THE THIRD HOMOTOPY GROUP AS A π_1–MODULE

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Abstract. It is well–known how to compute the structure of the second homotopy group of a space, X, as a module over the fundamental group, $\pi_1 X$, using the homology of the universal cover and the Hurewicz isomorphism. We describe a new method to compute the third homotopy group, $\pi_3 X$, as a module over $\pi_1 X$. Moreover, we determine $\pi_3 X$ as an extension of $\pi_1 X$–modules derived from Whitehead's Certain Exact Sequence. Our method is based on the theory of quadratic modules. Explicit computations are carried out for pseudo–projective 3–spaces $X = S^1 \cup e^2 \cup e^3$ consisting of exactly one cell in each dimension ≤ 3.

1. Introduction

Given a connected 3–dimensional CW–complex, X, with universal cover, $\widehat{X}$, Whitehead's Certain Exact Sequence [W2] yields the short exact sequence

$$\Gamma\pi_2 X \rightarrowtail \pi_3 X \twoheadrightarrow \mathrm{H}_3\widehat{X} \tag{1.1}$$

of π_1–modules, where $\pi_1 = \pi_1(X)$. As a group, the homology $\mathrm{H}_3\widehat{X}$ is a subgroup of the free abelian group of cellular 3–chains of $\widehat{X}$, and thus itself free abelian. Hence the sequence splits as a sequence of abelian groups. This raises the question whether (1.1) splits as a sequence of π_1–modules – there are no examples known in the literature.

It is well–known how to compute $\pi_2(X) \cong \mathrm{H}_2\widehat{X}$ as a π_1–module, using the Hurewicz isomorphism, and how to compute $\mathrm{H}_3\widehat{X}$ using the cellular chains of the universal cover. In this paper we compute $\pi_3(X)$ as π_1–module and (1.1) as an extension over π_1. We answer the question above by providing an infinite family of examples where (1.1) does not split over π_1, as well as an infinite family of examples where it does split over π_1. As a first surprising example we obtain

Theorem 1.1. *There is a connected 3–dimensional CW–complex X with fundamental group $\pi_1 = \pi_1 X = \mathbb{Z}/2\mathbb{Z}$, such that π_1 acts trivially on both $\Gamma\pi_2 X$ and $H_3\widehat{X}$, but non–trivially on $\pi_3 X$. Hence*

$$\Gamma\pi_2 X \rightarrowtail \pi_3 X \twoheadrightarrow H_3\widehat{X}$$

does not split as a sequence of π_1–modules.

Below we describe examples for all finite cyclic fundamental groups, π_1, of even order, where (1.1) does not split over π_1. The examples we consider are CW–complexes,

$$X = S^1 \cup e^2 \cup e^3,$$

with precisely one cell, e^i, in every dimension $i = 0, 1, 2, 3$. In general, we obtain such a CW–complex, X, by first attaching the 2–cell e_2 to S^1 via $f \in \pi_1 S^1 = \mathbb{Z}$. We assume $f > 0$. This yields the 2–skeleton of X, $X^2 = P_f$, which is a pseudo–projective plane, see [O]. Then $\pi_1 = \pi_1 X = \pi_1 P_f = \mathbb{Z}/f\mathbb{Z}$ is a cyclic group of order f. We write $R = \mathbb{Z}[\pi_1]$ for the integral group ring of π_1 and K for the kernel of the augmentation $\varepsilon : R \to \mathbb{Z}$. Then the pseudo–projective 3–space, $X = P_{f,x}$, is determined by the pair, (f, x), of attaching maps, where $x \in \pi_2 P_f = K$ is the attaching map of the 3–cell e_3. In this case

$$\pi_2(X) = \mathrm{H}_2(\widehat{X}) = K/xR,$$

and

$$\mathrm{H}_3\widehat{X} = \ker(d_x : R \to R, x \mapsto xy),$$

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where xy is the product of $x, y \in R$.

A splitting function u for the exact sequence (1.1) is a function between sets, $u : \mathrm{H}_3\widehat{X} \to \pi_3 X$, such that $u(0) = 0$ and the composite of u and the projection $\pi_3 X \twoheadrightarrow \mathrm{H}_3\widehat{X}$ is the identity. Such a splitting function determines maps

$$A = A_u : \mathrm{H}_3\widehat{X} \times \mathrm{H}_3\widehat{X} \to \Gamma(\pi_2 X) \quad \text{and} \quad B = B_u : \mathrm{H}_3\widehat{X} \to \Gamma(\pi_2 X),$$

by the cross–effect formulæ

$$A(y, z) = u(y + z) - (u(y) + u(z)) \quad \text{and} \quad B(y) = (u(y))^1 - u(y^1).$$

Here B is determined by the action of the generator 1 in the cyclic group π_1, denoted by $y \mapsto y^1$.

Remark 1.2. The functions A and B determine $\pi_3 X$ as a π_1–module. In fact, the bijection $\mathrm{H}_3\widehat{X} \times \Gamma(\pi_2 X) = \pi_3(P_{f,x})$, which assigns to (y, v) the element $u(y) + v$ is an isomorphism of π_1–modules, where the left hand side is an abelian group by

$$(y, v) + (z, w) = (y + z, v + w + A(y, z))$$

and a π_1–module by

$$(y, v)^1 = (y^1, v^1 + B(y)).$$

The cross–effect of B satisfies

$$B(y + z) - (B(y) + B(z)) = (A(y, z))^1 - A(y^1, z^1),$$

such that B is a homomorphism of abelian groups if $A = 0$.

In this paper we describe a method to determine a splitting function $u = u_x$, which, a priori, is not a homomorphism of abelian groups. We investigate the corresponding functions A and B and compute them for a family of examples.

Theorem 1.3. *Let $X = P_{f,x}$ be a pseudo–projective 3–space with $x = \tilde{x}([\bar{1}] - [\bar{0}]) \in K, \tilde{x} \in \mathbb{Z}, \tilde{x} \neq 0$ and $f > 1$. Let $N = \sum_{i=0}^{f-1} [\bar{i}]$ be the norm element in R. Then*

$$H_3(\widehat{P}_{f,x}) = \{\tilde{y}N \mid \tilde{y} \in \mathbb{Z}\} \cong \mathbb{Z}$$

is a π_1–module with trivial action of π_1, and

$$\pi_2(P_{f,x}) = (\mathbb{Z}/\tilde{x}\mathbb{Z}) \otimes_{\mathbb{Z}} K,$$

with the action of π_1 induced by the π_1–module K. There is a splitting function $u = u_x$ such that, for $y = \tilde{y}N$ and $z \in H_3(\widehat{P}_{f,x})$, the functions A and B are given by

$$\begin{aligned} A(y, z) &= 0 \\ B(y) &= -\tilde{x}\tilde{y}\gamma q([\bar{1}] - [\bar{0}]), \end{aligned}$$

where $\gamma : \pi_2(P_{f,x}) \to \Gamma(\pi_2(P_{f,x}))$ is the universal quadratic map for the Whitehead functor Γ and $q : K \to \pi_2(P_{f,x}), k \mapsto 1 \otimes k$. As in 1.2, the pair A, B computes $\pi_3 X$ as a π_1–module.

As $\mathrm{H}_3(\widehat{X})$ is free abelian, the exact sequence (1.1) always allows a splitting function which is a homomorphism of abelian groups. This leads, for $X = P_{f,x}$, to the injective function

$$\tau : \mathrm{Ext}_{\pi_1}(\mathrm{H}_3(\widehat{X}), \Gamma(\pi_2 X)) \rightarrowtail \mathrm{coker}(\beta),$$

with

$$\beta : \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_3(\widehat{X}), \Gamma(\pi_2 X)) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_3(\widehat{X}), \Gamma(\pi_2 X)), t \mapsto \beta_t,$$

given by

$$\beta_t(\ell) = -t(\ell^1) + (t(\ell))^1.$$

The function τ maps the equivalence class of an extension to the element in cokerβ represented by $B = B_u$, where u is a $\mathbb{Z}$–homomorphic splitting function for the extension. Hence the equivalence class, $\{\pi_3 X\}$, of the extension $\pi_3 X$ in (1.1) is determined by the image $\tau\{\pi_3 X\} \in \mathrm{coker}(\beta)$. For the family of examples in 1.3 we show

Theorem 1.4. *Let $X = P_{f,x}$ be a pseudo–projective 3–space with $x = \tilde{x}([\overline{1}]-[\overline{0}]), \tilde{x} \in \mathbb{Z}, \tilde{x} \neq 0$ and $f > 1$. Then $\beta : \Gamma((\mathbb{Z}/\tilde{x}\mathbb{Z}) \otimes_{\mathbb{Z}} K) \to \Gamma((\mathbb{Z}/\tilde{x}\mathbb{Z}) \otimes_{\mathbb{Z}} K)$ maps ℓ to $-\ell + \ell^1$ and $\tau\{\pi_3 X\} \in coker(\beta)$ is represented by $\tilde{x}\gamma q([\overline{1}] - [\overline{0}]) \in \Gamma(\pi_2)$. Hence $\tau\{\pi_3 X\} = 0$ if $\tilde{x}$ is odd, so that, in this case, $\pi_3 X$ in (1.1) is a split extension over π_1. If both $\tilde{x}$ and f are even, then $\tau\{\pi_3 X\}$ is a non–trivial element of order 2, and the extension $\pi_3 X$ in (1.1) does not split over π_1. Moreover, $\tau\{\pi_3 X\}$ is represented by B in 1.3. If $\tilde{x}$ is even and f is odd, then $\tau\{\pi_3 X\}$ is trivial and the extension $\pi_3 X$ in (1.1) does split over π_1.*

This result is a corollary of 1.3, the computations are contained at the end of Section 8.

Given a pseudo–projective 3–space, $P_{f,x}$, and an element $z \in \pi_3(P_{f,x})$, we obtain a pseudo–projective 4–space, $X = P_{f,x,z} = S^1 \cup e^2 \cup e^3 \cup e^4$, where z is the attaching map of the 4–cell e^4. For $n \geq 2$, the attaching map z of an $(n+1)$–cell in a CW–complex, X, is *homologically non–trivial* if the image of z under the Hurewicz homomomorphism is non–trivial in $\mathrm{H}_n \widehat{X}^n$.

