# On More Bent Functions From Dillon Exponents 

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#### Abstract

In this paper, we obtain a new class of $p$-ary binomial bent functions which are determined by Kloosterman sums. The bentness of another three classes of functions is characterized by some exponential sums and some results in [10] are generalized. Furthermore we obtain, in some special cases, some bent functions are determined by Kloosterman sums.


Key Words Binary bent function, p-ary bent function, Kloosterman sum.

## 1 Introduction

In 1976, Rothaus [13] introduced boolean bent functions which are maximally nonlinear boolean functions with even number of variables. That is, they achieve the maximal Hamming distance between boolean functions and affine functions. Boolean bent functions have attracted much attention due to their application in coding theory, cryptography and sequence design. Later, Kumar, Scholtz and Welch [8] generalized the notion of boolean bent functions to the case of functions over an arbitrary finite field $\mathbb{F}_{p^{n}}$, where $p$ is a prime integer and $n$ is a positive integer. Some results on constructions of bent functions on monomial, binomial and quadratic functions could be found in $1.7,9,10,12,14,15]$.

Throughout this paper, let $p$ be a prime integer and $m$ be a positive integer with $n=2 m$, $\mathbb{F}_{p^{n}}$ be the finite field with $p^{n}$ elements and $\mathbb{F}_{p^{n}}^{*}=\mathbb{F}_{p^{n}} \backslash\{0\}$. Let $\operatorname{Tr}_{1}^{n}(\cdot)$ be the trace function from $\mathbb{F}_{p^{n}}$ to $\mathbb{F}_{p}$, i.e. $\operatorname{Tr}_{1}^{n}(x)=\sum_{i=0}^{n-1} x^{p^{i}}$ for $x \in \mathbb{F}_{p^{n}}$. The bentness of boolean monomials with Dillon exponents was characterized by Dillon in [4] and Charpin et al. in [2]. The corrosponding $p$-ary case was investigated by Helleseth and Kholosha in [5]. Some multinomial bent functions with Dillon exponents were investigated in [7], 12], [14], [15]. Recently, Li et al. 10] investigated the bentness of several special classes of functions in the following form

$$
\begin{equation*}
f(x)=\sum_{i=1}^{p^{m}-1} \operatorname{Tr}_{1}^{n}\left(a_{i} x^{i\left(p^{m}-1\right)}\right)+\operatorname{Tr}_{1}^{o(d)}\left(b x^{\frac{p^{n}-1}{d}}\right) \tag{1.1}
\end{equation*}
$$

[^0]where $n=2 m, a_{i} \in \mathbb{F}_{p^{n}}, b \in \mathbb{F}_{p^{o(d)}}, d$ is a positive integer with $d \mid\left(p^{m}+1\right)$ and $o(d)$ is the smallest positive integer satisfying $o(d) \mid n$ and $d \mid\left(p^{o(d)}-1\right)$. The bentness of all these special classes of functions is determined by some exponential sums, most of which have close relations with Kloosterman sums.

The aim of this paper is that we further investigate four classes of bent functions in the form (1.1), which generalize some results in [10]. By applying the results on $S_{i}(a), i=0,1, d=2$ (see [7]), we establish a relationship between some partial exponential sums and Kloosterman sums. Based on this result, a new class of $p$-ary binomial bent functions are obtained (see Theorem 3.13). Moreover, we further investigate the bentness of another three classes of bent functions from Dillon exponents in the form (1.1). In particular, the bentness of some functions is determined by Kloosterman sums (see Theorem 3.13, Corollaries 3.10, 3.19, 3.20).

The remainder of this paper is organized as follows. Section 2 gives some preliminaries. In Section 3, the bentness of four classes of functions is characterized by some exponential sums. The concluding remarks are given in Section 4.

## 2 Preliminaries

In this section, we give some basic definitions and results.
Definition 2.1. Let $f: \mathbb{F}_{p^{n}} \rightarrow \mathbb{F}_{p}$ be a p-ary function. The Walsh transform of $f$ is defined by

$$
W_{f}(\lambda)=\sum_{x \in \mathbb{F}_{p^{n}}} \omega^{f(x)-\operatorname{Tr}_{1}^{n}(\lambda x)}, \lambda \in \mathbb{F}_{p^{n}}
$$

where $\omega=e^{\frac{2 \pi \sqrt{-1}}{p}}$ is a complex primitive $p$-th root of unity.
Definition 2.2. Let $f: \mathbb{F}_{p^{n}} \rightarrow \mathbb{F}_{p}$ be a p-ary function. Then $f(x)$ is called a bent function if $\left|W_{f}(\lambda)\right|^{2}=p^{n}$ for all $\lambda \in \mathbb{F}_{p^{n}}$. A p-ary bent function $f(x)$ is said to be regular if for all $\lambda \in \mathbb{F}_{p^{n}}$, $W_{f}(\lambda)=p^{\frac{n}{2}} \omega^{f^{*}(\lambda)}$ for some functions $f^{*}$ from $\mathbb{F}_{p^{n}}$ to $\mathbb{F}_{p}$. The function $f^{*}(x)$ is called the dual of $f(x)$.

Remark 2.3. In particular, for $p=2$, a boolean bent function is always regular.
Definition 2.4. The Dickson polynomial $D_{r}(x) \in \mathbb{F}_{2}[x]$ of degree $r$ is defined by

$$
D_{r}(x)=\sum_{i=0}^{\lfloor r / 2\rfloor} \frac{r}{r-i}\binom{r-i}{i} x^{r-2 i}, \quad r=2,3, \cdots
$$

where $\binom{k}{s}=\frac{\prod_{j=0}^{s-1}(k-j)}{\prod_{j=1}^{s} j}$ and $\left\lfloor\frac{r}{2}\right\rfloor= \begin{cases}r / 2, & \text { if } r \text { is even; } \\ (r-1) / 2, & \text { otherwise. }\end{cases}$
Definition 2.5. Let $\alpha \in \mathbb{F}_{p^{m}}$, the Kloosterman sum $K_{m}(\alpha)$ over $\mathbb{F}_{p^{m}}$ is defined as

$$
K_{m}(\alpha)=\sum_{x \in \mathbb{F}_{p^{m}}} \omega^{T r_{1}^{m}\left(\alpha x+x^{p^{m}-2}\right)}
$$

where $\omega=e^{\frac{2 \pi \sqrt{-1}}{p}}$ is a complex primitive $p$-th root of unity.

It is easy to see that $K_{m}(\alpha)$ is a real number, where $\alpha \in \mathbb{F}_{p^{m}}, m$ is a positive integer.
Let $d$ be a divisor of $p^{m}+1$ and $U=\left\{x \mid x^{p^{m}+1}=1, x \in \mathbb{F}_{p^{n}}\right\}$ be a cyclic subgroup of $\mathbb{F}_{p^{n}}^{*}$. It is easy to check that $U$ can be decomposed into $U=\bigcup_{k=0}^{d-1} V_{k}$, where $V_{0}=\left\{\xi^{d i} \left\lvert\, 0 \leq i<\frac{p^{m}+1}{d}\right.\right\}$, $V_{k}=\xi^{k} V_{0}$ for $1 \leq k \leq d-1$, and $\xi$ is a generator of the cyclic group $U$. For $i=0,1, \cdots, d-1$ and $a \in \mathbb{F}_{p^{n}}$, we define

$$
S_{i}(a)=\sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}\left(a \xi^{i} x\right)}
$$

It is well known that if $p=2$, then

$$
\sum_{x \in U} \omega^{\operatorname{Tr}_{1}^{n}(a x)}=\sum_{i=0}^{d-1} S_{i}(a)=1-K_{m}(a), \quad a \in \mathbb{F}_{2^{m}}
$$

If $p>2$, then

$$
\sum_{x \in U} \omega^{\operatorname{Tr}_{1}^{n}(a x)}=\sum_{i=0}^{d-1} S_{i}(a)=1-K_{m}\left(a^{p^{m}+1}\right), a \in \mathbb{F}_{p^{n}}
$$

which is given in [5]. In particular, for the case of $p=2$ and $d=3$, Mesnager [11] found a relationship between $S_{i}(a)$ and some well-known exponential sums, and then constructed a new class of binomial bent functions. Furthermore, for $p=2$ and $d=5, S_{i}(a)$ was determined by some well-known exponential sums in [14]. Using these results, they also characterized the bentness of a new class of binomial functions. For $p>2$, the only known results on $S_{i}(a)$ were given in [7], where $i=0,1$ and $d=2$, which were used to characterize a new class of binomial bent functions. Following this idea, the bentness of more functions can be characterized if $S_{i}(a)$ is obtained for some $p$ and $d$, where $0 \leq i \leq d-1$.

