HAMMING DISTANCES FROM A FUNCTION TO ALL CODEWORDS OF A GENERALIZED REED-MULLER CODE OF ORDER ONE

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Abstract. For any finite field $\mathbb{F}_q$ with q elements, we study the set $\mathcal{F}_{(q,m)}$ of functions from $\mathbb{F}_q^m$ into $\mathbb{F}^q$. We introduce a transformation that allows us to determine a linear system of q^{m+1} equations and q^{m+1} unknowns, which has for solution the Hamming distances of a function in $\mathcal{F}_{(q,m)}$ to all the affine functions.

1. Introduction

1.1. **Generalized Reed-Muller codes of order 1.** Let $\mathbb{F}_q$ be the finite field with q elements. For any integer $m \geq 1$, we will identify $\mathbb{F}_{q^m}$ with $\mathbb{F}_q^m$ as follows: consider a basis $\{e_1, \cdots, e_m\}$ of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$, then an element $u \in \mathbb{F}_{q^m}$ will be identified with the vector $(u_1, \cdots, u_m) \in \mathbb{F}_q^m$ if and only if, $u = \sum_{i=1}^m u_i e_i$.

If $u = (u_1, \cdots, u_m)$ and $v = (v_1, \cdots, v_m)$, are two elements of $\mathbb{F}_q^m$ we will denote by $u.v$ their product in the field $\mathbb{F}_{q^m}$ and by $\langle u, v\rangle$ their scalar product

$$\langle u, v\rangle = \sum_{i=1}^{m} u_i v_i.$$

We denote by $\mathcal{F}_{(q,m)} = \{f : \mathbb{F}_q^m \to \mathbb{F}_q\}$ the set of functions from $\mathbb{F}_q^m$ to $\mathbb{F}_q$. Each function $f \in \mathcal{F}_{(q,m)}$ can be identified with its image $\big(f(u)\big)_{u\in\mathbb{F}_q^m}$. We know that these functions are polynomial functions of m variables. The kernel of the map which associates to any polynomial the corresponding polynomial function is the ideal I generated by the m polynomials $X_i^q - X_i$. The reduced polynomials are the polynomials $P(X_1, \cdots, X_m)$ such that for each i, the partial degree $\deg_i(P(X_1, \cdots, X_m))$ of $P(X_1, \cdots, X_m)$ with respect to the variable X_i is $\leq q-1$. Then for any $f \in \mathcal{F}_{(q,m)}$ there exists an unique reduced

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polynomial $P(X_1, \cdots, X_m)$ for which f is the associated polynomial function. The total degree of $P(X_1, \cdots, X_m)$ is called the degree of f and denoted by $\deg(f)$.

With these notations, the Generalized Reed-Muller code of order 1 is the set

$$RM^{(1)}_{(q,m)} = \{(g(u))_{u\in\mathbb{F}_q^m} \mid g \in \mathcal{F}_{(q,m)} \text{ and } \deg(g) \leq 1\}.$$

If $f, g \in \mathcal{F}_{(q,m)}$, the Hamming distance between these two functions is defined by

$$d(f,g) = \text{card}\left(\{u \in \mathbb{F}_q^m \mid f(u) \neq g(u)\}\right).$$

1.2. **Organization of the article.** In this article we study the Hamming distances from a function $f \in \mathcal{F}_{(q,m)}$ to all the codewords $g \in RM^{(1)}_{(q,m)}$.

2. An adapted transform

It is known that every codeword $g \in RM^{(1)}_{(q,m)}$ can be characterized by a pair $(v,t) \in \mathbb{F}_q^m \times \mathbb{F}_q$ in the following sense:

$$g(u) = \langle u, v\rangle + t \qquad \forall u \in \mathbb{F}_q^m.$$

If $f \in \mathcal{F}_{(q,m)}$ and g as above, we have that

$$d(f,g) = \text{card}\left(\{u \in \mathbb{F}_q^m \mid f(u) \neq \langle u, v\rangle + t\}\right) = q^m - N_{v,t}(f),$$

where $N_{v,t}(f) = \text{card}\left(\{u \in \mathbb{F}_q^m \mid f(u) = \langle u, v\rangle + t\}\right)$.

