 2.

As pointed out previously, surfaces of revolution generated by a rational curve $(0, \phi_1(t), \phi_2(t))$ are included in the family of parametric swung surfaces. However, there are surfaces of revolution which are rational, although generated by non-rational curves, as the following example shows. We do not know yet if they are parametric swung surfaces.

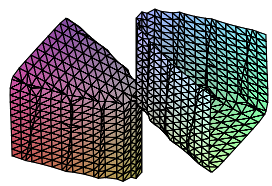

Figure 2: Another swung surface: $x^6z^2 - 8y^3z + 6x^4zy + x^6$

Example 2.3. Let us consider the offset of an ellipsoid of revolution. See Figure 3. As explained below, it is known to be rational, but it is also (cf. [18]) the revolution surface generated by the offset curve (which is non-rational) of an ellipse (cf. [5]). Therefore it is a rational and classical surface of revolution, which is parametrizable over the reals, yet its intersection with the $x = 0$ plane (the generatrix curve) is not rationally parametrizable.

Indeed, consider the ellipse

$$\frac{y^2}{4} + z^2 = 1$$

which can be parametrized as

$$y = \frac{4t}{t^2+1} \qquad z = \frac{t^2-1}{t^2+1}$$

We rotate it around the z-axis, so that we get the ellipsoid $\mathcal{S}_{\mathbb{C}}$ (as a surface in $\mathbb{C}^3$)

$$\frac{x^2}{4} + \frac{y^2}{4} + z^2 = 1$$

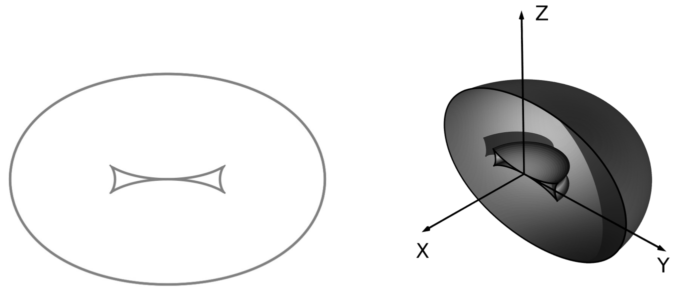

Figure 3: A rational revolution surface, not generated by a rational curve. Left: offset of the ellipse (profile curve of genus 1); Right: half offset of the ellipsoid (rational revolution surface).

which can be parametrized, using the ϕ, ψ scheme, as in the introduction, by

$$x = \frac{4t}{t^2+1}\,\frac{s^2-1}{s^2+1} \qquad y = \frac{4t}{t^2+1}\,\frac{2s}{s^2+1} \qquad z = \frac{t^2-1}{t^2+1}$$

or, alternatively (notice that the previous parametrization is not proper), as

$$x = \frac{4t}{t^2+s^2+1} \qquad y = \frac{4s}{t^2+s^2+1} \qquad z = \frac{t^2+s^2-1}{t^2+s^2+1}$$

For our purposes of constructing the offset of the ellipsoid, an even more suitable parametrization, although one defined over $\mathbb{C}$, is

$$\begin{aligned} x &= -2\,\frac{8\,t-16+s^2t^2-16\,s^2}{s\,(t^2+8\,t-32)} \\ y &= \frac{2\,\mathtt{i}\left(-8\,t+16+s^2t^2-16\,s^2\right)}{s\,(t^2+8\,t-32)} \\ z &= \frac{(t-8)\,t}{t^2+8\,t-32}. \end{aligned}$$

Indeed, a mechanical calculation shows that, with this parametrization, the norm of the normal vector to $\mathcal{S}_{\mathbb{C}}$ at a point $(x(t,s), y(t,s), z(t,s))$ is a rational fraction in t and s. Therefore it can be used in a straightforward way to construct the parametric equations of the offset $\mathcal{S}'_{\mathbb{C}}$ at distance 1 of the ellipsoid, which in this way results a rational surface (details on how to compute a rational parametrization of the offset of the ellipsoid can be found in [19], Theorem 5; alternatively, one may check [9]). Namely, we get the following, birational parametrization with complex coefficients of the offset $\mathcal{S}'_{\mathbb{C}}$:

$$x = -1/2\,\frac{\left(5\,t^2-8\,t+32\right)\left(8\,t-16+s^2t^2-16\,s^2\right)}{s\,(t^2+8\,t-32)\,(t^2-4\,t+16)}$$

$$y = -\mathtt{i}/10\,\frac{\left(-8\,t+16+s^2t^2-16\,s^2\right)\left(-25\,t^2+40\,t-160\right)}{s\,(t^2+8\,t-32)\,(t^2-4\,t+16)}$$

$$z = 2\,\frac{t\,(t+4)\,(t-2)\,(t-8)}{(t^2+8\,t-32)\,(t^2-4\,t+16)}.$$

Notice that this parametrization is not of the form (‡) of swung surfaces. Also, apparently, the property (known by construction) of $\mathcal{S}'_{\mathbb{C}}$ being real is hidden behind this birational parametrization. However, implicitization of the above parametrization gives as implicit equation of $\mathcal{S}'_{\mathbb{C}}$:

$-240\,y^2z^2x^2+66\,y^2z^4x^2+30\,y^4z^2x^2+30\,y^2z^2x^4+450\,z^2y^2-120\,y^4z^2-210\,y^2z^4-30\,y^4x^2-30\,y^2x^4-120\,z^2x^4-210\,z^4x^2+450\,z^2x^2+18\,x^2y^2+40\,y^2z^6+10\,y^6z^2+33\,y^4z^4+4\,y^6x^2+6\,y^4x^4+4\,y^2x^6+33\,z^4x^4+40\,z^6x^2+10\,z^2x^6-207\,z^4-324\,z^2+9\,x^4+9\,y^4+8\,z^6-10\,y^6-10\,x^6+16\,z^8+y^8+x^8=0,$

which of course is real. Moreover, we know that $\mathcal{S}'_{\mathbb{C}}$ is "real" in the sense that has many real points (see Section 3 for details on this concept), since $(1,0,0)$ is a real regular point in this surface.

Let us see how we can recover a real parametrization. For that purpose we use the construction of the Weil variety (cf. [3]). In the complex parametrization, we substitute $t = t_0 + \mathtt{i}t_1$, $s = s_0 + \mathtt{i}s_1$ and normalize the resulting expressions so that they have real denominators. The Weil variety is then defined as the zero set of the imaginary parts of this normal expression, minus the zero set of the denominator (see [3] for further details on this technique for reparametrizing these surfaces over the reals). In our example the Weil variety W is the tubular surface in the hyperplane $t_1 = 0$, described by

$$(t_0^2-16)s_0^2+(t_0^2-16)s_1^2-8(t_0-2)=0,$$

and we get an $\mathbb{R}$-birational map from it to the offset $\mathcal{S}'_{\mathbb{C}}$.

Now, by [16], Theorem 3, all tubular surfaces are real parametrizable and, therefore, by composing such parametrization with the mentioned birational map we get a parametrization of $\mathcal{S}'_{\mathbb{C}}$ over the reals. We claim that this real parametrization cannot be birational. Indeed, if it were, by the $\mathbb{R}$-birational map, our Weil variety W would have a birational parametrization. But following [17], it is easy to deduce that the tubular surface W can not be birationally parametrizable over the reals since its projectivization and desingularization has more that one connected component (an invariant for the real rational function field of the surface, cf. [6]).

As a consequence, it follows that the offset $\mathcal{S}'_{\mathbb{C}}$ cannot be birationally parametrized over the complexes as a swung surface. In fact, were it possible, then, we could apply the Remark 4.3, stating that, under the assumption of $\mathcal{S}'_{\mathbb{C}}$ having a birational complex parametrization as swung surface, the reality of $\mathcal{S}'_{\mathbb{C}}$ would imply the existence of a birational real parametrization for it, which is not possible as we just pointed out. We remark here that we do not know if there is a complex, non-proper parametrization of $\mathcal{S}'_{\mathbb{C}}$ as swung surface.

On the other hand, we know that, alternatively, $\mathcal{S}'_{\mathbb{C}}$ can be constructed by considering first the offset of the ellipse $(1/4)y^2 + z^2 = 1$ above, which is:

$-324\,z^2 + 9\,y^4 + 450\,z^2y^2 - 207\,z^4 - 10\,y^6 - 120\,y^4z^2 - 210\,y^2z^4 + 8\,z^6 + y^8 + 10\,y^6z^2 + 33\,y^4z^4 + 40\,y^2z^6 + 16\,z^8 = 0,$

and then rotating it around the z-axis. However, this curve has genus one (see [5]), so that it is not rational, although its revolution around the z-axis produces the offset $\mathcal{S}'_{\mathbb{C}}$, which, as we have seen, is rational.

In conclusion, $\mathcal{S}'_{\mathbb{C}}$ is a real rational surface of revolution with no rational profile curve for any possible revolution axis, and we do not know whether it can be presented as a parametrized swung surface (although we know that this parametrization can never be proper).

