

Secret sharing schemes based on additive codes over $GF(4)$

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Abstract A secret sharing scheme (SSS) was introduced by Shamir in 1979 using polynomial interpolation. Later it turned out that it is equivalent to an SSS based on a Reed-Solomon code. SSSs based on linear codes have been studied by many researchers. However there is little research on SSSs based on additive codes. In this paper, we study SSSs based on additive codes over $GF(4)$ and show that they require at least two steps of calculations to reveal the secret. We also define minimal access structures of SSSs from additive codes over $GF(4)$ and describe SSSs using some interesting additive codes over $GF(4)$ which contain generalized 2-designs.

Keywords access structure · additive codes · generalized t -design · minimal access structure · secret sharing scheme

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1 Introduction

A *secret sharing scheme* (SSS) is a method of distributing a secret to a finite set of participants such that only predefined subsets of the participants can recover the secret. All the participants receive a piece of the secret, known as

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a *share*, in such a way that only qualified subsets of the participants can have access to the secret by pooling the shares of their members.

The first construction of an SSS was done by Shamir [16] and Blakley [2] independently in 1979. Shamir used polynomial interpolation for constructing an SSS, while Blakley used hyperplane geometry. Later Shamir's SSS turned out to be equivalent to an SSS based on a Reed-Solomon code [15]. Some of SSSs were applied to various fields such as cloud computing, controlling nuclear weapons in military, recovering information from multiple servers, and controlling access in banking system.

Since an SSS plays an important role in protecting secret information, it has been studied by several authors (see [5], [7], [12], and [14]). In particular, Massey [14] used linear codes for secret sharing and pointed out the relationship between the access structure and the minimal codewords of the dual code of the underlying code in 1993. However there has been less attention to SSSs based on additive codes. Since additive codes include linear codes, we raise an intriguing question whether SSSs based on additive codes have more advantage than SSSs based on linear codes.

Ding et al. [5] remark the following. Normally the weight distribution of a code is very hard to determine and that of only a few classes of codes is known. As an SSS can be constructed from any error-correcting linear code, what matters is how we are going to determine the access structure. The access structure of SSSs based on error-correcting codes depends on the weight distribution of their dual codes. To determine the access structure for the SSSs we need more information than the weight distribution. This makes it difficult to determine the access structure of SSSs based on codes, as determining the weight distribution of codes is difficult.

In this paper we introduce SSSs based on linear codes with an example in Section 2. In Section 3.2, we define SSSs based on additive codes and show that they require two steps of calculations, while the SSSs based on linear codes require only one step of calculation. In Section 3.3, we define minimal access structures of SSSs based on additive codes over $GF(4)$, which is developed from Proposition 2 in [6]. We also describe SSSs based on a hexacode, a dodecacode over $GF(4)$ which is described and analyzed in [11], and S_{18} . We determine the access structure of the SSSs and prove their properties. The access structure for these SSSs is more abundant than that of the SSSs based on linear codes. We are able to determine the access structure of these SSSs because the structure of the underlying additive codes is thoroughly understood.

2 Some preliminaries

Let $GF(q)$ be a finite field with $q = p^r$ elements, where p is a prime and r is a positive integer. The *Hamming distance* between two vectors $\mathbf{x}, \mathbf{y} \in GF(q)^n$ is defined to be the number of coordinates in which \mathbf{x} and \mathbf{y} differ. Note that the *minimum distance* d of a code C is the smallest nonzero distance between two distinct codewords and is important in determining the error-correcting

capability of C ; the higher the minimum distance, the more errors the code can correct. In general, it can detect up to $d-1$ errors and correct up to $\lfloor \frac{d-1}{2} \rfloor$ errors.

The *Hamming weight* of a vector \mathbf{c} in $GF(q)^n$ denoted by $wt(\mathbf{c})$ is the total number of nonzero coordinates. Let A_i , also denoted by $A_i(C)$, be the number of codewords of weight i in C . The list of A_i for $0 \leq i \leq n$ is called the *weight distribution* of C . An $[n, k, d]$ code C is a linear subspace of $GF(q)^n$ with dimension k and minimum nonzero Hamming weight d . A generator matrix G for an $[n, k, d]$ code C is any $k \times n$ matrix G whose rows form a basis for C .

We refer to [17]. Let $G = (\mathbf{g}_0, \mathbf{g}_1, \dots, \mathbf{g}_{n-1})$ be a generator matrix of an $[n, k, d]$ code. We assume that none of \mathbf{g}_i 's is the zero vector. Let \mathbf{g}_i ($1 \leq i \leq n-1$) be a column vector. In an SSS constructed from an $[n, k, d]$ linear code C , the secret is an element of $GF(q)$ and there are $n-1$ participants P_1, P_2, \dots, P_{n-1} and a dealer P_0 . To compute the shares with respect to the secret s , the dealer randomly takes an element $\mathbf{u} = (u_0, u_1, \dots, u_{k-1}) \in GF(q)^k$ such that $s = \mathbf{u}\mathbf{g}_0$. The dealer treats \mathbf{u} as an information vector and computes the corresponding codeword

$$\mathbf{t} = (t_0, t_1, \dots, t_{n-1}) = \mathbf{u}G$$

and the dealer gives the share t_i to participant P_i as share for each $i \geq 1$.

Since $t_0 = \mathbf{u}\mathbf{g}_0 = s$, a set of shares $\{t_{i_1}, t_{i_2}, \dots, t_{i_m}\}$ determines the secret s if and only if the column \mathbf{g}_0 of the generating matrix G is a linear combination of the columns $\{g_{i_1}, g_{i_2}, \dots, g_{i_m}\}$ of G .

Lemma 1 ([17]) *Let C be an $[n, k, d]$ linear code over the finite field $GF(q)$ and let C^\perp be its dual code. In the secret sharing scheme based on C , a subset of shares $\{t_{i_1}, t_{i_2}, \dots, t_{i_m}\}$, $1 \leq i_1 \leq \dots \leq i_m \leq n-1$, determines the secret if and only if there is a codeword*

$$(1, 0, \dots, 0, c_{i_1}, 0, \dots, 0, c_{i_m}, 0, \dots, 0) \quad (1)$$

in C^\perp with $c_{i_j} \neq 0$ for at least one j .

We explain how the secret is recovered using (1). If there is a codeword of (1) in C^\perp , then the vector \mathbf{g}_0 is a linear combination of $\mathbf{g}_{i_1}, \dots, \mathbf{g}_{i_m}$, i.e.,

$$\mathbf{g}_0 = \sum_{j=1}^m x_j \mathbf{g}_{i_j},$$

where $x_j \in GF(q)$ for $1 \leq j \leq m$.

Then the secret s is recovered by computing

$$s = \sum_{j=1}^m x_j t_{i_j}.$$

Here we think about a case of some malicious behaviors lying among participants, called *cheaters*. They modify their shares in order to cheat. In this case, we make use of detection of errors up to $d - 1$ and correction of errors up to $\lfloor \frac{d-1}{2} \rfloor$. The errors being considered as modified shares, SSSs based on error-correcting codes are able to detect up to $d - 1$ cheaters and correct up to $\lfloor \frac{d-1}{2} \rfloor$ cheaters.

If a group of participants can recover the secret by pooling their shares, then any group of participants containing this group can also recover the secret.

Definition 1 An *access group* is a subset of a set of participants that can recover the secret from its shares. A collection Γ of access groups of participants is called an *access structure* of the scheme. An element $A \in \Gamma$ is called a *minimal access group* if no element of Γ is a proper subset of A . Hence a set is a minimal access group if it can recover the secret but no proper subset can recover the secret. We let $\bar{\Gamma} = \{A | A \text{ is a minimal access group}\}$. We call $\bar{\Gamma}$ the *minimal access structure*.