Now the problem is to study the integer numbers $N_{(v,t)}(f)$. In order to do that, we will introduce a transform on the group algebra $\mathbb{C}\mathbb{F}_q$ of the additive group $\mathbb{F}_q$ over the complex field $\mathbb{C}$ which is quite similar to a Fourier Transform.

More precisely, $\mathbb{C}\mathbb{F}_q$ is the algebra of formal linear combinations with coefficients in $\mathbb{C}$

$$\sum_{t\in\mathbb{F}_q} \alpha_t Z^t$$

where the operations are defined by

$$\sum_{t\in\mathbb{F}_q} \alpha_t Z^t + \sum_{t\in\mathbb{F}_q} \beta_t Z^t = \sum_{t\in\mathbb{F}_q} (\alpha_t + \beta_t) Z^t,$$

$$\lambda\left(\sum_{t\in\mathbb{F}_q} \alpha_t Z^t\right) = \sum_{t\in\mathbb{F}_q} (\lambda\alpha_t) Z^t,$$

$$\left(\sum_{t\in\mathbb{F}_q} \alpha_t Z^t\right)\left(\sum_{t\in\mathbb{F}_q} \beta_t Z^t\right) = \sum_{t\in\mathbb{F}_q} \left(\sum_{r+s=t} (\alpha_r \beta_s)\right) Z^t.$$

Let $\mathcal{G}_{(q,m)}$ be the algebra of functions from $\mathbb{F}_q^m$ (or from $\mathbb{F}_{q^m}$) into $\mathbb{C}\mathbb{F}_q$. It is a vector space of dimension q^{m+1} over $\mathbb{C}$. Let us define an order on $\mathbb{F}_{q^m} \times \mathbb{F}_q$ and define the family $(e_{u,t})_{(u,t)\in\mathbb{F}_{q^m}\times\mathbb{F}_q}$ of elements of $\mathcal{G}_{(q,m)}$ where

$$e_{u,t}(v) = \begin{cases} 0 & \text{if} \quad v \neq u \\ Z^t & \text{if} \quad v = u \end{cases} \tag{1}$$

This family is a basis of $\mathcal{G}_{(q,m)}$ and has q^{m+1} elements.

Define the operator $T_{(q,m)}$ of the $\mathbb{C}$-vector space $\mathcal{G}_{(q,m)}$ by

$$T_{(q,m)}(\phi)(v) = \sum_{u\in\mathbb{F}_q^m} \phi(u) Z^{-\langle u,v\rangle}.$$

Remark 2.1. In the case where the function ϕ is given by $\phi(v) = Z^{f(v)}$ for some $f \in \mathcal{F}_{(q,m)}$, then the transform introduced above is the same that the the one introduced by Ashikhmin and Litsyn (see [1]). We recall here some basic properties of this transform, for more details see [2].

Lemma 2.2. *The transform of $e_{u,t}$ by $T_{(q,m)}$ is given by*

$$\epsilon_{u,t}(v) = T_{(q,m)}(e_{u,t})(v) = \sum_{w\in\mathbb{F}_q^m} e_{u,t}(w) Z^{-\langle w,v\rangle} = Z^{t-\langle u,v\rangle},$$

then

$$\epsilon_{u,t} = \sum_{(v,\tau)\in E_{-u,t}} e_{v,\tau},$$

where $E_{-u,t}$ is the hyperplane of $\mathcal{F}_{(q,m)} \times \mathcal{F}_q$ defined by

$$E_{-u,t} = \{(v,\tau) \in \mathbb{F}_{q^m} \times \mathbb{F}_q \mid \tau = t - < u, v >\}.$$