Theorem 1.5. *Let $X = S^1 \cup e^2 \cup e^3 \cup e^4$ be a pseudo–projective 4–space with $\pi_1 X = \mathbb{Z}/2\mathbb{Z}$ and homologically non–trivial attaching maps of cells in dimension 3 and 4. Then the action of $\pi_1 X$ on $\pi_3 X$ is trivial.*

Theorem 1.5 is a corollary to Theorem 9.1.

2. Crossed Modules

We recall the notions of pre-crossed module, Peiffer commutator, crossed module and nil(2)–module, which are ingredients of algebraic models of 2– and 3–dimensional CW–complexes used in the proofs of our results, see [B] and [BHS]. In particular, Theorem 2.2 provides an exact sequence in the algebraic context of a nil(2)–module equivalent to Whitehead's Certain Exact Sequence (1.1).

A *pre–crossed module* is a homomorphism of groups, $\partial : M \to N$, together with an action of N on M, such that, for $x \in M$ and $\alpha \in N$,

$$\partial(x^\alpha) = -\alpha + \partial x + \alpha.$$

Here the action is given by $(\alpha, x) \mapsto x^\alpha$ and we use additive notation for group operations even where the group fails to be abelian. The *Peiffer commutator* of $x, y \in M$ in such a pre–crossed module is given by

$$\langle x, y \rangle = -x - y + x + y^{\partial x}.$$

The subgroup of M generated by all iterated Peiffer commutators $\langle x_1, \ldots, x_n \rangle$ of length n is denoted by $P_n(\partial)$ and a *nil(n)–module* is a pre–crossed module $\partial : M \to N$ with $P_{n+1}(\partial) = 0$. A *crossed module* is a nil(1)–module, that is, a pre–crossed module in which all Peiffer commutators vanish. We also consider nil(2)–modules, that is, pre–crossed modules for which $P_3(\partial) = 0$.

A morphism or map $(m, n) : \partial \to \partial'$ in the category of pre–crossed modules is given by a commutative diagram

$$\begin{tikzcd} M \arrow[r, "m"] \arrow[d, "\partial"'] & M' \arrow[d, "\partial'"] \\ N \arrow[r, "n"'] & N' \end{tikzcd}$$

in the category of groups, where m is n–equivariant, that is, $m(x^\alpha) = m(x)^{n(\alpha)}$, for $x \in M$ and $\alpha \in N$. The categories of crossed modules and nil(2)–modules are full subcategories of the category of pre–crossed modules.

Note that $P_{n+1}(\partial) \subseteq \ker \partial$ for any pre–crossed module, $\partial : M \to N$. Thus we obtain the *associated nil(n)–module* $r_n(\partial) : M/P_{n+1}(\partial) \to N$, where the action on the quotient is determined by demanding that the quotient map $q : M \to M/P_{n+1}(\partial)$ be equivariant. For $n = 1$ we write $\partial^{cr} = r_1(\partial) : M^{cr} = M/P_2(\partial) \to N$ for the *crossed module associated* to ∂.

Given a set, Z, let $\langle Z \rangle$ denote the free group generated by Z. Now take a group, N, and a group homomorphism, $f : F = \langle Z \rangle \to N$. Then the *free N–group* generated by Z is the free group, $\langle Z \times N \rangle$, generated by elements denoted by $x^\alpha = ((x, \alpha))$ with $x \in Z$ and $\alpha \in N$. These are elements in the product $Z \times N$ of sets. The action is determined by

$$((x, \alpha))^\beta = ((x, \alpha + \beta)). \tag{2.1}$$

Define the group homomorphism $\partial_f : \langle Z \times N \rangle \to N$ by $((x, \alpha)) \mapsto -\alpha + f(x) + \alpha$, for generators $((x, \alpha)) \in Z \times N$, to obtain the pre–crossed module ∂_f with associated nil(n)–module $r_n(\partial_f) : \langle Z \times N \rangle / P_{n+1}(\partial_f) \to N$. Note that $r_n(\partial_f)\iota = f$, where $\iota = p\iota_F$ is the composition of the inclusion $\iota_F : F = \langle Z \rangle \to \langle Z \times N \rangle$ and the projection $p : \langle Z \times N \rangle \to M = \langle Z \times N \rangle / P_{n+1}(\partial_f)$ onto the quotient.

Remark 2.1. The nil(n)–module, $r_n(\partial_f) : M = \langle Z \times N \rangle / P_{n+1}(\partial_f) \to N$, satisfies the following universal property: For every nil(n)–module, $\partial' : M' \to N'$, and every pair of group homomorphisms, $m_F : F = \langle Z \rangle \to M'$, and $n : N \to N'$ with $\partial' m_F = nf$, there is a unique group homomorphism, $m : M \to M'$, such that $m\iota = m_F$, and $(n, m) : r_n(\partial_f) \to \partial'$ is a map of nil(n)–modules.

$$\begin{tikzcd}
M \arrow[rr, dashed, "m"] \arrow[dd, "r_n(\partial_f)"'] & & M' \arrow[dd, "\partial'"] \\
& F \arrow[ul, "\iota"'] \arrow[ur, "m_F"'] \arrow[dl, "f"] & \\
N \arrow[rr, "n"'] & & N'
\end{tikzcd}$$

Thus $r_n(\partial_f)$ is called the *free nil(n)–module with basis* f. A free nil(n)–module is *totally free* if N is a free group.

Given a path connected space Y and a space X obtained from Y by attaching 2–cells, let Z_2 be the set of 2–cells in $X - Y$, and let $f : Z_2 \to \pi_1(Y)$ be the attaching map. J.H.C. Whitehead [W1] showed that

$$\partial : \pi_2(X, Y) \to \pi_1(Y) \tag{2.2}$$

is a free crossed module with basis f. Then $\ker \partial = \pi_2(X), \operatorname{coker} \partial = \pi_1(\mathrm{X})$ and ∂ is totally free if Y is a one–point union of 1–spheres. Whitehead also proved that the abelianisation of the group $\pi_2(X, Y)$ is the free R–module $\langle Z_2 \rangle_R$ generated by the set Z_2, where $R = \mathbb{Z}[\pi_1(X)]$ is the group ring [W1].

Now take a totally free nil(2)–module $\partial : M \to N$ with associated crossed module $\partial^{cr} : M^{cr} \to N$. Let

$$M \overset{q}{\twoheadrightarrow} M^{cr} \overset{h_2}{\twoheadrightarrow} C = (M^{cr})^{ab}$$

be the composition of projections. Put $K = h_2(\ker(\partial^{cr}))$. Further, let Γ be *Whitehead's quadratic functor* and $\tau : \Gamma(K) \rightarrowtail K \otimes K \subset C \otimes C$ the composition of the injective homomorphism induced by the quadratic map $K \to K \otimes K, k \mapsto k \otimes k$ and the inclusion. The *Peiffer commutator map*, $w : C \otimes C \to M$, is given by $w(\{x\} \otimes \{y\}) = \langle x, y \rangle$, for $x, y \in M$ with $\{x\} = h_2(q(x)), \{y\} = h_2(q(y))$. Lemma (IV 1.6) and Theorem (IV 1.8) in [B] imply

Theorem 2.2. *Let $\partial : M \to N$ be a totally free nil(2)–module. Then the sequence*

$$\Gamma(K) \overset{\tau}{\rightarrowtail} C \otimes C \xrightarrow{w} M \overset{q}{\twoheadrightarrow} M^{cr}$$

is exact and the image of w is central in M.

3. Pseudo–Projective Spaces in Dimensions 2 and 3

Real projective n–space $\mathbb{R}\mathrm{P}^n$ has a cell structure with precisely one cell in each dimension $\leq n$. More generally, a CW–complex,

$$X = S^1 \cup e^2 \cup \ldots \cup e^n,$$

with precisely one cell in each dimension $\leq n$, is called a *pseudo–projective n–space.* For $n = 2$ we obtain *pseudo–projective planes*, see [O]. In this section we fix notation and consider pseudo–projective spaces in dimensions 2 and 3. In particular, we determine the totally free crossed module associated with a pseudo–projective plane and begin to investigate the totally free nil(2)–module associated with a pseudo–projective 3–space.

The fundamental group of a pseudo–projective plane $P_f = S^1 \cup e^2$, with attaching map $f \in \pi_1(S^1) = \mathbb{Z}$, is the cyclic group $\pi_1 = \pi_1(P_f) = \mathbb{Z}/f\mathbb{Z}$. We obtain $\pi_1 = \mathbb{Z}$ for $f = 0$, $\pi_1 = \{0\}$ for $f = 1$, and the bijection of sets

$$\{0, 1, 2, \ldots, f-1\} \to \pi_1 = \mathbb{Z}/f\mathbb{Z}, \quad k \mapsto \overline{k} = k + f\mathbb{Z},$$

for $1 < f$. Addition in π_1 is given by

$$\overline{k} + \overline{\ell} = \begin{cases} \overline{k+\ell} & \text{for} \quad k + \ell < f; \\ \overline{k+\ell-f} & \text{for} \quad k + \ell \geq f. \end{cases}$$

Denoting the integral group ring of the cyclic group π_1 by $R = \mathbb{Z}[\pi_1]$, an element $x \in R$ is a linear combination

$$x = \sum_{\alpha \in \pi_1} x_\alpha[\alpha] = \sum_{k=0}^{f-1} x_{\overline{k}}[\overline{k}],$$

with $x_\alpha, x_{\overline{k}} \in \mathbb{Z}$. Note that $1_R = [\overline{0}]$ is the neutral element with respect to multiplication in R and, for $x = \sum_{\alpha\in\pi_1} x_\alpha[\alpha], y = \sum_{\beta\in\pi_1} y_\beta[\beta]$,

$$xy = \sum_{\alpha,\beta\in\pi_1} x_\alpha y_\beta\,[\alpha + \beta] = \sum_{\ell=0}^{f-1} \Big(\sum_{k=0}^{\ell} x_{\overline{k}}\, y_{\overline{\ell-k}} + \sum_{k=\ell+1}^{f-1} x_{\overline{k}}\, y_{\overline{f+\ell-k}} \Big) [\overline{\ell}].$$

The *augmentation* $\varepsilon = \varepsilon_R : R \to \mathbb{Z}$ maps $\sum_{\alpha\in\pi_1} x_\alpha[\alpha]$ to $\sum_{\alpha\in\pi_1} x_\alpha$. The *augmentation ideal*, K, is the kernel of ε. For a right R–module, C, we write the action of $\alpha \in \pi_1$ on $x \in C$ exponentially as $x^\alpha = x[\alpha]$.