In general, determining the minimal access structure is a difficult problem [5].

Definition 2 The *support* of a vector $\mathbf{c} = (c_0, \dots, c_{n-1}) \in GF(q)^n$ is defined by

$$\text{supp}(\mathbf{c}) = \{0 \leq i \leq n - 1 \mid c_i \neq 0\}.$$

Let \mathbf{c}_1 and \mathbf{c}_2 be two codewords of a code C . We say that \mathbf{c}_1 *covers* \mathbf{c}_2 if $\text{supp}(\mathbf{c}_2) \subseteq \text{supp}(\mathbf{c}_1)$.

If a nonzero codeword \mathbf{c} covers only its scalar multiples, but no other codewords, then it is called a *minimal codeword*.

Theorem 1 [6] *Let C be an $[n, k; q]$ code, and let $G = (\mathbf{g}_0, \mathbf{g}_1, \dots, \mathbf{g}_{n-1})$ be its generator matrix, where all \mathbf{g}_i 's are nonzero. If each nonzero codeword of C is minimal, then in the secret sharing scheme based on C^\perp , there are altogether q^{k-1} minimal access groups. In addition, we have the following:*

- *If \mathbf{g}_i is a scalar multiple of \mathbf{g}_0 , $1 \leq i \leq n - 1$, then participant P_i must be in every minimal access set. Such a participant is called a dictatorial participant.*
- *If \mathbf{g}_i is not a scalar multiple of \mathbf{g}_0 , $1 \leq i \leq n - 1$, then participant P_i must be in $(q - 1)q^{k-2}$ out of q^{k-1} minimal access groups.*

Definition 3 A t - (v, k, λ) *design* or briefly a t -*design*, is a pair $(\mathcal{P}, \mathcal{B})$ where \mathcal{P} is a set of v elements, called *points*, and \mathcal{B} is a collection of distinct subsets of \mathcal{P} of size k , called *blocks*, such that every subset of points of size t is contained in precisely λ blocks.

For linear codes with a special weight distribution, a powerful result of the Assmus-Mattson theorem guarantees that a set of codewords with a fixed weight holds a t -design [1]. The Assmus-Mattson Theorem has been the main tool in discovering designs in codes.

Theorem 2 (Assmus-Mattson [10]) *Let C be an $[n, k, d]$ code over $GF(q)$. Suppose C^\perp has minimum weight d^\perp . Let w be the largest integer with $w \leq n$ satisfying*

$$w - \lfloor \frac{w+q-2}{q-1} \rfloor < d.$$

(So $w = n$ when $q = 2$.) Define w^\perp analogously using d^\perp . Suppose that $A_i = A_i(C)$ and $A_i^\perp = A_i(C^\perp)$, for $0 \leq i \leq n$, are the weight distributions of C and C^\perp , respectively. Fix a positive integer t with $t < d$, and let s be the number of i with $A_i^\perp \neq 0$ for $0 < i \leq n - t$. Suppose $s \leq d - t$. Then:

- (i) the vectors of weight i in C hold a t -design provided $A_i \neq 0$ and $d \leq i \leq w$, and
- (ii) the vectors of weight i in C^\perp hold a t -design provided $A_i^\perp \neq 0$ and $d^\perp \leq i \leq \min\{n - t, w^\perp\}$.

A t - (v, k, λ) design is also an i - (v, k, λ_i) design for $0 \leq i \leq t$. We can get λ_i by the formula [10]:

$$\lambda_i = \lambda \binom{v-i}{t-i} / \binom{k-i}{t-i}. \quad (2)$$

Let $W_C = \sum A_i y^i$. Let D_i denote the 1-design formed from the vectors of weight i , and $\lambda_s(D_i)$ denote λ_s for that particular design. Here λ_s denotes the number of blocks that are incident with a given s -tuple of points for $s \leq t$.

Corollary 1 ([7]) *The access groups in the secret sharing scheme based on a binary self-dual code C have the following size distribution generating function when groups of each size form a 1-design :*

$$\sum_i \lambda_1(D_i) y^{i-1}. \quad (3)$$

Now let us consider the accessibility of an access structure. Let $P = \{P_1, \dots, P_m\}$ be a set of m participants and let \mathcal{A}_P be the set of all access structures on P .

Definition 4 ([3]) The *accessibility index* on P is the map $\delta_P : \mathcal{A}_P \rightarrow \mathbb{R}$ given by

$$\delta_P(\Gamma) = \frac{|\Gamma|}{2^m} \text{ for } \Gamma \in \mathcal{A}_P$$

where $m = |P|$. The number $\delta_P(\Gamma)$ will be called the *accessibility degree* of structure Γ .

$\delta_P(\Gamma)$ may be interpreted as the probability of a random coalition in P to be authorized when each participant has a probability $1/2$ to belong to it. As it is obvious, $\delta_P(\Gamma) = 0$ iff $\Gamma = \emptyset$. Otherwise, $0 < \delta_P(\Gamma) < 1$, and $|\Gamma| < |\Gamma'|$ implies $\delta_P(\Gamma) < \delta_P(\Gamma')$.

Since supports of each weight in a code holding a 1-design determine the size of the access structure Γ , the accessibility degree of Γ for the SSS based on the code can be defined as follows :

$$\delta_P(\Gamma) = \frac{1}{2^m} \sum_i \lambda_1(D_i).$$

Remark 1 The Gleason-Pierce-Ward Theorem [10] provides the main motivation for studying self-dual codes over $GF(2)$, $GF(3)$, and $GF(4)$ since these codes have the property that they are divisible. When a code C is divisible by $c > 1$, it implies that all codewords have weights divisible by an integer c , which is called a divisor of C .

Example 1 Here we introduce an SSS from [24, 12, 8] Golay code as given in [7]. The weight enumerator of the length 24 Golay code is:

$$1 + 759y^8 + 2576y^{12} + 759y^{16} + y^{24}. \quad (4)$$

Note that the supports of any nonzero weight of the [24, 12, 8] Golay code form a 5-design. It is easy to calculate for $\lambda_1(D_8) = 253$, $\lambda_1(D_{12}) = 1288$, and $\lambda_1(D_{16}) = 506$. These groups together with the entire group, comprise the 2048 elements of the access structure. Each of the 253 groups of size 8 must be in the minimal access structure. Additionally, each of the 1288 groups of size 12 must be in the minimal access structure because if the support of a weight 8 vector were a subset of the support of a weight 12 vector then the sum of these vectors would have weight 4, which is a contradiction. The group of size 24 is not in the minimal access structure. We note that no weight 16 vector can have a support containing the support of weight 12 vector since it would produce a weight 4 vector in the code, which is a contradiction. There are 253 weight 16 vectors whose support cannot be in the minimal access structure and 253 that are in the minimal access structure. This gives the following.