Lemma 2.3. *Let $\gamma_a \in \mathcal{G}_{(q,m)}$ be defined by $\gamma_a(u) = Z^{\langle a,u\rangle}$, then the transform of γ_a is given by*

$$T_{(q,m)}(\gamma_a)(v) = \begin{cases} q^m Z^0 & \text{if } v = a \\ q^{m-1} \sum_{t\in\mathbb{F}_q} Z^t & \text{if } v \neq a. \end{cases}$$

Proof. We have successively

$$\begin{aligned} T_{(q,m)}(\gamma_a)(v) &= \sum_{u\in\mathbb{F}_q^m} \gamma_a(u) Z^{-\langle u,v\rangle} \\ &= \sum_{u\in\mathbb{F}_q^m} Z^{\langle a,u\rangle} Z^{-\langle u,v\rangle} \\ &= \sum_{u\in\mathbb{F}_q^m} Z^{\langle a-v,u\rangle}. \end{aligned}$$

If $v = a$ we have that $T_{(q,m)}(\gamma_a)(v) = q^m Z^0$ and then, when $v \neq a$, for each $t \in \mathbb{F}_q$, the equation $\langle a - v, u\rangle = t$ defines a hyperplane and consequently has q^{m-1} solutions. □

Let ϕ be an element of $\mathcal{G}_{(q,m)}$, we denote by $\psi = T_{(q,m)}(\phi)$ its transform, and by $\theta = T_{(q,m)}(\psi)$ its double transform.

Theorem 2.4. *With the previous notations we have*

$$\theta(w) = q^{m-1}\sum_{t\in\mathbb{F}_q}(Z^0 - Z^t)\phi(-w) + \left(q^{m-1}\sum_{t\in\mathbb{F}_q} Z^t\right)\psi(0).$$

Proof. We have that

$$\begin{aligned}\theta(w) &= \sum_{v\in\mathbb{F}_q^m}\left(\sum_{u\in\mathbb{F}_q^m}\phi(u)Z^{-\langle u,v\rangle}\right)Z^{-\langle v,w\rangle}\\ &= \sum_{v\in\mathbb{F}_q^m}\sum_{u\in\mathbb{F}_q^m}\phi(u)Z^{-\langle u+w,v\rangle}\\ &= \sum_{u\in\mathbb{F}_q^m}\phi(u)\sum_{v\in\mathbb{F}_q^m}Z^{-\langle u+w,v\rangle}\end{aligned}$$

From Lemma 2.3 we obtain

$$\theta(w) = q^m\phi(-w) + \left(q^{m-1}\sum_{t\in\mathbb{F}_q} Z^t\right)\sum_{u\in\mathbb{F}_q^m\setminus\{-w\}}\phi(u).$$

The Lemma follows from the equality above and from the fact that:

$$\sum_{u\in\mathbb{F}_q^m\setminus\{-w\}}\phi(u) = \sum_{u\in\mathbb{F}_q^m}\phi(u) - \phi(-w) = \psi(0) - \phi(-w).$$

□

We want to characterize the kernel of $T_{(q,m)}$, in order to do that, we need the following lemma:

Lemma 2.5. *A function $\phi \in \mathcal{G}_{(q,m)}$ verifies*

$$\phi(w)\cdot\left(q - \sum_{t\in\mathbb{F}_q} Z^t\right) = 0$$

for each $w \in \mathbb{F}_q^m$ if, and only if,

$$\phi(w) = \lambda(w)\sum_{t\in\mathbb{F}_q} Z^t,$$

where λ is a function from $\mathbb{F}_q^m$ into $\mathbb{C}$.

Proof. Let ϕ be given by $\phi(w) = \sum_{t\in\mathbb{F}_q} C_t(\phi)(w)Z^t$, then we have

$$\phi(w)\cdot\left(q-\sum_{t\in\mathbb{F}_q} Z^t\right) = q\phi(w) - \left(\sum_{t\in\mathbb{F}_q} C_t(\phi)(w)\right)\left(\sum_{t\in\mathbb{F}_q} Z^t\right).$$

If this product is equal to zero, then

$$\phi(w) = (1/q)\left(\sum_{t\in\mathbb{F}_q} C_t(\phi)(w)\right)\left(\sum_{t\in\mathbb{F}_q} Z^t\right).$$

On the other hand a direct computation of

$$\left(\lambda(w)\sum_{t\in\mathbb{F}_q} Z^t\right).\left(q-\sum_{t\in\mathbb{F}_q} Z^t\right)$$

shows the converse. □

Now we can determine the kernel of $T_{(q,m)}$.