In particular, for $p=2$, Li et al. in [10] obtained a relation between $S_{0}(a)$ and some exponential sums as follows.

Lemma 2.6. 10] Let $p=2$, d be a divisor of $p^{m}+1$, $a=\bar{a} \xi^{k} \in \mathbb{F}_{2^{n}}, \bar{a} \in \mathbb{F}_{2^{m}}^{*}, 0 \leq k \leq 2^{m}$. If $k \equiv 0(\bmod d)$, then

$$
S_{0}(a)=\frac{1+2 E_{m, d}(\bar{a})-K_{m}(\bar{a})}{d}
$$

where $E_{m, d}(\bar{a})=\sum_{x \in \mathbb{F}_{2^{m}}}(-1)^{\operatorname{Tr}_{1}^{m}\left(\bar{a} D_{d}(x)\right)}$, $\xi$ is a generator of the cyclic group $U$.
For convenience, we give some notations. Let $p$ be an odd prime and $\alpha$ be a primitive element in $\mathbb{F}_{p^{n}}$. We define $\mathcal{C}_{t}=\left\{\alpha^{2 i+t} \mid i=0,1, \cdots, \frac{p^{n}-3}{2}\right\} \subseteq \mathbb{F}_{p^{n}}^{*}$ for $t=0$, 1 . For $a \in \mathbb{F}_{p^{n}}$ and for $b \in \mathcal{C}_{0}$, we define $R(a)$ and $Q(b)$ as follows:

$$
R(a)=\frac{1-K_{m}\left(a^{p^{m}+1}\right)}{2}, \quad Q(b)=2 \operatorname{Tr}_{1}^{m}\left(b^{\frac{p^{m}+1}{2}}\right)
$$

Lemma 2.7. [7] With the notations given above. Then for $d=2$, we have

$$
S_{0}(a)= \begin{cases}R(a)+I\left(\omega^{Q(a)}-\omega^{-Q(a)}\right), & a \in \mathcal{C}_{0}^{+} \\ R(a), & \text { otherwise }\end{cases}
$$

and

$$
S_{1}(a)= \begin{cases}R(a)-I\left(\omega^{Q(a)}-\omega^{-Q(a)}\right), & a \in \mathcal{C}_{0}^{+} \\ R(a), & \text { otherwise }\end{cases}
$$

where $I=\left\{\begin{array}{ll}\frac{(-1)^{\frac{3 m}{2}} p^{\frac{m}{2}}}{2}, & p \equiv 3(\bmod 4) ; \\ \frac{(-1)^{m} p^{\frac{m}{2}}}{2}, & \text { otherwise },\end{array}\right.$ and $\mathcal{C}_{0}^{+}=\left\{a \in \mathcal{C}_{0} \mid Q(a) \neq 0\right\}$.
In particular, let $p^{m} \equiv 3(\bmod 4), d=4$, then the following relationship between $S_{i}(a)$, $i=1,3$, and Kloosterman sum can be established.

Corollary 2.8. Let $\alpha$ be a primitive element of $\mathbb{F}_{p^{n}}, p^{m} \equiv 3(\bmod 4), d=4$ and $a=\alpha^{i\left(p^{m}+1\right)} \in$ $\mathbb{F}_{p^{m}}^{*}$, where $i$ is a positive integer with $0 \leq i \leq p^{m}-2$. Then

$$
S_{1}(a)=S_{3}(a)=\frac{R(a)-I\left(\omega^{Q(a)}-\omega^{-Q(a)}\right)}{2},
$$

where $R(a), Q(a), I$ are given in Lemma 2.7.
Proof. Since $p^{m} \equiv 3(\bmod 4)$, then we have $4 \mid\left(p^{m}+1\right)$ and

$$
\begin{aligned}
S_{1}(a) & =\sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}(a \xi x)}=\sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}\left(a \xi^{p^{m}} x^{p^{m}}\right)}=\sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}\left(a \xi^{3} \xi^{p^{m}-3} x\right)}=\sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}\left(a \xi^{3} x\right)} \\
& =S_{3}(a)
\end{aligned}
$$

Since $S_{1}(a)+S_{3}(a)=\sum_{x \in H} \omega^{\operatorname{Tr}_{1}^{n}(a \xi x)}$, where $H=\left\{\xi^{2 i} \left\lvert\, 0 \leq i \leq \frac{p^{m}-1}{2}\right.\right\}$, by Lemma 2.7, we have

$$
S_{1}(a)=S_{3}(a)= \begin{cases}\frac{R(a)-I\left(\omega^{Q(a)}-\omega^{-Q(a)}\right)}{2}, & a \in \mathcal{C}_{0}^{+} \\ \frac{R(a)}{2}, & \text { otherwise }\end{cases}
$$

where $R(a), Q(a), I$ are given in Lemma 2.7. Note that $a \in \mathcal{C}_{0}$, we have that $Q(a) \neq 0$ if $a \in \mathcal{C}_{0}^{+}$, and $Q(a)=0$ if $a \in \mathcal{C}_{0}$ and $a \notin \mathcal{C}_{0}^{+}$. This finishes the proof.

Let $\alpha$ be a primitive element of $\mathbb{F}_{p^{n}}$. For an odd prime $p$, every $x \in \mathbb{F}_{p^{n}}^{*}$ has a unique representation as $x=u y$, where $u \in \mathcal{U}=\left\{1, \alpha, \cdots, \alpha^{p^{m}}\right\}, y \in \mathbb{F}_{p^{m}}^{*}$. Then we get the following proposition.

Proposition 2.9. For $\lambda \in \mathbb{F}_{p^{n}}^{*}$, then there exists only one solution in $\mathcal{U}$ such that $\operatorname{Tr}_{m}^{n}(\lambda x)=0$. The following lemma can be found in [5].
Lemma 2.10. [5] Let $p$ be an odd prime, $f: \mathbb{F}_{p^{n}} \rightarrow \mathbb{F}_{p}$ be a regular bent function such that $f(x)=f(-x)$ and $f(0)=0$, then $f^{*}(0)=0$, where $f^{*}$ is the dual function of $f$.

A necessary and sufficient condition such that $f(x)$ defined by (1.1) is bent was given in [10]. We restate this result and give another proof.
Lemma 2.11. 10] Assume the notations given above. Then the function $f(x)$ defined by (1.1) is regular if and only if

$$
S\left(a_{1}, a_{2}, \cdots, a_{p^{m}-1}, b\right)=1
$$

where $S\left(a_{1}, \cdots, a_{p^{m}-1}, b\right)=\sum_{x \in U} \omega^{\sum_{i=1}^{p^{m}-1} \operatorname{Tr}_{1}^{n}\left(a_{i} x^{i}\right)+\operatorname{Tr}_{1}^{o(d)}\left(b x^{\frac{p^{m}+1}{d}}\right) \text {. } . ~ . ~ . ~}$

Proof. We first compute the walsh transform of $f(x)$. If $\lambda=0$, then

$$
\begin{align*}
W_{f}(0) & =\sum_{x \in \mathbb{F}_{p^{n}}} \omega^{f(x)} \\
& =1+\sum_{u \in \mathcal{U}} \sum_{y \in \mathbb{F}_{p}^{*}} \omega{ }^{p_{i=1}^{p^{m}}} \operatorname{Tr}_{1}^{n}\left(a_{i} u^{i\left(p^{m}-1\right)}\right)+\operatorname{Tr}_{1}^{o(d)}\left(b u u^{\frac{p^{n}-1}{d}}\right) \\
& =1+\left(p^{m}-1\right) \sum_{u \in \mathcal{U}} \omega \sum_{i=1}^{p^{m}-1} \operatorname{Tr}_{1}^{n}\left(a_{i} u^{i\left(p^{m}-1\right)}\right)+\operatorname{Tr}_{1}^{o(d)}\left(b u u^{\frac{p^{n}-1}{d}}\right) \\
& =1+\left(p^{m}-1\right) \sum_{x \in U} \omega \sum_{i=1}^{p^{m}-1} \operatorname{Tr}_{1}^{n}\left(a_{i} x^{i}\right)+\operatorname{Tr}_{1}^{o(d)}\left(b x^{p^{m}+1} \frac{1}{d}\right) \\
& =1+\left(p^{m}-1\right) S\left(a_{1}, a_{2}, \cdots, a_{p^{m}-1}, b\right) \tag{2.1}
\end{align*}
$$