Theorem 3 ([7]) In the secret sharing scheme produced from the extended Golay code we have the following :

- *The access structure consists of 253 groups of size 7, 1288 groups of size 11, 506 groups of size 15 and 1 group of size 23.*
- *The minimal access structure consists of the 253 groups of size 7, the 1288 groups of size 11, and 253 groups of size 15.*
- *No group of size less than 7 can determine the secret.*

3 SSSs based on additive codes over $GF(4)$

3.1 Introduction to additive codes over $GF(4)$

An *additive code* C over $GF(4)$ of length n is an additive subgroup of $GF(4)^n$ (see [11] for details). Since C is a vector space over $GF(2)$, it has a basis consisting of k ($0 \leq k \leq 2n$) vectors whose entries are in $GF(4)$. We call C

an $(n, 2^k)$ code. A *generator matrix* of \mathcal{C} is a $k \times n$ matrix with entries in $GF(4)$ whose rows are a basis of \mathcal{C} . The *weight* of \mathbf{c} , denoted as $\text{wt}(\mathbf{c})$, in \mathcal{C} is the number of nonzero components of \mathbf{c} . The minimum weight d of \mathcal{C} is the smallest weight of any nonzero codeword in \mathcal{C} . If \mathcal{C} is an $(n, 2^k)$ additive code of minimum weight d , \mathcal{C} is called an $(n, 2^k, d)$ code. In order to define an inner product on additive codes we define the *trace map*, i.e., for x in $GF(4)$, $\text{Tr}(x) = x + x^2 \in GF(2)$. We now define the *trace inner product* of two vectors $\mathbf{x} = (x_1 x_2 \cdots x_n)$ and $\mathbf{y} = (y_1 y_2 \cdots y_n)$ in $GF(4)^n$ to be

$$\mathbf{x} \star \mathbf{y} = \sum_{i=1}^n \text{Tr}(x_i \overline{y_i}) \in GF(2),$$

where $\overline{y_i}$ denotes the conjugate of y_i . Note that $\text{Tr}(x_i \overline{y_i}) = 1$ if and only if x_i and y_i are nonzero distinct elements in $GF(4)$.

If \mathcal{C} is an additive code, its *dual*, denoted by \mathcal{C}^\perp , is the additive code $\{\mathbf{x} \in GF(4)^n \mid \mathbf{x} \star \mathbf{c} = 0 \text{ for all } \mathbf{c} \in \mathcal{C}\}$. If \mathcal{C} is an $(n, 2^k)$ code, then \mathcal{C}^\perp is an $(n, 2^{2n-k})$ code. As usual, \mathcal{C} is called *self-dual* if $\mathcal{C} = \mathcal{C}^\perp$. We note that if \mathcal{C} is self-dual, \mathcal{C} is an $(n, 2^n)$ code.

3.2 SSSs based on additive codes over $GF(4)$

Let $G = (\mathbf{g}_0, \mathbf{g}_1, \dots, \mathbf{g}_{n-1})$ be a generator matrix of an $(n, 2^k)$ code over $GF(4)$, where \mathbf{g}_i denotes the generic column of G . We assume that none of \mathbf{g}_i 's is the zero vector. In an SSS constructed from an $(n, 2^k)$ code \mathcal{C} , the secret is an element of $GF(4)$, and $n - 1$ participants P_1, P_2, \dots, P_{n-1} and a dealer P_0 are involved. To compute the shares with respect to the secret s , the dealer randomly takes an element $\mathbf{u} = (u_0, u_1, \dots, u_{k-1}) \in GF(2)^k$ such that $s = \mathbf{u} \mathbf{g}_0$. There are altogether 2^{k-2} vectors $\mathbf{u} \in GF(2)^k$ if s is in $GF(4)$, or 2^{k-1} if s an element in $\{0, 1\}$, $\{0, \omega\}$, or $\{0, \overline{\omega}\}$. The dealer then treats \mathbf{u} as an information vector and computes the corresponding codeword

$$\mathbf{t} = (t_0, t_1, \dots, t_{n-1}) = \mathbf{u} G$$

and the dealer gives the share t_i to participant P_i as share for each $i \geq 1$.

Lemma 2 *Let \mathcal{C} be an $(n, 2^k)$ code over $GF(4)$ and \mathcal{C}^\perp its dual code defined by the trace inner product. Let*

$$\begin{aligned} H_1 &= \left\{ x \mid x = (1, \dots, 0, x_{i_1}, 0, \dots, 0, x_{i_m}, 0, \dots, 0) \in \mathcal{C}^\perp \right. \\ &\quad \left. \text{with } x_0 = 1, x_{i_j} \neq 0 \text{ for at least one } j \right\}, \\ H_2 &= \left\{ y \mid y = (\omega, \dots, 0, y_{i_1}, 0, \dots, 0, y_{i_l}, 0, \dots, 0) \in \mathcal{C}^\perp \right. \\ &\quad \left. \text{with } y_0 = \omega, y_{i_j} \neq 0 \text{ for at least one } j \right\}, \\ H_3 &= \left\{ z \mid z = (\overline{\omega}, \dots, 0, z_{i_1}, 0, \dots, 0, z_{i_r}, 0, \dots, 0) \in \mathcal{C}^\perp \right. \\ &\quad \left. \text{with } z_0 = \overline{\omega}, z_{i_j} \neq 0 \text{ for at least one } j \right\}. \end{aligned} \tag{5}$$

In the secret sharing scheme based on C , two subsets of shares $\{t_{i_1}, t_{i_2}, \dots, t_{i_m}\}$ and $\{t_{i_1}, t_{i_2}, \dots, t_{i_l}\}$, for $1 \leq i_1 < \dots < i_m \leq n-1$ and $1 \leq i_1 < \dots < i_l \leq n-1$, determine the secret if and only if there are at least two codewords from distinct sets among H_i 's, $1 \leq i \leq 3$.

Proof (\Leftarrow) Suppose there are at least two codewords in C^\perp as in (5). Then we get two of the three equations from trace inner product as follows:

$$\begin{aligned} (t_0 + t_0^2) + (t_{i_1}\bar{x}_1 + (t_{i_1}\bar{x}_1)^2) + \dots + (t_{i_m}\bar{x}_m + (t_{i_m}\bar{x}_m)^2) &= 0, \\ (t_0\bar{\omega} + (t_0\bar{\omega})^2) + (t_{i_1}\bar{y}_1 + (t_{i_1}\bar{y}_1)^2) + \dots + (t_{i_l}\bar{y}_l + (t_{i_l}\bar{y}_l)^2) &= 0, \\ (t_0\omega + (t_0\omega)^2) + (t_{i_1}\bar{z}_1 + (t_{i_1}\bar{z}_1)^2) + \dots + (t_{i_r}\bar{z}_r + (t_{i_r}\bar{z}_r)^2) &= 0. \end{aligned} \quad (6)$$

Since $s = ug_0 = t_0$, the equation can be rewritten as

$$\begin{aligned} s + s^2 &= \sum_{j=1}^m (t_{i_j}\bar{x}_j + (t_{i_j}\bar{x}_j)^2) \in GF(2), \\ s\bar{\omega} + (s\bar{\omega})^2 &= \sum_{j=1}^l (t_{i_j}\bar{x}_j + (t_{i_j}\bar{x}_j)^2) \in GF(2), \\ s\omega + (s\omega)^2 &= \sum_{j=1}^l (t_{i_j}\bar{x}_j + (t_{i_j}\bar{x}_j)^2) \in GF(2). \end{aligned}$$

Let $\alpha_1 = s + s^2$, $\alpha_2 = s\bar{\omega} + (s\bar{\omega})^2$, and $\alpha_3 = s\omega + (s\omega)^2$. Now the secret s can be recovered using two values of α_i 's based on Table 1. For example, if $\alpha_1 = 0$ and $\alpha_2 = 1$, then the secret s is uniquely determined as 1.

As we can see in Table 1, we do not need all the three values of α_i 's since two values of α_i 's are sufficient to determine the secret s . Now we can say we recover the secret s with two values of α_i 's, where $i = 1, 2$, or 3.

Table 1 Recovering the secret s from α_i 's

$\alpha_1 = s + s^2$	$\alpha_2 = s\bar{\omega} + (s\bar{\omega})^2$	$\alpha_3 = s\omega + (s\omega)^2$	s
0	0	0	0
0	1	1	1
1	0	1	ω
1	1	0	$\bar{\omega}$

(\Rightarrow) Suppose there are two subsets of shares $\{t_{i_1}, t_{i_2}, \dots, t_{i_m}\}$ and $\{t_{i_1}, t_{i_2}, \dots, t_{i_l}\}$, for $1 \leq i_1 < \dots < i_m \leq n-1$ and $1 \leq i_1 < \dots < i_l \leq n-1$, that determine the secret s .