Remark 2.4. More precisely, we have the following: a rational surface of revolution $\mathcal{S}_{\mathbb{C}}$, with the z-axis as the revolution axis, has a sectional curve, at the plane $x = 0$, which is rational if and only if it admits a parametrization

$$(\lambda_1(u,v), \lambda_2(u,v), \lambda_3(u,v))$$

where $\lambda_1(u,v)^2 + \lambda_2(u,v)^2$ is the square of a rational function. Indeed, assume that $\mathcal{S}_{\mathbb{C}}$ is the surface of revolution generated by rotating the planar curve $(0, \phi_1(u), \phi_2(u))$ around the z-axis. Then $\mathcal{S}_{\mathbb{C}}$ has a rational parametrization as

$$\left(\phi_1(u)\,\frac{v^2-1}{1+v^2}, \phi_1(u)\,\frac{2v}{1+v^2}, \phi_2(u)\right)$$

and we have $(\phi_1(u)(v^2-1)/(1+v^2))^2 + (\phi_1(u)2v/(1+v^2))^2 = \phi_1(u)^2$. Conversely, assume that we have a parametrization

$$(\lambda_1(u,v), \lambda_2(u,v), \lambda_3(u,v))$$

with $\lambda_1(u,v)^2 + \lambda_2(u,v)^2$ the square of a rational function and λ_3 not constant (otherwise $\mathcal{S}_{\mathbb{C}}$ is the plane $z = \lambda_3$). Then, consider a rational curve $(u(t), v(t))$ such that $\lambda_3(u(t), v(t))$ takes, when $t \in \mathbb{C}$, infinitely many values (a property that holds for almost every choice of $(u(t), v(t))$). Now, for almost every t_0, the point

$$\left(0, \sqrt{\lambda_1(u(t_0), v(t_0))^2 + \lambda_2(u(t_0), v(t_0))^2}, \lambda_3(u(t_0), v(t_0))\right)$$

lies in $\mathcal{S}_{\mathbb{C}}$, since so does the point

$$(\lambda_1(u(t_0), v(t_0)), \lambda_2(u(t_0), v(t_0)), \lambda_3(u(t_0), v(t_0)))$$

and $\mathcal{S}_{\mathbb{C}}$ contains every circle in a xy-parallel plane with center at $(0, 0, \lambda_3(u(t_0), v(t_0)))$ and passing through $(\lambda_1(u(t_0), v(t_0)), \lambda_2(u(t_0), v(t_0)), \lambda_3(u(t_0), v(t_0)))$.

Now, it is immediate to conclude that the intersection of $\mathcal{S}_{\mathbb{C}}$ with the plane $x = 0$ can be parametrized by

$$\left(0, \sqrt{\lambda_1(u(t), v(t))^2 + \lambda_2(u(t), v(t))^2}, \lambda_3(u(t), v(t))\right)$$

which is rational by our hypothesis on $\lambda_1(u, v)^2 + \lambda_2(u, v)^2$.

In particular it follows that the offset $\mathcal{S}'_{\mathbb{C}}$ of the previous example, can not have a parametrization $(\lambda_1(u, v), \lambda_2(u, v), \lambda_3(u, v))$, where $\lambda_1(u, v)^2 + \lambda_2(u, v)^2$ is the square of a rational function.

3 Reparametrizing: some basic issues

The starting point for our approach, our input, is a rational parametrization of a true surface over the complexes of the form,

$$(\phi_1(t)\psi_1(s), \phi_1(t)\psi_2(s), \phi_2(t))$$

with complex coefficients. We can imagine, from the context where the parametrization has risen or from the way it has been obtained, that this parametrizes a swung surface over $\mathbb{R}^3$. But, strictly speaking, our only mathematical data is the given parametrization. Since it has complex coefficients, all we can assert is that it parametrizes a surface $\mathcal{S}_{\mathbb{C}}$ in $\mathbb{C}^3$.

In this Section we will deal with two basic issues that we have already mentioned in the Introduction: *a*) the precise meaning of the word "real" when applied to a complex surface, since it will be a basic requirement for our results and, *b*) the proper versus improper character of the given parametrization.

We recall that a parametrization is called *proper* or *birational* if the map from parameters to points in the surface is generically one-to-one, i.e. it is possible to invert the parametrization and to obtain the parameters in terms of rational functions on the surface. Otherwise, that is, in the many-to-one case, we say that the parametrization is *improper* or *unirational*. For (real or complex) curves it is well known (Lüroth's theorem, see, for instance [13]) that the existence of an improper rational mapping implies –and it is algorithmically easy to find– the existence of a birational parametrization [2]. Castelnuovo theorem states that any complex unirational surface is also rational. But this is not true for real surfaces. In fact, Example 2.3 provides a real surface (although not properly parametrizable as swung surface over the complexes) that has a real unirational parametrization, but can not have a real birational parametrization.

We start we the following easy observation that will be used later:

Remark 3.1. Assume that a given plane curve parametrization $(p_1(t), p_2(t))$ is proper over $\mathbb{C}$. Then, for every scalars $\lambda, \mu \in \mathbb{C} \setminus \{0\}$, the parametrization $(\lambda\, p_1(t), \mu\, p_2(t))$ is also proper. Indeed, as field extensions, we have $\mathbb{C}(\lambda\, p_1(t), \mu\, p_2(t)) = \mathbb{C}(p_1(t), p_2(t)) = \mathbb{C}(t)$. Obviously, the result works for curves in any dimension.

Now observe that, given a swung surface parametrization

$$(\phi_1(t)\psi_1(s), \phi_1(t)\psi_2(s), \phi_2(t))$$

we may consider diverse candidates for our trajectory and profile curves, namely adjusting constants: $(\lambda\,\phi_1(t), \phi_2(t))$ and $((1/\lambda)\,\psi_1(s), (1/\lambda)\,\psi_2(s))$, for each non-zero, complex, value of λ. However, as a consequence of the previous observation, if for choice of λ the curves are proper, so they are for any other choice.

Bearing this in mind we can state the following

Lemma 3.2. *Assume that the parametrization of the surface*

$$(\phi_1(t)\,\psi_1(s), \phi_1(t)\,\psi_2(s), \phi_2(t))$$

is proper. Then the parametrizations of the curves $\phi(t) = (\phi_1(t), \phi_2(t))$ *and* $\psi(s) = (\psi_1(s), \psi_2(s))$ *are also proper.*

Proof. Indeed, suppose that $t = T_1(x, y, z)$, $s = T_2(x, y, z)$ is the inverse of the parametrization of the surface, i.e.,

$$t = T_1(\phi_1(t)\,\psi_1(s), \phi_1(t)\,\psi_2(s), \phi_2(t))$$
$$s = T_2(\phi_1(t)\,\psi_1(s), \phi_1(t)\,\psi_2(s), \phi_2(t))$$

in $\mathbb{C}(t, s)$. Take any s_0 such that $\psi_1(s_0)$, $\psi_2(s_0)$ and $T_1(y\psi_1(s_0), y\psi_2(s_0), z)$ are well defined. We claim that

$$\tilde{T}_1(y, z) := T_1(y\psi_1(s_0), y\psi_2(s_0), z)$$

is the inverse of the parametrization $(\phi_1(t), \phi_2(t))$. Indeed, note that

$$\tilde{T}_1(\phi_1(t), \phi_2(t)) = T_1(\phi_1(t)\,\psi_1(s_0), \phi_1(t)\,\psi_2(s_0), \phi_2(t)) = t$$

by the equations above, which shows that $\mathbb{C}(\phi_1(t), \phi_2(t)) = \mathbb{C}(t)$, that is, that the curve $(\phi_1(t), \phi_2(t))$ is birational. A similar (symmetric) argument shows that for a fixed t_0, the function

$$\tilde{T}_2(x, y) := T_2(x\phi_1(t_0), y\phi_1(t_0), \phi_2(t_0))$$

is the inverse of the parametrization $(\psi_1(s), \psi_2(s))$ so that this curve is birational too. □

Remark 3.3. Notice that the converse is false, that is, if both parametrizations $(\phi_1(t), \phi_2(t))$ and $(\psi_1(s), \psi_2(s))$ are birational, then it is not true, in general, that the parametrization

$$(\phi_1(t)\,\psi_1(s), \phi_1(t)\,\psi_2(s), \phi_2(t))$$

of the swung surface is also birational. For instance, in the Example 2.3 above, both the ellipse $(4t/(t^2 + 1), (t^2 - 1)/(t^2 + 1))$ and the circle $((s^2 - 1)/(s^2 +$

1), $2s/(s^2+1)$) parametrizations are birational, but the parametrization of the ellipsoid of revolution

$$\left(\frac{s^2-1}{s^2+1}\,\frac{4t}{t^2+1}, \frac{2s}{s^2+1}\,\frac{4t}{t^2+1}, \frac{t^2-1}{t^2+1}\right)$$

is not, since it is not injective; all points of the ellipsoid being covered twice by the parametrization mapping because we are rotating the whole ellipse around the z-axis rather than only half of it.

In the formulation of our main result (see Theorem 4.2) we have just required the strictly weaker assumption that the involved ϕ and ψ curves are given by a proper parametrization (over the complexes). We recall that, given any parametrization of a swung surface, it is algorithmically easy to obtain another one, describing the same surface, verifying this condition. See [2].

On the other technical issue –the notion of real surface– we can start by recalling that the concept of (algebraic) surface over $\mathbb{C}^3$ is simple and well established in algebraic geometry. It is just the solution set (over the complexes) of a non-constant polynomial in three variables, with complex coefficients: its implicit equation $F(x, y, z) = 0$. At every point, the surface is either locally diffeomorphic to an open ball of $\mathbb{C}^2$ (if we are at a regular point) or close, in the euclidean topology, to a regular point. This is the reason we say that a complex surface has (complex) dimension 2 (even considering that a ball in $\mathbb{C}^2$ is a 4-dimensional real object).

Given an algebraic surface $\mathcal{S}_{\mathbb{C}}$ in $\mathbb{C}^3$, its real points $\mathcal{S} = \mathcal{S}_{\mathbb{C}} \cap \mathbb{R}^3$ might yield a two dimensional subset of $\mathbb{R}^3$, but it could also be just some geometric object of smaller (real) dimension or even empty. This is clearly the case if its implicit equation involves non-real coefficients (such as the complex plane $x+y+\mathtt{i}z = 0$, describing just a real line in $\mathbb{R}^3$). Having a real implicit equation (i.e. being *real-defined*) is a necessary condition to avoid this phenomena and try to guarantee a two dimensional real part of a complex surface. But it is not sufficient. Think, for instance, of the surfaces defined by $x^2 + y^2 + z^2 + 1 = 0$ or by $x^2 + y^2 + z^2 = 0$. In the first case, the solution set over $\mathbb{R}^3$ is just empty. In the second case, just the origin of coordinates, while, over the complex affine space $\mathbb{C}^3$, both cases yield true surfaces (according to our definition above), in fact rational. Therefore, neither the solution set of $x^2 + y^2 + z^2 + 1 = 0$ nor of $x^2 + y^2 + z^2 = 0$ are parametrizable with real coefficients, since if such parametrization would exist, it would yield –for real values of the parameters– many real points in the surface. Since we are interested in learning when there is a reparametrization with real coefficients of a given complex parametric surface, it is natural that we rule out –at least– such cases.