Given a pseudo–projective plane $P_f = S^1 \cup e^2$ with attaching map $f \in \pi_1(S^1) = \mathbb{Z}$, Whitehead's results on the free crossed module (2.2) imply that

$$\partial : \pi_2(P_f, S^1) \to \pi_1(S^1) \tag{3.1}$$

is a totally free crossed module with one generator, e_i, in dimensions $i = 1, 2$, and basis $\tilde{f} : Z_2 = \{e_2\} \to \pi_1(S^1)$ given by $\tilde{f}(e_2) = fe_1$. Note that ∂ has cokernel $\pi_1(P_f) = \mathbb{Z}/f\mathbb{Z} = \pi_1$ and kernel $\pi_2(P_f)$.

Lemma 3.1. *The diagram*

$$\begin{tikzcd} \pi_2(P_f, S^1) \arrow[r, "\partial"] \arrow[d, "\cong"'] & \pi_1(S^1) \arrow[d, equal] \\ R \arrow[r, "f\cdot\varepsilon_R"'] & \mathbb{Z} \end{tikzcd}$$

is an isomorphism of crossed modules, where $\varepsilon_R : R \to \mathbb{Z}$ is the augmentation.

Proof. By Whitehead's results [W1] on the free crossed module (2.2), it is enough to show that $\pi_2(P_f, S^1)$ is abelian. As ∂ is a totally free crossed module with basis $\tilde{f}$, $\pi_2(P_f, S^1)$ is generated by elements $e^n = ((e_2, n))$, see (2.1). Note that we obtain e^n by the action of $n \in \mathbb{Z}$ on $\iota(e_2) = ((e_2, 0)) = e^0$ and $\partial(e^n) = -n + \partial e + n = \partial e = f$ as $\pi_1(S^1) = \mathbb{Z}$ is abelian. We obtain

$$\begin{aligned} \langle e^n, e^m\rangle - \langle e^m, e^m\rangle &= -e^n - e^m + e^n + (e^m)^{\partial(e^n)} - (-e^m - e^m + e^m + (e^m)^{\partial(e^m)}) \\ &= -e^n - e^m + e^n + (e^m)^f - (e^m)^f + e^m \\ &= (e^n, e^m), \end{aligned}$$

where $(a, b) = -a - b + a + b$ denotes the commutator of a and b. Thus commutators of generators are sums of Peiffer commutators which are trivial in a crossed module. $\square$

With the notation of Theorem 2.2 and $M = \pi_2(P_f, S^1)$, Lemma 3.1 shows that $M = M^{cr} = (M^{cr})^{ab} = R$ and that $\pi_2(P_f) = \ker\partial = \ker\partial^{cr} = \ker(f\cdot\varepsilon) = K$ is the augmentation ideal of R, for $f \neq 0$. Thus the homotopy type of a pseudo–projective 3–space,

$$P_{f,x} = S^1 \cup e^2 \cup e^3, \tag{3.2}$$

is determined by the pair (f, x) of attaching maps, $f \in \pi_1(S^1) = \mathbb{Z}$ of the 2-cell e^2, and $x \in \pi_2(P_f) = K \subseteq R$ of the 3–cell e^3. We obtain the totally free nil(2)–module

$$M = \pi_2(P_{f,x}, S^1) \xrightarrow{\partial} N = \pi_1(S^1). \tag{3.3}$$

In the next section we use Theorem 2.2 to describe the group structure of $\pi_2(P_{f,x}, S^1)$, as well as the action of N on $\pi_2(P_{f,x}, S^1)$. The formulæ we derive are required to compute the homotopy group $\pi_3(P_{f,x})$ as a π_1–module.

4. Computations in nil(2)–Modules

In this Section we consider totally free nil(2)–modules, $\partial : M \to N$, generated by one element, e_i, in dimensions $i = 1, 2$, with basis $\tilde{f} : \{e_2\} \to N \cong \mathbb{Z}$. Then $\pi_1 = \mathrm{coker}\partial = \mathbb{Z}/f\mathbb{Z}$ and, with $R = \mathbb{Z}[\pi_1]$, we obtain $(M^{cr})^{ab} = C = R$. Thus Theorem 2.2 yields the short exact sequence

$$(R \otimes R)/\Gamma(K) \rightarrowtail^{w} M \overset{q}{\twoheadrightarrow} R \tag{4.1}$$

with the image of $(R \otimes R)/\Gamma(K)$ central in M. This allows us to compute the group structure of M, as well as the action of $N = \mathbb{Z}$ on M, by computing the cross–effects of a set–theoretic splitting s of (4.1) with respect to addition and the action of N, even though here M need not be commutative.

The element $x \otimes y \in R \otimes R$ represents an equivalence class in $R \otimes R/\Gamma(K)$, also denoted by $x \otimes y$, so that $w(x \otimes y) = \langle \hat{x}, \hat{y}\rangle$ is the Peiffer commutator for $x, y \in R$, with $x = q(\hat{x})$ and $y = q(\hat{y})$. As a group, M is generated by elements $e^n = ((e_2, n))$, in particular, $e = e^0 = ((e_2, 0))$, see (2.1). We write

$$ke^n = \begin{cases} e^n + \ldots + e^n \quad (k\,\text{summands}) & \text{for}\, k > 0, \\ 0 & \text{for}\, k = 0\,\text{and} \\ -e^n - \ldots - e^n \quad (-k\,\text{summands}) & \text{for}\, k < 0, \end{cases}$$

and define the set-theoretic splitting s of (4.1) by

$$s : R \longrightarrow M, \quad \sum_{k=0}^{f-1} x_{\overline{k}}[\overline{k}] \longmapsto x_{\overline{0}}e^0 + x_{\overline{1}}e^1 + \ldots + x_{\overline{f-1}}e^{f-1}.$$

Then every $m \in M$ can be expressed uniquely as a sum $m = s(x) + w(m^{\otimes})$ with $x \in R$ and $m^{\otimes} \in (R \otimes R)/\Gamma(K)$. The following formulæ for the cross–effects of s with respect to addition and the action provide a complete description of the nil(2)–module M in terms of R and $R \otimes R/\Gamma(K)$.

Given a function, $f : G \to H$, between groups, G and H, we write

$$f(x|y) = f(x + y) - (f(x) + f(y)), \quad \text{for}\ x, y \in G. \tag{4.2}$$

Lemma 4.1. *Take* $x = \sum_{m=0}^{f-1} x_{\overline{m}}, y = \sum_{n=0}^{f-1} y_{\overline{n}}[\overline{n}] \in R$. *Then*

$$s(x|y) = w(\nabla(x, y)),$$

where

$$\nabla(x, y) = \sum_{m=1}^{f-1}\sum_{n=0}^{m-1} x_{\overline{m}}\, y_{\overline{n}} w([\overline{n}] \otimes [\overline{m}] - [\overline{m}] \otimes [\overline{m}]).$$

Thus $\nabla(x, y)$ *is linear in* x *and* y, *yielding a homomorphism* $\nabla : R \otimes R \to R \otimes R$.

Proof. First note that, by definition, $\nabla(k[\overline{m}], \ell[\overline{n}]) = 0$ unless $m > n$. To deal with the latter case, recall that commutators are central in M and use induction, first on k, then on ℓ, to show that

$$(ke^m, \ell e^n) = k\ell(e^m, e^n),$$

for $k, \ell > 0$. To show equality for negative k or ℓ, replace e^m or e^n by $-e^m$ and $-e^n$, respectively. Furthermore, note that the equality

$$(e^n, e^m) = -e^n - e^m + e^n + e^m = \langle e^n, e^m\rangle - \langle e^m, e^m\rangle \tag{4.3}$$

for commutators of generators of totally free cyclic crossed modules derived in the proof of Lemma 3.1 holds in any totally free nil(n)–module generated by one element in each dimension. Taking $x = \sum_{m=0}^{f-1} x_{\overline{m}}\,[\overline{m}]$ and $y = \sum_{n=0}^{f-1} y_{\overline{n}}\,[\overline{n}]$, we obtain

$$\begin{aligned}
&s(x+y)\\
&\quad = (x_{\overline{0}} + y_{\overline{0}})\,e + \ldots + (x_{\overline{m}} + y_{\overline{m}})\,e^m + \ldots + (x_{\overline{f-1}} + y_{\overline{f-1}})\,e^{f-1}\\
&\quad = (x_{\overline{i}}\,e + \ldots + x_{\overline{f-1}}\,e^{f-1}) + (y_{\overline{0}}\,e + \ldots + y_{\overline{f-1}}\,e^{f-1}) + \sum_{m=1}^{f-1}\sum_{n=0}^{m-1} x_{\overline{m}}\,y_{\overline{n}}\,(e^n, e^m)\\
&\quad = s(x) + s(y) + \sum_{m=1}^{f-1}\sum_{n=0}^{m-1} x_{\overline{m}}\,y_{\overline{n}}\,\big(\langle e^n, e^m\rangle - \langle e^m, e^m\rangle\big)\\
&\quad = s(x) + s(y) + \sum_{m=1}^{f-1}\sum_{n=0}^{m-1} x_{\overline{m}}\,y_{\overline{n}} w([\overline{n}] \otimes [\overline{m}] - [\overline{m}] \otimes [\overline{m}]).
\end{aligned}$$

□

Corollary 4.2. *Take $x \in R$ and $r \in \mathbb{Z}$. Then*

$$s(rx) = rs(x) + \binom{r}{2} w(\nabla(x,x)), \quad \textit{where} \quad \binom{r}{2} = \frac{r(r-1)}{2}.$$

As $N = \mathbb{Z}$ is cyclic, the action of N on M is determined by the action of the generator, $1 \in \mathbb{Z}$. The formula for general $k \in Z$ provided in the next lemma is required for the definition of the set–theoretic splitting u_x of (1.1) and the explicit computation of A and B in Theorem 1.3.