First, we note the following:

$$\begin{aligned}
(\mathbf{u}\mathbf{g}_i)^2 &= ((u_0, u_1, \dots, u_{k-1})(g_{0i}, g_{1i}, \dots, g_{(k-1),i})^T)^2 \\
&= (u_0g_{0i} + u_1g_{1i} + \dots + u_{k-1}g_{(k-1),i})^2 \\
&= u_0^2g_{0i}^2 + u_1^2g_{1i}^2 + \dots + u_{k-1}^2g_{(k-1),i}^2 \\
&= (u_0^2 + u_1^2 + \dots + u_{k-1}^2)(g_{0i}^2 + g_{1i}^2 + \dots + g_{(k-1),i}^2) \\
&= (u_0^2, u_1^2, \dots, u_{k-1}^2)(g_{0i}^2, g_{1i}^2, \dots, g_{(k-1),i}^2)^T \\
&= (u_0, u_1, \dots, u_{k-1})(g_{0i}^2, g_{1i}^2, \dots, g_{(k-1),i}^2)^T
\end{aligned} \tag{7}$$

Here $\mathbf{u} = (u_0, u_1, \dots, u_{k-1}) = (u_0^2, u_1^2, \dots, u_{k-1}^2)$ since $\mathbf{u} \in GF(2)^k$. Letting $\mathbf{g}_i^2 = (g_{0i}^2, g_{1i}^2, \dots, g_{(k-1),i}^2)^T$ for convenience, we have $(\mathbf{u}\mathbf{g}_i)^2 = \mathbf{u}\mathbf{g}_i^2$.

Now we can rewrite α_i 's in the following way :

$$\begin{aligned}
\alpha_1 &= \sum_{j=1}^m (t_{i_j}\bar{x}_j + (t_{i_j}\bar{x}_j)^2) = \mathbf{u} \sum_{j=1}^m (x_j\mathbf{g}_{i_j} + \bar{x}_j\mathbf{g}_{i_j}^2), \\
\alpha_2 &= \sum_{j=1}^l (t_{i_j}\bar{y}_j + (t_{i_j}\bar{y}_j)^2) = \mathbf{u} \sum_{j=1}^l (x_j\mathbf{g}_{i_j} + \bar{x}_j\mathbf{g}_{i_j}^2), \\
\alpha_3 &= \sum_{j=1}^r (t_{i_j}\bar{z}_j + (t_{i_j}\bar{z}_j)^2) = \mathbf{u} \sum_{j=1}^r (x_j\mathbf{g}_{i_j} + \bar{x}_j\mathbf{g}_{i_j}^2).
\end{aligned}$$

We can determine two of the values of α_i 's, $1 \leq i \leq 3$ by the two sets of shares, $\{t_{i_1}, t_{i_2}, \dots, t_{i_m}\}$ and $\{t_{i_1}, t_{i_2}, \dots, t_{i_l}\}$, for $1 \leq i_1 < \dots < i_m \leq n-1$, $1 \leq i_1 < \dots < i_l \leq n-1$, if and only if

$$\begin{aligned}
\mathbf{g}_0 + \mathbf{g}_0^2 &= \sum_{j=1}^m (x_j\mathbf{g}_{i_j} + \bar{x}_j\mathbf{g}_{i_j}^2), \quad \bar{\omega}\mathbf{g}_0 + \omega\mathbf{g}_0^2 = \sum_{j=1}^l (x_j\mathbf{g}_{i_j} + \bar{x}_j\mathbf{g}_{i_j}^2), \\
\omega\mathbf{g}_0 + \bar{\omega}\mathbf{g}_0^2 &= \sum_{j=1}^r (x_j\mathbf{g}_{i_j} + \bar{x}_j\mathbf{g}_{i_j}^2).
\end{aligned} \tag{8}$$

We can find x_j 's by solving the linear equations and get two values of α_i 's. Using these two values of α_i 's we can recover the secret s by Table 1. Hence there exist at least two codewords from distinct sets among H_i 's, $1 \leq i \leq 3$ if we recover the secret s using two subsets of shares, $\{t_{i_1}, t_{i_2}, \dots, t_{i_m}\}$ and $\{t_{i_1}, t_{i_2}, \dots, t_{i_l}\}$. \square

Now we need to define an access group and an access structure for SSS based on additive codes over $GF(4)$.

Let

$\Gamma_{H_1} = \{\text{the set of supports for } x \in H_1 \text{ excluding 1 from each support}\},$

$\Gamma_{H_2} = \{\text{the set of supports for } y \in H_2 \text{ excluding 1 from each support}\},$

$\Gamma_{H_3} = \{\text{the set of supports for } z \in H_3 \text{ excluding 1 from each support}\}.$

The access structure for a linear code based SSS is a set of supports of vectors in \mathcal{C}^\perp with $c_0 = 1$, which is same to Γ_{H_1} . The access structures from additive codes over $GF(4)$ are different from those from linear codes. To recover the secret s , we need at least two sets among Γ_{H_1} , Γ_{H_2} , or Γ_{H_3} for an access structure. We obtain the values of α_i and α_j from two elements of Γ_{H_i} and Γ_{H_j} , $i \neq j$, respectively. With the two values, we can recover the secret s using the table above.

Since this process requires at least two steps of calculations to reveal the secret, we call this process as a *2-step* SSS. On the other hand, the previous SSS can be regarded as a *1-step* SSS.

We need the Assmus-Mattson Theorem for additive codes over $GF(4)$ which gives designs with possibly repeated blocks to define our process.

Theorem 4 ([11]) *Let \mathcal{C} be an additive $(n, 2^k)$ code over $GF(4)$ with minimum weight d . Let \mathcal{C}^\perp be its dual $(n, 2^{2n-k})$ code with minimum weight d' . Let $0 < t < d$. Let s be the number of weights $B_i \neq 0$ in \mathcal{C}^\perp where $0 < i \leq n - t$. Suppose that $s \leq d - t$. Then the following hold.*

- (i) *For each weight u ($d \leq u \leq n$), the set of supports of codewords of weight u in \mathcal{C} holds a t -design with possibly repeated blocks.*
- (ii) *The set of supports of vectors of weight w in \mathcal{C}^\perp where $B_w \neq 0$ and $d' \leq w \leq n - t$ hold a t -design with possibly repeated blocks.*
- (iii) *The supports of minimum weight vectors are either simple blocks or have repetition number 3.*

Corollary 2 ([11]) *Let $n_i := 6m + 2(i - 1)$ with $m \geq 1$ any integer and $i = 1, 2$, or 3 . Let \mathcal{C} be an extremal additive even self-dual $(n_i, 2^{n_i})$ code over $GF(4)$ with minimum weight $d = 2m + 2 \geq 6$. Then the vectors of each weight w in \mathcal{C} where $A_w \neq 0$ and $d \leq w \leq n_i$ hold a $(7 - 2i)$ -design with possibly repeated blocks.*

Lemma 3 *Let \mathcal{C} be an additive even $(n, 2^k)$ self-dual code over $GF(4)$. Then the supports of codewords for all non-trivial weights hold a 1-design with possible repeated blocks if $d \geq \frac{n+2}{3}$.*

Proof An additive $(n, 2^k)$ self-dual code over $GF(4)$ has $\frac{n}{2} - 1$ possible non-trivial weights. Then $\frac{d}{2} - 1$ of these possible weights have no vectors since d is the minimum weight. Since $d \geq \frac{n+2}{3}$, from this we get the following inequality which satisfies Assmus-Mattson Theorem:

$$d - 1 \geq \left(\frac{n}{2} - 1\right) - \left(\frac{d}{2} - 1\right).$$

Thus the supports of codewords for all non-trivial weights hold a 1-design with possibly repeated blocks.