Theorem 2.6. *The kernel of $T_{q,m}$ is the subspace of the functions ϕ such that for each $w \in \mathbb{F}_q^m$*

$$\phi(w) = \lambda(w)\sum_{t\in\mathbb{F}_q} Z^t$$

where λ is any function from $\mathbb{F}_q^m$ into $\mathbb{C}$ verifying

$$\sum_{u\in\mathbb{F}_q^m} \lambda(u) = 0.$$

The dimension of the kernel is $q^m - 1$.

Proof. Note that if the transform of ϕ is the zero function, then using the Proposition 2.4 we get

$$\phi(w)\cdot\left(q-\sum_{t\in\mathbb{F}_q} Z^t\right) = 0,$$

and by Lemma 2.5

$$\phi(w) = \lambda(w)\sum_{t\in\mathbb{F}_q} Z^t.$$

Hence, for each $t \in \mathbb{F}_q$ we must have

$$C_t(\phi)(w) = \lambda(w).$$

If we denote by ψ the transform of ϕ we know that

$$\begin{aligned} C_t(\psi)(w) &= \sum_{u\in\mathbb{F}_q^m} C_{\langle u,w\rangle+t}(\phi)(u) \\ &= \sum_{u\in\mathbb{F}_q^m} \lambda(u) \end{aligned}$$

The result follows. Let us remark that the functions λ such that

$$\sum_{u\in\mathbb{F}_q^m} \lambda(u) = 0,$$

defines an hyperplane of the space of functions from $\mathbb{F}_q^m$ into $\mathbb{C}$ and then, the dimension of the kernel is $q^m - 1$. □

Proposition 2.7. *The functions*

$$\delta_a = \sum_{t\in\mathbb{F}_q} (e_{0,t} - e_{a,t})$$

with $a \in \mathbb{F}_q^m \setminus \{0\}$ *are a basis of the kernel* $\mathrm{Ker}(T_{(q,m)})$*, where*

$$e_{u,t}(v) = \begin{cases} Z^t & \text{if } v = u \\ 0 & \text{otherwise} \end{cases}$$

The functions $e_{a,t}$ *with*

$$(a \neq 0 \text{ and } t \neq 0) \text{ or } (a = 0)$$

is a basis of a complement of $\mathrm{Ker}(T_{q,m})$.

Proof. For any $a \in \mathbb{F}_q^m \setminus \{0\}$ the following holds:

$$\delta_a(v) = \lambda(v) \sum_{t\in\mathbb{F}_q} Z^t,$$

with $\lambda(0) = 1$, $\lambda(a) = -1$ and $\lambda(v) = 0$ for the other values of v, then by Theorem 2.6 δ_a is in the kernel of $T_{(q,m)}$. As the $e_{a,t}$ are linearly independent, the δ_a are linearly independent. We conclude that the δ_a constitute a basis of $\mathrm{Ker}(T_{(q,m)})$.

Let I be the set

$$I = \{(v,t) \mid (v \neq 0 \text{ and } t \neq 0) \text{ or } (v = 0)\}$$

and ϕ the function

$$\phi = \sum_{(v,t)\in I} \lambda_{v,t} e_{v,t}.$$

The following holds:

$$T_{(q,m)}(\phi)(v) = \sum_{\{t|(v,t)\in I\}} \lambda_{v,t} Z^t.$$

If $v \neq 0$ then

$$T_{(q,m)}(\phi)(v) = \sum_{t\in\mathbb{F}_q^*} \lambda_{v,t} Z^t.$$

Then $T_{(q,m)}(\phi)(v)$ cannot be a multiple of $\sum_{t\in\mathbb{F}_q} Z^t$ unless all the $\lambda_{v,t}$ are zero for $v \neq 0$ and in this case the coefficient of $\sum_{t\in\mathbb{F}_q} Z^t$ is 0. Now if $v = 0$ then