Lemma 4.3. *Take $x = \sum_{n=0}^{f-1} x_{\overline{n}}[\overline{n}] \in R$ and $\overline{k} \in \pi_1$. Write $R = \mathbb{Z}[\overline{0}, \ldots, \overline{f-1}] = R_k \times \widehat{R}_k$, where $R_k = \mathbb{Z}[\overline{0}, \ldots, \overline{f-k-1}]$ and $\widehat{R}_k = \mathbb{Z}[\overline{f-k}, \ldots, \overline{f-1}]$. Then*

$$\big(s(x)\big)^k = s(x^{\overline{k}}) + w(\overline{\nabla}_k(a,b)),$$

where $x = (a,b)$ and

$$\overline{\nabla}_k : R_k \times \widehat{R}_k \to R \otimes R, \quad (a,b) \mapsto Q_k(a,b) + L_k(b)$$

with

$$\begin{aligned}
Q_k(a,b) &= \sum_{p=0}^{f-\ell-1}\sum_{q=0}^{\ell-1} x_{\overline{p}}\,x_{\overline{q+f-\ell}}\,([\overline{p+\ell}] \otimes [\overline{q}] - [\overline{q}] \otimes [\overline{q}])\\
L_k(b) &= \sum_{q=0}^{\ell-1} x_{\overline{q+f-\ell}}\,[\overline{q}] \otimes [\overline{q}].
\end{aligned}$$

Thus Q_k is linear in a and b and L_k is linear in b.

Proof. For $\overline{j} \in \pi_1$ and $p \in \mathbb{Z}$,

$$\begin{aligned}
e^{j+f} &= (e^j)^{\partial(e)}\\
&= e^j + (e^j, e) + \langle e, e^j\rangle\\
&= e^j - (\langle e, e^j\rangle - \langle e^j, e^j\rangle) + \langle e, e^j\rangle\\
&= e^j + \langle e^j, e^j\rangle.
\end{aligned}$$

Thus, for $\overline{n}, \overline{k} \in \pi_1$, with $\overline{n} + \overline{k} = \overline{j}$,

$$\begin{aligned}\big(s([\overline{n}])\big)^k &= \begin{cases} e^j, & \text{for } 0 \le n < f-k, \\ e^j + \langle e^j, e^j \rangle, & \text{for } f-k \le n < f \end{cases} \\ &= \begin{cases} s([\overline{n}]^{\overline{k}}), & \text{for } 0 \le n < f-k, \\ s([\overline{n}]^{\overline{k}}) + w([\overline{j}] \otimes [\overline{j}]), & \text{for } f-k \le n < f. \end{cases}\end{aligned}$$

Hence, for $x = \sum_{p=0}^{f-1} x_{\overline{p}}\, [\overline{p}]$,

$$\begin{aligned}&\big(s(x)\big)^k \\ &= x_{\overline{0}}\, s([\overline{0}])^k + x_{\overline{1}}\, s([\overline{1}])^k + \ldots + x_{\overline{f-1}}\, s([\overline{f-1}])^k \\ &= x_{\overline{0}}\, s([\overline{0}]^{\overline{k}}) + x_{\overline{1}}\, s([\overline{1}]^{\overline{k}}) + \ldots + x_{\overline{f-1}}\, s([\overline{f-1}]^{\overline{k}}) + \sum_{n=f-k}^{f-1} x_{\overline{n}}\, w([\overline{n+k-f}] \otimes [\overline{n+k-f}]) \\ &= x_{\overline{f-k}}\, s([\overline{f-k}]^{\overline{k}}) + \ldots + x_{\overline{f-1}}\, s([\overline{f-1}]^{\overline{k}}) + x_{\overline{0}}\, s([\overline{0}]^{\overline{k}}) + \ldots + x_{\overline{f-k-1}}\, s([\overline{f-k-1}]^{\overline{k}}) \\ &\quad + \sum_{p=0}^{f-k-1} \sum_{n=f-k}^{f-1} (x_{\overline{p}} s([\overline{p} + \overline{k}]), x_{\overline{n}} s([\overline{n} + \overline{k}])) + \sum_{q=0}^{k-1} x_{\overline{q+f-k}}\, w([\overline{q}] \otimes [\overline{q}]) \\ &= s(x^{\overline{k}}) + \sum_{p=0}^{f-k-1} \sum_{q=0}^{k-1} x_{\overline{p}}\, x_{\overline{q+f-k}}\, w([\overline{p+k}] \otimes [\overline{q}] - [\overline{q}] \otimes [\overline{q}]) + \sum_{q=0}^{k-1} x_{\overline{q+f-k}}\, w([\overline{q}] \otimes [\overline{q}]).\end{aligned}$$

□

Remark 4.4. We use the final results of this section to define and establish the properties of the set–theoretic splitting u_x of (1.1). The next result shows how the cross–effects interact with multiplication in R.

Lemma 4.5. *Take $x, y \in R$. Then*

$$\sum_{i=0}^{f-1} y_{\overline{i}} \big(s(x)\big)^i = s(xy) + w(\mu(x,y)),$$

where $\mu : R \times R \to R \otimes R$ is given by

$$\mu(x,y) = -\sum_{i<j} y_{\overline{i}}\, y_{\overline{j}}\, \nabla(x^{\overline{i}}, x^{\overline{j}}) + \sum_{i=0}^{f-1} \big(\overline{\nabla}_i(y_{\overline{i}} x) - \binom{y_{\overline{i}}}{2} \nabla(x,x)^{\overline{i}}\big).$$

Proof. By Lemmata 4.1 and 4.3 and Corollary 4.2, we obtain, for $x, y \in R$,

$$\begin{aligned}\sum_{i=0}^{f-1} y_{\overline{i}} \big(s(x)\big)^i &= \sum_{i=0}^{f-1} \big(y_{\overline{i}} s(x)\big)^i \\ &= \sum_{i=0}^{f-1} \big(s(y_{\overline{i}} x) - \binom{y_{\overline{i}}}{2} w(\nabla(x,x))\big)^i \\ &= \sum_{i=0}^{f-1} s(y_{\overline{i}} x^{\overline{i}}) + w(\overline{\nabla}_i(y_{\overline{i}} x)) - \big(\binom{y_{\overline{i}}}{2} w(\nabla(x,x))\big)^i \\ &= s(\sum_{i=0}^{f-1} y_{\overline{i}}\, x^{\overline{i}}) - \sum_{i<j} w(\nabla(y_{\overline{i}}\, x^{\overline{i}}, y_{\overline{j}}\, x^{\overline{j}})) + \sum_{i=0}^{f-1} w(\overline{\nabla}_i(y_{\overline{i}} x)) - \binom{y_{\overline{i}}}{2} w(\nabla(x,x)^{\overline{i}}).\end{aligned}$$

□

Finally, the definitions and a simple calculation yield

Lemma 4.6. *For $x, y, z \in R$ and with the notation in (4.2),*

$$\mu(x,y|z) = -\sum_{i<j} (y_{\bar{i}}\, z_{\bar{j}} + z_{\bar{i}}\, y_{\bar{j}})\, \nabla(x^{\bar{i}}, x^{\bar{j}}) + 2\sum_{i=1}^{f-1} y_{\bar{i}} z_{\bar{i}} Q_i(x) - \sum_{i=0}^{f-1} y_{\bar{i}} z_{\bar{i}} \nabla(x,x)^{\bar{i}}.$$

Hence, for fixed $x \in R, \mu(x,\ ) : R \times R \to R \otimes R, (y,z) \mapsto \mu(x,y|z)$ is bilinear.

5. Quadratic Modules

In dimension 3, quadratic modules assume the role played by crossed modules in dimension 2. We recall the notion of quadratic modules and totally free quadratic modules, see [B], which we require for the description of the third homotopy group $\pi_3(P_{f,x})$ of a 3–dimensional pseudo–projective space $P_{f,x}$, as in (3.2).

A *quadratic module* $(\omega, \delta, \partial)$ consists of a commutative diagram of group homomorphisms

$$\begin{tikzcd} & C \otimes C \arrow[dl, "\omega"'] \arrow[d, "w"] & \\ L \arrow[r, "\delta"] & M \arrow[r, "\partial"] & N, \end{tikzcd}$$

such that

- $\partial : M \to N$ is a nil(2)–module with quotient map $M \twoheadrightarrow C = (M^{cr})^{ab}, x \mapsto \{x\}$, and Peiffer commutator map w given by $w(\{x\} \otimes \{y\}) = \langle x, y \rangle$;
- the *boundary homomorphisms* ∂ and δ satisfy $\partial\delta = 0$, and the *quadratic map* ω is a lift of w, that is, for $x, y \in M$,
$$\delta\omega(\{x\} \otimes \{y\}) = \langle x, y \rangle;$$
- N acts on L, all homomorphisms are equivariant with respect to the action of N and, for $a \in L$ and $x \in M$,
$$a^{\partial(x)} = a + \omega(\{\delta a\} \otimes \{x\} + \{x\} \otimes \{\delta a\}); \tag{5.1}$$
- finally, for $a, b \in L$,
$$(a,b) = -a - b + a + b = \omega(\{\delta a\} \otimes \{\delta b\}). \tag{5.2}$$

A *map* $\varphi : (\omega, \delta, \partial) \to (\omega', \delta', \partial')$ of quadratic modules is given by a commutative diagram

$$\begin{tikzcd} C \otimes C \arrow[r, "\omega"] \arrow[d, "\varphi_* \otimes \varphi_*"'] & L \arrow[r, "\delta"] \arrow[d, "l"] & M \arrow[r, "\partial"] \arrow[d, "m"] & N \arrow[d, "n"] \\ C' \otimes C' \arrow[r, "\omega'"] & L' \arrow[r, "\delta'"] & M' \arrow[r, "\partial'"] & N' \end{tikzcd}$$

where l is n–equivariant, and (m, n) is a map between pre–crossed modules inducing $\varphi_* : C \to C'$.

Given a nil(2)–module $\partial : M \to N$, a free group F and a homomorphism $\tilde{f} : F \to M$ with $\partial\tilde{f} = 0$, a quadratic module $(\omega, \delta, \partial)$ is *free with basis* $\tilde{f}$, if there is a homomorphism $i : F \to L$ with $\delta i = \tilde{f}$, such that the following universal property is satisfied: For every quadratic module $(\omega', \delta', \partial')$ and map $(m, n) : \partial \to \partial'$ of nil(2)–modules and every homomorphism $l_F : F \to L'$ with $m\tilde{f} = \delta' l_F$, there is a unique map (l, m, n) of quadratic modules with $li = l_F$.

$$\begin{tikzcd} L \arrow[rr, "\delta"] \arrow[dd, dashed, "l"'] & & M \arrow[r, "\partial"] \arrow[dd, "m"] & N \arrow[dd, "n"] \\ & F \arrow[ul, "i"] \arrow[ur, "\tilde{f}"'] \arrow[dl, "l_F"] & & \\ L' \arrow[rr, "\delta'"'] & & M' \arrow[r, "\partial'"'] & N' \end{tikzcd}$$

For $F = \langle Z \rangle$, the homomorphism $\tilde{f}$ is determined by its restriction $\tilde{f}|_Z$ which is then called a *basis* for $(\omega, \delta, \partial)$. A quadratic module $(\omega, \delta, \partial)$ is *totally free* if it is free, if ∂ is a free nil(2)–module and if N is a free group.