3.3 SSSs based on extremal additive even self-dual codes over $GF(4)$

Up to now we have introduced a different method of defining access structures based on additive codes. We have shown that the access structures are nicely constructed in this way. However there might be some repeated blocks in these access structures. We, hence, employ the notion of a generalized t -design from [4] to resolve this issue.

The generalized t -design is to count the number of groups in an access structure for an additive code based SSS. We will, first of all, redefine *covering* an element for the generalized t -design [4].

Definition 5 Let $G = GF(4)$, the set of n -tuples of $GF(4)$. An element \mathbf{a} of G is said to be *componentwisely covered* (abbr. *c-covered*) by an element \mathbf{b} of G if each nonzero component a_i of \mathbf{a} is equal to the corresponding component b_i of \mathbf{b} ; we denote this by $\mathbf{a} \leq \mathbf{b}$. For example, $\mathbf{a} = (1, 1, \omega, 0)$ is c-covered by $\mathbf{b} = (1, 1, \omega, \bar{\omega})$.

Definition 6 A subset S of G is called a *generalized t -design of type $q - 1$* , with parameters $t(n, k, \mu_t)$, $0 \leq t \leq k \leq n$, $\mu_t \geq 1$, if the following two conditions are satisfied:

- (i) all elements of S have the same weight k ,
- (ii) each element of weight t in G is c-covered by a constant number μ_t of elements of S . If a subset S of G holds a generalized t -design of type $q - 1$, then it holds a generalized $(t - 1)$ -design of type $q - 1$.

In the binary case ($q = 2$), this is the same as a classical t -design without repeated blocks.

For a given code \mathcal{C} of length n and for an element e of $G = GF(4)^n$, we denote by $\mu(p, e)$ the number of codewords of weight p that c-cover e and $\mu_i(p, e)$ the number of codewords of weight p in Γ_{H_i} that c-cover e . Trivially, if $p < wt(e)$, then $\mu(p, e) = 0$ and $\mu_i(p, e) = 0$.

Delsarte's theorem for any finite alphabet is given as follows.

Theorem 5 ([4]) *Let \mathcal{C} be a q -ary code of dual distance d' . Let t be an integer, $1 \leq t \leq d'$, such that the number of weights of \mathcal{C} that are at least equal to t is at most equal to $d' - t$. Then each set of codewords of a given weight $\geq t$ is a generalized t -design of type $q - 1$.*

Now we obtain generalized t -designs from additive codes over $GF(4)$.

Corollary 3 ([8]) *Let \mathcal{C} be an extremal even additive self-dual code over $GF(4)$ of length $n = 6m$ (respectively, $n = 6m + 2$). Then the set of codewords of weight w in \mathcal{C} with $A_w \neq 0$ forms a generalized 2-design (respectively, 1-design) of type 3.*

Since two elements from two different sets among Γ_{H_i} 's, $i \in \{1, 2, 3\}$, are sufficient to recover a secret, we are going to consider all the combinations of only two distinct Γ_{H_i} 's when defining access structures. That is, all the pairs

(s_i, s_j) for $s_i \in \Gamma_{H_i}$ and $s_j \in \Gamma_{H_j}$ with $i \neq j$, comprise all the elements of the access structure. An element $(s_i, s_j) \in \Gamma$ is called a *minimal access group* if neither s_i nor s_j c-covers any other elements in Γ_{H_i} and Γ_{H_j} , respectively. We let $\bar{\Gamma} = \{(s_i, s_j) | (s_i, s_j) \text{ is a minimal access group}\}$ and call it by the *minimal access structure*.

Theorem 6 *The access structure of this secret sharing scheme is given by*

$$\Gamma = \{(x, y) | x \in \Gamma_{H_i} \text{ and } y \in \Gamma_{H_j}, \text{ where } i \neq j \text{ and } i, j \in \{1, 2, 3\}\}. \quad (9)$$

The number of parties in the scheme is $n - 1$ and the access structure has the following properties:

- *Any group of size less than $d - 1$ cannot be used to recover the secret.*
- *There are $\mu_i(p, e1)\mu_j(q, e1)$ pairs of groups of size $(p-1, q-1)$ in $\Gamma_{H_i} \times \Gamma_{H_j}$, $i \neq j$, that can recover the secret, where $e1$ is any vector of weight 1 in $GF(4)^n$.*
- *When the parties come together, up to $\lfloor \frac{d-1}{2} \rfloor$ cheaters can be found in each group.*
- *Γ is a minimal access structure if for every element (x, y) in Γ , no element of Γ_{H_i} and Γ_{H_j} are subsets of x and y , respectively.*

Proof The first property is trivial from the definition of Γ_{H_i} 's. The minimum size of any group in Γ_{H_i} is greater or equal to $d - 1$. Thus any group of size less than $d - 1$ cannot be used to recover the secret. We can get the second property from the proof of Lemma 2. Since any element in $\Gamma_{H_i} \times \Gamma_{H_j}$, $i \neq j$, can recover the secret s , there are $\mu_i(p, e1)\mu_j(q, e1)$ pairs of groups of size $(p - 1, q - 1)$ that can recover the secret. The third property comes from the error-correcting capability of additive codes over $GF(4)$. The fourth property is from the definition of the minimal access structure. \square

Corollary 4 *The pairs of groups from $\Gamma_{H_i} \times \Gamma_{H_j}$, $i \neq j$, in this SSS based on an additive self-dual code \mathcal{C} have the following size distribution generating function :*

$$\sum_p \sum_q \mu_i(p, e1)\mu_j(q, e1)y^{(p-1, q-1)}, \quad (10)$$

where p and q denote the weights of codewords.

Furthermore, the size distribution generating function for the access structure is as follows:

$$\sum_{i \neq j} \sum_p \sum_q \mu_i(p, e1)\mu_j(q, e1)y^{(p-1, q-1)}. \quad (11)$$

Proof Since $|\Gamma_{H_i}| = \sum_p \mu_i(p, e1)$, $|\Gamma_{H_i} \times \Gamma_{H_j}| = \sum_p \sum_q \mu_i(p, e1)\mu_j(q, e1)$ for $i \neq j$. From this we get the size distribution generating function of $\Gamma_{H_i} \times \Gamma_{H_j}$. Since $|\cup_{i \neq j} \Gamma_{H_i} \times \Gamma_{H_j}| = \sum_{i \neq j} \sum_p \sum_q \mu_i(p, e1)\mu_j(q, e1)$, we get the size distribution generating function for the access structure as (11). \square

Note that we redefined minimal access group of SSSs from additive codes over $GF(4)$ considering its distinct way of recovering the secret s . Thus we can now develop Theorem 1 for SSSs based on additive codes over $GF(4)$.

Theorem 7 *Let C be an $(n, 2^k)$ code over $GF(4)$, and let $G = (\mathbf{g}_0, \mathbf{g}_1, \dots, \mathbf{g}_{n-1})$ be its generator matrix. If each nonzero codeword of C is a minimal vector, then in the secret sharing scheme based on C^\perp , there are altogether $3 \cdot 2^{2k-4}$ minimal access groups if the secret $s \in GF(4)$. In addition, we have the following:*

- If \mathbf{g}_i is the same vector to \mathbf{g}_0 , $1 \leq i \leq n-1$, then participant P_i must be in every Γ_{H_k} , $1 \leq k \leq 3$. Such a participant is called a dictatorial participant in SSS based on $GF(4)$.
- If \mathbf{g}_i is not same to \mathbf{g}_0 , $1 \leq i \leq n-1$, then participant P_i must be in $3^3 \cdot 2^{2k-8}$ out of $3 \cdot 2^{2k-4}$ minimal access groups.