Thus, given a complex algebraic surface $\mathcal{S}_{\mathbb{C}}$ in $\mathbb{C}^3$, we would like to name it as *real* if every (complex) polynomial vanishing over the set $\mathcal{S} = \mathcal{S}_{\mathbb{C}} \cap \mathbb{R}^3$ must also vanishes over $\mathcal{S}_{\mathbb{C}}$. That is, if, in this sense, the real part of $\mathcal{S}_{\mathbb{C}}$ is algebraically indistinguishable from the whole complex surface. More technically, this condition is expressed by saying that the closure of $\mathcal{S} = \mathcal{S}_{\mathbb{C}} \cap \mathbb{R}^3$ in the Zariski topology is equal to $\mathcal{S}_{\mathbb{C}}$, $\mathcal{S}_{\mathbb{C}} = \overline{\mathcal{S} \cap \mathbb{R}^3}$. Clearly, none of the surfaces $\mathcal{S}_{\mathbb{C}} = x^2+y^2+z^2+1 = 0$

or $\mathcal{S}_{\mathbb{C}} = x^2 + y^2 + z^2 = 0$ are real, since, in the first instance, 1 is a polynomial vanishing over $\mathcal{S}_{\mathbb{C}} \cap \mathbb{R}^3$, but not on $\mathcal{S}_C$ and, in the second, $x = 0$ is an equation holding over the real part, but not over the complex surface. For another example, let us consider a complex surface implicitly defined by a non-real polynomial, such as the plane $x + y + \mathtt{i}z = 0$. It has many real points, but they verify simultaneously the two equations $\{x + y + iz = 0, \quad x + y - iz = 0\}$ and, thus, the real part of this surface verifies the system $\{x+y = 0, \quad z = 0\}$, which, obviously, does not apply to the whole complex plane. We conclude that this plane is not real.

Although here we are attempting to reduce technicalities to a minimum, the study of geometric objects defined as real solutions of polynomial equations belongs to the field of real algebraic geometry and we would like to point out at least some references for further details on this subject, such as the foundational book [6], or the paper [15], which addresses the so-called *complexification* of a real algebraic set. With this terminology, we will say that a (complex) surface $\mathcal{S}_{\mathbb{C}}$ is *real* if it coincides with the complexification of its real part. For a *real* surface it is easy to prove that its real part is truly a surface, an object of real dimension two, in the sense of having points (in fact most of them) at which the surface is locally diffeomorphic to an open ball of $\mathbb{R}^2$. However, contrary to what happens in $\mathbb{C}^3$, this does not mean, in general, over $\mathbb{R}^3$, that such points are dense, in the euclidean topology, over the real part of the surface. For example, consider the (absolutely) irreducible *real* surface $\mathcal{S}_{\mathbb{C}}$ given by $x^2(1 - x) + y^2 + z^2 = 0$. Then, it happens that $\mathcal{S}$ is a 2-dimensional piece plus the origin, as an isolated (also in the Euclidean topology) real point.

Yet, with some simple algebraic considerations one can show that, for an irreducible complex surface, it is equivalent to be *real* and to have a two-dimensional real part (i.e. what one would expect to be "really" a real surface). From a computational point of view, there is an easy criterion to detect whether an irreducible complex surface (such as those given by a rational parametrization) is *real*. It is enough to detect the existence of a regular point which lies in $\mathbb{R}^3$. (A point is regular if it is not a zero simultaneously of the equation of the surface and the derivatives of this equation with respect to the three variables x, y, z). See Proposition 1 in [16] or the basic reference on the topic, [6]. This is the test we have performed in Example 2.3 to conclude the reality of the offset surface.

If a surface $\mathcal{S}_{\mathbb{C}}$ is parametrizable with real rational functions, say, $f_1(t, s)$, $f_2(t, s)$, $f_3(t, s)$ in $\mathbb{R}(t, s)$, then it is real. In fact if a polynomial $G(x, y, z)$ vanishes over $\mathcal{S} = \mathcal{S}_{\mathbb{C}} \cap \mathbb{R}^3$, it vanishes over all points $(f_1(t_0, s_0), f_2(t_0, s_0), f_3(t_0, s_0))$, with $t_0, s_0 \in \mathbb{R}$. Then $G(f_1(t, s), f_2(t, s), f_3(t, s))$ must be identically zero, hence, $G(x, y, z)$ vanishes over all $\mathcal{S}_{\mathbb{C}}$. As pointed out in the Introduction, it is unknown, in general, whether a complex parametrizable surface $\mathcal{S}_{\mathbb{C}}$ which is real, is also parametrizable by real rational functions. Our main result shows, that this is true in the particular case of parametrized swung surfaces.

For curves the situation is completely understood. As above, a (complex) curve is called real if every polynomial vanishing over all its real points must also vanish over the complex points of the curve, or, equivalently (in the irreducible case) the curve has infinitely many real points, or, equivalently, the subset of real points is one dimensional, or it contains a real regular point, etc. Contrary to the case of surfaces, it is well known that a complex parametrizable curve has a real parametrization if and only if it is real, and we know how to find such a parametrization, [12]. This is the basis for the proof of our main result.

4 Reparametrizing swung surfaces

This section is devoted to present the main reparametrization result for *swung* surfaces. The problem of reparametrizing $\mathcal{S}_{\mathbb{C}}$ with rational functions having only real coefficients will be reduced, in essence, to the case of reparametrizing the involved curves ϕ and ψ. Then, for these curves, we will apply the real version of Lüroth theorem, using hypercircles, as in [12, 13]:

Theorem 4.1 ([12])**.** *Let $\mathcal{C}$ be a rational curve (over the complexes) given by a proper parametrization $\phi(t)$ with complex coefficients. There are equivalent:*

1. *$\mathcal{C}$ is $\mathbb{R}$-parametrizable.*

2. *There exists a change of parameter $s \to t = \xi(s) = \frac{as+b}{cs+d}$, with $a, b, c, d \in \mathbb{C}$, and $ad - bc \neq 0$, such that $\phi(\xi(s))$ has real coefficients.*

3. *$\mathcal{C}$ is a real curve.*

Moreover, there is an algorithm that taking as input the given parametrization ϕ determines if these equivalent conditions hold and, if so, computes the change of variables $t = \xi(s)$.

However, some complications arise. Consider, for instance, the surface $\mathcal{S}_{\mathbb{C}} := \{yz + x^2 = 0\}$, parametrized by $\mathcal{P}(s,t) = (\mathtt{i}ts, ts^2, t)$. Then we may think of $\mathcal{P}$ as a swung surface as in (‡) with $\phi(t) = (\mathtt{i}t, t)$, $\psi(s) = (s, -\mathtt{i}s^2)$, so that neither ϕ nor ψ describes a real curve. However, we may also consider $\mathcal{P}$ as described by $\phi'(t) = (t,t)$, $\psi'(s) = (\mathtt{i}s, s^2)$ and, then, both curves are real (the latter is the parabola $y + x^2 = 0$) and, thus, $\mathcal{P}(s,t)$ will be reparametrizable over the reals. Luckily, this example shows the general way to proceed. Next statement is the main result in the article.

Theorem 4.2. *Let $\mathcal{S}_{\mathbb{C}}$ be a rational complex surface, other than a plane, parametrized by $\mathcal{P}(s,t)$. Let $(\phi_1(t), \phi_2(t)) \in \mathbb{C}(t)^2$ and $(\psi_1(s), \psi_2(s)) \in \mathbb{C}(s)^2$ be any proper parametrization of curves such that*

$$\mathcal{P}(s,t) = (\phi_1(t)\psi_1(s), \phi_1(t)\psi_2(s), \phi_2(t)) \in \mathbb{C}(s,t)^3.$$

Then, the following statements are equivalent:

1. *There exists $\lambda \in \mathbb{C} \setminus \{0\}$ such that the curves defined by the parametrizations $\phi_\lambda = (\lambda\phi_1(t), \phi_2(t))$ and $\psi_\lambda = (\frac{1}{\lambda}\psi_1(s), \frac{1}{\lambda}\psi_2(s))$ are $\mathbb{R}$-parameterizable.*

2. There exists a change of variables:

$$\begin{array}{cccc} \xi: & \mathbb{C}^2 & \to & \mathbb{C}^2 \\ & (u,v) & \mapsto & \left(\frac{a_1u+b_1}{c_1u+d_1}, \frac{a_2v+b_2}{c_2v+d_2}\right) \end{array}$$

where $a_ib_i - c_id_i \neq 0$, $i = 1, 2$, *such that* $\mathcal{P}(\xi(u,v)) \in \mathbb{R}(u,v)^3$.

3. $\mathcal{S}_{\mathbb{C}}$ *is* $\mathbb{R}$*-parametrizable.*

4. $\mathcal{S}_{\mathbb{C}}$ *is a real surface.*

The proof 4.2 requires some technical results related to the the construction of the *parametric variety of Weil* associated to the given parametrization of the swung surface. The detailed proof of the technical results has been included in an Appendix.