6. The Homotopy Group π_3 of a Pseudo–Projective 3–Space and the Associated Splitting Function u_x

In this section we return to pseudo–projective 3–spaces

$$P_{f,x} = S^1 \cup e^2 \cup e^3,$$

determined by the pair (f, x) of attaching maps, $f \in \pi_1(S^1) = \mathbb{Z}$ and $x \in \pi_2(P_f) = K \subseteq R$, as in (3.2). Using results on totally free quadratic modules in [B], we investigate the structure of the third homotopy group $\pi_3(P_{f,x})$ as a π_1–module by defining a set–theoretic splitting u_x of J.H.C. Whitehead's Certain Exact Sequence of the universal cover, $\widehat{P}_{f,x}$,

$$\begin{tikzcd} \Gamma(\pi_2(P_{f,x})) \arrow[r, tail] & \pi_3(P_{f,x}) \arrow[r, two heads, shift left] & \mathrm{H}_3(\widehat{P}_{f,x}). \arrow[l, shift left, "u_x"] \end{tikzcd} \tag{6.1}$$

Recall that $\pi_1 = \pi_1(P_f) = \mathbb{Z}/f\mathbb{Z}$ with augmentation ideal $K = \ker f\varepsilon$, and let B be the image of $d_x : R \to R, y \mapsto xy$. Then

$$\pi_2(P_{f,x}) = \mathrm{H}_2(\widehat{P}_{f,x}) = K/B = (\ker f\varepsilon)/xR. \tag{6.2}$$

The functor σ in (IV 6.8) in [B] assigns a totally free quadratic module $(\omega, \delta, \partial)$ to the pseudo–projective 3–space $P_{f,x}$ and we obtain the commutative diagram

$$\begin{tikzcd}
\Gamma(\pi_2(P_{f,x})) \arrow[r, tail] \arrow[d, tail] & R \otimes R/\Delta_B \arrow[r, two heads, "q"] \arrow[d, "\omega"] & R \otimes R/\Gamma(K) \arrow[d, "w"] & \\
\pi_3(P_{f,x}) \arrow[r, tail] \arrow[d, two heads, shift right] & L \arrow[r, "\delta"] \arrow[d, two heads, shift right] & M \arrow[r, "\partial"] \arrow[d, two heads, shift right] & N \arrow[d, equal] \\
\mathrm{H}_3(\widehat{P}_{f,x}) \arrow[r, tail] \arrow[u, shift right, "u_x"'] & R \arrow[r, "d_x"] \arrow[u, shift right, "t_x"'] & R \arrow[r, "f \cdot \varepsilon"] \arrow[u, shift right, "s"'] & \mathbb{Z}
\end{tikzcd}$$

of straight arrows. Here the generators $e_3 \in L$, $e_2 \in M$ and $e_1 = 1 \in N = \mathbb{Z}$ correspond to the cells of $P_{f,x}$ and ∂ is the totally free nil(2)–module of Lemma 3.1. The right hand column is the short exact sequence (4.1) with the set theoretic splitting s defined in Section 4. The short exact sequence in the middle column is described in (IV 2.13) in [B], where the product $[\alpha, \beta]$ of $\alpha \in K$ and $\beta \in B$ is given by $[\alpha, \beta] = \alpha \otimes \beta + \beta \otimes \alpha \in R \otimes R$ and

$$\Delta_B = \Gamma(B) + [K, B].$$

By Corollary (IV 2.14) in [B], taking kernels yields Whitehead's short exact sequence (6.1) in the left hand column of the diagram, that is, $\ker q = \Gamma(\pi_2(\widehat{P}_{f,x})), \ker \delta = \pi_3(P_{f,x})$ and $\ker d_x = \mathrm{H}_3(\widehat{P}_{f,x})$. As $(\omega, \delta, \partial)$ is a quadratic module associated to $P_{f,x}$, we may assume that $\delta(e_3) = s(x)$.

In Section 4 we determined the structure of M as an N–module by computing the cross–effects of the set–theoretic splitting s with respect to addition and the action. Analogously to the definition of s, we now define a set-theoretic splitting of the short exact sequence in the second column of this diagram by

$$t_x : R \longrightarrow L, \quad \sum_{k=0}^{f-1} y_{\overline{k}} \, [\overline{k}] \longmapsto y_{\overline{0}} \, e_3^0 + \ldots + y_{\overline{f-1}} \, e_3^{f-1}.$$

The cross–effects of t_x with respect to addition and the action determine the N–module structure of L, but we want to determine the module structure of $\pi_3(P_{f,x})$. To obtain a set-theoretic splitting of the first column which will allow us to do so, we must adjust t_x, such that the image of $\mathrm{H}_3(\widehat{P}_{f,x})$ under the new splitting is contained in $\ker \delta = \pi_3(P_{f,x})$. Recall that δ is a homomorphism which is equivariant with respect to the action of N and $\delta(e_3) = s(x)$. Thus Lemma 4.5 yields, for $y \in \mathrm{H}_3(\widehat{P}_{f,x}) = \ker d_x$, that is, for $d_x(y) = xy = 0$,

$$\begin{aligned}\delta(t_x(y)) &= \delta\big(\sum_{i=0}^{f-1} y_i e_3^{\overline{i}}\big) = \sum_{i=0}^{f-1} y_i \delta(e_3)^{\overline{i}} = \sum_{i=0}^{f-1} y_i \big(s(x)\big)^{\overline{i}} \\ &= s(xy) + w(\mu(x,y)) \\ &= \delta\omega\mu(x,y).\end{aligned}$$

Hence $t_x(y) - \omega\mu(x,y) \in \ker\delta = \pi_3(P_{f,x})$, giving rise to the set theoretic splitting

$$u_x : \mathrm{H}_3(\widehat{P}_{f,x}) \longrightarrow \pi_3(P_{f,x}), \quad y \longmapsto t_x(y) - \omega\mu(x,y)$$

of the Hurewicz map $\pi_3 \twoheadrightarrow \mathrm{H}_3$. The cross–effects of u_x with respect to addition and the action determine (6.1) as a short exact sequence of π_1–modules. In Section 7 we determine the cross–effects of t_x and investigate the properties of the functions A and B describing the cross–effects of u_x.

7. Computations in Free Quadratic Modules

The first two results of this Section describe the cross–effects of t_x with respect to addition and the action, respectively. We then turn to the properties of the cross–effects of u_x.

Lemma 7.1. *Take $z, y \in R$. Then, with the notation in (4.2),*

$$t_x(z|y) = \omega(\Psi(z,y)),$$

where

$$\Psi(z,y) = \sum_{m=1}^{f-1}\sum_{n=0}^{m-1} z_{\overline{m}}\, y_{\overline{n}}\, x[\overline{n}] \otimes x[\overline{m}].$$

Thus $\Psi(z,y)$ is linear in z and y, yielding a homomorphism $\Psi : R\otimes R \to R\otimes R$.

Proof. As in the proof of Lemma 4.1, we obtain

$$t_x(z|y) = \sum_{m=1}^{f-1}\sum_{n=0}^{m-1} z_{\overline{m}}\, y_{\overline{n}} (e_3^{\overline{n}}, e_3^{\overline{m}}).$$

Note that $\{\delta(e_3^{\overline{n}})\} = \{\delta(t_x([\overline{n}]))\} = d_x([\overline{n}]) = x[\overline{n}]$. Thus (5.2) yields

$$t_x(z|y) = \sum_{m=1}^{f-1}\sum_{n=0}^{m-1} z_{\overline{m}}\, y_{\overline{n}}\, \omega(\{\delta(e_3^{\overline{n}})\} \otimes \{\delta(e_3^{\overline{m}})\}) = \sum_{m=1}^{f-1}\sum_{n=0}^{m-1} z_{\overline{m}}\, y_{\overline{n}}\, \omega(x[\overline{n}] \otimes x[\overline{m}]).$$

□

As $N = \mathbb{Z}$ is cyclic, the action of N on L is determined by the generator $1 \in \mathbb{Z}$.

Lemma 7.2. *Take $x \in R$. Then*

$$\big(t_x(y)\big)^1 = t_x(y^{\overline{1}}) + \omega(\overline{\Psi}_1(a,b)),$$

where

$$\overline{\Psi}_1 = \sum_{p=0}^{f-2} y_{\overline{p}}\, y_{\overline{f-1}}\, x[\overline{p+1}] \otimes x[\overline{0}] + y_{\overline{f-1}}(x\otimes[\overline{0}] + [\overline{0}]\otimes x).$$

Proof. With $\{\delta(e_3^{\overline{n}})\} = x[\overline{n}]$ from above and (5.1), we obtain

$$\begin{aligned} e_3^{1+f} &= (e_3^1)^f = (e_3^1)^{\partial(e)} = e^1 + \omega(\{\delta(e_3^1)\}\otimes\{e\} + \{e\}\otimes\{\delta(e_3^1)\}) \\ &= t_x([\overline{n}]^{\overline{1}}) + \omega(x[\overline{1}]\otimes[\overline{0}] + [\overline{0}]\otimes x[\overline{1}]).\end{aligned}$$

Thus, for $\overline{n} \in \pi$,

$$\big(t_x([\overline{n}])\big)^1 = \begin{cases} \omega(t_x([\overline{n}]^{\overline{1}}) & \text{for} \quad 0 \le n < f-1, \\ \omega(t_x([\overline{n}]^{\overline{1}}) + x[\overline{1}]\otimes[\overline{0}] + [\overline{0}]\otimes x[\overline{1}]) & \text{for} \quad n = f-\ell. \end{cases}$$

With (5.2), we obtain, for $y = \sum_{n=0}^{f-1} y_{\overline{n}}[\overline{n}]$,

$$\begin{aligned}
\big(t_x(y)\big)^{1} &= y_{\overline{0}}\, e_3^1 + y_{\overline{1}}\, e_3^2 \ldots + y_{\overline{f-2}}\, e_3^{f-1} + y_{\overline{f-1}}\, e_3^f \\
&= y_{\overline{0}}\, t_x([\overline{0}]^{\overline{1}}) + \ldots + y_{\overline{f-2}}\, t_x([\overline{f-1}]^{\overline{1}}) + y_{\overline{f-1}}\, t_x([\overline{f-1}]^{\overline{1}}) + y_{\overline{f-1}}\, \omega(x \otimes [\overline{0}] + [\overline{0}] \otimes x) \\
&= t_x(y^{\overline{1}}) + \sum_{p=0}^{f-2} y_{\overline{p}}\, y_{\overline{f-1}}\, (e_3^{p+1}, e_3) + y_{\overline{f-1}} \omega(x \otimes [\overline{0}] + [\overline{0}] \otimes x) \\
&= t_x(y^{\overline{1}}) + \sum_{p=0}^{f-2} y_{\overline{p}}\, y_{\overline{f-1}}\, x[\overline{p+1}] \otimes x[\overline{0}] + y_{\overline{f-1}}\, (x \otimes [\overline{0}] + [\overline{0}] \otimes x)
\end{aligned}$$

□

The next two results concern the properties of the maps A and B which describe the cross–effects of u_x with respect to addition and the action, respectively.