If $\lambda \in \mathbb{F}_{p^{n}}^{*}$, then

$$
\left.\begin{array}{rl}
W_{f}(\lambda)= & \sum_{x \in \mathbb{F}_{p^{n}}} \omega^{f(x)-\operatorname{Tr}_{1}^{\mathrm{n}}(\lambda x)} \\
= & 1+\sum_{u \in \mathcal{U}} \sum_{y \in \mathbb{F}_{p^{m}}^{*}} \omega^{p_{i=1}^{m}} \operatorname{Tr}_{1}^{n}\left(a_{i} u^{i\left(p^{m}-1\right)}\right)+\operatorname{Tr}_{1}^{o(d)}\left(b u u^{\frac{p^{n}-1}{d}}\right)-\operatorname{Tr}_{1}^{\mathrm{n}}(\lambda u y) \\
= & 1+\sum_{u \in \mathcal{U}} \sum_{y \in \mathbb{F}_{p^{m}}} \omega^{p^{m} \sum_{i=1}} \operatorname{Tr}_{1}^{n}\left(a_{i} u^{i\left(p^{m}-1\right)}\right)+\operatorname{Tr}_{1}^{o(d)}\left(b u \frac{p^{n}-1}{d}\right)-\operatorname{Tr}_{1}^{\mathrm{n}}(\lambda u y) \\
& -\sum_{u \in \mathcal{U}} \omega \sum_{i=1}^{p^{m}-1} \operatorname{Tr}_{1}^{n}\left(a_{i} u^{i\left(p^{m}-1\right)}\right)+\operatorname{Tr}_{1}^{o(d)}\left(b u \frac{p^{n}-1}{d}\right) \\
= & 1+\sum_{u \in \mathcal{U}} \omega \sum_{i=1}^{p^{m}-1} \operatorname{Tr}_{1}^{n}\left(a_{i} u^{i\left(p^{m}-1\right)}\right)+\operatorname{Tr}_{1}^{o(d)}\left(b u \frac{p^{n}-1}{d}\right) \\
\sum_{y \in \mathbb{F}_{p^{m}}} \omega^{-\operatorname{Tr}_{1}^{\mathrm{m}}\left(y \operatorname{Tr}_{\mathrm{m}}^{\mathrm{n}}(\lambda u)\right)} \\
& -\sum_{x \in U} \omega \sum_{i=1}^{p^{m}-1} \operatorname{Tr}_{1}^{n}\left(a_{i} x^{i}\right)+\operatorname{Tr}_{1}^{o(d)}\left(b x^{\frac{p^{m}+1}{d}}\right)  \tag{2.2}\\
= & 1+p^{m} \omega^{p^{m}-1} \sum_{i=1} \operatorname{Tr}_{1}^{n}\left(a_{i} u_{\lambda}^{i\left(p^{m}-1\right)}\right)+\operatorname{Tr}_{1}^{o(d)}\left(b u_{\lambda}^{p^{n}-1} d\right.
\end{array}\right)-S\left(a_{1}, a_{2}, \cdots, a_{p^{m}-1}, b\right),
$$

where $u_{\lambda}$ satisfies $\operatorname{Tr}_{m}^{n}(\lambda u)=0$ and the last equality in (2.2) is obtained by Proposition 2.9,
Case I. $p=2$ : if $S\left(a_{1}, a_{2}, \cdots, a_{2^{m}-1}, b\right)=1$, it is easy to see that $f(x)$ is bent from equation (2.1) and (2.2). Conversely, if $f(x)$ is bent, then $W_{f}(0)=1+\left(2^{m}-1\right) S\left(a_{1}, a_{2}, \cdots, a_{2^{m}-1}, b\right)$ $\in\left\{ \pm 2^{m}\right\}$. Since $S\left(a_{1}, a_{2}, \cdots, a_{2^{m}-1}, b\right)$ is an integer, then $S\left(a_{1}, a_{2}, \cdots, a_{2^{m}-1}, b\right)=1$.

Case II. $p>2$ : if $f(x)$ is regular bent, then $W_{f}(0)=p^{m} \omega^{f^{*}(0)}$ by Definition 2.2. By Lemma 2.10, we have $W_{f}(0)=1+\left(p^{m}-1\right) S\left(a_{1}, a_{2}, \cdots, a_{p^{m}-1}, b\right)=p^{m}$. Therefore, we get $S\left(a_{1}, a_{2}, \cdots, a_{p^{m}-1}, b\right)=1$. Conversely, if $S\left(a_{1}, a_{2}, \cdots, a_{p^{m}-1}, b\right)=1$, it is easy to check that $f(x)$ is regular bent from equation (2.1) and (2.2).

## 3 Binary and $p$-ary Bent Functions

In this section, we study four classes of functions in the form (1.1), whose bentness are determined by some exponential sums.

### 3.1 Binary bent functions

In this subsection, we investigate two special classes of bent functions in the form (1.1).

### 3.1.1 First class of binary bent functions

In the following of this part, we always assume that $d$ and $l$ are positive integers with $\operatorname{gcd}\left(l, \frac{2^{m}+1}{d}\right)=$ 1. We consider the bentness of the following functions

$$
\begin{equation*}
f_{a_{0}, \cdots, a_{d-1}, b}(x)=\sum_{i=0}^{d-1} \operatorname{Tr}_{1}^{n}\left(a_{i} x^{\left(l+i \frac{2^{m}+1}{d}\right)\left(2^{m}-1\right)}\right)+\operatorname{Tr}_{1}^{o(d)}\left(b x^{\frac{2^{n}-1}{d}}\right), \tag{3.1}
\end{equation*}
$$

where $a_{i} \in \mathbb{F}_{2^{n}}, 0 \leq i \leq d-1, b \in \mathbb{F}_{2^{o(d)}}$.
Theorem 3.1. Assume the notations given above. Then the function $f_{a_{0}, \cdots, a_{d-1}, b}(x)$ defined by (3.1) is bent if and only if

$$
\sum_{j=0}^{d-1}(-1)^{\operatorname{Tr}_{1}^{o(d)}\left(b \xi^{\left.\frac{j\left(2^{m}+1\right)}{d}\right)}\right.} \sum_{x \in V_{0}}(-1)^{\sum_{i=0}^{d-1} \operatorname{Tr}_{1}^{n}\left(a_{i} \xi^{j\left(\frac{2^{m}+1}{d}\right)} \xi^{j l} x\right)}=1
$$

Proof. Note that

$$
\begin{align*}
& \sum_{x \in U}(-1) \sum_{i=0}^{d-1} \operatorname{Tr}_{1}^{n}\left(a_{i} x^{l+i \frac{2^{m}+1}{d}}\right)+\operatorname{Tr}_{1}^{o(d)}\left(b x \frac{2^{m}+1}{d}\right) \\
= & \sum_{x \in V_{0}}(-1)^{\sum_{i=0}^{d-1} \operatorname{Tr}_{1}^{n}\left(a_{i} x^{l}\right)+\operatorname{Tr}_{1}^{o(d)}(b)}+\sum_{x \in V_{0}}(-1)^{\sum_{i=0}^{d-1} \operatorname{Tr}_{1}^{n}\left(a_{i} \xi^{l+i \frac{2^{m}+1}{d}} x^{l}\right)+\operatorname{Tr}_{1}^{o(d)}\left(b \xi^{\frac{2^{m}+1}{d}}\right)} \\
& \left.+\cdots+\sum_{x \in V_{0}}(-1)^{\sum_{i=0}^{d-1} \operatorname{Tr}_{1}^{n}\left(a_{i} \xi^{(d-1)\left(l+i \frac{2^{m}+1}{d}\right)} x^{l}\right)+\operatorname{Tr}_{1}^{o(d)}\left(b \xi \frac{(d-1)\left(2^{m}+1\right)}{d}\right.}\right) \\
= & \sum_{x \in V_{0}}(-1)^{\sum_{i=0}^{d-1} \operatorname{Tr}_{1}^{n}\left(a_{i} x\right)+\operatorname{Tr}_{1}^{o(d)}(b)}+\sum_{x \in V_{0}}(-1)^{\sum_{i=0}^{d-1} \operatorname{Tr}_{1}^{n}\left(a_{i} \xi^{i^{2} \frac{m^{m}+1}{d}} \xi^{l} x\right)+\operatorname{Tr}_{1}^{o(d)}\left(b \xi^{\frac{2^{m}+1}{d}}\right)} \\
& +\cdots+\sum_{x \in V_{0}}(-1)^{\sum_{i=0}^{d-1} \operatorname{Tr}_{1}^{n}\left(a_{i} \xi^{(d-1)\left(i \frac{2^{m}+1}{d}\right)} \xi^{(d-1) l} x\right)+\operatorname{Tr}_{1}^{o(d)}\left(b \xi \frac{(d-1)\left(2^{m}+1\right)}{d}\right)} \\
= & \left.\sum_{j=0}^{d-1}(-1)^{\operatorname{Tr}_{1}^{o(d)}\left(b \xi \frac{j\left(2^{m}+1\right)}{d}\right.}\right) \sum_{x \in V_{0}}(-1)^{\sum_{i=0}^{d-1} \operatorname{Tr}_{1}^{n}\left(a_{i} \xi^{j\left(i^{m} \frac{2^{m}+1}{d}\right)} \xi^{j l} x\right)} . \tag{3.2}
\end{align*}
$$