Proof At the beginning of this section, we assumed that none of \mathbf{g}_i 's is the zero vector. Hence $\mathbf{g}_0 \neq 0$. Thus $\mathbf{u}\mathbf{g}_0$ takes on each element of $GF(4)$ exactly 2^{k-2} times when \mathbf{u} ranges over all elements of $GF(2)^k$. Hence there are $2^k - 2^{k-2}$ codewords in C with a nonzero component in its first coordinate. Since each nonzero codeword is a minimal vector, a codeword c -covers another one if and only if they are the same vector. Hence the total number of minimal codewords is $2^k - 2^{k-2} = 3 \cdot 2^{k-2}$.

Since

$$\begin{aligned} |\Gamma_{H_1} = \{\mathbf{c} | c_0 = \mathbf{u}\mathbf{g}_0 = 1, \mathbf{c} \in C\}| &= |\Gamma_{H_2} = \{\mathbf{c} | c_0 = \mathbf{u}\mathbf{g}_0 = \omega, \mathbf{c} \in C\}| \\ &= |\Gamma_{H_3} = \{\mathbf{c} | c_0 = \mathbf{u}\mathbf{g}_0 = \bar{\omega}, \mathbf{c} \in C\}| = \frac{3 \cdot 2^{k-2}}{3} = 2^{k-2}, \end{aligned}$$

the number of minimal access groups is

$$\sum_{i \neq j} |\Gamma_{H_i} \times \Gamma_{H_j}| = 3 \times 2^{k-2} \times 2^{k-2} = 3 \cdot 2^{2k-4}.$$

If $\mathbf{g}_i = \mathbf{g}_0$, $1 \leq i \leq n-1$, then $\mathbf{u}\mathbf{g}_0 = a \neq 0$ implies $\mathbf{u}\mathbf{g}_i = a$. Thus the participant P_i is involved in every Γ_{H_k} , $1 \leq k \leq 3$. If \mathbf{g}_0 and \mathbf{g}_1 are linearly independent, then $(\mathbf{u}\mathbf{g}_0, \mathbf{u}\mathbf{g}_i)$ takes on each element of $GF(4)^2$ exactly $3^2 \cdot 2^{k-4}$ when the vector \mathbf{u} ranges over $GF(4)^k$. Hence

$$|\{\mathbf{u} | \mathbf{u}\mathbf{g}_0 \neq 0, \text{ and } \mathbf{u}\mathbf{g}_i \neq 0\}| = 3^2 \cdot 2^{k-4}$$

and

$$\begin{aligned} |\{\mathbf{u} | \mathbf{u}\mathbf{g}_0 = 1, \text{ and } \mathbf{u}\mathbf{g}_i \neq 0\}| &= |\{\mathbf{u} | \mathbf{u}\mathbf{g}_0 = \omega, \text{ and } \mathbf{u}\mathbf{g}_i \neq 0\}| \\ &= |\{\mathbf{u} | \mathbf{u}\mathbf{g}_0 = \bar{\omega}, \text{ and } \mathbf{u}\mathbf{g}_i \neq 0\}| = 3 \cdot 2^{k-4}. \end{aligned}$$

Thus the number of minimal groups in which P_i is involved is

$$3 \times 3 \cdot 2^{k-4} \times 3 \cdot 2^{k-4} = 3^3 \cdot 2^{2k-8}.$$

□

The accessibility degree for SSSs based on additive codes can be defined as the follows.

Definition 7 The *accessibility index* on P is the map $\delta_P : \mathcal{A}_P \rightarrow \mathbb{R}$ given by

$$\delta_P(\Gamma) = \frac{|\Gamma|}{2^{2m}} \text{ for } \Gamma \in \mathcal{A}_P$$

where $m = |P|$, the number of participants. The number $\delta_P(\Gamma)$ will be called the *accessibility degree* of structure Γ .

Here we have to divide $|\Gamma|$ by 2^{2m} since this is 2-step SSS and we have to pool the participants' shares twice.

Let Γ be the access structure above. Then we can determine the accessibility degree of access structure Γ for SSS based on an additive code over $GF(4)$ by

$$\delta_P(\Gamma) = \frac{1}{2^{2n-2}} \sum_{i \neq j} \sum_p \sum_q \mu_i(p, e1) \mu_j(q, e1).$$

Example 2 We will describe SSS using the $(6, 2^6)$ hexacode. Let \mathcal{G}_6 be a linear $[6, 3, 4]$ hexacode over $GF(4)$ whose generator matrix is as follows :

$$\begin{bmatrix} 1 & 0 & 0 & 1 & \omega & \omega \\ 0 & 1 & 0 & \omega & 1 & \omega \\ 0 & 0 & 1 & \omega & \omega & 1 \end{bmatrix}.$$

The weight enumerator of the hexacode \mathcal{G}_6 is :

$$1 + 45y^4 + 18y^6.$$

The vectors of weight 4 in \mathcal{G}_6 hold a 2-design by Theorem 2 and the vectors of weight 6 hold 1-design by Lemma 3.2 in [7]. Note that $45 = \lambda_2 \binom{6}{2} / \binom{4}{2}$, whence $\lambda_2 = 18$. Thus $\lambda_1 = 18 \binom{5}{1} / \binom{3}{1} = 30$. It is easy to see that there are 10 supports of blocks in Γ_{H_1} , considering scalar multiplication. That is, these numbers can be obtained by dividing λ for the 1-design held by these vectors by 3. The following is the size distribution of the access structure of the hexacode \mathcal{G}_6 .

$$\sum_{i \in \{4,6\}} \lambda_1(D_i) y^{i-1} = 10y^3 + 6y^5. \quad (12)$$

The accessibility degree of the access structure for the linear hexacode \mathcal{G}_6 is

$$\delta_P(\Gamma) = \frac{|\Gamma|}{2^m} = \frac{16}{2^5} = \frac{1}{2} = 0.5.$$

Now let us think of \mathcal{G}_6 as an additive code. Then it has the following generator matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & \omega & \omega \\ \omega & 0 & 0 & \omega & \bar{\omega} & \bar{\omega} \\ 0 & 1 & 0 & \omega & 1 & \omega \\ 0 & \omega & 0 & \bar{\omega} & \omega & \bar{\omega} \\ 0 & 0 & 1 & \omega & \omega & 1 \\ 0 & 0 & \omega & \bar{\omega} & \bar{\omega} & \omega \end{bmatrix}.$$

Note that there are two kinds of blocks for extremal additive even self-dual codes : one is a simple block and the other is of multiplicity 3. By Theorem 4, it is easy to check that the supports of weight 4 vectors form a 2-design with possibly repeated blocks and the supports of weights 4 and 6 vectors form 1-designs by Lemma 3. Since the hexacode \mathcal{G}_6 is extremal even additive self-dual, the weight 4 and 6 codewords hold generalized 2-designs of type 3 by Corollary 3. These codewords eventually hold generalized 1-designs of type 3 by the definition of generalized t -designs. It implies that there are same number of blocks in each Γ_{H_i} . For example, when $\lambda_1(D_4) = 10$, $\mu_i(4, e1) = 10$. Similarly, $\mu_i(6, e1) = 6$ when $\lambda_1(D_6) = 6$. The supports of the vectors are described in Table 1.