Lemma 7.3. *For $x \in K$ the map*

$$A : H_3\widehat{P}_{f,x} \times H_3\widehat{P}_{f,x} \to \Gamma(\pi_2 P_{f,x}), (y,z) \mapsto u_x(y|z)$$

is bilinear.

Proof. Take $x \in K$ and $y, z \in \mathrm{H}_3\widehat{P}_{f,x}$. By definition

$$A(y,z) = u_x(y|z) = t_x(y|z) - \omega\mu(x,y|z) = \omega\big(\Psi(y,z) - \mu(x,y|z)\big).$$

Thus Lemmata 4.6 and 7.1 imply that A is bilinear. □

Lemma 7.4. *For $x \in K$ define*

$$B : H_3\widehat{P}_{f,x} \to \Gamma(\pi_2 P_{f,x}), y \mapsto (u_x(y))^1 - u_x(y^1)$$

Then

$$H_3\widehat{P}_{f,x} \times H_3\widehat{P}_{f,x} \to \Gamma(\pi_2 P_{f,x}), (y,z) \mapsto B(y|z)$$

is bilinear.

Proof. Take $x \in K$ and $y, z \in \mathrm{H}_3\widehat{P}_{f,x}$. Then

$$\begin{aligned}
(A(y,z))^1 &= (u_x(y+z) - (u_x(y) + u_x(z))^1 \\
&= (u_x(y+z))^1 - (u_x(y))^1 - (u_x(z))^1 \\
&= B(y+z) + u_x((y+z)^1) - (B(y) + u_x(y^1) + B(z) + u_z(z^1)). \\
&= B(y|z) + A(y^1, z^1)
\end{aligned}$$

Thus

$$B(y|z) = (A(y,z))^1 - A(y^1, z^1) \tag{7.1}$$

and bilinearity follows from that of A and the properties of an action. □

8. Examples of Pseudo–Projective 3–Spaces

In this Section we provide explicit computations for examples of pseudo–projective 3–spaces, including proofs for Theorem 1.1, Theorem 1.3 and Theorem 1.4.

Note that, as abelian group, the augmentation ideal K of a pseudo–projective 3–space $P_{f,x}$, as in (3.2), is freely generated by $\{[\overline{1}]-[\overline{0}], \ldots, [\overline{f-1}]-[\overline{0}]\}$. We consider pseudo–projective 3–spaces, $P_{f,x}$, with $x = \tilde{x}([\overline{1}] - [\overline{0}])$ and $\tilde{x} \in \mathbb{Z}$. We compute $\pi_2(P_{f,x}), \mathrm{H}_3(\widehat{P}_{f,x})$, as well as the cross–effects of u_x for this special case. For $f = 2$, the general case coincides with the special case and provides an example where π_1 acts trivially on $\Gamma\pi_2(P_{2,\tilde{x}})$ and on $\mathrm{H}_3(\widehat{P}_{2,\tilde{x}})$, but non–trivially on $\pi_3(P_{2,\tilde{x}})$.

Lemma 8.1. *For $x = \tilde{x}([\overline{1}] - [\overline{0}])$ with $\tilde{x} \in \mathbb{Z}$,*

$$H_3(\widehat{P}_{f,x}) = \{\tilde{y}N \mid \tilde{y} \in \mathbb{Z}\} \cong \mathbb{Z},$$

is generated by the norm element $N = \sum_{k=0}^{f-1}[\overline{k}]$. Hence π_1 acts trivially on $H_3(\widehat{P}_{f,x})$. Furthermore,

$$\pi_2(P_{f,x}) = (\mathbb{Z}/\tilde{x}\mathbb{Z}) \otimes_{\mathbb{Z}} K.$$

Hence $\tilde{x}^2\ell = 0$ for every $\ell \in \Gamma(\pi_2(P_{f,x}))$.

Proof. Take $x = \tilde{x}([\overline{1}] - [\overline{0}])$ with $\tilde{x} \in \mathbb{Z}$ and $y = \sum_{k=0}^{f-1} y_{\overline{k}}[\overline{k}] \in \ker d_x$. Then

$$\begin{aligned} d_x(y) = xy = 0 \quad &\iff \quad \tilde{x}\sum_{k=0}^{f-1} y_{\overline{k}}([\overline{k}+\overline{1}] - [\overline{k}]) = 0 \\ &\iff \quad y_{\overline{f-1}} = y_{\overline{0}} = y_{\overline{1}} = y_{\overline{2}} = \ldots = y_{\overline{f-2}} = \tilde{y}, \end{aligned}$$

for some $\tilde{y} \in \mathbb{Z}$. Hence $y = \tilde{y}N$.

By (6.2), $\pi_2(P_{f,x}) = K/xR$. As abelian group, $K = \ker\varepsilon$ is freely generated by $\{[\overline{k}] - [\overline{0}]\}_{1\le k\le f-1}$ and hence also by $\{[\overline{k}] - [\overline{k-1}]\}_{1\le k\le f-1}$. For $y = \sum_{i=0}^{f-1} y_{\overline{i}}[\overline{i}] \in R$ we obtain

$$\begin{aligned} xy &= \tilde{x}\sum_{i=1}^{f-1} y_{\overline{i}}([\overline{i}] - [\overline{i-1}]) + \tilde{x}y_{\overline{f-1}}([\overline{0}] - [\overline{f-1}]) \\ &= \tilde{x}\sum_{i=1}^{f-1} y_{\overline{i}}([\overline{i}] - [\overline{i-1}]) - \tilde{x}y_{\overline{f-1}}\sum_{i=1}^{f-1}([\overline{i}] - [\overline{i-1}]) \\ &= \tilde{x}\sum_{i=1}^{f-1}(y_{\overline{i}} - y_{\overline{f-1}})([\overline{i}] - [\overline{i-1}]). \end{aligned}$$

As $\tilde{x}K \subseteq xR$, we obtain $xR = \tilde{x}K$ and hence

$$\pi_2(P_{f,x}) = K/xR = K/\tilde{x}K = (\mathbb{Z}/\tilde{x}\mathbb{Z}) \otimes_{\mathbb{Z}} K.$$

If $\tilde{x}$ is odd, then every element $\ell \in \Gamma(\pi_2(P_{f,x}))$ has order $\tilde{x}$. If $\tilde{x}$ is even, an element $\ell \in \Gamma(\pi_2(P_{f,x}))$ has order $2\tilde{x}$ or $\tilde{x}$. In either case, $\tilde{x}^2\ell = 0$ for every $\ell \in \Gamma(\pi_2(P_{f,x}))$. □

Lemma 8.2. *Take $x = \tilde{x}([\overline{1}] - [\overline{0}])$ and $y, z \in H_3(\widehat{P}_{f,x})$. Then*

$$A(y,z) = 0.$$

Proof. By definition,

$$A(y,z) = u_x(y|z) = t_x(y|z) - \omega\mu(x,y|z) = \omega(\Psi(y,z) - \mu(x,y|z)).$$

The definition of Ψ and Lemma 4.6 yield

$$\Psi(y,z) - \mu(x,y|z)) = \tilde{y}\tilde{z}\Big(\sum_{p=1}^{f-1}\sum_{q=0}^{p-1} x[\overline{q}] \otimes x[\overline{p}] + 2\sum_{q=1}^{f-1}\sum_{p=0}^{p-1}\nabla(x^{\overline{p}}, x^{\overline{q}}) - 2\sum_{p=1}^{f-1} Q_p(x) + \sum_{p=0}^{f-1}(\nabla(x,x))^{\overline{p}}\Big).$$

Recall that $\tilde{x}^2\ell = 0$ for every $\ell \in \Gamma(\pi_2(P_{f,x}))$ and note that, by the properties of Q and ∇, each summand in the above sum has a factor of $\tilde{x}^2$. □

Lemma 8.3. *Let $\gamma : \pi_2(P_{f,x}) \to \Gamma(\pi_2(P_{f,x}))$ be the universal quadratic map for the Whitehead functor Γ. Take $q : K \to \pi_2(P_{f,x}), k \mapsto 1 \otimes k, x = \tilde{x}([\overline{1}] - [\overline{0}])$ and $y = \tilde{y}N$. Then*

$$B(y) = -\tilde{x}\tilde{y}\gamma q([\overline{1}] - [\overline{0}]).$$

Proof. Note that $y^\beta = y$ for $\beta \in \pi_1$. As $\tilde{x}^2\ell = 0$ for every $\ell \in \Gamma(\pi_2(P_{f,x}))$, any summand with a factor $\tilde{x}^2$ is equal to 0. By Lemma 7.2,

$$\begin{aligned}\overline{\Psi}_1(y) &= \sum_{p=0}^{f-2} \tilde{y}^2\big(\tilde{x}([\overline{1}]-[\overline{0}])[\overline{p+1}]\otimes(\tilde{x}[\overline{1}]-[\overline{0}])\big) + \tilde{y}\big(\tilde{x}([\overline{1}]-[\overline{0}])\otimes[\overline{0}] + [\overline{0}]\otimes\tilde{x}([\overline{1}]-[\overline{0}])\big)\\ &= \tilde{x}\tilde{y}(([\overline{1}]-[\overline{0}])\otimes[\overline{0}] + [\overline{0}]\otimes([\overline{1}]-[\overline{0}])).\end{aligned}$$