$$T_{(q,m)}(\phi)(0) = \sum_{t\in\mathbb{F}_q} \lambda_{0,t} Z^t.$$

Then $T_{(q,m)}(\phi)(0)$ cannot be a multiple of $\sum_{t\in\mathbb{F}_q} Z^t$ unless all the $\lambda_{0,t}$ have the same value λ_0 and in this case the coefficient of $\sum_{t\in\mathbb{F}_q} Z^t$ is λ_0. Hence, if $T_{(q,m)}(\phi)(v)$ can be written $\lambda(v)\sum_{t\in\mathbb{F}_q} Z^t$, we have $\sum_{v\in\mathbb{F}_q^m} \lambda(v) = \lambda_0$. Then, if $\phi \in \mathrm{Ker}(T_{(q,m)})$, for any $(v,t) \in I$ we have $\lambda_{v,t} = 0$. We conclude that the $q^{m+1} - (q^m - 1)$ linearly independent vectors $(e_{v,t})_{(v,t)\in I}$ constitute a basis of a complement of $\mathrm{Ker}(T_{(q,m)})$. □

Corollary 2.8. *The vectors $\epsilon_{v,t} = T_{(q,m)}(e_{v,t})$ with $(v,t) \in I$ are linearly independent. They constitute a basis of the image $T_{(q,m)}(\mathcal{G}_{(q,m)})$.*

3. Application to the Hamming distances from a function to all codewords of a Generalized Reed-Muller code of order 1

3.1. **System of equations satisfied by the distances of a function to all codewords.** Coming back to our problem, if $f \in \mathcal{F}_{(q,m)}$ let us associate to it the function $F \in \mathcal{G}_{(q,m)}$ defined by

$$F(u) = Z^{f(u)}.$$

The transform $T_{(q,m)}(F)$ is given by

$$\begin{aligned} T_{(q,m)}(F)(v) &= \textstyle\sum_{u\in\mathbb{F}_q^m} Z^{f(u)-\langle u,v\rangle} \\ &= \textstyle\sum_{t\in\mathbb{F}_q} N_{v,t}(f) Z^t, \end{aligned} \tag{2}$$

where $N_{v,t}(f) = \sharp\{u \in \mathbb{F}_q^m \mid f(u) - \langle u, v\rangle = t\}$ as defined before.

Lemma 3.1. *Pour any $v \in \mathbb{F}_q^m$ the following formula holds:*

$$\sum_{t\in\mathbb{F}_q} N_{v,t}(f) = q^m.$$

Proof. It is a direct consequence of the equalities (2). Indeed the total sum of coefficients in the first expression is q^m and in the second one it is $\sum_t N_{v,t}$. □

As one can see, the numbers $N_{v,t}(f)$ are exactly the coefficients of $T_{(q,m)}(F)$ where F is associated to f as above.

For each $w \in \mathbb{F}_{q^m}$ we consider the linear form L_w defined over $\mathbb{F}_{q^m} \times \mathbb{F}_q$ by

$$L_w(v,t) = -\langle w, v\rangle + t,$$

and for each $w \in \mathbb{F}_{q^m}$ and each $\tau \in \mathbb{F}_q$ we consider the hyperplane $E_{w,\tau}$ of $\mathbb{F}_{q^m} \times \mathbb{F}_q$ defined by

$$E_{w,\tau} = \{(v,t) \in \mathbb{F}_{q^m} \times \mathbb{F}_q \mid L_w(v,t) = \tau\}.$$

Theorem 3.2. *Let $f \in \mathcal{F}_{(q,m)}$, then $N_{v,t}(f)$ are solutions of the following linear system with q^{m+1} equations on q^{m+1} variables where the equation numbered (w,τ) is:*

$$(w,\tau) \qquad \sum_{(v,t)\in E_{w,\tau}} x_{v,t} = \begin{cases} q^{2m-1} - q^{m-1} & if \quad f(-w) \neq \tau \\ q^{2m-1} - q^{m-1} + q^m & if \quad f(-w) = \tau \end{cases}$$