Proof. 1. $\to$ *2.* If there is a λ such that the curves $(\lambda\phi_1(t), \phi_2(t))$ and $(\frac{1}{\lambda}\psi_1(s), \frac{1}{\lambda}\psi_2(s))$ are $\mathbb{R}$-parametrizable, then, using the real Lüroth theorem [13] there exists a change of parameters $u \to s(u) = \frac{a_1u+b_1}{c_1u+d_1}$, $v \to t(v) = \frac{a_2v+b_2}{c_2v+d_2}$, with $a_ib_i - c_id_i \neq 0$, $i = 1, 2$, such that $(\lambda\phi_1(t(v)), \phi_2(t(v)))$, $(\frac{1}{\lambda}\psi_1(s(u)), \frac{1}{\lambda}\psi_2(s(u)))$ are real parametrizations, so we take $\xi(u,v) = (s(u), t(v))$ and

$$\mathcal{P}(s(u),t(v)) = \left(\lambda\phi_1(t(v))\frac{1}{\lambda}\psi_1(s(u)), \lambda\phi_1(t(v))\frac{1}{\lambda}\psi_2(s(u)), \phi_2(t(v))\right) \in \mathbb{R}(u,v)^3$$

is real.
It is clear that *2.*$\to$*3.* and *3.*$\to$*4.*, so we are left with proving that if the surface is real, then, for a suitable $\lambda \neq 0$, $(\lambda\phi_1(t), \phi_2(t))$ and $(\frac{1}{\lambda}\psi_1(s), \frac{1}{\lambda}\psi_2(s))$ define $\mathbb{R}$-parametrizable curves.

In this direction, we will consider the specific parametric variety of Weil V (see the Appendix) associated to the parametrization $\mathcal{P}(t,s)$. By definition, this variety is obtained as follows. First, in the parametrization of $\mathcal{S}_{\mathbb{C}}$, perform the substitution $s := s_0 + \mathtt{i}s_1$ and $t := t_0 + \mathtt{i}t_1$, where s_0, s_1, t_0, t_1 are new variables. Then, after some normalization, we get $\mathcal{P}(s_0 + \mathtt{i}s_1, t_0 + \mathtt{i}t_1) = (\mathcal{P}_1(\bar{s},\bar{t}), \mathcal{P}_2(\bar{s},\bar{t}), \mathcal{P}_3(\bar{s},\bar{t}))$, where

$$\begin{aligned} \mathcal{P}_1(\bar{s},\bar{t}) = &\ \frac{[A_0(\bar{t}) + \mathtt{i}A_1(\bar{t})]}{A(\bar{t})}\frac{[C_0(\bar{s}) + \mathtt{i}C_1(\bar{s})]}{C(\bar{s})} \\ \mathcal{P}_2(\bar{s},\bar{t}) = &\ \frac{[A_0(\bar{t}) + \mathtt{i}A_1(\bar{t})]}{A(\bar{t})}\frac{[D_0(\bar{s}) + \mathtt{i}D_1(\bar{s})]}{D(\bar{s})} \\ \mathcal{P}_3(\bar{s},\bar{t}) = &\ \frac{[B_0(\bar{t}) + \mathtt{i}B_1(\bar{t})]}{B(\bar{t})} \end{aligned}$$

with $A_i(\bar{t})$, $B_i(\bar{t})$, $A(\bar{t})$, $B(\bar{t}) \in \mathbb{R}[\bar{t}]$, $C_i(\bar{s})$, $D_i(\bar{s})$, $C(\bar{t})$, $D(\bar{t}) \in \mathbb{R}[\bar{s}]$, $\bar{s} = (s_0, s_1)$ and $\bar{t} = (t_0, t_1)$. Notice that the A's and B's arise from the substitution in ϕ_1 and ϕ_2 and likewise the C's and D's come from the substitution in ψ_1 and ψ_2.

Second, we take the Zariski closure V of the open set given by:

$$\begin{aligned} A_0(\bar{t})C_1(\bar{s}) + A_1(\bar{t})C_0(\bar{s}) &= 0 \\ A_0(\bar{t})D_1(\bar{s}) + A_1(\bar{t})D_0(\bar{s}) &= 0 \\ B_1(\bar{t}) &= 0 \\ A(\bar{t}) \neq 0,\ B(\bar{t}) \neq 0,\ C(\bar{s}) \neq 0,\ D(\bar{s}) &\neq 0 \end{aligned}$$

where the first three equations correspond to the vanishing of the imaginary parts of the numerators of $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$. Notice that V does not depend on the precise choice of ϕ and ψ (by adjusting constants), but only on their product.

We have naturally the map:

$$\begin{array}{cccccc} \mathcal{P}^*: & V & \to & \mathcal{S}_{\mathbb{C}} & \\ & (s_0, s_1, t_0, t_1) & \mapsto & (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3) & = \mathcal{P}(s_0 + \mathtt{i}s_1, t_0 + \mathtt{i}s_1) \end{array}$$

From the definition of V, it is clear that $\mathcal{P}^*$ carries real points of V to real points of $\mathcal{S}_{\mathbb{C}}$. Now, since $\mathcal{S}_{\mathbb{C}}$ is real, Theorem A.2 assures the existence of a real 2-dimensional component U of V such that $\mathcal{P}^* : U \to \mathcal{S}_{\mathbb{C}}$ is dominant.

Then, consider the matrix

$$M = \begin{pmatrix} A_0(t_0, t_1) & A_1(t_0, t_1) \\ -C_0(s_0, s_1) & C_1(s_0, s_1) \\ -D_0(s_0, s_1) & D_1(s_0, s_1) \end{pmatrix}$$

Notice that no row of M can be identically zero in U, since $\mathcal{P}^*$ is dominant and $\mathcal{S}_{\mathbb{C}}$ is not a plane. For any point $p = (a_0, a_1, b_0, b_1)$ in the nonempty open subset of U such that $(A_0, A_1)(p) \neq (0, 0)$ we have that $rank(M) = 1$. Thus, if $A_1 \equiv 0$ in U, it follows that, $M \cdot \binom{0}{1} = \left(\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}\right)$ in U. If $A_1 \not\equiv 0$ in U, then $(A_0/A_1)(t_0, t_1) = (-C_0/C_1)(s_0, s_1) = (-D_0/D_1)(s_0, s_1)$ is a real rational function in U. By Theorem A.4, U is a Cartesian product of two irreducible curves, so, by Lemma A.5, $(A_0/A_1)(t_0, t_1) = (-C_0/C_1)(s_0, s_1) = (-D_0/D_1)(s_0, s_1) = r \in \mathbb{R}$ and $M \cdot \binom{1}{-r} = \left(\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}\right)$ in U.

In any case, there is a vector $(r_1, r_0) \in \mathbb{R}^2$, $(r_1, r_0) \neq (0, 0)$, such that $M \cdot \binom{r_1}{r_0} = \left(\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}\right)$ in U.

Let $\lambda = r_0 + \mathtt{i}r_1$. We are going to prove that the curves defined by $\phi_\lambda = (\lambda\phi_1(t), \phi_2(t))$ and $\psi_\lambda = (\frac{1}{\lambda}\psi_1(s), \frac{1}{\lambda}\psi_2(s))$ are $\mathbb{R}$-parametrizable. Indeed,

$$(r_0 + \mathtt{i}r_1)\frac{A_0(\bar{t}) + \mathtt{i}A_1(\bar{t})}{A(\bar{t})} = \frac{r_0A_0(\bar{t}) - r_1A_1(\bar{t}) + \mathtt{i}(r_1A_0(\bar{t}) + r_0A_1(\bar{t}))}{A(\bar{t})}$$

$$\left(\frac{r_0 - \mathtt{i}r_1}{r_0^2 + r_1^2}\right)\frac{C_0(\bar{s}) + \mathtt{i}C_1(\bar{s})}{C(\bar{s})} = \frac{r_0C_0(\bar{s}) + r_1C_1(\bar{s}) + \mathtt{i}(r_0C_1(\bar{s}) - r_1C_0(\bar{s}))}{(r_0^2 + r_1^2)C(\bar{s})}$$

$$\left(\frac{r_0 - \mathtt{i}r_1}{r_0^2 + r_1^2}\right)\frac{D_0(\bar{s}) + \mathtt{i}D_1(\bar{s})}{C(\bar{s})} = \frac{r_0D_0(\bar{s}) + r_1D_1(\bar{s}) + \mathtt{i}(r_0D_1(\bar{s}) - r_1D_0(\bar{s}))}{(r_0^2 + r_1^2)D(\bar{s})}$$

If $p = (a_0, a_1, b_0, b_1) \in U$ then, $(r_1A_0 + r_0A_1)(a_0, a_1) = 0$, $B_1(a_0, a_1) = 0$, $A(a_0, a_1) \neq 0$, $B(a_0, a_1) \neq 0$ and $(r_0C_1 - r_1C_0)(b_0, b_1) = 0$, $(r_0D_1 - r_1D_0)(b_0, b_1) = 0$, $C(b_0, b_1) \neq 0$, $D(b_0, b_1) \neq 0$. Hence $U = U_1 \times U_2$, where U_1 is contained in the parametric variety of Weil of ϕ_λ and U_2 is contained in the parametric variety of Weil of ψ_λ.

Since U is real, U_1, U_2 are real curves. It follows from the theory of hypercircles [11, 13, 3] that ϕ_λ and ψ_λ are real curves and, hence, real parametrizable curves. □

Remark 4.3. If the given swung parametrization $\mathcal{P}$ is proper (see Section 3) and if some of the equivalent conditions of Theorem 4.2 hold, then the real parametrization $\mathcal{P}(\xi(s,t))$ described in item *2.* is also proper.

Remark 4.4. In the hypotheses of the theorem we have explicitly discarded the case of planes. We can easily check whether the given parametrization $\mathcal{P}(s,t)$ of a surface corresponds to a plane by considering four generic points $\mathcal{P}(s_i,t_i), i = 1 \ldots 4$, and verifying, by computing a determinant, if they are coplanar. On the other hand, if $\mathcal{S}_\mathbb{C}$ is a plane, it is clear that we can parametrize it over the reals if and only if it is real. However, items 1 and 2 in the statement above need not hold, see Example 5.4.