By Lemma 2.11, we finish the proof.
Moreover, for some case of $a_{i}$ 's, the bentness of $f_{a_{0}, \cdots, a_{d-1}, b}(x)$ defined by (3.1) can be characterized by some well-known exponential sums.

Theorem 3.2. Let $a_{0} \in \mathbb{F}_{2^{m}}^{*}, a_{1}=a_{2}=\cdots=a_{d-1} \in \mathbb{F}_{2^{m}}$ and $a_{0} \neq a_{1}$. Then $f_{a_{0}, \cdots, a_{d-1}, 0}(x)$ be defined by (3.1) is bent if and only if

$$
K_{m}\left(a_{0}\right)+(d-1) K_{m}\left(a_{0}+a_{1}\right)= \begin{cases}2\left(E_{m, d}\left(a_{0}\right)+(d-1) E_{m, d}\left(a_{0}+a_{1}\right)\right), & \text { if } d \mid l ; \\ 2\left(E_{m, d}\left(a_{0}\right)-E_{m, d}\left(a_{0}+a_{1}\right)\right), & \text { if } \operatorname{gcd}(d, l)=1,\end{cases}
$$

where $E_{m, d}(a)$ is given in Lemma 2.6.
Proof. Since $\xi^{j^{\frac{2^{m}+1}{d}}}$ is a root of $1+z+z^{2}+\cdots+z^{d-1}=0$ for each $1 \leq j \leq d-1$, and $a_{1}=$ $a_{2}=\cdots=a_{d-1}$, we get $\sum_{i=0}^{d-1} \operatorname{Tr}_{1}^{n}\left(a_{i} \xi^{i\left(j \frac{2^{m}+1}{d}\right)} \xi^{j l} x\right)=\operatorname{Tr}_{1}^{n}\left(\left(a_{0}+a_{1}\right) \xi^{j l} x\right)$ for each $1 \leq j \leq d-1$. Note that $b=0$ and $\operatorname{gcd}\left(l, \frac{2^{m}+1}{d}\right)=1$, then Equation (3.2) is

$$
\begin{align*}
& \left.\sum_{x \in U}(-1)^{\sum_{i=0}^{d-1} \operatorname{Tr}_{1}^{n}\left(a_{i} x^{l+i} \frac{2^{m}+1}{d}\right.}\right) \\
= & \sum_{x \in V_{0}}(-1)^{\operatorname{Tr}_{1}^{n}\left(a_{0} x\right)}+\sum_{x \in V_{0}}(-1)^{\operatorname{Tr}_{1}^{n}\left(\left(a_{0}+a_{1}\right) \xi^{l} x\right)}+\cdots+\sum_{x \in V_{0}}(-1)^{\operatorname{Tr}_{1}^{n}\left(\left(a_{0}+a_{1}\right) \xi^{(d-1) l} x\right)} . \tag{3.3}
\end{align*}
$$

In the following, we discuss Equation (3.3) in two cases.

1) If $d \mid l$, then by Lemma 2.6, we have

$$
\begin{aligned}
& \left.\sum_{x \in U}(-1)^{\sum_{i=0}^{d-1} \operatorname{Tr}_{1}^{n}\left(a_{i} x^{l+i} \frac{2^{m}+1}{d}\right.}\right) \\
= & \sum_{x \in V_{0}}(-1)^{\operatorname{Tr}_{1}^{n}\left(a_{0} x\right)}+(d-1) \sum_{x \in V_{0}}(-1)^{\operatorname{Tr}_{1}^{n}\left(\left(a_{0}+a_{1}\right) x\right)} \\
= & \frac{1+2 E_{m, d}\left(a_{0}\right)-K_{m}\left(a_{0}\right)}{d}+(d-1) \frac{1+2 E_{m, d}\left(a_{0}+a_{1}\right)-K_{m}\left(a_{0}+a_{1}\right)}{d} .
\end{aligned}
$$

Hence, by Theorem 3.1, $f_{a_{0}, \cdots, a_{d-1}, b}(x)$ is bent if and only if

$$
K_{m}\left(a_{0}\right)+(d-1) K_{m}\left(a_{0}+a_{1}\right)=2\left(E_{m, d}\left(a_{0}\right)+(d-1) E_{m, d}\left(a_{0}+a_{1}\right)\right) .
$$

2) If $\operatorname{gcd}(d, l)=1$, it is easy to verify that $\{l(\bmod d), 2 l(\bmod d), \cdots,(d-1) l(\bmod d)\}=$ $\{1,2, \cdots, d-1\}$. By Lemma 2.6, we have

$$
\begin{aligned}
& \left.\sum_{x \in U}(-1)^{\sum_{i=0}^{d-1} \operatorname{Tr}_{1}^{n}\left(a_{i} x^{l+i} \frac{2^{m}+1}{d}\right.}\right) \\
= & \sum_{x \in V_{0}}(-1)^{\operatorname{Tr}_{1}^{n}\left(a_{0} x\right)}+\sum_{x \in V_{0}}(-1)^{\operatorname{Tr}_{1}^{n}\left(\left(a_{0}+a_{1}\right) \xi x\right)}+\cdots+\sum_{x \in V_{0}}(-1)^{\operatorname{Tr}_{1}^{n}\left(\left(a_{0}+a_{1}\right) \xi^{d-1} x\right)} \\
= & \sum_{x \in V_{0}}(-1)^{\operatorname{Tr}_{1}^{n}\left(a_{0} x\right)}+\sum_{x \in V_{1}}(-1)^{\operatorname{Tr}_{1}^{n}\left(\left(a_{0}+a_{1}\right) x\right)}+\cdots+\sum_{x \in V_{d-1}}(-1)^{\operatorname{Tr}_{1}^{n}\left(\left(a_{0}+a_{1}\right) x\right)} \\
= & \sum_{x \in V_{0}}(-1)^{\operatorname{Tr}_{1}^{n}\left(a_{0} x\right)}+\sum_{x \in U}(-1)^{\operatorname{Tr}_{1}^{n}\left(\left(a_{0}+a_{1}\right) x\right)}-\sum_{x \in V_{0}}(-1)^{\operatorname{Tr}_{1}^{n}\left(\left(a_{0}+a_{1}\right) x\right)} \\
= & 1-K_{m}\left(a_{0}+a_{1}\right)+\frac{1+2 E_{m, d}\left(a_{0}\right)-K_{m}\left(a_{0}\right)}{d}-\frac{1+2 E_{m, d}\left(a_{0}+a_{1}\right)-K_{m}\left(a_{0}+a_{1}\right)}{d} .
\end{aligned}
$$

Therefore, by Theorem 3.1, $f_{a_{0}, \cdots, a_{d-1}, b}(x)$ is bent if and only if

$$
K_{m}\left(a_{0}\right)+(d-1) K_{m}\left(a_{0}+a_{1}\right)=2 E_{m, d}\left(a_{0}\right)-2 E_{m, d}\left(a_{0}+a_{1}\right) .
$$

This finishes the proof.
If we take $d=3$ in Theorem 3.2, and combine the results on $S_{i}(a)$ in [11], $i=0,1,2$, we obtain the following result, which is exactly Corollary 1 in [10].