Proof. Computing $T^2_{(q,m)}(F)$, where $F = Z^f$ and by using the result of Theorem 2.4, we obtain

$$T^2_{(q,m)}(F)(w) = q^{m-1}(q - \sum_{t\in\mathbb{F}_q} Z^t)F(-w) + q^{m-1}\sum_{t\in\mathbb{F}_q} Z^t \sum_{u\in\mathbb{F}_{q^m}} F(u).$$

Denoting $K(Z) = \sum_{t\in\mathbb{F}_q} Z^t$ and observing that

$$K(Z)\sum_{t\in\mathbb{F}_q} \alpha_t Z^t = (\sum_{t\in\mathbb{F}_q} \alpha_t)K(Z),$$

we obtain

$$\begin{aligned} T^2_{(q,m)}(F)(w) &= q^m Z^{f(-w)} + K(Z)(q^{2m-1} - q^{m-1}) \\ &= (q^{2m-1} - q^{m-1} + q^m)Z^{f(-w)} \\ &\quad + (q^{2m-1} - q^{m-1}) \sum_{t\neq f(-w)} Z^t. \end{aligned}$$

On the other hand, if we compute $T^2_{(q,m)}(F)$ using that

$$T_{(q,m)}(F)(v) = \sum_{t\in\mathbb{F}_{q^m}} N_{v,t} Z^t,$$

we obtain

$$T^2_{(q,m)}(F)(w) = \sum_{\tau\in\mathbb{F}_q} \left(\sum_{(v,t)\in E_{w,\tau}} N_{v,t} \right) Z^\tau.$$

The theorem follows by comparing the two expressions obtained for $T^2_{(q,m)}(F)$. □

Remark 3.3. The system presented in Theorem 3.2 has the following structure: it is constituted by q^m blocks $\mathcal{B}_w$ of q equations. The block $\mathcal{B}_w$ contains the q equations numbered (w,τ) where w is fixed and τ takes the q possible values in $\mathbb{F}_q$. Each equation of a block involves q^m variables, namely the variables indexed by the points (v,t) of the hyperplane $E_{w,\tau}$ of $\mathbb{F}_{q^m}\times\mathbb{F}_q$. The q hyperplanes $E_{w,\tau}$ (w fixed, $\tau\in\mathbb{F}_q$) are parallel, then each variable $x_{v,t}$ is in one and only one equation of each block $\mathcal{B}_w$.

Let us consider the basis defined in section 2 by (1). Remark that the matrix of the system (3.2) is the matrix $\mathcal{T}_{(q,m)}$ of $T_{(q,m)}$ with respect to the considered basis. Namely by construction (see the proof of Theorem 3.2), the system can be written

$$\mathcal{T}_{(q,m)}X = B,$$

where X is the column

$$X = \begin{pmatrix} \vdots \\ x_{v,t} \\ \vdots \end{pmatrix},$$

and B the column

$$B = \begin{pmatrix} \vdots \\ b_{w,\tau} \\ \vdots \end{pmatrix},$$

where

$$b_{w,\tau} = \begin{cases} q^{2m-1} - q^{m-1} & if \quad f(-w) \neq \tau \\ q^{2m-1} - q^{m-1} + q^m & if \quad f(-w) = \tau \end{cases}.$$

The system has a solution because we know that the values $N_{v,t}(f)$ constitute a solution. But, as the linear map $T_{(q,m)}$ has a kernel, the system has not a unique solution. However, if we add some "normalization" conditions we obtain the desired solution.