Lemma 4.5 yields

$$\begin{aligned}\mu(x,y) &= -\sum_{q=0}^{f-1}\sum_{p=0}^{q-1}\tilde{x}^2\tilde{y}^2\nabla\big(([\overline{p+1}]-[\overline{p}]),([\overline{q+1}]-[\overline{q}])\big) + \sum_{p=0}^{f-1}\overline{\nabla}_p\big(\tilde{y}\tilde{x}([\overline{1}]-[\overline{0}])\big)\\ &\qquad -\tilde{x}^2\binom{\tilde{y}}{2}\big(\nabla(([\overline{1}]-[\overline{0}]),([\overline{1}]-[\overline{0}]))\big)^{\overline{p}}\\ &= \overline{\nabla}_{f-1}\big(\tilde{x}\tilde{y}([\overline{1}]-[\overline{0}])\big)\\ &= -\tilde{x}^2\tilde{y}^2\big([\overline{f-1}]\otimes[\overline{0}]-[\overline{0}]\otimes[\overline{0}]\big) + \tilde{x}\tilde{y}\,[\overline{0}]\otimes[\overline{0}]\\ &= \tilde{x}\tilde{y}\,[\overline{0}]\otimes[\overline{0}].\end{aligned}$$

Thus

$$B(y) = (u_x(y))^1 - u_x(y^{\overline{1}}) = \omega\big(\overline{\Psi}_1(y) - (\mu(x,y))^1 + \mu(x,y)\big) = -\tilde{x}\tilde{y}\gamma q([\overline{1}]-[\overline{0}]).$$

□

Together Lemmata 8.1, 8.2 and 8.3 provide a proof of Theorem 1.3.

For $f = 2$ the special case coincides with the general case and we obtain

Theorem 8.4. *Let $X = P_{2,x}$ be a pseudo–projective 3–space with $x = \tilde{x}([\overline{1}]-[\overline{0}])$, for $\tilde{x}\in\mathbb{Z}$ and $\tilde{x}\neq 0$. Then u_x is a homomorphism and the fundamental group $\pi_1 = \mathbb{Z}/2\mathbb{Z}$ acts trivially on $\Gamma(\pi_2 P_{2,x})$ and on $H_3\widehat{P}_{2,x}$. The action of π_1 on $\pi_3 P_{2,x}$ is non–trivial if and only if $\tilde{x}$ is even.*

Proof. For $f = 2$ the augmentation ideal K is generated by $k = [\overline{1}]-[\overline{0}]$. Since $k[\overline{1}] = -k$, the action of $\pi_1 = \mathbb{Z}/2\mathbb{Z}$ on K and hence on $\pi_2 P_{2,x} = K/xR = \mathbb{Z}/\tilde{x}\mathbb{Z}$ is multiplication by -1. As the Γ–functor maps multiplication by -1 to the identity morphism, the action on π_1 on $\Gamma(\pi_2 P_{2,x})$ is trivial. The group $\mathrm{H}_3\widehat{P}_{2,x}$ is generated by the norm element $N = [\overline{0}]+[\overline{1}]$. As $N[\overline{1}] = N$, π_1 acts trivially on $\mathrm{H}_3\widehat{P}_{2,x}$. As $\pi_2 = \mathbb{Z}/\tilde{x}\mathbb{Z}$ is cyclic, $\Gamma\pi_2 = \pi_2$ if $\tilde{x}$ is odd and $\Gamma\pi_2 = \mathbb{Z}/2\tilde{x}\mathbb{Z}$ if $\tilde{x}$ is even, that is,

$$\Gamma\pi_2 = \mathbb{Z}/\gcd(\tilde{x},2)\tilde{x}\,\mathbb{Z}. \tag{8.1}$$

By Lemma 8.3 and (8.1), the action of π_1 on $\pi_3 X$ is non–trivial if and only if $\tilde{x}$ is even. □

Theorem 1.1 is a corollary to Theorem 8.4.

Proof of 1.4. Note that $\mathbb{Z}/\tilde{x}\mathbb{Z}\otimes_{\mathbb{Z}} K$ is generated by $\{\alpha_k = q([\overline{k}]-[\overline{k-1}])\}_{0<k<f}$, where $q : K \to \mathbb{Z}/\tilde{x}\mathbb{Z}\otimes_{\mathbb{Z}} K, k \mapsto 1\otimes k$. Thus $\Gamma(\pi_2(P_{f,x})) = \Gamma(\mathbb{Z}/\tilde{x}\mathbb{Z}\otimes K) \subseteq (\mathbb{Z}/\tilde{x}\mathbb{Z}\otimes_{\mathbb{Z}} K)\otimes(\mathbb{Z}/\tilde{x}\mathbb{Z}\otimes_{\mathbb{Z}} K)$ is generated by $\{\gamma q(\alpha_k), [q(\alpha_j), q(\alpha_k)]\}_{0<j<k,0<k<f}$. With $\alpha_k^1 = \alpha_{k+1}$ for $1 < k < f-1$ and $\alpha_{f-1}^1 = [\overline{0}]-[\overline{f-1}] = -\sum_{i=1}^{f-1}\alpha_i$, we obtain, for $\ell = \sum_{k=1}^{f-1}\ell_k\gamma(\alpha_k) + \sum_{k=2}^{f-1}\sum_{j=1}^{k-1}\ell_{j,k}[\alpha_j,\alpha_k] \in$

$\Gamma(\pi_2(P_{f,\tilde{x}}))$,

$$\begin{aligned}
\ell^1 - \ell &= \sum_{k=1}^{f-1} \ell_k \gamma q(\alpha_k)^1 + \sum_{k=2}^{f-1}\sum_{j=1}^{k-1} \ell_{j,k}\ [q(\alpha_j), q(\alpha_k)]^1 - \sum_{k=1}^{f-1} \ell_k \gamma q(\alpha_k) - \sum_{k=2}^{f-1}\sum_{j=1}^{k-1} \ell_{j,k}[q(\alpha_j), q(\alpha_k)] \\
&= \sum_{k=1}^{f-2} \ell_k \gamma q(\alpha_{k+1}) + \ell_{f-1}\gamma q(-\sum_{i=1}^{f-1}\alpha_i) + \sum_{k=2}^{f-2}\sum_{j=1}^{k-1} \ell_{j,k}\ [q(\alpha_{j+1}), q(\alpha_{k+1})] \\
&\quad + \sum_{j=1}^{f-1} \ell_{j,f-1}[\gamma q(\alpha_{j+1}), \gamma q(-\sum_{i=1}^{f-1}\alpha_i)] - \sum_{k=1}^{f-1} \ell_k \gamma q(\alpha_k) - \sum_{k=2}^{f-1}\sum_{j=1}^{k-1} \ell_{j,k}[q(\alpha_j), q(\alpha_k)] \\
&= (\ell_{f-1} - \ell_1)\gamma q(\alpha_1) + \sum_{k=2}^{f-1}(\ell_{k-1} - \ell_k + \ell_{f-1} - 2\ell_{k-1,f-1})\gamma q(\alpha_k) \\
&\quad + \sum_{k=2}^{f-1}(\ell_{f-1} - \ell_{1,k} - \ell_{k-1,f-1})[q(\alpha_1, q(\alpha_k)] \\
&\quad + \sum_{k=3}^{f-1}\sum_{j=2}^{k-1}(\ell_{f-1} + \ell_{j-1,k-1} - \ell_{j,k} - \ell_{j-1,f-1} - \ell_{k-1,f-1})[q(\alpha_j), q(\alpha_k)].
\end{aligned}$$

Thus the sequence (1.1) splits if and only if there is at least one solution of the system of equations

$$\begin{array}{lll}
(A) & 0 = \ell_{f-1} - \ell_1 & \bmod 2\tilde{x} \\
(B_k) & 0 = \ell_{k-1} - \ell_k + \ell_{f-1} - 2\ell_{k-1,f-1} & \bmod 2\tilde{x}\ \text{ for } 2 \le k \le f-1 \\
(C_k) & 0 = \ell_{f-1} - \ell_{1,k} - \ell_{k-1,f-1} & \bmod \tilde{x}\ \text{ for } 2 \le k \le f-1 \\
(D_{j,k}) & 0 = \ell_{f-1} + \ell_{j-1,k-1} - \ell_{j,k} - \ell_{j-1,f-1} - \ell_{k-1,f-1} & \bmod \tilde{x}\ \text{ for } 2 \le j \le k, 2 < k < f-1.
\end{array}$$

For odd f, a solution of the system is given by $\ell_{j,k} = 0$ for $1 \le j \le k-1, 1 < k < f-1, \ell_k = 0$ for k odd, and $\ell_k = \tilde{x}$ for k even. Hence (1.1) splits if f is odd. It remains to show that there are no solutions for even $f > 2$.

For $2 \le j < \frac{1}{2}(f-2)$, subtract the equation $(D_{i,f-j+i})$ from the equation $(D_{i,f-j+i-1})$ for $2 \le i < j$. Add $(D_{j,f-1})$ and (C_{f-j}), then subtract (C_{f-j+1}). Adding the resulting equations yields

$$(E_j) \quad 0 = \ell_{f-1} - \ell_{j,f-1} - \ell_{f-j-1,f-1} \mod \tilde{x}.$$

Multiplying the equations (C_{f-1}) and $(E_j), 2 \le j \le \frac{1}{2}(f-2)$ by 2 and adding them we obtain

$$0 = (f-2)\ell_{f-1} - 2\sum_{j=1}^{f-2} \ell_{j,f-1} \mod 2\tilde{x}.$$

On the other hand, adding the equations (A) and $(B_k), 1 < k < f-1$, the resulting equation is

$$\tilde{x} = (f-2)\ell_{f-1} - 2\sum_{j=1}^{f-2} \ell_{j,f-1} \mod 2\tilde{x}.$$