Corollary 3.3. Let $f_{a_{0}, \cdots, a_{d-1}, 0}(x)$ be defined by (3.1) with $b=0, d=3, a_{0} \in \mathbb{F}_{2^{m}}^{*}, a_{1}=a_{2} \in$ $\mathbb{F}_{2^{m}}$ and $a_{0} \neq a_{1}$. Then $f_{a_{0}, \cdots, a_{d-1}, 0}(x)$ is bent if and only if

$$
K_{m}\left(a_{0}\right)+2 K_{m}\left(a_{0}+a_{1}\right)= \begin{cases}2\left(C_{m}\left(a_{0}\right)+2 C_{m}\left(a_{0}+a_{1}\right)\right), & \text { if } 3 \mid l ; \\ 2\left(C_{m}\left(a_{0}\right)-C_{m}\left(a_{0}+a_{1}\right)\right), & \text { otherwise }\end{cases}
$$

where $C_{m}(a)=\sum_{a \in \mathbb{F}_{2^{m}}}(-1)^{\operatorname{Tr}_{1}^{m}\left(a x^{3}+a x\right)}$.
Example 3.4. Let $n=2 m=6, d=9$ and $l=1, a_{0} \in \mathbb{F}_{2^{3}}^{*}, a_{1}=a_{2}=\cdots=a_{8} \in \mathbb{F}_{2^{3}}^{*}$, then $f_{a_{0}, \cdots, a_{8}, 0}(x)=\operatorname{Tr}_{1}^{6}\left(a_{0} x^{7}\right)+\sum_{i=1}^{8} \operatorname{Tr}_{1}^{6}\left(a_{1} x^{7(1+i)}\right)$. By using Maple, we get that there exist 9 pairs $\left(a_{0}, a_{1}\right)$ such that $f_{a_{0}, \cdots, a_{8}, 0}(x)$ is bent.

If $b \neq 0$, by a similar discussion as that in Theorem 3.2, we obtain the following theorem.
Theorem 3.5. Let $f_{a_{0}, \cdots, a_{d-1}, b}(x)$ be defined by (3.1) with $b \neq 0, d \mid l, a_{0} \in \mathbb{F}_{2^{m}}^{*}, a_{1}=a_{2}=$ $\cdots=a_{d-1} \in \mathbb{F}_{2^{m}}$ and $a_{0} \neq a_{1}$. Then $f_{a_{0}, \cdots, a_{d-1}, b}(x)$ is bent if and only if

$$
\rho K_{m}\left(a_{0}\right)+\sigma K_{m}\left(a_{0}+a_{1}\right)=2\left(\rho E_{m, d}\left(a_{0}\right)+\sigma E_{m, d}\left(a_{0}+a_{1}\right)\right)+\rho+\sigma-d,
$$

where $\rho=(-1)^{\operatorname{Tr}_{1}^{o(d)}(b)}, \sigma=\sum_{j=1}^{d-1}(-1)^{\left.\operatorname{Tr}_{1}^{o(d)\left(b \xi^{j} \frac{2^{m}+1}{d}\right.}\right)}$ and $E_{m, d}(a)$ is given in Lemma 2.6.
If we take $a_{1}=a_{2}=\cdots=a_{d-1}=0$, then we have the following result by Theorem 3.5. which is exactly Theorem 3 in [10].

Corollary 3.6. Let $d \mid l, a_{0} \in \mathbb{F}_{2^{m}}^{*}$ and $a_{1}=\cdots=a_{d-1}=0$. Then $f_{a_{0}, 0, \cdots, 0, b}(x)$ defined by (3.1) is bent if and only if

$$
\sum_{j=0}^{d-1}(-1)^{\operatorname{Tr}_{1}^{o(d)}\left(b \xi^{\frac{2}{}^{m}+1}\right.} d=\frac{d}{1+2 E_{m, d}\left(a_{0}\right)-K_{m}\left(a_{0}\right)}
$$

where $E_{m, d}(a)$ is given in Lemma 2.6 .
Example 3.7. Let $n=2 m=4, d=5$ and $l=5$, $\alpha$ be a primitive element of $\mathbb{F}_{2^{4}}, a_{0} \in \mathbb{F}_{2^{2}}^{*}$, $a_{0} \neq a_{1}, a_{1}=a_{2}=a_{3}=a_{4} \in \mathbb{F}_{2^{2}}^{*}, b \in \mathbb{F}_{2^{4}}^{*}$, then $f_{a_{0}, \cdots, a_{4}, b}(x)$ defined by (3.1) is equal to $\operatorname{Tr}_{1}^{4}\left(a_{0} x^{15}\right)+\sum_{i=1}^{4} \operatorname{Tr}_{1}^{4}\left(a_{1} x^{3(5+i)}\right)+\operatorname{Tr}_{1}^{4}\left(b x^{3}\right)$. By using Maple, the number of $\left(a_{0}, a_{1}, b\right)$ such that $f_{a_{0}, \cdots, a_{4}, b}(x)$ is bent function is 60 .

### 3.1.2 Second class of binary bent functions

In this part, we always assume that $s, k, r$ are integers with $r \mid\left(2^{m}+1\right)$. We investigate the bentness of

$$
\begin{equation*}
f_{a, r, s}(x)=\sum_{i=1}^{\frac{2^{m}+1}{r}-1} \operatorname{Tr}_{1}^{n}\left(a x^{(r i+s)\left(2^{m}-1\right)}\right), \tag{3.4}
\end{equation*}
$$

where $a \in \mathbb{F}_{2^{n}}^{*}$ and $f(0)=0$.
Theorem 3.8. Assume the notations given above. Then

1. if $\operatorname{gcd}\left(s, 2^{m}+1\right)=1,0 \leq k \leq 2^{m}$ and $a=\bar{a} \xi^{k} \in \mathbb{F}_{2^{n}}$ with $\bar{a} \in \mathbb{F}_{2^{m}}^{*}$, then $f_{a, r, s}(x)$ is bent if and only if

$$
K_{m}(\bar{a})=r-\sum_{x^{r}=1, x \in U}(-1)^{\operatorname{Tr}_{1}^{n}(a x)}
$$

2. if $\operatorname{gcd}\left(s, 2^{m}+1\right)=d, 0 \leq k<\frac{2^{m}+1}{d}$ and $a=\bar{a} \xi^{k d} \in \mathbb{F}_{2^{n}}$ with $\bar{a} \in \mathbb{F}_{2^{m}}^{*}$, then $f_{a, r, s}(x)$ is bent if and only if

$$
d S_{0}(\bar{a})=\sum_{x^{r}=1, x \in U}(-1)^{\operatorname{Tr}_{1}^{n}\left(a x^{s}\right)}+1-r,
$$

where $S_{0}(\bar{a})$ is given in Lemma 2.6.
Proof. By Lemma 2.11, $f_{a, r, s}(x)$ is bent if and only if $\sum_{x \in U}(-1) \sum_{i=1}^{\frac{2^{m}+1}{r}-1} \operatorname{Tr}_{1}^{n}\left(a x^{r i+s}\right)=1$. On the other hand, $\quad \sum_{i=1}^{\frac{2^{m}+1}{r}-1} \operatorname{Tr}_{1}^{n}\left(a x^{(r i+s)}\right)=\operatorname{Tr}_{1}^{n}\left(a x^{s}\right)$ when $x^{r} \neq 1$ and $x \in U$. Since $\frac{2^{m}+1}{r}-1$ is even, we get

$$
\begin{aligned}
\sum_{x \in U}(-1) \sum_{i=1}^{\frac{2^{m}+1}{r}-1} \operatorname{Tr}_{1}^{n}\left(a x^{\left.r^{r i+s}\right)}\right. & =\sum_{x \in U \backslash x^{r}=1}(-1)^{\operatorname{Tr}_{1}^{n}\left(a x^{s}\right)}+r \\
& =\sum_{x \in U}(-1)^{\operatorname{Tr}_{1}^{n}\left(a x^{s}\right)}+r-\sum_{x^{r}=1, x \in U}(-1)^{\operatorname{Tr}_{1}^{n}\left(a x^{s}\right)} .
\end{aligned}
$$

If $\operatorname{gcd}\left(s, 2^{m}+1\right)=1$, then $\sum_{x \in U}(-1)^{\operatorname{Tr}_{1}^{n}\left(a x^{s}\right)}=\sum_{x \in U}(-1)^{\operatorname{Tr}_{1}^{n}(a x)}=1-K_{m}(\bar{a})$ and $\operatorname{gcd}(s, r)=1$. Thus $f_{a, r, s}(x)$ is bent if and only if

$$
K_{m}(\bar{a})=r-\sum_{x^{r}=1, x \in U}(-1)^{\operatorname{Tr}_{1}^{n}(a x)} .
$$

If $\operatorname{gcd}\left(s, 2^{m}+1\right)=d$ and $a=\bar{a} \xi^{k d}$, we have

$$
\sum_{x \in U}(-1)^{\operatorname{Tr}_{1}^{n}\left(a x^{s}\right)}=d \sum_{x \in V_{0}}(-1)^{\operatorname{Tr}_{1}^{n}(a x)}=d S_{0}(\bar{a}) .
$$

Thus $f_{a, r, s}(x)$ is bent if and only if

$$
d S_{0}(\bar{a})=\sum_{x^{r}=1, x \in U}(-1)^{\operatorname{Tr}_{1}^{n}\left(a x^{s}\right)}+1-r .
$$

In particular, if we take $r=1$ in Theorem 3.8, we have the following result, which is exactly Theorem 4 in [10].