Corollary 4.5. *Let $\mathcal{S}_\mathbb{C}$ be a rational revolution surface, parametrized by*

$$\mathcal{P}(s,t) = (\phi_1(t)\frac{s^2-1}{s^2+1}, \phi_1(t)\frac{2s}{s^2+1}, \phi_2(t)) \in \mathbb{C}(s,t)^3,$$

where $(\phi_1(t), \phi_2(t))$ is a proper parametrization of a curve. The following statements are equivalent:

1. *The curve defined by $\phi(t) = (\phi_1(t), \phi_2(t))$ is $\mathbb{R}$-parametrizable (equivalently, it is real).*
2. *There exists a change of parameters with complex coefficients $\xi : \mathbb{C} \longrightarrow \mathbb{C}$, where $\xi(t) = \dfrac{at+b}{ct+d}$ and $ad - bc \neq 0$, such that $\mathcal{P}(s, \xi(t)) \in \mathbb{R}(s,t)^3$.*
3. *$\mathcal{S}_\mathbb{C}$ is $\mathbb{R}$-parametrizable (but, perhaps, not necessarily with a proper parametrization)*
4. *$\mathcal{S}_\mathbb{C}$ is a real surface.*

Proof. The only nontrivial implication is *4.* $\to$ *1.* Notice that, in this case, the parametrization determines uniquely the curve ϕ and $\psi = (\frac{s^2-1}{s^2+1}, \frac{2s}{s^2+1})$. Assume first that $\mathcal{S}_\mathbb{C}$ is not a plane. By Theorem 4.2, from *4.* it follows that there is a $\lambda \in \mathbb{C}^*$ with ϕ_λ and ψ_λ, $\mathbb{R}$-parametrizable. Now, observe that for any $\lambda \in \mathbb{C}^*$, ψ_λ parametrizes the circle $x^2 + y^2 = 1/\lambda^2$, that is real if and only if λ is real. And ϕ_λ, with $\lambda \in \mathbb{R}^*$, is $\mathbb{R}$-parametrizable if and only if ϕ is $\mathbb{R}$-parametrizable.

On the other hand, suppose that $\mathcal{S}_\mathbb{C}$ is a real plane defined by the real equation $ax + by + cz = d$. Then $\phi_1(t)(a\,\frac{s^2-1}{s^2+1} + b\,\frac{2s}{s^2+1}) = d - c\,\phi_2(t)$. Now

$(a\,\frac{s^2-1}{s^2+1} + b\,\frac{2s}{s^2+1})$ must be a constant. Otherwise, since the second term of the equality above does not involve the s variable, it will imply that $d - c\,\phi_2(t)$ is zero and, then, $\phi_1(t)$ must be zero (but then $\mathcal{S}_{\mathbb{C}}$ is not a surface). Now, if $(a\,\frac{s^2-1}{s^2+1} + b\,\frac{2s}{s^2+1})$ is a constant, it must be $a = b = 0$. Thus $d - c\phi_2(t)$ is zero. But c can not be zero (since then $a = b = c = 0$, and do not have a plane). Therefore $\phi_2(t) = d/c \in \mathbb{R}$ and, since the parametrization ϕ is proper, it can be reparametrized to $(t, d/c)$. $\square$

5 The algorithm and examples

In this section we present how to derive an algorithm to check whether a swung parametrization defines a real surface $\mathcal{S}$ and, if it is the case, to compute a real parametrization of $\mathcal{S}$.

Since we already have algorithms to reparametrize real curves ([12]) given by complex parametrizations, we base the algorithm on the characterization (1) of Theorem 4.2. Given two curve parametrizations $\phi(t) = (\phi_1(t), \phi_2(t))$ and $\psi(s) = (\psi_1(s), \psi_2(s))$, the only problem left is computing, if it exists, a $\lambda \in \mathbb{C}^*$ such that $\phi_\lambda = (\lambda\psi_1, \psi_2)$ and $\psi_\lambda = (\frac{1}{\lambda}\psi_1, \frac{1}{\lambda}\psi_2)$ are real curves. One possible naive approach could be implicitizing one of the curves, by considering λ a parameter, and then adjusting the possible values of λ that make such implicit equation real. But that procedure would not guarantee (unless we use some Cylindric Algebraic Decomposition techniques, see [6]) that the curve is real (only that it is real-defined) and, anyway, we would like to avoid the implicitization computation, preferring to work directly with the given parametric input.

Our approach relies on the following key observation. Let $\phi(t) = (\phi_1(t), \phi_2(t))$ be a complex parametrization and t_0, t_1 new variables. Write $\phi_2(t_0 + \mathtt{i}t_1) = \frac{B_0(t_0,t_1)+\mathtt{i}B_1(t_0,t_1)}{B(t_0,t_1)}$. If there is a λ such that ϕ_λ parametrizes a real curve, then the corresponding hypercircle Z_1 of ϕ_λ is a real circle or line and its implicit equation is a factor of $B_1(t_0, t_1)$ in $\mathbb{R}[t_0, t_1]$. This provides an algorithm to reparametrize $\mathcal{P}$ over the reals.

Algorithm 5.1.

- **Input:** A complex swung parametrization $\mathcal{P}$ of a surface $\mathcal{S}_{\mathbb{C}}$, different from a plane, such that there exists $\eta(t) = (\eta_1(t), \eta_2(t)) \in \mathbb{C}(t)^2$ and $\mu(s) = (\mu_1(s), \mu_2(s)) \in \mathbb{C}(s)^2$ parametrizations of curves such that

$$\mathcal{P}(t, s) = (\eta_1(t)\mu_1(s), \eta_1(t)\mu_2(s), \eta_2(t)) \in \mathbb{C}(t, s)^3.$$

  Output: A real parametrization $\mathcal{P}'(t, s)$ of $\mathcal{S}_{\mathbb{C}}$ or *"The surface is not real"*

1. **Compute** a pair $\eta(t)$, $\mu(s)$ from $\mathcal{P}$, verifying the input structure.
2. **Reparametrize** $\eta(t)$ and $\mu(s)$ to proper parametrizations $\phi(t)$ and $\psi(s)$ of the same curves.

3. **Write** $\phi_2(t_0 + \mathtt{i}t_1) = \frac{B_0(t_0,t_1)+\mathtt{i}B_1(t_0,t_1)}{B(t_0,t_1)}$
4. **Compute** the factors of degree 1 and/or of degree 2 (that correspond to circles) of $B_1(t_0, t_1)$ in $\mathbb{R}[t_0, t_1]$.
5. **For each** factor f from step 4. **do**
   (a) **Compute** a real parametrization $(v_0(t), v_1(t))$ of the line or circle defined by f.
   (b) Let $v(t) = v_0(t) + \mathtt{i}v_1(t)$
   (c) **If** there exists a $\lambda_f \in \mathbb{C}^*$ such that $(\lambda_f\, \phi_1(v(t)), \phi_2(v(t)))$ is real **then**:
      i. Apply the real reparametrization algorithm for curves to $\psi_{\lambda_f} = (1/\lambda_f \psi_1, 1/\lambda_f \psi_2)$.
      ii. **If** ψ_{λ_f} is real and $u(s)$ is an invertible linear fraction such that $\psi_{\lambda_f}(u(s))$ is real **then return** $(u(s), v(t))$.
6. **If** no factor f works **then return** *"The surface is not real"*.

We remark that the computations in steps 1 and 5 (c) are straightforward. For instance, λ_f can be taken as the inverse of the leading coefficient of the numerator of $\phi_1(v(t))$ when this fraction is written with monic denominator. Step 2 can be carried out by standard techniques ([2]).

The main difficulty in this approach is step 4, in which we have to factor a bivariate polynomial in $\mathbb{R}[t_0, t_1]$. We present an alternative that needs only to manipulate the complex roots of a univariate polynomial.

If $\phi_\lambda = (\lambda\phi_1, \phi_2)$, $\lambda \in \mathbb{C}^*$ parametrizes a real curve $\mathcal{C}_\lambda$, then the complex conjugate parametrization $\overline{\phi_\lambda} = (\overline{\lambda\phi_1}, \overline{\phi}_2)$ is also a proper parametrization of $\mathcal{C}_\lambda$. Hence, there is a linear fraction $v' \in \mathbb{C}(t)$ such that $\overline{\phi}_2(v'(t)) = \phi_2(t)$, $\overline{\lambda\phi}_1(v'(t)) = \lambda\phi_1(t)$. For all but finitely many values t_0 of t, we have that $\lambda/\overline{\lambda} = \overline{\phi}_1(v'(t_0))/\phi_1(t_0)$.

The idea to compute the possible values of $\lambda/\overline{\lambda}$ is the following. First, we choose a $t_0 \in \mathbb{C}$. Compute $\phi_2(t_0) = a_0$. The possible values of $u'(t_0)$ are the solutions b_j in $\mathbb{C}$ of the univariate equation $\overline{\phi}_2(x) = a_0$. This will give a set $A_{t_0} = \{b_1, \ldots, b_d\}$. Now, the possible values of $\lambda/\overline{\lambda}$ are $S_{t_0} = \{\overline{\phi}_1(b_1)/\phi_1(a_0), \ldots, \overline{\phi}_1(b_d)/\phi_1(a_0)\}$. Note that $\lambda/\overline{\lambda}$ always has norm 1 so we can take in S_{t_0} only those values of norm 1. On the other hand, from $\lambda/\overline{\lambda}$ we can recover λ up to a real constant and thus, we get a finite set of candidates to a λ verifying item *1.* in Theorem 4.2. This description alone already provides an algorithm. For every candidate λ, we apply the reparametrization algorithm for ϕ_λ and ψ_λ.