Table 2 The supports of each weight for the hexacode \mathcal{G}_6

	$\Gamma_{H_1}(x_0 = 1)$	$\Gamma_{H_2}(y_0 = \omega)$	$\Gamma_{H_3}(z_0 = \bar{\omega})$
wt4	{2, 3, 4}	{2, 3, 4}	{2, 3, 4}
	{2, 3, 5}	{2, 3, 5}	{2, 3, 5}
	{2, 3, 6}	{2, 3, 6}	{2, 3, 6}
	{2, 4, 5}	{2, 4, 5}	{2, 4, 5}
	{2, 4, 6}	{2, 4, 6}	{2, 4, 6}
	{2, 5, 6}	{2, 5, 6}	{2, 5, 6}
	{3, 4, 5}	{3, 4, 5}	{3, 4, 5}
	{3, 4, 6}	{3, 4, 6}	{3, 4, 6}
	{3, 5, 6}	{3, 5, 6}	{3, 5, 6}
	{4, 5, 6}	{4, 5, 6}	{4, 5, 6}
# of wt 4	10	10	10
wt 6	{2, 3, 4, 5, 6}	{2, 3, 4, 5, 6}	{2, 3, 4, 5, 6}
# of wt 6	6	6	6
Total #	16	16	16

The size distribution of the access structure of the hexacode \mathcal{G}_6 by Corollary 4 is

$$\begin{aligned}
& \sum_{\substack{i \neq j \\ 1 \leq i, j \leq 3}} \sum_{p \in \{4, 6\}} \sum_{q \in \{4, 6\}} \mu_i(p, e1) \mu_j(q, e1) y^{(p-1, q-1)} \\
& = 100y^{(3,3)} + 60y^{(3,5)} + 60y^{(5,3)} + 36y^{(5,5)}.
\end{aligned} \tag{13}$$

These pairs of groups comprise the 256 elements of the access structure. Additionally, a group of size 6 do not c -cover any group of size 4 . If a weight 4 vector were c -covered by a weight 6 vector, then the sum of the two vectors will yield a weight 2 vector, which is a contradiction. Thus 256 pairs of supports form the minimal access structure. Note that Γ_{H_1} in Table 1 is the access structure for linear hexacode \mathcal{G}_6 .

We summarize the following properties of SSS using the hexacode.

Summary :

We summarize the following properties of SSS using the hexacode.

- (i) The access structure consists of 100 pairs of sets of size (3,3), 60 pairs of groups of size (3,5), 60 pairs of groups of size (5,3), 36 pairs of groups of size (5,5).
- (ii) All the pairs of groups constitute the minimal access structure.
- (iii) No group of size less than 3 can be used in recovering the secret.

Moreover, the accessibility degree for the access structure of the additive hexacode \mathcal{G}_6 is

$$\delta_{\mathcal{P}}(\Gamma) = \frac{1}{2^{2m}} \sum_{i \neq j} \sum_p \sum_q \mu_i(p, e1) \mu_j(q, e1) = \frac{256}{2^{10}} = \frac{1}{4} = 0.25.$$

Example 3 Let us consider a self-dual [12,6,4] code E_{12} with a generator matrix [13]

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

The weight enumerator of E_{12} is :

$$1 + 45y^4 + 216y^6 + 1755y^8 + 1800y^{10} + 279y^{12}. \quad (14)$$

Since we cannot apply Theorem 2 to E_{12} , we have to get the access structure by MAGMA, which is one of the commonly used computer languages. Using this, we obtain the following size distribution of access structure for E_{12} :

$$5y^3 + 36y^5 + 390y^7 + 500y^9 + 93y^{11}. \quad (15)$$

The accessibility degree of the access structure for the SSS based on E_{12} is

$$\delta_{\mathcal{P}}(\Gamma) = \frac{|\Gamma|}{2^m} = \frac{1024}{2^{11}} = \frac{1}{2} = 0.5.$$

Now we will describe an SSS based on an extremal additive even self-dual $(12, 2^{12})$ dodecacode QC_{12} (see [8], [9], [11]). It has the following generator matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega & \omega & \omega & \omega & \omega & \omega \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \omega & \omega & \omega & \omega & \omega & \omega & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \omega & \bar{\omega} & 0 & 0 & 0 & 1 & \omega & \bar{\omega} \\ 0 & 0 & 0 & \omega & \bar{\omega} & 1 & 0 & 0 & 0 & \omega & \bar{\omega} & 1 \\ 1 & \bar{\omega} & \omega & 0 & 0 & 0 & 1 & \bar{\omega} & \omega & 0 & 0 & 0 \\ \omega & 1 & \bar{\omega} & 0 & 0 & 0 & \omega & 1 & \bar{\omega} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \bar{\omega} & \omega & \omega & \bar{\omega} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega & 1 & \bar{\omega} & 1 & \omega & \bar{\omega} & 0 & 0 & 0 \\ 1 & \omega & \bar{\omega} & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\omega} & \omega & 1 \\ \bar{\omega} & 1 & \omega & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \bar{\omega} & \omega \end{bmatrix}.$$

The weight enumerator of QC_{12} is :

$$1 + 396y^6 + 1485y^8 + 1980y^{10} + 234y^{12}.$$

By Corollary 2, the supports of weight 6 vectors forms a 5-design with possibly repeated blocks. We remark that there are 18 codewords of weight 6 whose supports repeat three times. Among them, exactly 12 codewords are such that their scalar multiples are also codewords. Since $A_6 = 396$, there are $396-18=378$ codewords of weight 6 whose supports are simple blocks. Note that $396 = \lambda_5 \binom{12}{5} / \binom{6}{5}$, whence $\lambda_5 = 3$. Allowing repeated blocks we obtain a 5-(12, 6, $\lambda_5 = 3$) design in QC_{12} . Each 5-set is either contained in one block repeated three times or in three distinct blocks. We see that vectors of other weights 8, 10, and 12 hold 5-designs with possibly repeated blocks with $\lambda_5 = 105, 630$ and 234 , respectively [11].

Since QC_{12} is extremal even additive self-dual, the codewords of all the nonzero weights hold generalized 2-designs of type 3 by Corollary 3 and also hold generalized 1-designs of type 3 by the definition of generalized t -designs. Thus we have the following numbers $\mu_i(p, e1)$ by dividing λ_1 by 3. When $\lambda_5 = 3$ for weight 6 codewords, $\lambda_1 = 3 \binom{11}{4} / \binom{5}{4} = 198$. It implies that $\mu_1(6, e1) = \mu_2(6, e1) = \mu_3(6, e1) = 66$. Repeating the calculation for weight 8 with $\lambda_5 = 105$, we get $\lambda_1 = 105 \binom{11}{4} / \binom{7}{4} = 990$. Thus $\mu_1(8, e1) = \mu_2(8, e1) = \mu_3(8, e1) = 330$. For weight 10 with $\lambda_5 = 630$, $\lambda_1 = 630 \binom{11}{4} / \binom{9}{4} = 1650$. Thus $\mu_i(10, e1) = 550$. For weight 12 with $\lambda_5 = 234$, $\lambda_1 = 234 \binom{11}{4} / \binom{11}{4} = 234$. Thus $\mu_i(12, e1) = 78$.

By Corollary 4, the size distribution of the access structure for the dodecacode QC_{12} is

$$\begin{aligned} & \sum_{\substack{i \neq j \\ 1 \leq i, j \leq 3}} \sum_{p \in \{6, 8, 10, 12\}} \sum_{q \in \{6, 8, 10, 12\}} \mu_i(p, e1) \mu_j(q, e1) y^{(p-1, q-1)} \\ &= 4356y^{(5,5)} + 21780y^{(5,7)} + 36300y^{(5,9)} + 5148y^{(5,11)} + 21780y^{(7,5)} \\ &+ 108900y^{(7,7)} + 181500y^{(7,9)} + 25740y^{(7,11)} + 36300y^{(9,5)} + 181500y^{(9,7)} \\ &+ 302500y^{(9,9)} + 42900y^{(9,11)} + 5148y^{(11,5)} + 25740y^{(11,7)} + 42900y^{(11,9)} \\ &+ 6084y^{(11,11)}. \end{aligned} \tag{16}$$

A vector of weight 8 does not c-cover any vector of weight 6. If a weight 6 vector were c-covered by a weight 8 vector, then the sum of the two vectors will yield a weight 2 vector, which is a contradiction. Likewise, a vector of weight 10 does not c-cover any vector of weight 6 or 8. If a weight 6 or 8 vector were c-covered by a weight 10 vector, then the sum of the two vectors will yield a weight 4, or 2 vector, respectively, which is a contradiction.