Corollary 3.9. Assume the notation given above. Then we have

1. if $\operatorname{gcd}\left(s, 2^{m}+1\right)=1,0 \leq k \leq 2^{m}$ and $a=\bar{a} \xi^{k} \in \mathbb{F}_{2^{n}}$ with $\bar{a} \in \mathbb{F}_{2^{m}}$, then $f_{a, 1, s}(x)$ is bent if and only if

$$
K_{m}(\bar{a})=1-(-1)^{\operatorname{Tr}_{1}^{n}(a)}
$$

2. if $\operatorname{gcd}\left(s, 2^{m}+1\right)=d, 0 \leq k<\frac{2^{m}+1}{d}$ and $a=\bar{a} \xi^{k d} \in \mathbb{F}_{2^{n}}$ with $\bar{a} \in \mathbb{F}_{2^{m}}$, then $f_{a, 1, s}(x)$ is bent if and only if

$$
d S_{0}(\bar{a})=(-1)^{\operatorname{Tr}_{1}^{n}(a)}
$$

where $S_{0}(\bar{a})$ is given in Lemma 2.6.
In particular, let $r=3, \operatorname{gcd}\left(s, 2^{m}+1\right)=1$, then we have the following corollary.
Corollary 3.10. Assume the notation given above. Let $\operatorname{gcd}\left(s, 2^{m}+1\right)=1$, $a=\bar{a} \xi^{k} \in \mathbb{F}_{2^{n}}$ with $\bar{a} \in \mathbb{F}_{2^{m}}$ and $0 \leq k \leq 2^{m}, f(0)=0$ and $r=3$. Then $f_{a, 3, s}(x)$ defined by (3.4) is bent if and only if

$$
K_{m}(\bar{a})=3-\sum_{j=0}^{2}(-1)^{\operatorname{Tr}_{1}^{n}\left(a \xi^{j} \frac{2^{m}+1}{3}\right)}
$$

Furthermore, if $f_{a, 3, s}(x)$ is bent, then $K_{m}(\bar{a})=0$ when $\operatorname{Tr}_{1}^{n}(a)=\operatorname{Tr}_{1}^{n}\left(a \xi^{\frac{2^{m}+1}{3}}\right)=\operatorname{Tr}_{1}^{n}\left(a \xi^{2 \frac{2^{m}+1}{3}}\right)=$ 0 , otherwise, $K_{m}(\bar{a})=4$.

Proof. By Theorem [3.8, we have that $f_{a, 3, s}(x)$ defined by (3.4) is bent if and only if

$$
K_{m}(\bar{a})=3-\sum_{j=0}^{2}(-1)^{\operatorname{Tr}_{1}^{n}\left(a \xi^{j^{\frac{2^{m}+1}{3}}}\right)}
$$

Note that $\operatorname{Tr}_{1}^{n}(a)=\operatorname{Tr}_{1}^{n}\left(a \xi^{\frac{2^{m}+1}{3}}\right)+\operatorname{Tr}_{1}^{n}\left(a \xi^{\frac{2^{m}+1}{3}}\right)$, since $\xi^{\frac{2^{m}+1}{3}}+\xi^{2 \frac{2^{m}+1}{3}}=1$. It is easy to check that $K_{m}(\bar{a})=0$ if $\operatorname{Tr}_{1}^{n}(a)=\operatorname{Tr}_{1}^{n}\left(a \xi^{\frac{2^{m}+1}{3}}\right)=\operatorname{Tr}_{1}^{n}\left(a \xi^{2 \frac{2^{m}+1}{3}}\right)=0$ and in other cases, $K_{m}(\bar{a})=4$. This completes the proof.

Example 3.11. Let $\alpha$ be a primitive element of $\mathbb{F}_{2^{6}}, n=2 m=6, r=3, s=1, a \in \mathbb{F}_{2^{6}}^{*}$, then $f_{a, 3,1}(x)=\operatorname{Tr}_{1}^{6}\left(a x^{28}\right)+\operatorname{Tr}_{1}^{6}\left(a x^{49}\right)$. By applying Maple, the number of this class of binomial regular bent functions is 36 .

## 3.2 p-ary bent functions

In this subsection, we always assume that $p$ is an odd prime. By Corollary 2.8, the bentness of a new class of binomial $p$-ary functions (see Theorem 3.13) is characterized by Kloosterman sums. Following the similar idea of construction of the second class of binary bent functions, we also obtain another class of regular bent functions (see Theorem 3.17), and in some special cases, we get more bent functions which are determined by Kloosterman sums.

### 3.2.1 First class of $p$-ary bent functions

We have established a connection between $S_{i}(a), i=1,3$, and Kloosterman sum in Corollary 2.8 and use this result to characterize the bentness of the following binomial functions

$$
\begin{equation*}
f_{a, b}(x)=\operatorname{Tr}_{1}^{n}\left(a x^{l\left(p^{m}-1\right)}\right)+\operatorname{Tr}_{1}^{2}\left(b x^{\frac{p^{n}-1}{4}}\right) \tag{3.5}
\end{equation*}
$$

where $p^{m} \equiv 3(\bmod 4), a \in \mathbb{F}_{p^{n}}^{*}, b \in \mathbb{F}_{p^{2}}^{*}, l$ is an integer with $\operatorname{gcd}\left(l, \frac{p^{m}+1}{4}\right)=1$.
Theorem 3.12. Assume the notations given above. Then $f_{a, b}(x)$ defined by (3.5) is regular bent if and only if

$$
\left.\sum_{j=0}^{3} \omega^{\operatorname{Tr}_{1}^{2}\left(b \xi^{j} \frac{p^{m}+1}{4}\right.}\right) \sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}\left(a \xi^{j l} x\right)}=1
$$

Proof. Since

$$
\begin{aligned}
& \sum_{x \in U} \omega^{\operatorname{Tr}_{1}^{n}\left(a x^{l}\right)+\operatorname{Tr}_{1}^{2}\left(b x^{\frac{p^{m}+1}{4}}\right)} \\
= & \sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}\left(a x^{l}\right)+\operatorname{Tr}_{1}^{2}(b)}+\sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}\left(a \xi^{l} x^{l}\right)+\operatorname{Tr}_{1}^{2}\left(b \xi^{\frac{p^{m}+1}{4}}\right)} \\
& +\sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}\left(a \xi^{2 l} x^{l}\right)-\operatorname{Tr}_{1}^{2}(b)}+\sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}\left(a \xi^{3 l} x^{l}\right)+\operatorname{Tr}_{1}^{2}\left(b \xi^{\frac{3}{} \frac{m^{m}+1}{4}}\right)} \\
= & \omega^{\operatorname{Tr}_{1}^{2}(b)} \sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}(a x)}+\omega^{\operatorname{Tr}_{1}^{2}\left(b \xi^{\frac{p^{m}+1}{4}}\right)} \sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}\left(a \xi^{l} x\right)} \\
& \left.+\omega^{\operatorname{Tr}_{1}^{2}(-b)} \sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}\left(a \xi^{2 l} x\right)}+\omega^{\operatorname{Tr}_{1}^{2}\left(b \xi^{3} \frac{p^{m}+1}{4}\right.}\right) \sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}\left(a \xi^{3 l} x\right)} \\
= & \left.\sum_{j=0}^{3} \omega^{\operatorname{Tr}_{1}^{2}\left(b \xi^{j} \frac{p^{m}+1}{4}\right.}\right) \sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}\left(a \xi^{j l} x\right)} .
\end{aligned}
$$

Then, by Lemma 2.11, we have that $f_{a, b}(x)$ is regular bent if and only if

$$
\left.\sum_{j=0}^{3} \omega^{\operatorname{Tr}_{1}^{2}\left(b \xi^{j} \frac{p^{m}+1}{4}\right.}\right) \sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}\left(a \xi^{j l} x\right)}=1
$$

In particular, we obtain the following result.