Hence there are no solutions for f even. □

9. Pseudo–Projective Spaces in Dimension 4

In the final section we consider 4–dimensional pseudo–projective spaces and provide a proof of Theorem 1.5. We begin by constructing a 4–dimensional pseudo–projective space associated to given algebraic data. Namely, take $f \in \mathbb{Z}$ with $f \ge 0, x, y \in R = \mathbb{Z}[\mathbb{Z}/f\mathbb{Z}]$ with $xy = 0$ and $f\varepsilon(x) = 0$, where ε is the augmentation of the group ring, R, so that $xR \subseteq \ker \varepsilon$. Finally, take $\gamma \in \Gamma((\ker f\varepsilon)/xR)$. Given such data, (f, x, y, α), take a 3–dimensional pseudo–projective space $P_{f,x}$ as in (3.2). Then the set–theoretic splitting u_x of the short exact sequence

$$\Gamma(\pi_2(P_{f,x})) \rightarrowtail \pi_3(P_{f,x}) \twoheadrightarrow \mathrm{H}_3(\widehat{P}_{f,x})$$

implies that every element of $\pi_3(P_{f,x})$ may be expressed uniquely as a sum $u_x(v) + \beta$ with $v \in \mathrm{H}_3(\widehat{P}_{f,x})$, that is, $xv = 0$, and $\beta \in \Gamma(\pi_2(P_{f,x})) = \Gamma((\ker f\varepsilon)/xR)$, see (6.2). Using $u_x(y) + \alpha \in \pi_3(P_{f,x})$ to attach a 4–cell to $P_{f,x}$ we obtain the 4–dimensional pseudo–projective space,

$$P = P_{f,x,y,\alpha} = S_1 \cup e^2 \cup e^3 \cup e^4.$$

Note that the homotopy type of $P = P_{f,x,y,\alpha}$ is determined by (f, x, y, α) and that every 4–dimensional pseudo–projective space is of this form. The cellular chain complex, $C_*(\widehat{P})$, of the universal cover, $\widehat{P} = \widehat{P}_{f,x,y,\alpha}$, is the complex of free R–modules,

$$\langle e_4\rangle_R \xrightarrow{d_4} \langle e_3\rangle_R \xrightarrow{d_3} \langle e_2\rangle_R \xrightarrow{d_2} \langle e_1\rangle_R \xrightarrow{d_1} \langle e_0\rangle_R,$$

given by $d_1(e_1) = e_0([\overline{1}] - [\overline{0}]), d_2(e_2) = e_1N$, that is, multiplication by the *norm element*, $N = \sum_{i=0}^{f-1}[\overline{i}], d_3(e_3) = e_2x$, and $d_4(e_4) = e_3y$. Let $\bar{b} : R \to \pi_3P_{f,x}$ be the homomorphism of R–modules which maps the generator $[\overline{0}] \in R$ to $\bar{b}([\overline{0}]) = u_x(y) + \alpha$, so that composition with the projection onto $\mathrm{H}_3\widehat{P}_{f,x}$ yields the homomorphism of R–modules induced by the boundary operator d_4. Thus we obtain the commutative diagram

$$\begin{tikzcd}
\mathrm{H}_4\widehat{P} \arrow[r, "b"] \arrow[d] & \Gamma\pi_2P \arrow[d] \arrow[dr, "j"] & \\
R \arrow[r, "\bar{b}"] \arrow[dr, "\bar{d}_4"'] & \pi_3P_{f,x} \arrow[r, two heads] \arrow[d] & \pi_3P \arrow[d, "h"] \\
& \mathrm{H}_3\widehat{P}_{f,x} \arrow[r, two heads] & \mathrm{H}_3\widehat{P}
\end{tikzcd}$$

in the category of R–modules, where the middle column is the short exact sequence (6.1) and

$$\mathrm{H}_4\widehat{P} \xrightarrow{b} \Gamma\pi_2P \xrightarrow{j} \pi_3P \xrightarrow{h} \mathrm{H}_3\widehat{P} \tag{9.1}$$

is Whitehead's Certain Exact Sequence of the universal cover, $\widehat{P} = \widehat{P}_{f,x,y,\alpha}$.

Now we restrict attention to the case $f = 2$. Then $\pi_1 = \pi_1P = \mathbb{Z}/2\mathbb{Z}$ and the augmentation ideal, K is generated by $[\overline{1}] - [\overline{0}]$. Thus

$$x = \tilde{x}([\overline{1}] - [\overline{0}]) \quad \text{and} \quad y = \tilde{y}([\overline{1}] + [\overline{0}]), \quad \text{for some } \tilde{x}, \tilde{y} \in \mathbb{Z}.$$

We assume that x and y are non–trivial, that is, $\tilde{x}, \tilde{y} \neq 0$.

Theorem 9.1. *For $P = P_{2,x,y,\alpha}$, with x and y as above, $\pi_1P = \mathbb{Z}/2\mathbb{Z}$ acts on $\pi_2P = \mathbb{Z}/\tilde{x}Z$ via multiplication by -1, trivially on $H_3\widehat{P} = \mathbb{Z}/\tilde{y}\mathbb{Z}$ and via multiplication by -1 on $H_4\widehat{P} = \mathbb{Z} = \langle[\overline{1}] - [\overline{0}]\rangle$. The exact sequence (9.1) is given by*

$$H_4\widehat{P} = \mathbb{Z} \xrightarrow{b} \Gamma\pi_2P = \Gamma(\mathbb{Z}/\tilde{x}\mathbb{Z}) \xrightarrow{j} \pi_3P \xrightarrow{h} H_3\widehat{P} = \mathbb{Z}/\tilde{y}\mathbb{Z}. \tag{9.2}$$

Denoting the generator of $\Gamma\pi_2P$ by ξ, the boundary b is determined by

$$b([\overline{1}] - [\overline{0}]) = \tilde{x}\tilde{y}\xi,$$

and the action of π_1P on π_3P is trivial. As abelian group, π_3P is the extension of $H_3\widehat{P}$ by $\mathit{coker}\, b$ given by the image of $-\alpha \in \Gamma\pi_2$ under the homomorphism

$$\tau : \Gamma\pi_2 \twoheadrightarrow \mathit{coker}\, b \twoheadrightarrow \mathit{coker}\, b/\tilde{y}\,\mathit{coker}\, b = \mathit{Ext}(\mathbb{Z}/\tilde{y}\mathbb{Z}, \operatorname{coker} b).$$

Hence the extension π_3P over $\mathbb{Z}$ determines α modulo $\ker\tau$.

Theorem 1.5 is a corollary to Theorem 9.1.

Proof. As the augmentation ideal $K \cong \mathbb{Z}$ is generated by $k = [\overline{1}] - [\overline{0}]$, the action of $\pi_1 = \mathbb{Z}/2\mathbb{Z}$ on $K = \pi_2 P_2$ and hence on $\pi_2 P = K/xR = \mathbb{Z}/\tilde{x}\mathbb{Z}$ is multiplication by -1, since $k[\overline{1}] = -k$. But the Γ–functor maps mutliplication by -1 to the identity morphism, so that π_1 acts trivially on $\Gamma(\pi_2 P)$.

As $d_3(e_3) = e_2 x$, we obtain $\mathrm{H}_3\widehat{P}_{2,x} \cong \mathbb{Z}$, generated by the norm element $N = [\overline{1}] + [\overline{0}]$. Since $N[\overline{1}] = N$, the action of π_1 on $\mathrm{H}_3\widehat{P}_{2,x}$ is trivial.

As $d_4(e_4) = e_3 y$, we obtain $\mathrm{H}_3\widehat{P} \cong \mathbb{Z}/\tilde{y}\mathbb{Z}$ and $\mathrm{H}_4\widehat{P} \cong \mathbb{Z}$, generated by $k = [\overline{1}] - [\overline{0}]$. Hence the action of π_1 on $\mathrm{H}_4\widehat{P}$ is multiplication by -1.

Now let $\xi = ([\overline{1}] - [\overline{0}]) \otimes ([\overline{1}] - [\overline{0}])$ be the generator of $\Gamma(K)$. Note that $v[\overline{1}] = v$ and $\beta[\overline{1}] = \beta$, for $v \in \mathrm{H}_3\widehat{P}_{2,x}$ and $\beta \in \Gamma(\pi_2 P)$, since π_1 acts trivially on both $\mathrm{H}_3\widehat{P}_{2,x}$ and $\Gamma(\pi_2 P)$. Lemma 8.3 implies

$$(u(v) + \beta)[\overline{1}] = -\tilde{x}\tilde{y}\,\omega(\xi) + u(v[\overline{1}]) + \omega(\beta)[\overline{1}] = -\tilde{x}\tilde{y}\,\omega(\xi) + u(v) + \omega(\beta).$$

We obtain

$$\begin{aligned} \bar{b}(e_4([\overline{1}] - [\overline{0}])) &= (u(y) + \omega(\alpha))([\overline{1}] - [\overline{0}]) \\ &= -\tilde{x}\tilde{y}\,\omega(\xi) + u(y) + \omega(\alpha) - (u(y) + \omega(\alpha)) \\ &= -\tilde{x}\tilde{y}\,\omega(\xi). \end{aligned}$$

By definition of $\bar{b}$,

$$\pi_3 P = \pi_3 P_{2,x}/\mathrm{im}\,\bar{b}.$$

Hence π_1 acts trivially on $\pi_3(P)$.

Sequence (9.1) yields the short exact sequence

$$G = \operatorname{coker} b \rightarrowtail \pi_3 P \overset{h}{\twoheadrightarrow} \mathrm{H}_3\widehat{P} \cong \mathbb{Z}/\tilde{y}\mathbb{Z}, \tag{9.3}$$

which represents $\pi_3 P$ as an extension of $\mathbb{Z}/\tilde{y}\mathbb{Z}$ by $G = \operatorname{coker} b$. Thus the extension $\pi_3 P$ over $\mathbb{Z}$ determines γ modulo the kernel of the map

$$\tau : \Gamma\pi_2 \twoheadrightarrow \operatorname{coker} b \twoheadrightarrow \operatorname{coker} b/\tilde{y}\operatorname{coker} b = \operatorname{Ext}(\mathbb{Z}/\tilde{y}\mathbb{Z}, \operatorname{coker} b)\ .$$

□

References

[B] H.J. Baues, *Combinatorial Homotopy and 4–dimenisonal CW–complexes*, de Gruyter expositions in mathematics (1991).

[BHS] R. Brown, P. J. Higgins and R. Sivera, *Nonabelian Algebraic Topology*, EMS Publishing House (2011).

[O] P. Olum, *Self–equivalences and pseudo–projective planes*, Topology **4**, (1965), 109–127.

[W1] J.H.C. Whitehead, *Combinatorial homotopy II*, Bull. AMS 55 (1949), 213–245.

[W2] J.H.C. Whitehead, *A certain exact sequence*, Ann. of Math. 52 (1950), 51–110.

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