Summary :

We summarize the following properties of SSS using QC_{12} .

- (i) The access structure consists of the pairs of groups as in (16).
- (ii) All the pairs of groups with the sizes $\in \{5, 7, 9\}$ are contained in the minimal access structure.
- (iii) No group of size less than 5 can be used in recovering the secret.

The accessibility degree for the SSS based on the dodecacode QC_{12} is

$$\delta_{\mathcal{P}}(\Gamma) = \frac{1}{2^{2m}} \sum_{i \neq j} \sum_p \sum_q \mu_i(p, e1) \mu_j(q, e1) = \frac{1,048,576}{2^{22}} = \frac{1}{4} = 0.25.$$

Example 4 Now we are going to describe an SSS based on S_{18} which is an $(18, 2^{18})$ extremal additive even self-dual code. The weight enumerator of S_{18} is [13]:

$$1 + 2754y^8 + 18360y^{10} + 77112y^{12} + 110160y^{14} + 50949y^{16} + 2808y^{18}. \quad (17)$$

Since all the non-zero weights in S_{18} hold 5-designs with possibly repeated blocks by Corollary 2, we can easily calculate λ_1 for each weight using (2).

Note that $2754 = \lambda_5 \binom{18}{5} / \binom{8}{5}$, whence $\lambda_5 = 18$. Allowing repeated blocks we obtain a 5- $(18, 8, \lambda_5 = 18)$ design in S_{18} . Each 5-set is either contained in one block repeated three times or in three distinct blocks. We see that vectors of other weights 10, 12, 14, 16, and 18 hold 5-designs with possibly repeated blocks with $\lambda_5 = 540, 7128, 25740, 25974,$ and $2808,$ respectively.

Since S_{18} is extremal even additive self-dual, the codewords of all the nonzero weights hold generalized 2-designs of type 3 by Corollary 3 and also hold generalized 1-designs of type 3 by the definition of generalized t -designs. Thus we have the following numbers $\mu_i(p, e1)$ by dividing λ_1 by 3. When $\lambda_5 = 18$ for weight 8 codewords, $\lambda_1 = 18 \binom{17}{4} / \binom{7}{4} = 1224$. It implies that $\mu_1(8, e1) = \mu_2(8, e1) = \mu_3(8, e1) = 408$. Repeating the calculation for weight 10 with $\lambda_5 = 540$, we get $\lambda_1 = 540 \binom{17}{4} / \binom{9}{4} = 10200$. Thus $\mu_1(10, e1) = \mu_2(10, e1) = \mu_3(10, e1) = 3400$. For weight 12 with $\lambda_5 = 7128$, $\lambda_1 = 7128 \binom{17}{4} / \binom{11}{4} = 51408$. Thus $\mu_i(12, e1) = 17136$. For weight 14 with $\lambda_5 = 25740$, $\lambda_1 = 25740 \binom{17}{4} / \binom{13}{4} = 85680$. Thus $\mu_i(14, e1) = 28560$. For weight 16 with $\lambda_5 = 25974$, $\lambda_1 = 25974 \binom{17}{4} / \binom{15}{4} = 45288$. Thus $\mu_i(16, e1) = 15096$. For weight 18 with $\lambda_5 = 2808$, $\lambda_1 = 2808 \binom{17}{4} / \binom{17}{4} = 2808$. Thus $\mu_i(18, e1) = 936$.

By Corollary 4, the size distribution of the access structure for S_{18} is

$$\begin{aligned} & \sum_{\substack{i \neq j \\ 1 \leq i, j \leq 3}} \sum_{p \in \{8, 10, 12, 14, 16, 18\}} \sum_{q \in \{8, 10, 12, 14, 16, 18\}} \mu_i(p, e1) \mu_j(q, e1) y^{(p-1, q-1)} \\ &= 166464y^{(7,7)} + 1387200y^{(7,9)} + 6991488y^{(7,11)} + 11652480y^{(7,13)} + 6159168y^{(7,15)} \\ &+ 381888y^{(7,17)} + 1387200y^{(9,7)} + 11560000y^{(9,9)} + 58262400y^{(9,11)} + 97104000y^{(9,13)} \\ &+ 51326400y^{(9,15)} + 3182400y^{(9,17)} + 6991488y^{(11,7)} + 58262400y^{(11,9)} \\ &+ 293642496y^{(11,11)} + 489404160y^{(11,13)} + 258685056y^{(11,15)} + 16039296y^{(11,17)} \\ &+ 11652480y^{(13,7)} + 97104000y^{(13,9)} + 489404160y^{(13,11)} + 815673600y^{(13,13)} \\ &+ 431141760y^{(13,15)} + 26732160y^{(13,17)} + 6159168y^{(15,7)} + 51326400y^{(15,9)} \\ &+ 258685056y^{(15,11)} + 431141760y^{(15,13)} + 227889216y^{(15,15)} + 14129856y^{(15,17)} \\ &+ 381888y^{(17,7)} + 3182400y^{(17,9)} + 16039296y^{(17,11)} + 26732160y^{(17,13)} \\ &+ 14129856y^{(17,15)} + 876096y^{(17,17)}. \end{aligned} \quad (18)$$

A vector of weight 10 does not c -cover any vector of weight 8. If a weight 8 vector were c -covered by a weight 10 vector, then the sum of the two vectors will yield a weight 2 vector, which is a contradiction. Likewise, a vector of weight 12 does not c -cover any vector of weight 8 or 10. If a weight 8 or 10 vector were c -covered by a weight 12 vector, then the sum of the two vectors will yield a weight 4, or 2 vector, respectively, which is a contradiction. A vector of weight 14 does not c -cover any vector of weight 8, 10 or 12. If a weight 8, 10 or 12 vector were c -covered by a weight 14 vector, then the sum of the two vectors will yield a weight 6, 4, or 2 vector, respectively, which is a contradiction.

Summary :

We summarize the following properties of SSS using S_{18} .

- (i) The access structure consists of the pairs of groups as in (18).
- (ii) All the pairs of groups with the sizes $\in \{7, 9, 11, 13\}$ are contained in the minimal access structure.
- (iii) No group of size less than 7 can be used in recovering the secret.

The accessibility degree for the SSS based on S_{18} is

$$\delta_{\mathcal{P}}(\Gamma) = \frac{1}{2^{2m}} \sum_{i \neq j} \sum_p \sum_q \mu(p, e1) \mu(q, e1) = \frac{4294967296}{2^{34}} = \frac{1}{4} = 0.25.$$

The accessibility degree of the access structure for the SSS based on S_{18} is $\frac{1}{4}$ which is same as that of the dodecacode QC_{12} .

4 Conclusion

In this paper, we introduce two contrasting access structures, one from linear codes and the other from additive codes. The new results we obtained are mainly stated in Section 3.2 and 3.3. In Section 3.2, we newly defined SSSs based on additive codes over $GF(4)$. The access structure from additive codes over $GF(4)$ is described in a distinct way requiring at least two steps of calculations. In Section 3.3, we determined the access structure for SSSs based on a hexacode, a dodecacode over $GF(4)$ and S_{18} using the notion we introduced in Section 3.2.

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