In practice, except for rare cases, S_{t_0} is either empty (and $\mathcal{S}_\mathbb{C}$ is not real) or it is already the complete set of valid $\lambda/\overline{\lambda}$. Moreover, it is, typically, a singleton. If $r \in S_{t_0}$, $r = r_0 + \mathtt{i}r_1 \in \mathbb{C}$ then $r_0^2 + r_1^2 = 1$ and $\lambda/\overline{\lambda} = r_0 + \mathtt{i}r_1$. If $r = 1$ a solution is $\lambda = 1$. If $r \neq 1$ a solution is $\lambda = r_1 + \mathtt{i}(1 - r_0) \in \mathbb{C}^*$.

There are only two possible kinds of t_0 values where this procedure to compute λ does not work. First, when $\phi(t_0)$ is not defined (because the denominator vanishes). The other case is if $\phi_2(t_0) = \overline{\phi_2}(\infty)$. But these are $2d$ cases that can be discarded easily.

Once the possible $\lambda's$ are computed, we only have to check, for each λ, if ϕ_λ and ψ_λ are real and, if so, to compute a real reparametrization.

We can use this discussion to derive an algorithm that either checks that ϕ_λ is never a real curve or returns the values λ such that ϕ_λ is real. We must point out that this approach will not work if ϕ is a horizontal or vertical line. But these are corner cases that can be easily solved by direct means.

The description of this alternative algorithm (without emphasizing corner cases) could be:

Algorithm 5.2.

- **Input:** A complex swung parametrization $\mathcal{P}$ of a surface $\mathcal{S}_{\mathbb{C}}$, different from a plane, such that there exists $\eta(t) = (\eta_1(t), \eta_2(t)) \in \mathbb{C}(t)^2$ and $\mu(s) = (\mu_1(s), \mu_2(s)) \in \mathbb{C}(s)^2$ parametrizations of curves such that

$$\mathcal{P}(s,t) = (\eta_1(t)\mu_1(s), \eta_1(t)\mu_2(s), \eta_2(t)) \in \mathbb{C}(t,s)^3.$$

  Output: A real parametrization $\mathcal{P}'(t,s)$ of $\mathcal{S}_{\mathbb{C}}$ or *"The surface is not real"*

1. **Compute** a pair $\eta(t)$, $\mu(s)$ from $\mathcal{P}$, verifying the input structure.
2. **Reparametrize** $\eta(t)$ and $\mu(s)$ to proper parametrizations $\phi(t)$ and $\psi(s)$ of the same curves.
3. **Compute** the complex conjugates $\overline{\phi_1}$, $\overline{\phi_2}$ of ϕ.
4. **Compute** $a_\infty = \overline{\phi_2}(\infty) \in \mathbb{C} \cup \{\infty\}$
5. $S \leftarrow \mathbb{C}$
6. **while** $S = \mathbb{C}$ **do**
   (a) $a \leftarrow random(\mathbb{C})$
   (b) $b \leftarrow \phi_2(a)$
   (c) **If** $b \neq \infty$ **and** $b \neq a_\infty$ **then**
      i. $T \leftarrow \{t \in \mathbb{C} \mid \overline{\phi_2(t)} = b\}$
      ii. $S \leftarrow S \cap \{s = \overline{\phi_1(t)}/\phi_1(a) \quad | \; t \in T, |s| = 1\}$
7. **If** $S = \emptyset$ **then return** *"ϕ_λ is never real"*
8. $\Lambda \leftarrow \emptyset$
9. **For each** $r = r_0 + ir_1 \in S$ **do**
   (a) **If** $r = 1$ **then** $\lambda \leftarrow 1$ **else** $\lambda \leftarrow r_1 + i(1 - r_0)$.
   (b) $\Lambda = \Lambda \cup \{\lambda\}$

10. **for each** λ in Λ **do**

    (a) Compute (if possible) u, v such that $\phi_\lambda(v)$, $\psi_\lambda(s)$ are real.

11. No pair (u, v) is found **then return** *"ϕ_λ is never real"* **else return** "pairs $(u(s), v(t))$ found".

This alternative algorithm has some advantages over the first one. Along the paper, including the algorithms, it is assumed that we are working in a field $\mathbb{K}(\mathtt{i})$ were computations are exact (infinite precision). However, the case that the input is given by a floating point approximation is also interesting. In this context, if we apply Algorithm 5.1, we should be dealing with an approximate factorization of $B_1(t_0, t_1)$ over the reals. On the other hand, Algorithm 5.2 would have to compute all complex roots of some univariate polynomials, a more common problem. We have made experiments with the math software Sage using both algorithms for inputs in $\mathbb{Q}(\mathtt{i})$ and with floating point arithmetic. The running times are described in Table 1. Case 1 is Algorithm 5.1 in $\mathbb{Q}(\mathtt{i})$ and exact computations. Case 2 is Algorithm 5.2 also in $\mathbb{Q}(\mathtt{i})$ and exact computations. Finally, Case 3 is Algorithm 5.2 using floating point arithmetic. The tests are performed as following. First, we construct two random rational parametrizations $\phi_r = (\phi_1(t), \phi_2(t))$ and $\psi_r = (\psi_1(s), \psi_2(s))$, of degree d and coefficients over $\mathbb{Q}$. The tested degrees for ϕ and ψ have been $d = 1, 2, 5, 10, 25$. Then we compute random linear fractions $u(s), v(t)$ with coefficients in $\mathbb{Q}(\mathtt{i})$. Finally, the input is $\mathcal{P} = (\phi_1(v(t))\psi_1(u(s)), \phi_1(v(t))\psi_2(u(s)), \phi_2(v(t)))$. We have prepared three tables considering a bound for the size of the integers in ϕ and ψ, with bounds $2^8, 2^{16}$ and 2^{32} respectively. In all cases, the coefficients of u and v are bounded by 100, so we know before hand that in all cases there are solutions with *small height*. Note that these figures are not the bound of the input $\mathcal{P}$, since we have to perform a composition and a multiplication. For instance, the bigger case is degree 25 and initial coefficients bounded by 2^{32}, yielding the final size of the coefficients of the input $\mathcal{P}$ around 2^{1700}.

By looking into the tables we observe that Algorithm 5.2 behaves similarly to Algorithm 5.1 for reasonable degrees. But, for very big degrees or very big coefficients, Algorithm 5.2 performs better.

On the other hand, we notice that using floating point arithmetic is much faster. What we get as output in this case is a couple of linear fractions $(u(s), v(t))$ such that, for (s, t) real parameters, $\mathcal{P}(u(s), v(t))$ has a very small imaginary part (i.e. as if it were real, in practice). In the floating point case, as the degree grows, the numerical error increases to the point that, for degree 25, our implementation sometimes fail. Each case has been executed ten times and we display, in the corresponding entry of the table, both the best and worst obtained time in seconds.

size 2^8	deg. 1	deg. 3	deg. 5	deg. 10	deg. 25
Case 1	0.25-0.42	0.49-0.52	0.64-0.78	1.22-1.36	5.45-14.15
Case 2	0.52-0.55	0.65-0.69	0.77-1.01	1.11-1.19	2.88-3.12
Case 3	<0.01	<0.01	0.013-0.015	0.02-0.03	0.15-0.19

size 2^{16}	deg. 1	deg. 3	deg. 5	deg. 10	deg. 25
Case 1	0.72-0.78	0.88-0.91	1.06-1.34	2.01-2.18	41.78-52.7
Case 2	0.39-0.41	0.51-0.66	0.64-0.68	1.06-1.11	3.71-3.94
Case 3	<0.01	0.01	0.013-0.015	0.02-0.03	0.14-0.15

size 2^{32}	deg. 1	deg. 3	deg. 5	deg. 10	deg. 25
Case 1	0.96-0.99	0.72 - 1.16	1.43-1.47	3.31-3.59	>60
Case 2	0.53-0.56	0.75-0.79	0.91-0.97	1.51-1.56	6.02-6.71
Case 3	<0.01	<0.01	0.013-0.015	0.03	0.14-0.16

Table 1: Running time of the algorithms

5.1 Examples

Example 5.3. Let $\mathcal{S}_{\mathbb{C}}$ be the classical revolution surface given by the parametrization

$$\left(\frac{3-t^2}{4-2t}\frac{s^2-1}{s^2+1}, \frac{3-t^2}{4-2t}\frac{2s}{s^2+1}, \frac{-\mathtt{i}t^2+4\mathtt{i}t-3\mathtt{i}}{2t-4}\right)$$

If we take the ϕ-curve parametrized by $\left(\frac{3-t^2}{4-2t}, \frac{-\mathtt{i}t^2+4\mathtt{i}t-3\mathtt{i}}{2t-4}\right)$ and perform the method in [12], we obtain that we have to parametrize the circle $x^2+y^2-4x+3=0$ (and, thus, the given curve is real), yielding the associated unit $\xi(t) = (t+3\mathtt{i})/(t+\mathtt{i})$. If we apply this unit to the original parametrization we get the following real parametrization of $\mathcal{S}_{\mathbb{C}}$:

$$\left(\frac{t^2+3}{t^2+1}\frac{s^2-1}{s^2+1}, \frac{t^2+3}{t^2+1}\frac{2s}{s^2+1}, \frac{2t}{t^2+1}\right)$$

Example 5.4. We now show that Theorem 4.2 does not work for planes. Consider the plane given by the parametrization

$$\mathcal{P} = ((\mathtt{i}t+1)s, (\mathtt{i}t+1)s, t)$$

Of course, this is the plane $\{x=y\}$, but if one computes the parametric variety of Weil as in the proof of 4.2, one gets $V = U = \{t_1 = 0, t_0 s_0 + s_1 = 0\}$, so U does not have the shape announced in Theorem A.4. This happens because $\mathcal{S}_{\mathbb{C}}$ is a plane, so items (1) and (2) of Theorem 4.2 do not apply. There is no $\lambda \in \mathbb{C}^*$ such that $(\lambda(\mathtt{i}t+1), t)$, $(1/\lambda s, 1/\lambda s)$ are real curves. Still, U is $\mathbb{R}$-parametrizable by $t_0 = v, t_1 = 0, s_0 = u, s_1 = -uv$, so $\mathcal{P}(u, v - \mathtt{i}uv) = (u^2 v + v, u^2 v + v, u) \in \mathbb{R}(u,v)^3$.