Theorem 3.13. Assume the notations given above. Let $k$ be a positive integer with $k \equiv$ 1 or $3(\bmod 4), a=\bar{a} \xi^{k}$, where $\bar{a} \in \mathbb{F}_{p^{m}}^{*}$, and $4 \mid l$. Then $f_{a, b}(x)$ defined by (3.5) is regular bent if and only if

$$
K_{m}\left(\bar{a}^{2}\right)=1-4 I \sqrt{-1} \sin \frac{2 \pi Q(\bar{a})}{p}-\frac{2}{\cos \frac{2 \pi \operatorname{Tr}_{1}^{2}(b)}{p}+\cos \frac{2 \pi \operatorname{Tr}_{1}^{2}\left(b \xi \frac{p^{m}+1}{4}\right)}{p}}
$$

where $Q(\bar{a}), I$ are given in Lemma 2.7.
Proof. By Theorem 3.12, we have that $f_{a, b}(x)$ is regular bent if and only if

$$
\left.\sum_{j=0}^{3} \omega^{\operatorname{Tr}_{1}^{2}\left(b \xi^{j} \frac{p^{m}+1}{4}\right.}\right) \sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}\left(a \xi^{j l} x\right)}=1
$$

On the other hand, since $4 \mid l$, then

$$
\left.\left.\sum_{j=0}^{3} \omega^{\operatorname{Tr}_{1}^{2}\left(b \xi^{j} \frac{p^{m}+1}{4}\right.}\right) \sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}\left(a \xi^{j l} x\right)}=\sum_{j=0}^{3} \omega^{\operatorname{Tr}_{1}^{2}\left(b \xi^{j} \frac{p^{m}+1}{4}\right.}\right) \sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}(a x)}
$$

Since $a=\bar{a} \xi^{k}, \bar{a} \in \mathbb{F}_{p^{m}}^{*}, k \equiv 1$ or $3(\bmod 4)$, we have that

$$
\sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}(a x)}=\sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}(\bar{a} \xi x)}=\sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}\left(\bar{a} \xi^{3} x\right)}=S_{1}(\bar{a})=S_{3}(\bar{a})
$$

Thus, by Corollary 2.8, we have

$$
\sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}(a x)}=\frac{R(\bar{a})-I\left(\omega^{Q(\bar{a})}-\omega^{-Q(\bar{a})}\right)}{2}
$$

To sum up, $f_{a, b}(x)$ is regular bent if and only if

$$
\left.\sum_{j=0}^{3} \omega^{\operatorname{Tr}_{1}^{2}\left(b j^{j} \frac{p^{m}+1}{4}\right.}\right)=\frac{4}{1-K_{m}\left(\bar{a}^{2}\right)-4 I \sqrt{-1} \sin \frac{2 \pi Q(\bar{a})}{p}},
$$

where $Q(\bar{a}), I$ are given in Lemma 2.7. Note that

$$
\sum_{j=0}^{3} \omega^{\operatorname{Tr}_{1}^{2}\left(b \xi^{j^{\frac{p^{m}+1}{4}}}\right)}=2\left(\cos \frac{2 \pi \operatorname{Tr}_{1}^{2}(b)}{p}+\cos \frac{2 \pi \operatorname{Tr}_{1}^{2}\left(b \xi^{\frac{p^{m}+1}{4}}\right.}{p}\right),
$$

we finish the proof.
Corollary 3.14. If there exist $(a, b) \in \mathbb{F}_{3^{n}}^{*} \times \mathbb{F}_{3^{2}}^{*}$ such that $f_{a, b}(x)$ defined by (3.5) is a regular bent function. Then the number of these regular bent functions is divided by 4 .

Proof. Since $b \in \mathbb{F}_{3^{2}}^{*}$, then we can get $b \in\left\{\left.\alpha^{\frac{i^{n}-1}{8}} \right\rvert\, 0 \leq i \leq 7\right\}$, where $\alpha$ is a primitive element in $\mathbb{F}_{3^{n}}$. Since $\xi$ is the generator of $U$, so $\xi^{\frac{3^{m}+1}{4}}=\alpha^{\frac{3^{n}-1}{4}}$. Then $b, b \alpha^{\frac{3^{n}-1}{4}},-b, b \alpha^{\frac{3^{n^{n}-1}}{4}}$ have the same value of $\cos \frac{2 \pi \operatorname{Tr}_{1}^{2}(b)}{3}+\cos \frac{2 \pi \operatorname{Tr}_{1}^{2}\left(b \xi^{\frac{p^{m}+1}{4}}\right)}{3}$. This completes the proof.

Example 3.15. Let $l=4$, $a=\bar{a} \xi, \bar{a} \in \mathbb{F}_{3^{3}}^{*}, b \in \mathbb{F}_{3^{2}}^{*}$, $\xi$ be a generator of cyclic $U=\{x \in$ $\left.\mathbb{F}_{3^{6}} \mid x^{3^{3}+1}=1\right\}$, then we have $3^{3} \equiv 3(\bmod 4)$ and $f_{a, b}(x)=\operatorname{Tr}_{1}^{6}\left(a x^{144}\right)+\operatorname{Tr}_{1}^{2}\left(b x^{182}\right)$. By using Maple, the number of this binomial bent functions is 48 .

Remark 3.16. Following the similar construction of bent functions in Theorem 3.2 and Theorem 3.5, we can also investigate the bentness of $f(x)=\sum_{i=0}^{3} \operatorname{Tr}_{1}^{n}\left(a_{i} x^{\left(l+i \frac{p^{m}+1}{4}\right)\left(p^{m}-1\right)}\right)+\operatorname{Tr}_{1}^{2}\left(b x^{\frac{p^{n}-1}{4}}\right)$. In particular, by Corollary 2.8, the bentness of this class of functions can also be characterized by some exponential sums, which have close relations with Kloosterman sums.

### 3.2.2 Second class of $p$-ary bent functions

In this part, we always assume $s, r$ are integers with $\operatorname{gcd}\left(s, p^{m}+1\right)=1$ and $r \mid\left(p^{m}+1\right)$. Similar to the second class of binary bent functions, we investigate the bentness of the following function

$$
\begin{equation*}
f_{a, b, r}(x)=\sum_{i=1}^{\frac{p^{m}+1}{r}-1} \operatorname{Tr}_{1}^{n}\left(a x^{(r i+s)\left(p^{m}-1\right)}\right)+b x^{\frac{p^{n}-1}{2}} \tag{3.6}
\end{equation*}
$$

where $a \in \mathbb{F}_{p^{n}}^{*}, b \in \mathbb{F}_{p}^{*}$ and $f(0)=0$.

Theorem 3.17. Assume the notations given above. Then $f_{a, b, r}(x)$ defined by (3.6) is regular bent if and only if

$$
\left(1-K_{m}\left(a^{p^{m}+1}\right)\right) \cos \frac{2 \pi b}{p}= \begin{cases}4 I \sin \frac{2 \pi b}{p} \sin \frac{2 \pi Q(-a)}{p}+\epsilon, & -a \in \mathcal{C}_{0}^{+} \\ \epsilon, & \text { otherwise }\end{cases}
$$

where $\epsilon=\sum_{x^{r}=1, x \in U} \omega^{\operatorname{Tr}_{1}^{n}\left(-a x^{s}\right)+b x^{\frac{p^{m}+1}{2}}}-\sum_{x^{r}=1, x \in U} \omega^{\left(\frac{p^{m}+1}{r}-1\right) \operatorname{Tr}_{1}^{n}\left(a x^{s}\right)+b x^{\frac{p^{m}+1}{2}}}+1$.
Proof. Note that if $x^{r} \neq 1$ and $x \in U$, then $\sum_{i=1}^{\frac{p^{m}+1}{r}-1} \operatorname{Tr}_{1}^{n}\left(x^{(r i+s)}\right)=-\operatorname{Tr}_{1}^{n}\left(x^{s}\right)$. By Lemma 2.11, $f_{a, b, r}(x)$ is regular bent if and only if

$$
\sum_{x \in U} \omega \sum_{i=1}^{\frac{p^{m}+1}{r}-1} \operatorname{Tr}_{1}^{n}\left(a x^{(r i+s)}\right)+b x^{\frac{p^{m}+1}{2}}=1
$$

which is equivalent to

$$
\begin{equation*}
\sum_{x^{r}=1, x \in U} \omega^{\left(\frac{p^{m}+1}{r}-1\right) \operatorname{Tr}_{1}^{n}\left(a x^{s}\right)+b x^{\frac{p^{m}+1}{2}}}+\sum_{x^{r} \neq 1, x \in U} \omega^{\operatorname{Tr}_{1}^{n}\left(-a x^{s}\right)+b x^{\frac{p^{m}+1}{2}}}=1 \tag{3.7}
\end{equation*}
$$