Theorem 3.4. *The numbers $N_{v,t}(f)$ are the unique solution of the system that appears on the Theorem 3.2 if we join the following q^m equations*

$$\sum_{t\in\mathbb{F}_q} x_{v,t} = q^m. \quad \forall v \in \mathbb{F}_{q^m}.$$

Proof. We know that any other solution is obtained from the previous solution $(N_{v,t})_{v,t}$ by adding an element in the kernel of the transformation, that is any other solution has the form $\big(N_{v,t} + \lambda(v)\big)_{v,t}$ with $\sum_v \lambda(v) = 0$. For any v fix, we have that $\sum_{t\in\mathbb{F}_q} N_{v,t} = q^m$ and the result follows from it. □

3.2. **Transformation into a Cramer linear system.**

Theorem 3.5. *The system (S) constructed in the following way:*

1. *suppress from the system (3.2) the $q^m - 1$ lines numbered $(w, 0)$ with $w \neq 0$,*
2. *replace these equations by the $q^m - 1$ equations $\sum_{t\in\mathbb{F}_q} x_{w,t} = q^m$, where $w \neq 0$,*

is a Cramer linear system and has $\big(N_{v,t}(f)\big)_{v,t}$ for unique solution.

Proof. Let $\mathcal{T}_{(q,m)}$ the matrix of the original system. The columns are the vectors $T_{(q,m)}(e_{v,t}) = \epsilon_{v,t}$ decomposed on the basis $(e_{v,t})_{(v,t)\in\mathbb{F}_q\times\mathbb{F}_q}$.

Let us consider the columns (v, t) for which one of the two following conditions holds:

1. $v = 0$;
2. $v \neq 0$ and $t \neq 0$.

Denote by I these indexes. We know by Lemma 2.8 that these $q^{m+1} - (q^m - 1)$ columns are linearly independent.

Denote by $a_{(w,\tau),(v,t)}$ the coefficient of $\mathcal{T}_{(q,m)}$ which is at the line indexed by (w, τ) and the column indexed by (v, t). This coefficient is the component of $\delta_{v,t}$ on $e_{w,\tau}$, namely by Lemma 2.2:

$$a_{(w,\tau),(v,t)} = \begin{cases} 1 & \text{if} \quad (w,\tau) \in E_{-v,t} \\ 0 & \text{if} \quad (w,\tau) \notin E_{-v,t} \end{cases}.$$

But as the relation $(w, \tau) \in E_{-v,t}$ is equivalent to $(v, t) \in E_{w,\tau}$ we have

$$a_{(w,\tau),(v,t)} = a_{(v,t),(-w,\tau)}.$$

Then the elements of line $((w, \tau)$ are the elements of the column $(-w, \tau)$ By Proposition 2.7 the $q^{m+1} - (q^m - 1)$ lines indexed by (w, τ) where $w \neq 0$ and $t \neq 0$, or $w = 0$, are linearly independent.

Remark that the original system has a vector space of dimension $q^m - 1$ of solutions $(x_{v,t})_{(v,t)\in\mathbb{F}_q^m\times\mathbb{F}_q}$. Adding all equations of the system gives the following equality:

$$\sum_{v,t} x_{v,t} = q^{2m}.$$

Then if we suppose that the $q^m - 1$ conditions

$$\sum_{t\in\mathbb{F}_q} x_{v,t} = q^m,$$

where $v \neq 0$, are satisfied, the last condition

$$\sum_{t\in\mathbb{F}_q} x_{0,t} = q^m$$

is also satisfied. Now, using Theorem 3.4, we conclude that (S) is a Cramer linear System. □

Remark 3.6. From the definition it follows that

$$N_{v,t} = \operatorname{card}\left(\{w \in \mathbb{F}_{q^m} \mid (v,t) \in E_{w,f(-w)}\}\right).$$

So, it would be interesting to consider the arrangement of hyperplanes $\mathcal{A}(f)$, consisting of the q^m hyperplanes $E_{w,f(-w)}$ and to relate the geometric and combinatorial properties of $\mathcal{A}(f)$ to the properties of the distance between f and the affine functions. A very simple example is the following: if the arrangement $\mathcal{A}(f)$ is centered, then there is a (v,t) such that $N_{v,t} = q^m$ and consequently the function f is affine.

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