Example 5.5. Consider now the surface $xz - y^4$ given by the parametrization

$$\mathcal{P}(s,t) = (\mathtt{i}ts^4, \mathtt{i}ts, -\mathtt{i}t^3)$$

The parametrization is not proper, but $(t, -\mathtt{i}t^3)$, $(\mathtt{i}s^4, \mathtt{i}s)$ are both proper. If we perform our method we get in V three valid components in the sense of Theorem A.1:

$$U_1 = \{t_0 = 0, s_1 = 0\}, \lambda_1 = \mathtt{i}$$

$$U_2 = \{t_0 - \sqrt{3}t_1 = 0, s_0 - \sqrt{3}/3s_1 = 0\}, \lambda_2 = \frac{\sqrt{3} - \mathtt{i}}{2}$$

$$U_3 = \{t_0 + \sqrt{3}t_1 = 0, s_0 + \sqrt{3}/3s_1 = 0\}, \lambda_3 = \frac{-\sqrt{3} - \mathtt{i}}{2}$$

Each U_i is a plane, parametrizable as

$$U_1 : (0, t, s, 0)$$

$$U_2 : (\sqrt{3}t, t, \sqrt{3}/3s, s)$$

$$U_3 : (-\sqrt{3}t, t, -\sqrt{3}/3s, s)$$

Thus, we get three different reparametrizations of the original surface:

$$\mathcal{P}_1 = \left(-ts^4, -ts, -t^3\right)$$

$$\mathcal{P}_2 = \left(\frac{32}{9}ts^4, -\frac{4}{\sqrt{3}}ts, 8t^3\right)$$

$$\mathcal{P}_3 = \left(\frac{32}{9}ts^4, \frac{4}{\sqrt{3}}ts, 8t^3\right)$$

Example 5.6. Similarly, if we start with the parametrization

$$(\mathtt{i}ts^8, \mathtt{i}ts, -\mathtt{i}t^7)$$

and perform the algorithm, we find that there are seven valid components U. If we take $\phi = (t, -\mathtt{i}t^7)$, $\psi = (\mathtt{i}s^8, \mathtt{i}s)$, one of the components of U is associated to the value $\lambda = \mathtt{i}$ and the change of variables is $(u(s) = s, v(t) = \mathtt{i}t)$.

However, for the rest of components, we have that the other six values of λ are the complex roots of $x^6 - 5\mathtt{i}x^5 - 11x^4 + 13\mathtt{i}x^3 + 9x^2 - 3\mathtt{i}x - 1$. Each of these $\lambda's$ corresponds to the change of variables

$$u(s) = (G(\lambda) + I)s, v(t) = (F(\lambda) + I)t$$

where

$$F(\lambda) = (2144\lambda^{11} + 6096\lambda^9 + 18187\lambda^7 - 5532\lambda^5 + 52746\lambda^3 - 29068\lambda)/2059,$$

$$G = (564\lambda^{11} + 1788\lambda^9 + 5687\lambda^7 + 404\lambda^5 + 18462\lambda^3 - 10520\lambda)/14413$$

Example 5.7. This is an example of floating point computation. Let $\mathcal{P} = (((-0.235869421766 +0.00479979499514i)\ t^2s^2 +(-1.06313828776 -0.166407418395i)\ t^2s +(-0.2298109337 -0.194094699602i)\ ts^2 +(-0.549385710585 -0.417008231694i)\ t^2 +(-0.877430457459 -1.05483464219i)\ ts +(1.66137786935 + 0.435653733693i)\ s^2 +(-0.174582398271 -0.861933962222i)\ t +(7.11424400952 +3.28051289442i)\ s +3.0177826152 +4.01338551785i)\ /\ (t^2s^2 +(1.86773892267 +0.815610295477i)\ t^2s +(0.246950616172 +0.659953272957i)\ ts^2 +(-0.629990211803 +0.819708831258i)t^2 +(-0.0770254061546 +1.43403588007i)\ ts +(-0.637235543476 -0.0986590151152i)\ s^2 +(-0.696545997048 -0.213336501249i)\ t +(-1.10972231899 -0.704005152506i)\ s +0.482323820976 -0.46019338875i),\ ((0.043748011838 +0.000454800948882i)\ t^2s^2 +(-0.0131269921629 +0.0392333791882i)\ t^2s +(0.0414912865933 +0.037287262802i)\ ts^2 +(-0.142191088123 +0.00237065960661i)\ t^2 +(-0.0456137529341 +0.0264953515089i)\ ts +(-0.305468984051 -0.0902226718149i)\ s^2 +(-0.138098376489 -0.117508139573i)\ t +(0.169985946752 -0.248640737912i)\ s +1.00050177551 +0.266290220991i)/ (t^2s^2 +(1.86773892267 +0.815610295477i)\ t^2s +(0.246950616172 +0.659953272957i)\ ts^2 +(-0.629990211803 +0.819708831258i)\ t^2 +(-0.0770254061546 +1.43403588007i)\ ts +(-0.637235543476 -0.0986590151152i)\ s^2 +(-0.696545997048 -0.213336501249i)\ t +(-1.10972231899 -0.704005152506i)\ s +0.482323820976 -0.46019338875i),\ ((-2.01273043888 +0.00917837700067i)\ t^2 +(-2.39821706934 -1.65257355305i)\ t+ 3.18517172695 +0.210190888739i)/(t^2 +(0.246950616172 +0.659953272957i)\ t -0.637235543476 -0.0986590151152i))$

This is an approximate parametrization of a real surface. If we perform Algorithm 5.2, we get, as λ,

$\lambda = -0.999993922197720 + 0.00348648356104579\mathtt{i}$, $u = ((121.322126428429 -103.7452830536660\mathtt{i})t - 103.745283053666 + 88.1900509458403\mathtt{i})/(t - \mathtt{i})$, $v = ((75.1892967277426 - 78.1929832049560\mathtt{i})s - 78.1929832049560 + 80.4349108110022\mathtt{i})/(s - \mathtt{i})$.

With this unit, we get, for instance:
$\mathcal{P}(u, v)(0, 2) = (-0.247210104423103 +3.75195846613607 \times 10^{-11}\mathtt{i},\ 0.0416569932380774 +5.64823188220487 \times 10^{-12}\mathtt{i},\ -2.00183575113046 +3.09979819590467 \times 10^{-12}\mathtt{i})$
which is "practically" real.

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A The parametric variety of Weil

The *parametric Weil* construction and the theory of *hypercircles and ultraquadrics*, are tools developed in [11], [3]. Here we will consider the specific parametric variety of Weil V associated to the parametrization $\mathcal{P}(t, s)$ defined in the proof of Theorem 4.2 and the map

$$\begin{array}{cccc} \mathcal{P}^* : & V & \rightarrow & \mathcal{S}_{\mathbb{C}} \\ & (t_0, t_1, s_0, s_1) & \mapsto & \mathcal{P}(t_0 + \mathtt{i} t_1, s_0 + \mathtt{i} s_1) \end{array}$$

Recall that, by construction, $\mathcal{P}^*$ carries real points of V to real points of $\mathcal{S}_{\mathbb{C}}$.

The importance of this variety V is that it encodes the fact that $\mathcal{S}_{\mathbb{C}}$ is real-defined or real parametrizable.

Theorem A.1. *Let V be the parametric variety of Weil associated to $\mathcal{P}$. If $\mathcal{S}_{\mathbb{C}}$ is a real-defined surface then there is (at least) one surface U that is an irreducible component of V such that $\mathcal{P}^* : U \rightarrow \mathcal{S}_{\mathbb{C}}$ is a dominant map. Moreover, if $\tau(u, v)$ is a real parametrization of U, then $\mathcal{P}^*(\tau(u, v))$ is a real parametrization of $\mathcal{S}_{\mathbb{C}}$.*

Proof. This is a direct consequence of Theorem 10 in [3]. $\square$

Note that, in the theorem, the surface U needs not be real-defined. By [3], Corollary 13, if $\mathcal{S}_{\mathbb{C}}$ is real-defined we know that there exists a real-defined surface W such that $\mathcal{P}^* : W \rightarrow \mathcal{S}_{\mathbb{C}}$ is dominant, but W needs not to be irreducible.

In our particular case we want to explore with more detail the surfaces U_i, those components of V such that the map $\mathcal{P}^* : U_i \to \mathcal{S}_{\mathbb{C}}$ is dominant. Specially, we would like to understand the projections of such components into the (t_0, t_1) and (s_0, s_1) planes.

Theorem A.2. *If $\mathcal{S}_{\mathbb{C}}$ is a real surface, then there is a real irreducible surface U, a component of V, such that the map $\mathcal{P}^* : U \to \mathcal{S}_{\mathbb{C}}$ is dominant and $\mathcal{P}^*$ takes real points of U to real points of $\mathcal{S}_{\mathbb{C}}$.*

Proof. By [3], $\mathcal{P}^* : V \to \mathcal{S}_{\mathbb{C}}$ is generically (over an nonempty open subset of $\mathcal{S}_{\mathbb{C}}$) finite to one. So, if U_i is a component of V of dimension different from 2, then $\mathcal{P}^* : U_i \to \mathcal{S}_{\mathbb{C}}$ is not dominant. Let U' be the union of all the components W of V such that the map $\mathcal{P}^* : W \to \mathcal{S}_{\mathbb{C}}$ is not dominant. In particular, U' contains all components of V that are not surfaces. Then $\mathcal{P}^*(U')$ is contained in a 1-dimensional subset of $\mathcal{S}_{\mathbb{C}}$. Let $\{U_1, \ldots, U_k\}$ be the remaining components of V. Each U_i is a surface and $\mathcal{P}^* : U_i \to \mathcal{S}_{\mathbb{C}}$ is dominant. By Theorem A.1 there is at least one such surface U_i.