On the other hand, we have

$$
\begin{align*}
& \sum_{x^{r} \neq 1, x \in U} \omega^{\operatorname{Tr}_{1}^{n}\left(-a x^{s}\right)+b x^{\frac{p^{m}+1}{2}}} \\
= & \sum_{x \in U} \omega^{\operatorname{Tr}_{1}^{n}\left(-a x^{s}\right)+b x^{\frac{p^{m}+1}{2}}}-\sum_{x \in U, x^{r}=1} \omega^{\operatorname{Tr}_{1}^{n}\left(-a x^{s}\right)+b x^{\frac{p^{m}+1}{2}}} \\
= & \omega^{b} \sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}\left(-a x^{s}\right)}+\omega^{-b} \sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}\left(-a \xi^{s} x^{s}\right)}-\sum_{x^{r}=1, x \in U} \omega^{\operatorname{Tr}_{1}^{n}\left(-a x^{s}\right)+b x^{\frac{p^{m}+1}{2}}} . \tag{3.8}
\end{align*}
$$

Since $\operatorname{gcd}\left(s, p^{m}+1\right)=1$, then

$$
\begin{align*}
& \omega^{b} \sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}\left(-a x^{s}\right)}+\omega^{-b} \sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}\left(-a \xi^{s} x^{s}\right)} \\
= & \omega^{b} \sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}(-a x)}+\omega^{-b} \sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}\left(-a \xi^{s} x\right)} \\
= & \omega^{b} \sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}(-a x)}+\omega^{-b} \sum_{x \in V_{0}} \omega^{\operatorname{Tr}_{1}^{n}\left(-a \xi\left(\xi^{s-1} x\right)\right)} \\
= & \omega^{b} S_{0}(-a)+\omega^{-b} S_{1}(-a) . \tag{3.9}
\end{align*}
$$

From (3.7), (3.8), (3.9) and Lemma 2.7, we complete this proof.
In particular, we have the following result, which is exactly Theorem 10, 11 in [10].
Corollary 3.18. Assume the notations given above. We have

1. if $b=0, r=1$, then $f_{a, 0,1}(x)$ is regular bent if and only if $K_{m}\left(a^{p^{m}+1}\right)=1-\omega^{\operatorname{Tr}_{1}^{n}(-a)}$.
2. if $b \neq 0, r=1$, then $f_{a, b, 1}(x)$ is regular bent if and only if

$$
\left(1-K_{m}\left(a^{p^{m}+1}\right)\right) \cos \frac{2 \pi b}{p}= \begin{cases}4 I \sin \frac{2 \pi b}{p} \sin \frac{2 \pi Q(-a)}{p}+\epsilon, & -a \in \mathcal{C}_{0}^{+} \\ \epsilon, & \text { otherwise }\end{cases}
$$

where $\epsilon=\omega^{\operatorname{Tr}_{1}^{n}(-a)+b}-\omega^{b}+1$.
Note that if $p=3, r=2$, then $\frac{p^{m}+1}{r}-1=\frac{3^{m}+1}{2}-1=1+3+\cdots+3^{m-1} \equiv 1(\bmod 3)$. Together with $\operatorname{gcd}\left(s, 3^{m}+1\right)=1$ and $b=0$, we have

$$
\begin{aligned}
\epsilon & =\sum_{x^{2}=1, x \in U} \omega^{\operatorname{Tr}_{1}^{n}\left(-a x^{s}\right)}-\sum_{x^{2}=1, x \in U} \omega^{\left(\frac{3^{m}+1}{2}-1\right) \operatorname{Tr}_{1}^{n}\left(a x^{s}\right)}+1 \\
& =\sum_{x= \pm 1} \omega^{\operatorname{Tr}_{1}^{n}\left(-a x^{s}\right)}-\sum_{x= \pm 1} \omega^{\operatorname{Tr}_{1}^{n}\left(a x^{s}\right)}+1 \\
& =\sum_{x= \pm 1} \omega^{\operatorname{Tr}_{1}^{n}(-a x)}-\sum_{x= \pm 1} \omega^{\operatorname{Tr}_{1}^{n}(a x)}+1 \\
& =1 .
\end{aligned}
$$

Therefore, we have the following result immediately.

Corollary 3.19. Assume the notations given above. Let $p=3, r=2$ and $b=0$. Then $f_{a, 0,2}(x)$ defined by (3.6) is regular bent if and only if

$$
K_{m}\left(a^{3^{m}+1}\right)=0 .
$$

Note that if $3^{m} \equiv 3(\bmod 4)$, one has $\frac{3^{m}+1}{2}-1=1+3+\cdots+3^{m-1} \equiv 1(\bmod 3)$ and $\frac{3^{m}+1}{2}$ is an even integer. Together with $\operatorname{gcd}\left(s, 3^{m}+1\right)=1$, we have

$$
\begin{aligned}
\epsilon & =\sum_{x^{2}=1, x \in U} \omega^{\operatorname{Tr}_{1}^{n}\left(-a x^{s}\right)+b x^{\frac{3^{m}+1}{2}}}-\sum_{x^{2}=1, x \in U} \omega^{\left(\frac{3^{m}+1}{2}-1\right) \operatorname{Tr}_{1}^{n}\left(a x^{s}\right)+b x^{\frac{3^{m}+1}{2}}}+1 \\
& =\sum_{x= \pm 1} \omega^{\operatorname{Tr}_{1}^{n}\left(-a x^{s}\right)+b x^{\frac{3^{m}+1}{2}}}-\sum_{x= \pm 1} \omega^{\operatorname{Tr}_{1}^{n}\left(a x^{s}\right)+b x^{\frac{3^{m}+1}{2}}}+1 \\
& =\omega^{b} \sum_{x= \pm 1} \omega^{\operatorname{Tr}_{1}^{n}(-a x)}-\omega^{b} \sum_{x= \pm 1} \omega^{\operatorname{Tr}_{1}^{n}(a x)}+1 \\
& =1 .
\end{aligned}
$$

Therefore, we have the following result.
Corollary 3.20. Assume the notations given above. Let $b \neq 0, p=3,3^{m} \equiv 3 \bmod (4)$ and $r=2$. Then $f_{a, b, 2}(x)$ defined by (3.6) is regular bent if and only if

$$
K_{m}\left(a^{p^{m}+1}\right)=1-\frac{1}{\cos \frac{2 \pi b}{p}} .
$$

Proof. Since $\epsilon=1$, by Theorem 3.17, we have that $f_{a, b, 2}(x)$ is regular bent if and only if

$$
\left(1-K_{m}\left(a^{p^{m}+1}\right)\right) \cos \frac{2 \pi b}{p}= \begin{cases}4 I \sin \frac{2 \pi b}{p} \sin \frac{2 \pi Q(-a)}{p}+1, & -a \in \mathcal{C}_{0}^{+} \\ 1, & \text { otherwise }\end{cases}
$$

Since $p=3,3^{m} \equiv 3 \bmod (4)$, then $I$ is a complex number in Lemma 2.7 and note that $\left(1-K_{m}\left(a^{p^{m}+1}\right)\right) \cos \frac{2 \pi b}{p}$ is a real number. Thus

$$
\left(1-K_{m}\left(a^{p^{m}+1}\right)\right) \cos \frac{2 \pi b}{p}=-4 I \sqrt{-1} \sin \frac{2 \pi b}{p} \sin \frac{2 \pi Q(-a)}{p}+1
$$

if and only if $Q(-a)=0$, which contradicts with $-a \in \mathcal{C}_{0}^{+}$. That is to say $f_{a, b, 2}(x)$ can not be bent if $-a \in \mathcal{C}_{0}^{+}$. This finishes the proof.

## 4 Concluding Remarks

In this paper, several new classes of binary and $p$-ary bent functions with Dillon exponents are obtained. The bentness of all these functions are characterized by some exponential sums. Moreover, some of results obtained in this paper generalize the work of [10].

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