Consider now the set $\mathcal{S}'_{\mathbb{C}} = \mathcal{P}^*(V) - \mathcal{P}^*(U') \subseteq \mathcal{S}_{\mathbb{C}}$. This is a subset of $\mathcal{S}_{\mathbb{C}}$ that contains a non-empty open Zariski subset of $\mathcal{S}_{\mathbb{C}}$ (Shafarevich, Chapter 1, §5, Theorem 6). It follows that the set of real points of $\mathcal{S}'_{\mathbb{C}}$ is Zariski-dense in $\mathcal{S}_{\mathbb{C}}$.

Let $p = (p_1, p_2, p_3)$ be a real point of $\mathcal{S}'_{\mathbb{C}}$. Since $p \in \mathcal{P}^*(V)$, then $p = \mathcal{P}(a, b)$, for some $a = a_0 + \mathtt{i}a_1$, $b = b_0 + \mathtt{i}b_1$, $a_0, a_1, b_0, b_1 \in \mathbb{R}$. Now

$$\frac{A_0(a_0, a_1) + \mathtt{i}A_1(a_0, a_1)}{A(a_0, a_1)} \cdot \frac{C_0(b_0, b_1) + \mathtt{i}C_1(b_0, b_1)}{C(b_0, b_1)} = \phi_1(a)\psi_1(a) = p_1 \in \mathbb{R},$$

so

$$A(a_0, a_1) \neq 0, C(b_0, b_1) \neq 0$$

and

$$A_0(a_0, a_1)C_1(b_0, b_1) + A_1(a_0, a_1)C_0(b_0, b_1) = 0.$$

Analogously,

$$D(b_0, b_1) \neq 0, B(a_0, a_1) \neq 0$$

and

$$A_0(a_0, a_1)D_1(b_0, b_1) + A_1(a_0, a_1)D_0(b_0, b_1) = 0.$$

Thus, $(a_0, a_1, b_0, b_1) \in V \cap \mathbb{R}^4$. Moreover, $(a_0, a_1, b_0, b_1) \notin U'$, by our choice of p; and $(a_0, a_1, b_0, b_1) \in U_1 \cup \ldots \cup U_k$. Therefore, we have proved that any real point of $\mathcal{S}'_{\mathbb{C}}$ comes from at least one real point in $(a_0, a_1, b_0, b_1) \in U_1 \cup \ldots \cup U_k$. If no U_i were real, then the set of real points $U_{i,\mathbb{R}}$ of each U_i would be contained in a 1-dimensional subset R_i of U_i. Then, the set of real points of $\mathcal{S}'_{\mathbb{C}}$ would be contained in $\mathcal{P}^*(R_1) \cup \ldots \cup \mathcal{P}^*(R_k)$, which is included in a dimension 1 subset of $\mathcal{S}'_{\mathbb{C}}$, contradicting the fact that this set is Zariski dense in $\mathcal{S}_{\mathbb{C}}$.

So, there is at least one component U_i that is real. The fact that any real point of U_i maps to a real point of $\mathcal{S}_{\mathbb{C}}$ follows from the definition of V and $\mathcal{P}^*$. □

With this result and bearing in mind the special shape of non planar swung surfaces, we can analyze the structure of the surfaces U_i in this case: they turn to be either planes, cylinders or tori. First, we need the following technical lemma:

Lemma A.3. *Consider the polynomial $f = C_0D_1 - C_1D_0 \in \mathbb{R}[s_0, s_1]$. If f is identically zero, then $\mathcal{S}_{\mathbb{C}}$ is a real plane.*

Proof. Since $\psi(t) = (\psi_1, \psi_2)$ is a proper parametrization of a curve, both components cannot be constants. Assume, without loss of generality, that ψ_2 is not constant, so D_0 and D_1 are not zero. Now, suppose that $C_0D_1 - C_1D_0 = 0$. Then $C_0/D_0 = C_1/D_1 = k(s_0, s_1)$. But, then, $C_0 + \mathtt{i}C_1 = k \cdot (D_0 + \mathtt{i}D_1)$ and

$$\psi_1(s_0 + \mathtt{i}s_1) = \frac{C_0 + \mathtt{i}C_1}{C} = \frac{D_0 + \mathtt{i}D_1}{D} \cdot \frac{k \cdot D}{C} = \psi_2(s_0 + \mathtt{i}s_1) \cdot \frac{k \cdot D}{C}$$

So, $\frac{k \cdot D}{C} = \psi_1(s_0 + \mathtt{i}s_1)/\psi_2(s_0 + \mathtt{i}s_1)$ is both an $\mathtt{i}$-analytic rational function (i.e., the expansion in terms of real and imaginary parts of the complex function $\psi_1(s)/\psi_2(s)$, after decomposing the variable s in real and imaginary terms, cf. [14]) and a real rational function. By the well known Cauchy-Riemann conditions for analiticity (cf. [14]), kD/C must be, then, a real constant r. Thus, $\psi_1 = r\psi_2$ and $\mathcal{S}_{\mathbb{C}}$ is the real plane $\{ry - x = 0\}$ in $\mathbb{C}^3$. $\square$

Theorem A.4. *Let $\mathcal{S}_{\mathbb{C}}$ be a real swung surface, different from a plane, given by the parametrization $\mathcal{P}$. Let U be any irreducible surface in V such that $\mathcal{P}^* : U \to \mathcal{S}_{\mathbb{C}}$ is dominant. Then, there are irreducible curves $Z_1, Z_2 \subseteq \mathbb{C}^2$ such that $U = Z_1 \times Z_2$. Moreover, U is real if and only if both Z_1, Z_2 are real.*

Proof. Consider the two projections $\pi_1 : \mathbb{C}^4 \to \mathbb{C}^2$, $\pi_2 : \mathbb{C}^4 \to \mathbb{C}^2$, so that $\pi_1(t_0, t_1, s_0, s_1) = (t_0, t_1)$ and $\pi_2(t_0, t_1, s_0, s_1) = (s_0, s_1)$. Let Z_i be the Zariski closure of $\pi_i(U)$, $i = 1, 2$. Clearly, Z_1, Z_2 are irreducible varieties of $\mathbb{C}^2$. If $\dim(Z_i)$ were 0, then $\mathcal{P}^*(U)$ would not be dense in $\mathcal{S}_{\mathbb{C}}$, contradicting the hypothesis. If $U = Z_1 \times Z_2$, then it is clear that U is real if and only if Z_1 and Z_2 are real. Since always $U \subseteq Z_1 \times Z_2$ and both varieties are irreducible, to prove the theorem, it suffices to show that they have the same dimension, i.e. that $dim(Z_i) \leq 1$, $i = 1, 2$.

Since $\mathcal{S}_{\mathbb{C}}$ is not a plane, $\phi_2(t)$ is not a constant, so, by [14], $B_1(t_0, t_1)$ is not a constant and $Z_1 \subseteq \{B_1(t_0, t_1) = 0\}$ has dimension at most 1.

Now, since $\psi = (\psi_1, \psi_2)$ is a curve, one of the components is not a constant. Assume, without loss of generality, that ψ_1 is not constant. Then, neither C_0 nor C_1 are constants.

Now, we distinguish three cases. First, if $A_0 \equiv 0$ in U, then $A_1 \not\equiv 0$ in U, because $\mathcal{P}^*(U)$ is dense in $\mathcal{S}_{\mathbb{C}}$. Since $A_0C_1 + A_1C_0 \equiv 0$ in U, it must happen that $C_0 \equiv 0$ in U, yielding $Z_2 \subseteq \{C_0 = 0\}$ and, thus, $\dim(Z_2) \leq 1$.

Analogously, if $A_1 \equiv 0$ in U, then $A_0 \not\equiv 0$ in U and $C_1 \equiv 0$ in U. Hence $Z_2 \subseteq \{C_1 = 0\}$ and $\dim(Z_2) \leq 1$.

Finally, assume that neither A_0 nor A_1 are zero in U, then

$$A_0A_1(C_0D_1 - C_1D_0) = A_0D_1(A_1C_0 + A_0C_1) - A_0C_1(A_1D_0 + A_0D_1)$$

is zero in U. It follows that $C_0D_1 - C_1D_0 \equiv 0$ in U and $Z_2 \subseteq \{C_0D_1 - C_1D_0 = 0\}$. Since $\mathcal{S}_{\mathbb{C}}$ is not a plane, $C_0D_1 - C_1D_0$ is not identically zero (in $\mathbb{C}^2$) by Lemma A.3 and, thus, $\dim(Z_2) \leq 1$. $\square$

Finally, we show another technical result:

Lemma A.5. *Let $U \subseteq \mathbb{C}^{n+m}$ be a real irreducible variety such that $U = U_1 \times U_2$ is the Cartesian product of two irreducible varieties $U_1 \subseteq \mathbb{C}^n$, $U_2 \subseteq \mathbb{C}^m$. Let $F(\overline{x}, \overline{y}) \in \mathbb{R}(U)$ be a real rational function (i.e. $F(p) \in \mathbb{R}$, for any real point where F is defined) such that it has two different representations $F(\overline{x}, \overline{y}) = G(\overline{x}) = H(\overline{y})$. Then F is a real constant function equal to some $c \in \mathbb{R}$.*

Proof. Let $p_{x_0} \in U_1$ be a point such that $G(p_{x_0}) = c$ is defined. The fiber $\{p_{x_0}\} \times U_2 \subseteq U$ is isomorphic to U_2 and, for any $p = (p_{x_0}, p_y) \in \{p_{x_0}\} \times U_2$, we have that $F(p) = H(p_y) = G(p_{x_0}) = c$. Hence H is constant in U_2 and $c = H(\overline{y}) = F(\overline{x}, \overline{y})$ is constant in U. Since both F and U are real, $c \in \mathbb{R}$. $\square$